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The gravitational energy-momentum pseudo-tensor in higher-order theories of gravity

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Abstract. The problem of non-localizability and the non-uniqueness of gravitational energy in general relativity has been considered by many authors. Several gravitational pseudo-tensor prescriptions have been proposed by physicists, such as Einstein, Tolman, Landau, Lifshitz, Papapetrou, Møller, and Weinberg. We examine here the energy–momentum complex in higher-order theories of gravity applying the Noether theorem for the invariance of gravitational action under rigid translations. This, in general, is not a tensor quantity because it is not a covariant object but only an affine tensor, that is, a pseudo-tensor. Therefore we propose a possible prescription of gravitational energy and momentum density for $\Box^k$ gravity governed by the gravitational Lagrangian $L_g = (\overline{R} + a_0R^2 + \sum_{k=1}^{p} a_kR\Box^kR)\sqrt{-g}$ and generally for n-order gravity described by the gravitational Lagrangian $L = L\left(g_{\mu\nu}, g_{\mu\nu,i_1}, g_{\mu\nu,i_1i_2}, g_{\mu\nu,i_1i_2i_3}, \cdots, g_{\mu\nu,i_1i_2i_3\cdots i_n}\right)$. The extended pseudo-tensor reduces to the one introduced by Einstein in the limit of general relativity where corrections vanish. Then, we explicitly show a useful calculation, i.e., the power per unit solid angle Ω emitted by a massive system and carried by a gravitational wave in the direction $\hat{x}$ for a fixed wave number $\mathbf{k}$. We fix a suitable gauge, by means of the average value of the pseudo-tensor over a spacetime domain and we verify that the local pseudo-tensor conservation holds. The gravitational energy–momentum pseudo-tensor may be a useful tool to search for possible further gravitational modes beyond the two standard ones of general relativity. Their finding could be a possible observable signatures for alternative theories of gravity.

1. Introduction

The problem of gravitational energy density in curved spacetime has been debated for decades. Bondi wrote "In relativity a non-localizable form of energy is inadmissible, because any form of energy contributes to gravitation and so its location can in principle be found". Several prescriptions for the pseudo-tensor have been suggested by Einstein, Tolman, Landau and Lifshitz, Papapetrou, Møller, and Weinberg [1–11]. These prescriptions have been formulated thanks to the introduction of a super-potential or through the expansion of Ricci tensor in the metric perturbation h or by manipulating the field equations. Thus, the

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geometric object describing the gravitational part of the energy–momentum density transforms as an affine tensor not as a covariant tensor, i.e. a pseudo-tensor and its affine property makes the gravitational energy–momentum density not localizable. However, the gravitational energy–momentum yields a four-vector, if we integrate the density over a suitable fixed-time spatial region under asymptotically flat coordinate transformations, that is, it becomes quasi independent of the coordinate system over the whole integration space. A generalization of the Einstein pseudo-tensor to Extended Theories of Gravity [12–15] is proposed by imposing the invariance of higher-order gravitational Lagrangian under an infinitesimal rigid translation and by using the Noether theorem. An associated Noether current and Noether charge can be derived. They correspond to the gravitational energy and momentum density and the gravitational energy and momentum respectively, both of which locally conserved. In the weak-field limit, the metric tensor can be weakly perturbed, and the perturbed pseudo-tensor of gravitational energy–momentum can be obtained for a Lagrangian of order n^{th}. Integrating over a suitable spacetime domain, we can calculate the power emitted by an astrophysical source, carried away by gravitational waves.

For more details on the problem of the energy–momentum localization in modified or alternative theories of curvature-based gravity, such as $f(R)$, $f(R, \Box R, \ldots, \Box^k R)$, see [14, 17, 18], while, for teleparallel gravity and its extended version $f(T)$, see Ref. [19]. For a study of wavelike solutions in modified teleparallel gravity, see references [20, 21].

The present paper is organized as follows. In Sec. 2, definitions of gravitational pseudo-tensors in general relativity are reported. In Sec. 3, the gravitational energy–momentum pseudo-tensor for a n^{th} order general Lagrangian is derived and it is shown how it transforms as a tensor under linear transformations but not under diffeomorphisms. Therefore, it is a pseudo-tensor. Furthermore, the pseudo-tensor for the Lagrangian $L_g = (\overline{R} + a_0 R^2 + \sum_{k=1}^{p} a_k R \Box^k R)\sqrt{-g}$ is found. In Sec. 4, the weak-field limit is derived. Hence, in Sec. 5, we show an application of pseudo-tensor to the calculation of gravitational radiation transported power. Conclusions are summarized in Sec. 6. The metric signature of $g_{\mu\nu}$ is $(+\ ,-\ ,-\ ,-)$. The Ricci and Riemann tensors are defined as $R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}$ and $R^{\alpha}{}_{\beta\mu\nu} = \Gamma^{\alpha}_{\beta\nu,\mu} + \ldots$, respectively.

