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On general solutions of equidistant vector fields on two-dimensional (pseudo-) Riemannian spaces

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Abstract. The paper is devoted to studying equidistant two-dimensional (pseudo-) Riemannian spaces. Embeddings of these spaces in three-dimensional Euclidean and Minkowski spaces as revolution or helical surfaces are given. The general solution of equidistant equations is found beyond these spaces under minimal requirements for the differentiability of the studied objects. These vector fields are associated with Killing vector fields on those spaces.

1. Introduction

The geometric properties of Riemann manifolds are frequently studied in connection with the existence of certain vector fields. For completeness, we start here with a fairly general class of vector fields and will specify it as a subclass in the sequel, which refers to the so-called equidistant manifolds, a class of (pseudo-) Riemann spaces.

Torse-forming and concircular vector fields were introduced by K. Yano [40, 41] in 1944. Special types of these manifolds were studied before by T. Levi-Civita, V.F. Kagan, P.A. Shirokov, H.W. Brinkmann, H.L. Vries, A.D. Fialkow, A.S. Solodovnikov etc. The spaces in which concircular vector fields exist are called equidistant manifolds. N.S. Sinyukov introduced this concept [36, 37], see [25, 38]. In several other papers, these manifolds are denoted as almost warped product manifolds [6]. Most equidistant spaces and similar spaces with a dimension greater than two were studied, e.g. [2–6, 8–23, 25, 26, 31–35, 42].

Some properties also carry over to dimension two, but due to the specificity of this dimension, we devote this paper to its study. Equidistant two-dimensional spaces are closely related to rotational and helical surfaces as well as to Killing vector fields. The differentiability of metrics plays an important role in studying these spaces. Among other things, precedents for the bifurcation of geodesics have been established for these spaces [27–30].

In the article, we deal with the question of the general solution of equidistant fields in a two-dimensional (pseudo-) Riemannian space under minimal conditions of differentiability of the metric. The achieved results and methods are carried out locally.

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2. Equidistant space

An n-dimensional (pseudo-) Riemannian space $\mathbb{V}_n$ with a metric tensor g is called *equidistant* (N.S. Sinyukov [36], see [38, p. 92], [25, p. 168]) if there exists in it a linear form $\varphi \neq 0$ satisfying the equations

$$\nabla\varphi = \rho\, g, \tag{1}$$

where ρ is a function and ∇ is a connection on $\mathbb{V}_n$. Since the form φ is necessarily a locally gradient (locally, there exists a function Φ for which $\varphi(X) = \nabla_X\Phi$), it defines a normal congruence in $\mathbb{V}_n$, which we will call *equidistant*. The origin of this the term will be explained next.

Spaces satisfying conditions (1) were encountered by H.W. Brinkmann [2] in the study of conformal mappings of Riemannian spaces. Later also, e.g. A. Fialkov, H.L. Vries, K. Yano etc. [4, 16, 25, 38, 40]. The special significance of equidistant spaces in the theory of geodesic and *HP*- mappings was discovered by N.S. Sinyukov [36–38], J. Mikeš [17–23, 25], I. Shandra [31–34], and others [3, 8–15, 26].

Among equidistant Riemannian spaces, there are some cosmological models. Modern cosmological models are based on the so-called Copernican principle (Bondi [1], 1960), stating that no position in the universe is distinguished in any way. A suitable concrete realization of this principle is the assumption of spatial homogeneity. At a sufficiently large scale, the universe's structure is essentially the same everywhere. Such a space is invariant under translations. The latter forms an isometry group, i.e., a group of transformations that leaves the metric invariant. A further interpretation of the Copernican principle is isotropy – the universe appears in every direction approximately the same. The associated isometry group is the group of rotations. Isotropy at every point ascertains homogeneity. The reverse is not true. Homogenous and isotropic models are the simplest cosmological models characterized by constant spatial curvature. They were introduced and studied by Friedmann, Lemaître, Robertson and Walker.

