Filomat 37:25 (2023), 8603–8609
https://doi.org/10.2298/FIL2325603S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Eigenproblem of tensors - a geometrical viewpoint

Jelena Stojanov[a], Vladimir Balan[b]

[a]*University of Novi Sad, Technical Faculty "Mihajlo Pupin", Serbia*
[b]*University Politehnica of Bucharest, Faculty of Applied Sciences, Department of Mathematics-Informatics, Romania*

Abstract. The classical eigenproblem focuses on eigenvalues and eigenvectors of linear operators acting on a vector space. The matrix representation of the problem has been extended towards multidimensional arrays, with various applications. Another extension addresses invariant subspaces of multilinear operators in Banach spaces. The generalization of the eigenproblem for tensors is still a challenging issue. We investigate eigenproblems of supersymmetric tensors on Riemannian manifolds, emerging from the initial proper definitions, with the proposed extensions.

1. Introduction

The classical eigenproblem $Ax = \lambda x$ concerns a real matrix $A_{n\times n}$ which stands for a linear operator, its eigenvalues $\lambda \in \mathbb{R}(\mathbb{C})$ and the corresponding eigenvectors $x \in \mathbb{R}^n(\mathbb{C}^n)$. The problem, extended toward higher order tensors in various manners, provides notable and widespread applications. The tensors are considered as:

- multidimensional data, i.e., hypermatrices - with focus on the data numerical analysis, decompositions and approximations [4, 9, 12, 19, 21];
- multilinear operators in Eucledean vector spaces, whose associate multivariate homogeneous forms are of great interest in image processing [5, 17, 24], Physics [3, 18] and in the stability study of nonlinear autonomous systems [7, 15, 23]. The positive definiteness of multivariate homogeneous forms is a notable feature, useful in applications [8].

The numerous effective algorithms for solving extended eigenproblems and analyzing their solution, especially for locating its eigenvalues, were successfully implemented [10, 11, 13, 22], underlining the richness and importance of this topic.

The basic seminal works on the subject in higher order tensor spectral analysis (HOTSA) were done in 2005 by L. Qi [16] and by L.-H. Lim [14], independently. Though developed by different (algebraic and variational) approaches, they assume that:

· the tensor is regarded as a multidimensional array which represents data organized as a multilinear operator on the flat vector space;
· the tensor is real and supersymmetric.

2020 *Mathematics Subject Classification*. 53A45; 15A18, 58C40, 53B50.
Keywords. Tensor eigenvalue problem; Best rank-one approximation; Z-eigenvalue; Homogeneity.
Received: 15 December 2022; Accepted: 01 March 2023
Communicated by Zoran Rakić and Mića Stanković
Email addresses: jelena.stojanov@uns.ac.rs (Jelena Stojanov), vladimir.balan@upb.ro (Vladimir Balan)

Both approaches produce compatible definitions of the extension, and emphasize two (the mostly used) Z-eigenproblem and H-eigenproblem. Both these extensions have important properties within the particular classical eigenproblem, providing homogeneity properties and geometrically meaningful eigenpairs (λ, x).

The spectral analysis for third- and fourth-order tensors associated to geometrical (non-Euclidean) structures was developed in [1, 2, 20]. The extended eigenproblem adjusted to the non-Euclidean space and the geometric relevance of the obtained spectral data, were provided.

The aim of the present work is to adjust the definition of the Z-eigenproblem in an appropriate way, to keep its full geometrical meaning. The proposed adjustment is based on an idea from [6], where invariant subspaces for nonlinear operators in infinite dimensional Banach spaces, are determined. A particular homeomorphism has been used instead of the identical map in considering the spectral operator $(A - \lambda I)$, for overcoming the disparity in homogeneity.

2. Eigenproblem for tensors - the algebraic viewpoint

The subjects presented in this section are based on the work initially stated in [16] and comprehensively exposed in [18]. An m-th order totally symmetric covariant tensor A with components $A = (A_{ii_2\cdots i_m})$ determines a multilinear map on the flat n-dimensional real vector space $\mathbb{R}^n$,

$$A : \mathbb{R}^n \otimes \ldots \otimes \mathbb{R}^n \to \mathbb{R}, \quad (x_1, x_2, \ldots, x_m) \mapsto Ax^m := A_{ii_2\cdots i_m} x_1^i x_2^{i_2} \cdots x_m^{i_m}.$$

Since the Euclidean metric $\langle x, y\rangle_\delta = \delta_{ij} y^i y^j$ is imposed, the components of vectors and of their corresponding covectors are equated, $x^i = \delta_{ij} x_j = x_i$.