2. Definitions of gravitational energy–momentum pseudo-tensor in general relativity

Below we list some of the most important definitions of gravitational energy–momentum pseudo-tensor in general relativity, for details see [11].

2.1. *The Einstein pseudo-tensor*

In general relativity, for the principle of general covariance, the conservation of matter energy–momentum tensor $T^{\mu\nu}$, given by

$$\frac{\partial T^{\mu\nu}}{\partial x^{\mu}} = 0\,, \tag{1}$$

is a relation not corresponding to any conservation law of physical quantities. It is

$$\nabla_{\mu} T^{\mu\nu} = 0\,. \tag{2}$$

Einstein formulated a conservation law where the sum of matter energy-momentum and gravitational energy-momentum are conserved. So the total conservation is

$$\frac{\partial \theta_{\mu}{}^{\nu}}{\partial x^{\nu}} = \frac{\partial}{\partial x^{\nu}}\left(\sqrt{-g}\left(T_{\mu}{}^{\nu} + t_{\mu}{}^{\nu}\right)\right) = 0\,, \tag{3}$$

where $t_{\mu}{}^{\nu}$ is the pseudo-tensor associated with the gravitational field, defined as

$$\sqrt{-g}\, t_{\mu}{}^{\nu} = \frac{1}{16\pi}\left(\delta^{\nu}_{\mu} L - \frac{\partial L}{\partial g^{\rho\sigma}{}_{,\nu}} g^{\rho\sigma}{}_{,\mu}\right)\,. \tag{4}$$

with the following Lagrangian density which is a non-covariant scalar density

$$L = \sqrt{-g}g^{\mu\nu}\left(\Gamma^{\sigma}_{\mu\nu}\Gamma^{\rho}_{\sigma\rho} - \Gamma^{\sigma}_{\mu\rho}\Gamma^{\rho}_{\nu\sigma}\right) . \tag{5}$$

The pseudo-tensoriality of ${t_\mu}^\nu$ makes it dependent on coordinates and the gravitational energy becomes non-localizable.

2.2. The Landau–Lifshitz energy–momentum pseudo-tensor

The gravitational energy–momentum pseudo-tensor proposed by Landau and Lifshitz is (see Ref. [15] for details)

$$\begin{aligned}16\pi\,(-g)\,t^{\mu\nu} =& {\mathfrak{g}^{\mu\nu}}_{,\rho}{\mathfrak{g}^{\rho\sigma}}_{,\sigma} - {\mathfrak{g}^{\mu\rho}}_{,\rho}{\mathfrak{g}^{\nu\sigma}}_{,\sigma} + \frac{1}{2}g^{\mu\nu}g_{\rho\sigma}{\mathfrak{g}^{\rho\alpha}}_{,\beta}{\mathfrak{g}^{\beta\sigma}}_{,\alpha} \\ &- \left(g^{\mu\rho}g_{\sigma\alpha}{\mathfrak{g}^{\nu\alpha}}_{,\beta}{\mathfrak{g}^{\sigma\beta}}_{,\rho} + g^{\nu\rho}g_{\sigma\alpha}{\mathfrak{g}^{\mu\alpha}}_{,\beta}{\mathfrak{g}^{\sigma\beta}}_{,\rho}\right) + g_{\rho\sigma}g^{\alpha\beta}{\mathfrak{g}^{\mu\rho}}_{,\alpha}{\mathfrak{g}^{\nu\sigma}}_{,\beta} \\ &+ \frac{1}{8}\left(2g^{\mu\rho}g^{\nu\sigma} - g^{\mu\nu}g^{\rho\sigma}\right)\left(2g_{\alpha\beta}g_{\gamma\lambda} - g_{\beta\gamma}g_{\alpha\lambda}\right){\mathfrak{g}^{\alpha\lambda}}_{,\rho}{\mathfrak{g}^{\beta\gamma}}_{,\sigma} ,\end{aligned} \tag{6}$$

where $\mathfrak{g}^{\mu\nu} = \sqrt{-g}g^{\mu\nu}$.