The vector field ξ, which is associated with the equidistant field φ: $g(\xi, X) = \varphi(X)$ for all tangent vector X, will also be called *equidistant*. This vector field is a special case of torse-forming and concircular vector fields introduced by K. Yano [40, 41]. *Torse-forming* vector fields ξ are characterized by the following equation $\nabla\xi = a\cdot\xi + \rho\cdot Id$, where a is a linear form, and ρ is a function on $\mathbb{V}_n$. In the case that a is a gradient, the vector field ξ is called *concircular*. We note that for any concircular vector field ξ, there exists a function f on $\mathbb{V}_n$ such that $f\cdot\xi$ is equidistant. Therefore, equidistant and concircular vector fields are often equated. In detail are the torse-forming, concircular and equidistant vector fields described in monographs [25, p. 168]. A lot of work is devoted to the above vector fields in various directions, for example, [3, 5, 11, 26, 31, 34, 35]. The majority of these works concern with spaces of dimension $n > 2$. We devote our work to the study of equidistant 2-dimensional spaces $\mathbb{V}_2$.

Next, we will supose that equidistant vector field ξ is not isotropic, i.e. lenght $\|\xi\| \neq 0$. For these equidistant space there exists local coordinate system $x = (x^1, x^2, \dots, x^n)$ for which metric form of $\mathbb{V}_n$ has the following form (see [38, p. 96], [25, p. 179]): $ds^2 = {dx^1}^2 + f(x^1)\cdot d\tilde{s}^2$, where f is a differentiable function of x^1 and $d\tilde{s}^2$ is a metric form $d\tilde{s}^2 = \tilde{g}_{\alpha\beta}(x^2, \dots, x^n)\, dx^\alpha dx^\beta$ of $(n-1)$-dimensional (pseudo-) Riemannian space $\overline{\mathbb{V}}_{n-1}$. We can consider the metrics ds^2 and $-ds^2$ to be equivalent.

The metric of 2-dimensional equidistant space $\mathbb{V}_2$ has the form

$$ds^2 = du^2 + f(u)\cdot dv^2. \tag{2}$$

For clarity, here and hereafter $u \equiv x^1$ and $v \equiv x^2$.

In local coordinate system equation (1) of equidistant field φ have the following shape

$$\varphi_{i,j} = \rho\cdot g_{ij}, \tag{3}$$

where φ_i and g_{ij} are components of φ and g and " , " denotes a covariant derivative. In detail, the left-hand side of equation (3) is $\partial_j\varphi_i - \varphi_\alpha\Gamma^\alpha_{ij} = \rho\, g_{ij}$, where $\Gamma^h_{ij} = \Gamma_{ij\alpha}g^{\alpha h}$, $\Gamma_{ijk} = \frac{1}{2}\,(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ik})$ are the Christoffel symbols, $\|g^{ij}\| = \|g_{ij}\|^{-1}$, and $\partial_i \equiv \partial/\partial x^i$.

By direct substitution, we make sure that $\varphi = c\cdot(\sqrt{|f(u)|}, 0, 0, \dots, 0)$, c – const, is a particular solution of the equations (1) and therefore φ is an equidistant vector field.

In the later Sections 5 and 6, we proved the following theorem.

Theorem 2.1. *In $\mathbb{V}_2 \in C^1$ of non-constant curvature the general solution of the equation* (1) *has at most one linearly independent solution φ. If $\mathbb{V}_2 \in C^k$, $k \geq 1$, then $\varphi_i(x) \in C^k$ holds and the general solution has the form* $\mathrm{const}\cdot\varphi$.

This theorem holds globally even if $\mathbb{V}_2$ has local regions with constant curvature. Furthermore, we can assume a priori that $\varphi_i(x)$ are differentiable functions, and $\varrho(x)$ is continuous. Higher differentiability of $\varphi_i(x)$ and $\varrho(x)$ follows from the results of I. Hinterleitner and J. Mikeš [12], see [25, p. 171].

3. Surfaces of revolution

It is known that the surfaces of revolution $\mathbb{S}_2$ on Euclidean space $\mathbb{E}_3$ can be described by the equation in Cartesian coordinates (x, y, z) in this way:

$$x = r(u)\cdot\cos v, \quad y = r(u)\cdot\sin v, \quad z = z(u), \tag{4}$$

where $r(u)$ (≥ 0) and $z(u)$ are functions of parameter $u \in \mathbb{R}$, and $v \in \mathbb{R}$. These functions and parameters fulfill additional natural conditions.