An incomplete action of the tensor A on a single vector x produces a one-dimensional tensor comparable with the scaled vector $Ax^{m-1} := A_{ii_2\cdots i_m} x^{i_2} \cdots x^{i_m} \xleftrightarrow{\lambda} x^i$. This comparison disregards the nature of the objects (witch is acceptable within flat spaces) but has no homogeneity feature. Homogeneity is achieved in the comparison with the componentwise appropriate power of the vector, $A_{ii_2\cdots i_m} x^{i_2} \cdots x^{i_m} \xleftrightarrow{\lambda} (x^i)^{m-1}$, where the vectorial character of the right-hand side is lost.

The exact definitions of the eigenproblem consider a scalar-vector pair (λ, x), where $\lambda \in \mathbb{C}$ and $x = (x^1, \ldots, x^n) \in \mathbb{C}^n$.

Definition 2.1. If λ and x are real solutions of the following Z-eigenproblem

$$\begin{cases} A_{ii_2\cdots i_m} x^{i_2} \cdots x^{i_m} = \lambda\, x^i \\ x^T x = 1, \end{cases} \tag{1}$$

then λ is a Z-eigenvalue and x is a corresponding Z-eigenvector of the tensor A; briefly, (λ, x) is a Z-eigenpair. If the solution vector of the problem is not a real vector, but $x \in \mathbb{C}^n$, then the objects are labeled as E-eigenvalue and E-eigenvector, respectively.

Definition 2.2. If λ and x are real solutions of the following H-eigenproblem

$$A_{ii_2\cdots i_m} x^{i_2} \cdots x^{i_m} = \lambda\, (x^i)^{m-1},$$

then λ is an H-eigenvalue and x is corresponding H-eigenvector of the tensor A, i.e., (λ, x) is an H-eigenpair. Optionally, if x is not a real solution, then (λ, x) is an N-eigenpair.

Though the H-eigenproblem is an algebraic genuine extension with notable algebraic flavor, we shall further mainly consider the Z-eigenproblem, which is considerably more relevant to Geometry.

Definition 2.3. A univariate polynomial of λ is the E-characteristic polynomial of the tensor A if it is the resultant of the following system

$$\text{for } m\text{-even:} \quad Ax^{m-1} - \lambda \left(x^T x\right)^{\frac{m-2}{2}} x = 0; \qquad \text{for } m\text{-odd:} \quad Ax^{m-1} - \lambda t^{m-2} x = 0,\ x^T x = t^2.$$

The tensor A is regular if the system $\{Ax^{m-1} = 0, \;\; x^T x = 0\}$ admits only the trivial complex solution.

Definition 2.4. The best rank one approximation of the tensor A is a rank one tensor $B = \alpha u \otimes u \otimes \ldots \otimes u = \alpha \otimes^m u$, such that

$$(\alpha, u) \in \text{ argmin } \{\|A - \alpha \otimes^m u\|_F \, : \, \alpha \in \mathbb{R}, u \in \mathbb{R}^n, \|u\|_2 = 1\},$$

where $\|\cdot\|_F$ denotes the Frobenius tensor norm.

The following Theorem is relevant for the main properties of Z-eigenvalues [18].

Theorem 2.1. *Let A be a real totally symmetric tensor of order m in an n-dimensional flat space. Then:*

1. *When A is regular, a complex number is an E-eigenvalue of A if and only if it is a root of its E-characteristic polynomial.*
2. *Z-eigenvalues always exist. An even order symmetric tensor is positive definite if and only if all of its Z-eigenvalues are positive.*
3. *If A and B are orthogonally similar, then they have the same E-eigenvalues and Z-eigenvalues.*
4. *If λ is the Z-eigenvalue of A with the largest absolute value and x is a Z-eigenvector associated with it, then λx^m is the best rank-one approximation of A,*

$$\|A - \lambda x^m\|_F = \sqrt{\|A\|_F^2 - \lambda^2} = \min\{\|A - \alpha u^m\|_F \, : \, \alpha \in \mathbb{R}, u \in \mathbb{R}^n, \|u\|_2 = 1\}.$$

The tensor $B = (b_{i_1 i_2 \cdots i_m})$ is orthogonally similar to A if there exists an orthogonal matrix $P = (p_{ij})_{n\times n}$ such that $b_{i_1 i_2 \cdots i_m} = \sum\limits_{j_1, j_2, \cdots, j_m = 1}^{n} p_{i_1 j_1} p_{i_2 j_2} \cdots p_{i_m j_m} a_{j_1 j_2 \cdots j_m}$. The matrix P is associated to coordinate transformations within flat spaces, hence the third statement of Theorem 2.1 implies that E-eigenvalues and Z-eigenvalues are **invariants** of the tensor.