2.3. The Møller energy–momentum complex

In 1958, Møller proposed an energy–momentum complex tensor ${\mathcal{T}_\mu}^\nu = {\theta_\mu}^\nu + {S_\mu}^\nu$, where $\theta^{\mu\nu} = T^{\mu\nu} + t^{\mu\nu}$ is the pseudo-tensor including matter plus gravity and $S^{\mu\nu}$ is a divergenceless quantity. To demonstrate this property, it is worth noticing that ${S_\mu}^\nu$, such that ${\mathcal{T}_\mu}^\nu$, transforms as a tensor only for spatial transformations. It is

$${\mathcal{T}_\mu}^\nu = \frac{1}{8\pi}\partial_\rho\left[\sqrt{-g}\left(g_{\mu\sigma,\lambda} - g_{\mu\lambda,\sigma}\right)g^{\lambda\nu}g^{\sigma\rho}\right] , \tag{7}$$

where the expression in square brackets is the antisymmetric super-potential ${U_\mu}^{\nu\rho} = -{U_\mu}^{\rho\nu}$ such that

$$\partial_\nu {\mathcal{T}_\mu}^\nu = 0 . \tag{8}$$

2.4. The Papapetrou energy–momentum pseudo-tensor

Papapetrou, in 1948, used the generalized Belifante method to derive his pseudo-tensor $\Omega^{\mu\nu}$ which can be written as (see Ref. [15] for details)

$$\Omega^{\mu\nu} = \frac{1}{16\pi}\frac{\partial^2}{\partial x^\rho x^\sigma}\left[\sqrt{-g}\,(g^{\mu\nu}\eta^{\rho\sigma} - g^{\mu\rho}\eta^{\nu\sigma} - g^{\rho\sigma}\eta\mu\nu - g^{\nu\sigma}\eta^{\mu\rho})\right] . \tag{9}$$

This geometric object is symmetric with respect to the first two indices μ e ν.

2.5. The Weinberg gravitational energy–momentum pseudo-tensor

Weinberg [22] derived the gravitational energy–momentum pseudo-tensor $t_{\mu\nu}$ expanding the Ricci tensor $R_{\mu\nu}$ in terms of powers of $h_{\mu\nu}$ up to second order, i.e. (see Ref. [15] for details)

$$t_{\mu\nu} = \frac{1}{8\pi G}\left(-\frac{1}{2}h_{\mu\nu}R^{(1)} + \frac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}{R^{(1)}}_{\rho\sigma} + {R^{(2)}}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}{R^{(2)}}_{\rho\sigma}\right) + O\left(h^3\right) , \tag{10}$$

where

$${R^{(1)}}_{\mu\nu} = \frac{1}{2}\left(\frac{\partial^2 {h^\lambda}_\lambda}{\partial x^\mu \partial x^\nu} - \frac{\partial^2 {h^\lambda}_\mu}{\partial x^\lambda \partial x^\nu} - \frac{\partial^2 {h^\lambda}_\nu}{\partial x^\lambda \partial x^\mu} + \frac{\partial^2 h_{\mu\nu}}{\partial x^\lambda \partial x_\lambda}\right) , \tag{11}$$

and

$$R^{(2)}{}_{\mu\nu} = -\frac{1}{2}h^{\lambda\rho}\left(\frac{\partial^2 h_{\lambda\rho}}{\partial x^\nu \partial x^\mu} - \frac{\partial^2 h_{\mu\rho}}{\partial x^\nu \partial x^\lambda} - \frac{\partial^2 h_{\lambda\nu}}{\partial x^\rho \partial x^\mu} + \frac{\partial^2 h_{\mu\nu}}{\partial x^\rho \partial x^\lambda}\right) + \frac{1}{4}\left(2\frac{\partial h^\rho{}_\sigma}{\partial x^\rho} - \frac{\partial h^\rho{}_\rho}{\partial x^\sigma}\right)\left(\frac{\partial h^\sigma{}_\mu}{\partial x^\nu} + \frac{\partial h^\sigma{}_\nu}{\partial x^\mu} - \frac{\partial h_{\mu\nu}}{\partial x_\sigma}\right) - \frac{1}{4}\left(\frac{\partial h_{\sigma\nu}}{\partial x^\lambda} + \frac{\partial h_{\sigma\lambda}}{\partial x^\nu} - \frac{\partial h_{\lambda\nu}}{\partial x^\sigma}\right)\left(\frac{\partial h^\sigma{}_\mu}{\partial x_\lambda} + \frac{\partial h^{\sigma\lambda}}{\partial x^\mu} - \frac{\partial h^\lambda{}_\mu}{\partial x_\sigma}\right) . \tag{12}$$