The first quadratic form of $\mathbb{S}_2$ (when $r(u)$ and $z(u)$ are differentiable functions) has the following form

$$ds^2 = (r'^2 + z'^2)\,du^2 + r^2\,dv^2. \tag{5}$$

After reparametrization of parameter u we can get the mentioned form of (2). In this case, $f(u) > 0$ and metric (2) defines a two-dimension equidistant Riemannian space $\mathbb{V}_2$. It is natural that the metrics in the form (2) have surfaces locally isometric to the surfaces of revolution. It is very easy to see that $r(u) = \sqrt{f(u)}$ and $z(u) = \int\sqrt{1 - r'(u)^2}\,du$. However, these formulas can only be implemented if $1 - r'^2 \geq 0$. This means that the positive function f must also satisfy the following inequality $f'^2 \geq 4f$. For the inequality $f'^2 < 4f$, there are no surfaces of revolution in Euclidean space.

On the other hand, for $f(u) > 0$ and without any additional requirements there are surfaces of revolution (4), where $r(u) = \sqrt{f(u)}$ and $z(u) = \int\sqrt{1 + r'(u)^2}\,du$, in Minkowski space with metric $ds^2 = dx^2 + dy^2 - dz^2$ has metric form (2).

In the case $f(u) < 0$, metric (2) defines the two-dimensional pseudo-Riemannian space $\mathbb{V}_2$. In this case, this metric is realized on the surface $\mathbb{S}_2$ embedded in the Minkowski space with metric $ds^2 = dx^2 + dy^2 - dz^2$. The equations of this surface have the form (5) at $r(u) = \sqrt{-f(u)}$ and $z(u) = \int\sqrt{1 + r'(u)^2}\,du$.

The above means that equidistant spaces $\mathbb{V}_2$ can be realized locally as surfaces of revolution. For example, even a helix is equidistant space because it is isometric to a catenoid, which is the surface of revolution.

In a Minkowski space with metric $ds^2 = \pm dx^2 \mp dy^2 + dz^2$, resp. $ds^2 = \pm dx^2 \mp dy^2 - dz^2$, the equations $x = r(u)\cdot\cosh v$, $y = r(u)\cdot\sinh v$, $z = z(u)$, generate also equidistant spaces $\mathbb{V}_2$ with metric $ds^2 = (\mp r'^2 + z'^2)\,du^2 \pm r^2\,dv^2$, resp. $ds^2 = (\mp r'^2 - z'^2)\,du^2 \pm r^2\,dv^2$. After reparametrization of parameter u we can also get the mentioned form (2) of the metric. Analogously, we can find above surfaces for metrics (2).

From the general theory of surfaces, there are surfaces $\mathbb{S}_2 \subset \mathbb{E}_3$ with metric (2) and the inequality $f'^2 \leq 4f$ does not apply everywhere. To construct surface $\mathbb{S}_2$ with metric (2), it is sufficient to find the second quadratic form $II = b_{ij}\,dx^i dx^j$, that holds the Gauss and Peterson-Codazzi equations

$$R_{1212} = b_{11}b_{22} - b_{12}^2, \quad \nabla_2 b_{11} = \nabla_1 b_{12}, \quad \nabla_2 b_{21} = \nabla_1 b_{22}, \tag{6}$$

where $R_{hijk} = g_{h\alpha}R^{\alpha}_{ijk}$ and $R^h_{ijk} = \partial_k\Gamma^h_{ij} + \Gamma^{\alpha}_{ij}\Gamma^h_{\alpha k} - \partial_j\Gamma^h_{ik} + \Gamma^{\alpha}_{ik}\Gamma^h_{\alpha j}$ are components of curvature tensor of $\mathbb{V}_2$.

We verify that for the metric (2), components $b_{11} = K\cdot f/b_{22}$, $b_{12} = 0$, $b_{22} = \pm 1/2\,\sqrt{cf - f'^2}$, are a particular solution of equations (6), where $K = -\dfrac{f''}{2f} + \left(\dfrac{f'}{2f}\right)^2$ is the Gaussian curvature and c is constant for which $cf - f'^2 > 0$.

We note that above are the general solution of the equations (6) at $b_{12} = 0$. Under this assumption from the Gaussian equation, $b_{11} = K\cdot f/b_{22}$ holds, and the Peterson-Codazzi equations have the form of the

Bernoulli equations for the function $b_{22}(u)$: $2\,\partial_1 b_{22} = b_{22}\, f'/f - K\, f'/b_{22}$. Since $g_{12} = b_{12} = 0$, the coordinate system (u, v) of the searched surfaces $\mathbb{S}_2$ is normal.