2.1. *Eigenproblem for tensors - the variational viewpoint*

The eigenvalues and eigenvalues of the higher order tensor A are regarded in [14] as critical values and critical points of the tensorial Rayleigh quotient, and equivalently of the polynomial form constrained to unit vectors,

$$(\lambda, x) - \text{eigenpair} \longleftrightarrow \max \frac{|Ax^m|}{\|x\|_{l^p}^m} \longleftrightarrow \min\left[Ax^m - \lambda(\|x\|_{l^p} - 1)\right].$$

One readily notes that the l^2-eigenvalues coincide with the Z-eigenvalues, while l^m-eigenvalues are the H-eigenvalues.

2.2. *Invariant subspaces for multilinear operators*

The invariant subspaces for a vectorial multilinear operator in an infinite dimensional Banach space $E \quad T : E \times E \times \ldots \times E \to E$, were studied in [6]. Namely, the following m-homogeneous polynomial map $p : E \to E$ is associated, $\quad p(x) = T(x, x, \ldots, x)$. Then one may define the invariance/strong invariance, as follows:

Definition 2.5. Let M be a closed linear subspace of E. If $p(M) \subset M$, then the subspace M is invariant for p and it is also invariant for T. If $T(M, \ldots, M) \subset M$, then M is strongly invariant for T.

The problem of finding the invariant subspaces for p is studied by using the eigenvalue problem for p, i.e., the existence of nonzero vectors $v \in E$ such that $p(v) = \lambda v$. The motivation there is the generalization of invariant subspace for nonlinear operators; however, this is not of interest in the present work. We shall therefore present only the relevant achievements, especially for finite n-dimensional Banach spaces. In such case ($dimE = n$), the multilinear operator can be regarded as a collection of n tensors of order $m + 1$, where the very first index indicates the component of the resulting vector (and should be neglected in comparison with the tensor A from above). Similary, p will appear as an array of n polynomials.

To achieve the necessary homogeneity for invariant subspaces, the adjusted eigenproblem is proposed: find $0 \neq v \in E$ and $\lambda \in \mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, such that

$$p(v) = \lambda J_m(v), \quad J_m(v) = \|v\|^{m-1} v. \tag{2}$$

By choosing the homeomorphism J_m, the mapping $\lambda J_m - p$ becomes homogeneous and appropriate for our further developments for Banach spaces. We further give an account for the main results.

Lemma 2.2. *Let p be associated to the given bounded operator T, where $dim E \geq 2$. If for some $\lambda \in \mathbb{K}$, $0 \neq v \in E$ we have $p(v) = \lambda J_m(v)$, then $lin\{v\}$ is invariant for p.*

Theorem 2.3. *Let E be real and p be compact. Then for $m \in \{2k-1 : k \in \mathbb{N}\}$ and $\lambda \neq 0$, if $p(v) = \lambda J_m(v)$ has nontrivial solutions for v, then λ is an eigenvalue of p.*

3. Eigenproblem for tensors - the geometrical viewpoint

In [1] were studied the Z and H-spectral data for tensors attached to the relativistic metrics (Berwald-Moor, Chernov and Bogoslovsky), in locally Minkowski Finsler geometric structures of m-th root type. The geometric relevance of the spectral data of the structural tensors was pointed out, and relations between spectra, polyangles and Riesz-type associated 1-forms of the corresponding geometric models, as well as the best rank-one approximation, were derived.

In [20], the Z- and H-spectra were studied in low dimensions for Finsler structures of Randers type, for the associated Cartan tensor, and it was emphasized that the Z-eigendata produces global information, while the H-eigendata exhibits a strong local character.

A selection of illustrative applications and formal coordinate-free Finslerian extensions of the Z and $H-$spectral problems, were also covered in the survey [2].

4. The geometrical adjustment of the Z-eigenproblem

The Z-eigendata are close to the geometry, but lack homogeneity and independence on the local representation. Hence, the following adjustment of the Z-eigenproblem attempts to compensate these drawbacks.

For better comparison with previous considerations of the eigenproblem, we shall consider for now only totally symmetric covariant tensors.