3. The gravitational energy–momentum pseudo-tensor in curvature-based gravity

3.1. The gravitational energy–momentum "tensor" of n^{th} order Lagrangian

First, let us analyze the energy–momentum complex for a gravitational Lagrangian of fourth order, i.e., depending up to the fourth derivative of the metric tensor $g_{\mu\nu}$ as $L = L(g_{\mu\nu}, g_{\mu\nu,\rho}, g_{\mu\nu,\rho\lambda}, g_{\mu\nu,\rho\lambda\xi}, g_{\mu\nu,\rho\lambda\xi\sigma})$. The related field equations are of eighth order (see [23]), if we include in the gravitational action all possible curvature invariants, not only the $\Box$ operators. Then, it is possible to generalize the procedure to a gravitational Lagrangian of n-th order, i.e., which depends up to n^{th} derivatives of metric tensor. It is then possible to derive the energy–momentum tensor, using the Noether theorem and imposing that gravitational action is invariant under global translations [1]. Let us vary the gravitational action both with respect to metric $g_{\mu\nu}$ and to coordinates x^μ [12]

$$I = \int_\Omega d^4x L \rightarrow \tilde{\delta} I = \int_{\Omega'} d^4x' L' - \int_\Omega d^4x L = \int_\Omega d^4x \left[\delta L + \partial_\mu \left(L \delta x^\mu\right)\right] , \tag{13}$$

where $\tilde{\delta}$ andδ stand for the local and total variation, respectively, the latter keeping the value of coordinate x fixed. From the following infinitesimal transformations

$$x'^\mu = x^\mu + \epsilon^\mu (x) , \tag{14}$$

the total variation of the metric tensor reads

$$\delta g_{\mu\nu} = g'_{\mu\nu}(x) - g_{\mu\nu}(x) = -\epsilon^\alpha \partial_\alpha g_{\mu\nu} - g_{\mu\alpha}\partial_\nu \epsilon^\alpha - g_{\nu\alpha}\partial_\mu \epsilon^\alpha . \tag{15}$$

Under the global transformation, $\partial_\lambda \epsilon^\mu = 0$, the functional variation of the metric becomes $\delta g_{\mu\nu} = -\epsilon^\alpha \partial_\alpha g_{\mu\nu}$. If we also require that the action is invariant under this transformation, that is, $\tilde{\delta} I = 0$, from arbitrariness of domain of integration Ω, we have

$$0 = \delta L + \partial_\mu \left(L\delta x^\mu\right) = \left(\frac{\partial L}{\partial g_{\mu\nu}} - \partial_\rho \frac{\partial L}{\partial g_{\mu\nu,\rho}} + \partial_\rho \partial_\lambda \frac{\partial L}{\partial g_{\mu\nu,\rho\lambda}} - \partial_\rho \partial_\lambda \partial_\xi \frac{\partial L}{\partial g_{\mu\nu,\rho\lambda\xi}} + \partial_\rho \partial_\lambda \partial_\xi \partial_\sigma \frac{\partial L}{\partial g_{\mu\nu,\rho\lambda\xi\sigma}}\right)\delta g_{\mu\nu} + \partial_\eta \left(2\chi\sqrt{-g}\tau^\eta_\alpha\right)\epsilon^\alpha , \tag{16}$$

where the explicit expression of gravitational energy–momentum tensor, that we will see being an affine pseudo-tensor tensor, is

$$\tau^\eta_\alpha = \frac{1}{2\chi\sqrt{-g}}\Bigg[\left(\frac{\partial L}{\partial g_{\mu\nu,\eta}} - \partial_\lambda \frac{\partial L}{\partial g_{\mu\nu,\eta\lambda}} + \partial_\lambda \partial_\xi \frac{\partial L}{\partial g_{\mu\nu,\eta\lambda\xi}} - \partial_\lambda \partial_\xi \partial_\sigma \frac{\partial L}{\partial g_{\mu\nu,\eta\lambda\xi\sigma}}\right) g_{\mu\nu,\alpha} + \left(\frac{\partial L}{\partial g_{\mu\nu,\rho\eta}} - \partial_\xi \frac{\partial L}{\partial g_{\mu\nu,\rho\eta\xi}} + \partial_\xi \partial_\sigma \frac{\partial L}{\partial g_{\mu\nu,\rho\eta\xi\sigma}}\right) g_{\mu\nu,\alpha\rho} + \left(\frac{\partial L}{\partial g_{\mu\nu,\rho\lambda\eta}} - \partial_\sigma \frac{\partial L}{\partial g_{\mu\nu,\rho\lambda\eta\sigma}}\right) g_{\mu\nu,\rho\lambda\alpha} + \frac{\partial L}{\partial g_{\mu\nu,\rho\lambda\eta\sigma}} g_{\mu\nu,\rho\lambda\xi\alpha} - \delta^\eta_\alpha L\Bigg], \tag{17}$$