The embedding of $\mathbb{V}_2$ as surface $\mathbb{S}_2 : r = r(u, v)$ is realized by solving Gaussian and Weiengarten equations

$$\partial_i r = r_i, \quad \partial_j r_i = \Gamma^k_{ij}\, r_k + b_{ij}\, m, \quad \partial_i m = -b^k_i\, r_k, \tag{7}$$

where r_i $(i = 1, 2)$ are tangent vectors of $\mathbb{S}_2$ and m is unit normal vector of $\mathbb{S}_2$, $b^k_i = g^{kj} b_{ij}$, and $\|g^{ij}\| = \|g_{ij}\|^{-1}$.

If the conditions (7) are met at the point $x_0 = (u_0, v_0)$, then the equations (6) have a local solution in the neighborhood of x_0 for the initial conditions $r(x_0) = r^0$, $r_i(x_0) = r^0_i$, $m(x_0) = m^0$. Clearly $r^0_i \cdot r^0_j = g_{ij}(x_0)$, and $m^0 = r^0_1 \times r^0_2 / |r^0_1 \times r^0_2|$, where $\times$ is vector product.

Therefore, for any value of c that is bounded by the natural condition $c > f'^2/f$ (i.e. $c\, f - f'^2 > 0$), a surface $\mathbb{S}^{(c)}_2$ can be constructed in the above manner. All these surfaces are mutually isometric. Above surfaces $\mathbb{S}^{(c)}_2$ exists for parameter c that lies on some interval $I \in \mathbb{R}$ and defines a one-parameter isometric deformation of these surfaces that is not a mere motion of the surface in Euclidean space $\mathbb{E}_3$. If the interval I contains 4, then these surfaces are isometric to surfaces of revolution, since $\mathbb{S}^{(4)}_2$ can be realized as a surface (4) with $r = \sqrt{f(u)}$.

4. Equidistant and Killing vector fields

Isometric mappings of a (pseudo-) Riemannian space $\mathbb{V}_n = (M, g)$ onto itself are called *isometric transformations* on $\mathbb{V}_n$ or *motions* of $\mathbb{V}_n$. A vector field ξ on $\mathbb{V}_n$ is called an *infinitesimal isometry* or a *Killing vector field* if for each point $p \in M$ there is a neighborhood U of p such that the local one-parameter group f_t determined by the vector field preserves the metric, that is, the mapping f_t: $M \to M$ is an isometric transformation. In a special coordinate system (x^i) in which $\xi = \partial_1$, the isometric transformation is characterized by $\partial_1 g_{ij}(x) = 0$.

It is known [25, p. 228], vector field ξ is the Killing vector field on $\mathbb{V}_n$ if it satisfies the *Killing equations*: $L_\xi g_{ij} = \xi_{i,j} + \xi_{j,i} = 0$, where L_ξ is the Lie derivation with respect to ξ, $\xi_i = g_{i\alpha}\xi^\alpha$, and ξ^h are components of ξ.

Since the metric tensor g in the metric (2) does not depend on the variable v, the vector field $\xi = \partial_v$ is Killing on $\mathbb{V}_2$. Therefore, equidistant spaces $\mathbb{V}_2$, and also surfaces of revolution $\mathbb{S}_2$ admit Killing vector fields. It is easy to prove the following Theorem for $\mathbb{V}_2$.

Theorem 4.1. *Let ξ be a Killing vector field then $F\xi$ is an equidistant vector field.*
Let φ be a equidistant vector field then $F\varphi$ is an Killing vector field.

Here F is an operator for which its components $F^h_i = g^{h\alpha}\varepsilon_{\alpha i}$, and ε_{ij} are components of discriminant tensor on $\mathbb{V}_2$ for which $\varepsilon_{11} = \varepsilon_{22} = 0$, $\varepsilon_{12} = -\varepsilon_{21} = \sqrt{|g_{11}g_{22} - g_{12}{}^2|}$, see [25, p. 159]. For this operator is true $F^2 = \varepsilon \cdot Id$, $g_{i\alpha}F^\alpha_j + g_{j\alpha}F^\alpha_i = 0$, $F^h_{i,j} = 0$. For definite form ds^2 we lay $\varepsilon = -1$, and for indefinite $\varepsilon = +1$.