Throughout this section, the following is implied:

- (M, g) is an n-dimensional differentiable manifold with regular positive definite metric structure g (a Riemannian manifold);
- the metric $g \in \mathcal{T}_2^0(M)$ is a covariant tensor field defining a collection of symmetric bilinear forms $g_x : T_xM \times T_xM \to \mathbb{R}$, $\langle z, v\rangle_g = g_x(z, v)$ determined by the matrix $\left(g_{ij}(x)\right)$ of inverse $g^{-1} \in \mathcal{T}_0^2$ with components $\left(g^{ij}(x)\right)$;
- we denote by $z, v \in T_xM$ two tangent vectors on M at some point $x \in M$, and the norm of a vector z is $\|z\| = \sqrt{\langle z, z\rangle_g}$;
- $A \in \mathcal{T}_m^0(M)$ is a totally symmetric tensor field on M;
- the metric tensor field will be used for raising and lowering indexes, i.e., to sharpen and flatten indices of tensors, particularly $A^\sharp = g^{-1}A$ and $z^\flat = gz$;
- locally, A has the components $A_x = (A_{ii_2\cdots i_m}(x))$ and defines an m-homogeneous polynomial map $A_x : T_xM \to \mathbb{R}$, $A_x z^m = A_x(z, z, \ldots, z) = A_{ii_2\cdots i_m}(x) z^i z^{i_2} \cdots z^{i_m}$ and an endomorphism $A_x^\sharp : T_xM \to T_xM$, $A_x^\sharp(z) = A_x z^{m-1} = A_{ii_2\cdots i_m}(x) z^{i_2} \cdot z^{i_m} g^{ij}$; if the tangent space T_xM is obvious or irrelevant, the subscript x will be omitted;
- globally, the tensor field A and a vector field (or a contravariant tensor field of order one in the case of a non-smooth section) $z \in \mathcal{T}_0^1$ provide a scalar field $Az^m \in \mathcal{T}_0^0(M)$ and the first order tensor fields $A^\sharp z^{m-1} \in \mathcal{T}_0^1(M)$ and $Az^{m-1} \in \mathcal{T}_1^0(M)$.

Definition 4.1. The geometrical eigenproblem (G-eigenproblem) for A consists of solving the system

$$A^{\sharp} z^{m-1} = \lambda \|z\|^{m-2} z, \tag{3}$$

which locally writes as the vectorial / covectorial equation

$$g^{ij} A_{ii_2 \cdots i_m} z^{i_2} \cdots z^{i_m} = \lambda \|z\|^{m-2} z^j, \quad j = 1, \dots, n \qquad \Leftrightarrow \qquad A_{ii_2 \cdots i_m} z^{i_2} \cdots z^{i_m} = \lambda \|z\|^{m-2} z^j g_{ij}, \quad i = 1, \dots, n. \tag{4}$$

The real solutions λ and z of the G-eigenproblem at the point $x \in M$ are respectively called G-eigenvalue and associated G-eigenvector, while (λ, z) is called G-eigenpair at x.

The announced adjustment of this definition was inspired by both the work presented in Subsection 2.2 and the homogeneity of the mappings Az^{m-1} and $\|z\|^{m-2} z$.

First of all, it is important to check the invariant character of the definition.

Theorem 4.1. *The geometrical eigenproblem is well defined.*

Proof. *The left hand side of (3) is a contravariant tensor of order one, obtained by the multiple contraction of an outer product of three tensor fields; the same occurs with the right hand side, which is a product of vector field and two scalar fields (λ and $\|z\|^{m-2}$). Both sides changes by the same rule under a change of local coordinates.* □

As a direct consequence, we have the following analogue of Theorem 2.1, 3:

Corollary 4.2. *The eigendata of the tensor A are invariants.*

Theorem 4.3. *If the tensor order is even ($m = 2k$) then the geometrical eigenproblem is homogeneous. Otherwise (for $m = 2k + 1$), the geometrical eigenproblem is positively homogeneous.*

Proof. *Substituting $\alpha z = (\alpha z^1, \alpha z^2, \dots, \alpha z^n)$ into (4), it is obvious that the scalar factor in the left hand side is α^{m-1}, while in the right hand side this is $|\alpha^{m-2}| \cdot \alpha$. The analysis of this relation for even / odd cases and positive / negative α, confirms the statement.* □

Having in mind the homogeneity, it is important to emphasize the following result.