where $\chi = 8\pi G/c^4$ is the gravitational coupling constant. If the metric tensor $g_{\mu\nu}$ satisfies the Euler–Lagrange equations for our gravitational Lagrangian

$$\frac{\delta L}{\delta g_{\mu\nu}} = \frac{\partial L}{\partial g_{\mu\nu}} - \partial_\rho \frac{\partial L}{\partial g_{\mu\nu,\rho}} + \partial_\rho\partial_\lambda \frac{\partial L}{\partial g_{\mu\nu,\rho\lambda}} - \partial_\rho\partial_\lambda\partial_\xi \frac{\partial L}{\partial g_{\mu\nu,\rho\lambda\xi}} + \partial_\rho\partial_\lambda\partial_\xi\partial_\sigma \frac{\partial L}{\partial g_{\mu\nu,\rho\lambda\xi\sigma}} = 0\,, \tag{18}$$

for an arbitrary ϵ^α, we get a local continuity equation for the Noether current

$$\partial_\eta\left(\sqrt{-g}\tau^\eta_\alpha\right) = 0\,. \tag{19}$$

In a more compact form, the gravitational energy–momentum tensor takes the following form

$$\tau^\eta_\alpha = \frac{1}{2\chi\sqrt{-g}}\Bigg[\sum_{m=0}^{3}(-1)^m\left(\frac{\partial L}{\partial g_{\mu\nu,\eta i_0\cdots i_m}}\right)_{,i_0\cdots i_m} g_{\mu\nu,\alpha} + \sum_{j=0}^{2}\sum_{m=j+1}^{3}(-1)^j\left(\frac{\partial L}{\partial g_{\mu\nu,\eta i_0\cdots i_m}}\right)_{,i_0\cdots i_j} g_{\mu\nu,i_{j+1}\cdots i_m\alpha} - \delta^\eta_\alpha L\Bigg]\,, \tag{20}$$

where we used the following notation

$$()_{,i_0} = \mathbb{I}; \qquad ()_{,i_0\cdots i_m} = \begin{cases} ()_{,i_1} & \text{if} \quad m=1 \\ ()_{,i_1i_2} & \text{if} \quad m=2 \\ ()_{,i_1i_2i_3} & \text{if} \quad m=3 \\ \text{and so on} & \end{cases}; \qquad ()_{,i_ki_k} = ()_{,i_k}$$

Let us now generalize the approach considering a general Lagrangian density depending up to n^{th} derivative of $g_{\mu\nu}$, that is, $L = L\left(g_{\mu\nu}, g_{\mu\nu,i_1}, g_{\mu\nu,i_1i_2}, g_{\mu\nu,i_1i_2i_3}, \cdots, g_{\mu\nu,i_1i_2i_3\cdots i_n}\right)$. The total variation of Lagrangian L and its Euler–Lagrange equations yield

$$\delta L = \sum_{m=0}^{n}\frac{\partial L}{\partial g_{\mu\nu,i_0\cdots i_m}}\delta g_{\mu\nu,i_0\cdots i_m} = \sum_{m=0}^{n}\frac{\partial L}{\partial g_{\mu\nu,i_0\cdots i_m}}\partial_{i_0\cdots i_m}\delta g_{\mu\nu}\,, \tag{21}$$

$$\frac{\delta L}{\delta g_{\mu\nu}} = \sum_{m=0}^{n}(-1)^m\,\partial_{i_0\cdots i_m}\frac{\partial L}{\partial g_{\mu\nu,i_0\cdots i_m}} = 0\,, \tag{22}$$