Proof. We will perform the proof at a fixed point $x_0 \in \mathbb{V}_2$ in which the metric g has the form $g_{ij} = \mathrm{diag}\,\{1, -\varepsilon\}$. Let ξ be the Killing vector field. Therefore, from Killing equation follows $\xi_{1,1} = \xi_{2,2} = 0$ and $\xi_{1,2} = -\xi_{2,1}$. We will put $\varphi = F\xi$, i.e. $\varphi^h = F^h_\alpha \xi^\alpha$, from which follows $\varphi_i = \varepsilon_{i\beta}g^{\alpha\beta}\xi_\alpha$. Then $\varphi_{i,j} = \varepsilon_{i\beta}g^{\alpha\beta}\xi_{\alpha,j}$ and by direct calculation we get $\varphi_{1,2} = \varphi_{2,1} = 0$, $\varphi_{1,1} = \varepsilon_{12} \cdot (-\varepsilon) \cdot \xi_{2,1}$ and $\varphi_{2,2} = \varepsilon_{21} \cdot \varepsilon \cdot \xi_{1,2}$. Hence the formula (3) is true when $\varrho = \varepsilon \cdot \varepsilon_{12} \cdot \xi_{1,2}$.

Finaly, let φ be the equidistant vector field. We will put $\xi = F\varphi$, i.e. $\xi^h = F^h_\alpha \varphi^\alpha$, from which follows $\xi_i = \varepsilon_{i\beta}g^{\alpha\beta}\varphi_\alpha$. Based on formula (3) it is true $\xi_{i,j} = \varrho \cdot \varepsilon_{ij}$. Because discriminant tensor ε_{ij} is skew the Killing equation is fulfilled. □

Evidently, there exist two **Criteria of isomery of $\mathbb{V}_2$ with surfaces of "revolution"** with metric (2):

1. *on $\mathbb{V}_2$ there exists the equidistant vector field;*
2. *on $\mathbb{V}_2$ there exists the Killing vector field.*

Let us remind that, in addition to surfaces of revolution, metric (2) have also helical surfaces. Interesting properties of these surfaces were studied, for example, in [39]. These surfaces are defined in $\mathbb{E}_3$ by equations $x = u \cdot \cos v$, $\;y = u \cdot \sin v$, $\;z = q(u) + mv$, where $q(u)$ is a function and m is non zero constant. The metric form of these surfaces does not depend on coordinate v, so there is a Killing vector on them.

5. General solution of equation of equidistant vector field in $\mathbb{V}_2 \in C^k$, $k = 3, 4$

We will search general solution of equation of equidistant vector field in $\mathbb{V}_2 \in C^k$, $k = 3,4$. It is well-known that in $\mathbb{V}_2$ the Gauss formula holds: $R_{hijk} = K \cdot (g_{hj}g_{ik} - g_{hk}g_{ij})$, where K is the Gauss curvature. We will first assume that $\mathbb{V}_2 \in C^4$, i.e. $g_{ij}(x)$ are differentiable functions of class C^4 ($g_{ij}(x) \in C^4$). From this it follows that $\Gamma^h_{ij}(x) \in C^3$, and the Gauss curvature K is two differentiable.

Let φ be an equidistant field for which (1), and, locally, (3): $\varphi_{i,j} = \varrho\, g_{ij}$ holds. Now, we calculate the integrability conditions: $\varphi_\alpha R^\alpha_{ijk} = \varphi_{i,jk} - \varphi_{i,kj}$. From this and (3) it follows $K \cdot (\varphi_j g_{ik} - \varphi_k g_{ij}) = \varrho_{,k} g_{ij} - \varrho_{,j} g_{ik}$. After contraction with g^{ij}, we find out that $\varrho_{,k} = -K \cdot \varphi_k$. It yields that integrability conditions of (3) are identically fulfilled. From the previous, we have the following system of equations:

$$\varphi_{i,j} = \varrho\, g_{ij}, \quad \varrho_{,k} = -K\, \varphi_k. \tag{8}$$

This system is a complete system of linear equations respective unknown functions $\varphi_i(x)$ and $\varrho(x)$. It means that it has only one solution with initial conditions $\varphi_i(x_0) = \varphi_i^0$ and $\varrho(x_0) = \varrho^0$. Thus, the general solution (8) depends only on 3 real parameters. The integrability conditions of the second equation of (8) has the form $\varrho_{,kl} = \varrho_{,lk}$, i.e. after using (8), we find

$$K_{,l}\varphi_k - K_{,k}\varphi_l = 0. \tag{9}$$

If $K_{,l} \equiv 0$, i.e. K is constant, then conditions (9) are identically fulfilled. In this case, equations (8) are completely integrable, therefore in space of constant curvature K these equations have solution for arbitrary initial conditions.