Corollary 4.4. *If (λ, z) is a G-eigenpair of the even order G-eigenproblem, then for any $\alpha \in \mathbb{R}$, $(\lambda, \alpha z)$ is a G-eigenpair, too. If (λ, z) is a G-eigenpair of the odd order G-eigenproblem, then for any positive $\alpha \in \mathbb{R}$, $(\lambda, \alpha z)$ is a G-eigenpair, too.*

We note a technical relief in the solving procedure: instead of answering the problem (4), the following system can be solved

$$\begin{cases} g^{ij} A_{ii_2 \cdots i_m} z^{i_2} \cdots z^{i_m} = \lambda \|z\|^{m-2} z^j, \quad j = 1, \dots, n \\ \|z\| = 1 \end{cases} \tag{5}$$

and then the previous corollary can be applied on the solutions.

The collections of G-eigenvectors corresponding to a G-eigevalue can be also characterized as follows.

Definition 4.2. Let (λ, z) be a G-eigenpair for the tensor A at the point $x \in M$. If m is even, then the set $span\{z\} = \{\alpha z \,:\, \alpha \in \mathbb{R}\} \subset T_x M$ is the invariant subspace generated by z. If m is odd, then the set $ray\{z\} = \{\alpha z \,:\, \alpha \in \mathbb{R}, \alpha \geq 0\} \subset T_x M$ is the invariant ray generated by z.

The G-eigenproblem is a proper extension of the original Z-eigenproblem, due to the system (5) and to the fact that in a flat space, raising and lowering indices is redundant ($g_{ij} = \delta_{ij}$), so $z_i = g_{ia} z^a = z^i$ and $A^{\sharp}_x = A_x$. Moreover, the following theorem is straightforward.

Theorem 4.5. *A pair (λ, z) is an Z-eigenpair of the tensor A considered in a flat real vector space if and only if it is a unit length solution of the G-eigenproblem (5) considered locally in a tangent space $T_x M$ at some point $x \in M$, with the local coordinates chosen such that $g_{ij} = \delta_{ij}$.*

This theorem actually confirms the existence of the G-eigendata for tensors; however, it should be noted that within a tangent vector space T_xM, the relation $Az^{m-1} = \lambda \|z\|_g^{m-2} z$ is an optimality condition for minimizing and maximizing continuous function $f(z) = Az^m$ on the compact feasible set of unit length vectors $\{z : \|z\|_g = 1\}$. The global extrema always exist, and hence (5) always admits real solutions.

Theorem 4.6. *(analogous to the statement 4 of Theorem 2.1)*
If (λ, z) is a G-eigenpair of the tensor A and z is of unit length, then

$$Az^m = \lambda \qquad \textit{and} \qquad \|A - \lambda(z^\flat)^m\|_F^2 = \|A\|_F^2 - \lambda^2 \geq 0,$$

where the Frobenius norm of tensor $T_{i_1 \cdots i_m}$ at point is defined by the relation $\|T\|_F^2 = g^{i_1 j_1} \cdots g^{i_m j_m} T_{i_1 \cdots i_m} T_{j_1 \cdots j_m}$. Moreover, if λ is the G-eigenvalue with the largest absolute value and z is a corresponding unit length G-eigenvector, then $\lambda(z^\flat)^m$ is the best rank-one approximation of A,

$$\|A - \lambda(z^\flat)^m\|_F = \min\left\{\|A - \alpha(u^\flat)^m\|_F \,|\, \alpha \in \mathbb{R}, u \in T_xM, \|u\|_g = 1\right\}.$$

Proof. *The proof relies on locally expressing the geometric objects of the Riemannian G-eigenproblem (5), in (Gaussian) Riemannian normal coordinates where the metric is flattened. This locally reduces the G-eigenproblem to the Z-Euclidean framework.* □

We conclude by pointing out several features of extending the Z-eigenproblem to the G-eigenproblem:
· invariance and homogeneity;
· both sides of the equation have the same tensor types;
· invariance of the sets spanned by Z-eigenvectors;
· versatile for all tensor types, mod the canonical isomorphism;
· it emphasizes the analogy between the metric tensor and unit matrix/identity mapping in flat space: $g_{ij} z^j = 1 \cdot \|z\|^0 g_{ia} z^a$; all such vectors are G-eigenvectors corresponding to the G-eigenvalue 1.