where $\delta/\delta g_{\mu\nu}$ is the functional derivative It is possible to exchange the variation δ with the derivatives $\delta g_{\mu\nu,i_0\cdots i_m} = \partial_{i_0\cdots i_m}\delta g_{\mu\nu}$, because we are varying keeping x fixed. So, we can find the most general local continuity equation which allows us to define the energy–momentum pseudo-tensor (which is an affine tensor as it will be proved later) for the gravitational field of $2n^{th}$ order gravity

$$\tau^\eta_\alpha = \frac{1}{2\chi\sqrt{-g}}\Bigg[\sum_{m=0}^{n-1}(-1)^m\left(\frac{\partial L}{\partial g_{\mu\nu,\eta i_0\cdots i_m}}\right)_{,i_0\cdots i_m} g_{\mu\nu,\alpha} + \Theta_{[2,+\infty[}(n)\sum_{j=0}^{n-2}\sum_{m=j+1}^{n-1}(-1)^j\left(\frac{\partial L}{\partial g_{\mu\nu,\eta i_0\cdots i_m}}\right)_{,i_0\cdots i_j} g_{\mu\nu,i_{j+1}\cdots i_m\alpha} - \delta^\eta_\alpha L\Bigg]\,, \tag{23}$$

where Θ is the Heaviside function

$$\Theta_{[a,+\infty[}(n) = \begin{cases} 1 & \text{if} \quad n \in [a,+\infty[\\ 0 & \text{otherwise} \end{cases}\,. \tag{24}$$

If fields and their derivatives vanish on the boundary of the spatial region or rapidly decrease to the spatial infinite on an infinity spacelike hypersurface, the gravitational energy–momentum tensor is totally conserved and satisfies a general conservation law. An alternative way to obtain the tensor (23) is the procedure developed by Landau [1]. For example, we can start by deriving the tensor (20), because its generalization to higher order Lagrangians is the same. First of all, let us impose the stationary condition and vary the action with respect to the metric to find the field equations under the hypothesis that both $\delta g_{\mu\nu}$ and the variation of derivative $\delta\partial^n g$ vanish on the boundary of integration domain, canceling the surface integrals. Hence, the following variation occurs:

$$\delta I = \delta \int_\Omega d^4x L\left(g_{\mu\nu}, g_{\mu\nu,\rho}, g_{\mu\nu,\rho\lambda}, g_{\mu\nu,\rho\lambda\xi}, g_{\mu\nu,\rho\lambda\xi\sigma}\right) = 0\,, \tag{25}$$

$$\Updownarrow$$

$$\frac{\partial L}{\partial g_{\mu\nu}} - \partial_\rho \frac{\partial L}{\partial g_{\mu\nu,\rho}} + \partial_\rho\partial_\lambda \frac{\partial L}{\partial g_{\mu\nu,\rho\lambda}} - \partial_\rho\partial_\lambda\partial_\xi \frac{\partial L}{\partial g_{\mu\nu,\rho\lambda\xi}} + \partial_\rho\partial_\lambda\partial_\xi\partial_\sigma \frac{\partial L}{\partial g_{\mu\nu,\rho\lambda\xi\sigma}} = 0\,. \tag{26}$$

Now, we perform the derivative of Lagrangian with respect to metric tensor and then we put it into the field equations (25). We obtain