We differentiate equation (9) and on the base of equations (3) we have: $\varrho(K_l g_{km} - K_k g_{lm}) + K_{l,m}\varphi_k - K_{k,m}\varphi_l = 0$. If $K_{,l} \neq 0$, from this, we can see that ϱ can be linearly expressed via components φ_1, φ_2. From (9), the one component φ_i is expressed via the second one. The initial condition φ_i, ϱ linearly depends on one real parameter. In this way, we have confirmed with standard methods that the Theorem 2.1 for $k = 4$ is true.

Moreover, with a simple modification of the proof, Theorem 2.1 also holds for $\mathbb{V}_2 \in C^3$. Under these conditions, the integrability conditions (9) apply. If K is non constant, for example $K_{,1} \neq 0$, then from (9) implies $\varphi_1 = c \cdot \varphi_2$, where $c = K_{,2}/K_{,1}$. Since $\varphi_i(x)$ satisfies the equations of (3), the function c must be differentiable. Furthermore, substituting into these equations shows that the function $\varrho(x)$ is linearly expressed in terms of $\varphi_i(x)$. This means that $\varrho(x)$ will also be expressed $\varrho(x) = \tilde{c} \cdot \varphi_2$, where $\tilde{c}$ is a function on $\mathbb{V}_2$. Therefore, the initial conditions at point x_0 depend on one real parameter and therefore the general solution $\varphi_i(x)$ of equations (1) depends on only one parameter.

6. General solution of equation of equidistant field in $\mathbb{V}_2$ with minimal differentiability conditions

As we have already mentioned, in the space $\mathbb{V}_2$ in which equidistant vector fields exist, there is a coordinate system (3): $ds^2 = du^2 + f(u)\, dv^2$, $f(u) \neq 0$. If $f \in C^3$, then based on Theorem 2.1 the general solution is the previously announced solution $\varphi_i = \mathrm{const} \cdot (\sqrt{|f(u)|}, 0)$ in the space of non-constant curvature.

The above calculations were performed under the assumption of sufficient differentiability of the studied functions. We suppose that $f(u) \in C^1$. In this case, there may be no Gaussian curvature and the methodology from the previous Section 5 is inapplicable.

Equation of equdistant field (3): $\varphi_{i,j} = \varrho g_{ij}$ we can write in expanded form $\partial_j \varphi_i - \varphi_\alpha \Gamma^\alpha_{ij} = \rho\, g_{ij}$. The minimum requirements for differentiability in equation (3) are as follows $\varphi_i \in C^1$ and $\varrho \in C^0$.

From equations (3) for $i = 2$ and $j = 1$ we obtain $\partial_1 \varphi_2 = 1/2\, \varphi_2 \cdot f'/f$. After integration, $\varphi_2 = A(v) \cdot \sqrt{|f(u)|}$, where $A(v)$ is a function of parameter v.

From equations (3) for $i = j = 1$ we have $\partial_1 \varphi_1 = \varrho$ and for $i = j = 2$ we have $\partial_2 \varphi_2 = -1/2\, \varphi_1 \cdot f' + \varrho \cdot f$. From this we have $\partial_1 \varphi_1 = 1/2\, \varphi_1\, f'/f - \partial_2 \varphi_2 / f$. Substituting φ_2 here and then integrating respective unknown function φ_1, we obtain $\varphi_1 = -A'(v)\, \sqrt{|f|} \int (1/f)\, du + B(v)\, \sqrt{|f|}$, where $B(v)$ is a function of parameter v.

Finaly, from equations (3) for $i = 1$ and $j = 2$ we get $\partial_2 \varphi_1 = 1/2\, \varphi_2 \cdot f'/f$. We will substitute the functions φ_1 and φ_2, and after division $\sqrt{|f(u)|}$ we obtain $-A'' \cdot \int (1/f)\, du + B' = f'/(2f) \cdot A$.

If $A(v) \neq 0$ for some v, then the above formula for v implies $f' = \alpha f \int (1/f)\, du + \beta\, f$, where α, β are constant. It is obvious that the solution of this differential-integral equation is analytical. This case is fully processed in the previous Section, and expressed in Theorem 2.1. Apparently, it is enough to assume that $A(v) \equiv 0$. Evidently, $B' = 0$, i.e. B is constant. It follows from the above that the general solution in this case is $\varphi_i = \text{const}\,(\sqrt{|f|}, 0)$. This means that the Theorem 2.1 also holds for $\mathbb{V}_2 \in C^2$.

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