References

[1] V. Balan, Spectra of symmetric tensors and m-root Finsler models, Linear Algebra and Its Applications 436 (2012) 152-162.
[2] V. Balan, On spectral data and tensor decompositions in Finslerian framework, AUT Journal of Mathematics and Computing 2(2) (2021) 153-163.
[3] Y. Chen, A. Jakli, L. Qi, The C-eigenvalue of third order tensors and its application in crystals, Journal of Industrial and Management Optimization, 2023, 19(1): 265-281. doi: 10.3934/jimo.2021183
[4] A. Cichocki, N. Lee, I. V. Oseledets, A. H. Phan, Q. Zhao and D. P. Mandic, Tensor networks for dimensionality reduction and large-scale optimization: part 1 low-rank tensor decompositions, Found. Trends Mach. Learn. 9 (4-5) (2016) 249-429.
[5] M. D. Cirillo, R. Mirdell, F. Sjoberg and T. Pham, Tensor decomposition for colour image segmentation of burn wounds, Sci. Rep. 9 (2019) 3291.
[6] J. Emenyu, An invariant subspace problem for multilinear operators on Banach spaces and algebras, Journal of Inequalities and Applications (2016) 179.
[7] L. Gao, Z. Cao and G. Wang, Almost sure stability of discrete-time nonlinear Markovian jump delayed systems with impulsive signals, Nonlinear Analysis: Hybrid Systems 34 (2019) 248-263.
[8] J. He, Y. Liu, J. Tian and Z. Zhang, New sufficient condition for the positive definiteness of fourth order tensors, Mathematics, 303 (2018) 1-10.
[9] D. Hong, T. G. Kolda, and J. A. Duersch, Generalized canonical polyadic tensor decomposition, SIAM Review: 62(1) (2020) 133-163, doi: 10.1137/18M1203626
[10] Z. Huang, L. Wang, Z. Xu and J. Cui, Some new inclusion sets for eigenvalues of tensors with application, Filomat 32(11) (2018) 3899-3916.
[11] Z. Huang, L.Wang, Z. Xu and J. Cui, A modified S-type eigenvalue localization set of tensors applications, Filomat 32(18) (2018) 6395-6416.
[12] B. Jiang, S. Ma and S. Zhang, Tensor principal component analysis via convex optimization, Mathematical Programming 150, 2 (2015) 423-457.
[13] T. Kolda and J. Mayo, Shifted power method for computing tensor eigenpairs, SIAM Journal on Matrix Analysis and Applications 34 (2011) 1095-1124.
[14] L.H. Lim, Singular values and eigenvalues of tensors: a variational approach, CAMSAP'05: Proceedings of the IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing, 2005.
[15] Q. Ni, L. Qi and F. Wang, An eigenvalue method for testing the positive definiteness of a multivariate form, IEEE Transactions on Automatic Control 53 (2008) 1096-1107.

[16] L. Qi, Eigenvalues of a real supersymmetric tensor, Journal of Symbolic Computation 40 (2005) 1302-1324.
[17] L. Qi, G. Yu and E. Wu, Higher order positive semi-definite diffusion tensor imaging, SIAM Journal on Imaging Sciences 3 (2010) 416-433.
[18] L. Qi, H. Chen and Y. Chen, Tensor Eigenvalues and Their Applications, Advances in Mechanics and Mathematics 39, Springer, Singapore, 2018, $https://doi.org/10.1007/978-981-10-8058-6_1$
[19] S. Sherman and T. G. Kolda, Estimating higher-order moments using symmetric tensor decomposition, SIAM Journal on Matrix Analysis and Applications 41(3) (2020) 1369-1387, doi: 10.1137/19m1299633
[20] J. Stojanov, V. Balan, Spectral Cartan properties in Randers-type spaces, BSG Proceedings 25 (2018) 96-110.
[21] Y. Wang, L. Qi, On the successive supersymmetric rank-1 decomposition of higher-order supersymmetric tensors, Numerical Linear Algebra with Applications 14 (2007) 503-519.
[22] G. Wang, L. Sun, Y. Wang, Sharp Y-eigenvalue inclusion set-based method for testing the positive definiteness of multivariate homogeneous forms, Filomat 34:9 (2020) 3131-3139.
[23] L. Xiong, Z. Jiang, J. Liu, Q. Qin, New Z-eigenvalue localization set for tensor and its application in entanglement of Multipartite Quantum States, Mathematics 2022, 10, 2624. https://doi.org/10.3390/math10152624
[24] S. Zhang, X. Guo, X. Xu, L. Li andC-C Chang, A video watermark algorithm based on tensor decomposition, Math. Biosci. Eng. 16(5) (2019) 3435-3449.