$$\begin{aligned}
\frac{\partial L}{\partial x^\alpha} &= \frac{\partial L}{\partial g_{\mu\nu}}\frac{\partial g_{\mu\nu}}{\partial x^\alpha} + \frac{\partial L}{\partial g_{\mu\nu,\rho}}\frac{\partial g_{\mu\nu,\rho}}{\partial x^\alpha} + \frac{\partial L}{\partial g_{\mu\nu,\rho\lambda}}\frac{\partial g_{\mu\nu,\rho\lambda}}{\partial x^\alpha} \\
&+ \frac{\partial L}{\partial g_{\mu\nu,\rho\lambda\xi}}\frac{\partial g_{\mu\nu,\rho\lambda\xi}}{\partial x^\alpha} + \frac{\partial L}{\partial g_{\mu\nu,\rho\lambda\xi\sigma}}\frac{\partial g_{\mu\nu,\rho\lambda\xi\sigma}}{\partial x^\alpha} \\
&= \partial_\rho \frac{\partial L}{\partial g_{\mu\nu,\rho}} g_{\mu\nu,\alpha} - \partial_\rho\partial_\lambda \frac{\partial L}{\partial g_{\mu\nu,\rho\lambda}} g_{\mu\nu,\alpha} + \partial_\rho\partial_\lambda\partial_\xi \frac{\partial L}{\partial g_{\mu\nu,\rho\lambda\xi}} g_{\mu\nu,\alpha} - \partial_\rho\partial_\lambda\partial_\xi\partial_\sigma \frac{\partial L}{\partial g_{\mu\nu,\rho\lambda\xi\sigma}} g_{\mu\nu,\alpha} \\
&+ \frac{\partial L}{\partial g_{\mu\nu,\rho}} g_{\mu\nu,\rho\alpha} + \frac{\partial L}{\partial g_{\mu\nu,\rho\lambda}} g_{\mu\nu,\rho\lambda\alpha} + \frac{\partial L}{\partial g_{\mu\nu,\rho\lambda\xi}} g_{\mu\nu,\rho\lambda\xi\alpha} + \frac{\partial L}{\partial g_{\mu\nu,\rho\lambda\xi\sigma}} g_{\mu\nu,\rho\lambda\xi\sigma\alpha} \\
&= \partial_\rho\left(\frac{\partial L}{\partial g_{\mu\nu,\rho}} g_{\mu\nu,\alpha}\right) - \partial_\rho\left(\partial_\lambda \frac{\partial L}{\partial g_{\mu\nu,\rho\lambda}} g_{\mu\nu,\alpha}\right) + \partial_\lambda\left(\frac{\partial L}{\partial g_{\mu\nu,\rho\lambda}} g_{\mu\nu,\rho\alpha}\right) \\
&+ \partial_\rho\left(\partial_\lambda\partial_\xi \frac{\partial L}{\partial g_{\mu\nu,\rho\lambda\xi}} g_{\mu\nu,\alpha}\right) + \partial_\lambda\left(\frac{\partial L}{\partial g_{\mu\nu,\rho\lambda\xi}} g_{\mu\nu,\rho\xi\alpha}\right) \\
&- \partial_\xi\left(\partial_\lambda \frac{\partial L}{\partial g_{\mu\nu,\rho\lambda\xi}} g_{\mu\nu,\alpha\rho}\right) - \partial_\rho\left(\partial_\lambda\partial_\xi\partial_\sigma \frac{\partial L}{\partial g_{\mu\nu,\rho\lambda\xi\sigma}} g_{\mu\nu,\alpha}\right) \\
&+ \partial_\lambda\left(\frac{\partial L}{\partial g_{\mu\nu,\rho\lambda\xi\sigma}} g_{\mu\nu,\rho\xi\sigma\alpha}\right) - \partial_\xi\left(\partial_\lambda \frac{\partial L}{\partial g_{\mu\nu,\rho\lambda\xi\sigma}} g_{\mu\nu,\rho\sigma\alpha}\right) \\
&+ \partial_\sigma\left(\partial_\xi\partial_\lambda \frac{\partial L}{\partial g_{\mu\nu,\rho\lambda\xi\sigma}} g_{\mu\nu,\rho\alpha}\right).
\end{aligned} \tag{27}$$

Grouping together terms and renaming dumb indices, we obtain

$$\partial_\eta\left(\sqrt{-g}\,\tau^\eta_\alpha\right) = 0\,, \tag{28}$$

that is, the pseudo-tensor is locally conserved, where τ^η_α is the tensor defined in (20).

The energy–momentum complex, instead, can be derived considering the material Lagrangian $L_m = 2\chi\sqrt{-g}\mathcal{L}_m$ with the stress–energy tensor given by

$$T^{\eta\alpha} = \frac{2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathcal{L}_m\right)}{\delta g_{\eta\alpha}}\,. \tag{29}$$

Thus, we can use the field equations in presence of matter, namely

$$P^{\eta\alpha} = \chi T^{\eta\alpha} \ , \tag{30}$$

where

$$P^{\eta\alpha} = -\frac{1}{\sqrt{-g}}\frac{\delta L_g}{\delta g_{\eta\alpha}} \ . \tag{31}$$

By field equations (30), we obtain

$$\begin{aligned}&\left(2\chi\sqrt{-g}\tau^{\eta}_{\alpha}\right)_{,\eta} = -\sqrt{-g}P^{\rho\sigma}g_{\rho\sigma,\alpha} = -\chi\sqrt{-g}T^{\rho\sigma}g_{\rho\sigma,\alpha}\\ &= 2\chi\sqrt{-g}T^{\eta}_{\alpha;\eta} - \left(2\chi\sqrt{-g}T^{\eta}_{\alpha}\right)_{,\eta} \ ,\end{aligned} \tag{32}$$

$$\partial_\eta\left[\sqrt{-g}\left(\tau^{\eta}_{\alpha} + T^{\eta}_{\alpha}\right)\right] = \sqrt{-g}T^{\eta}_{\alpha;\eta} \ , \tag{33}$$

being

$$\begin{aligned}&\delta L + \partial_\mu\left(L\delta x^\mu\right) = -P^{\mu\nu}\sqrt{-g}\delta g_{\mu\nu} + \partial_\eta\left(2\chi\sqrt{-g}\tau^{\eta}_{\alpha}\right)\epsilon^\alpha\\ &= \left[\sqrt{-g}P^{\mu\nu}g_{\mu\nu,\alpha} + \partial_\eta\left(2\chi\sqrt{-g}\tau^{\eta}_{\alpha}\right)\right]\epsilon^\alpha = 0 \ ,\end{aligned} \tag{34}$$

and because the symmetry of tensor T^{η}_{α}, one gets

$$\sqrt{-g}T^{\eta}_{\alpha;\eta} = \left(\sqrt{-g}T^{\eta}_{\alpha}\right)_{,\eta} - \frac{1}{2}g_{\rho\sigma,\alpha}T^{\rho\sigma}\sqrt{-g} \ . \tag{35}$$

The relation (33) tells us that the conservation law of energy–momentum complex, i.e., the sum of two stress–energy tensors due to matter plus gravitational fields, is related to the covariant derivative of the only matter part. From the contracted Bianchi identities, we get the total conservation law and conversely

$$G^{\eta\alpha}_{;\eta} = 0 \leftrightarrow P^{\eta\alpha}_{;\eta} = 0 \leftrightarrow T^{\eta\alpha}_{;\eta} = 0 \leftrightarrow \partial_\eta\left[\sqrt{-g}\left(\tau^{\eta}_{\alpha} + T^{\eta}_{\alpha}\right)\right] = 0 \ , \tag{36}$$

where $G^{\eta\alpha} = R^{\eta\alpha} - \frac{1}{2}g^{\eta\alpha}R$ is the Einstein tensor and the locally conserved energy–momentum complex is given by

$$\mathcal{T}^{\eta}_{\alpha} = \sqrt{-g}\left(\tau^{\eta}_{\alpha} + T^{\eta}_{\alpha}\right) \ . \tag{37}$$

In a nutshell, the contracted Bianchi identities lead to the local conservation of energy–momentum complex or, viceversa, the local conservation of matter and gravitational fields involves the contracted Bianchi identities (see [24] for a detailed discussion on modified and extended gravity).

From the local continuity equation (36), it is possible to derive some conserved quantities, the Noether charges, such as the total 4-momentum of matter plus gravitational field. If we require that the metric tensor derivatives up to the n^{th} order vanish on the 3-dimensional space-domain Σ, the surface integrals are zero over the boundary $\partial\Sigma$, that is

$$\partial_0\int_\Sigma d^3x\sqrt{-g}\left(T^{\mu 0} + \tau^{\mu 0}\right) = -\int_{\partial\Sigma} d\sigma_i\sqrt{-g}\left(T^{\mu i} + \tau^{\mu i}\right) = 0 \ , \tag{38}$$

where Σ is a slice of 4-dimensional manifold of spacetime at fixed t and $\partial\Sigma$ is its boundary. Such conditions are fulfilled when we are in presence of localized objects, where we can take a spatial domain becoming flat at infinity, i.e, an asymptotically flat spacetime. In this case, the energy and total momentum become [25]

$$P^\mu = \int_\Sigma d^3x\sqrt{-g}\left(T^{\mu 0} + \tau^{\mu 0}\right) \ . \tag{39}$$

These quantities are very useful for astrophysical applications [26].

3.2. The non-covariance of gravitational energy–momentum tensor

Let us begin by proving that, in the case of $n = 2$, the quantity τ_α^η is a non-covariant object but an affine object, that is, it evolves like a tensor only under affine transformations [27], i.e., a pseudo-tensor, but not under general transformations. The tensor (23) becomes

$$\tau_\alpha^\eta = \frac{1}{2\chi\sqrt{-g}}\left[\left(\frac{\partial L}{\partial g_{\mu\nu,\eta}} - \partial_\lambda \frac{\partial L}{\partial g_{\mu\nu,\eta\lambda}}\right) g_{\mu\nu,\alpha} + \frac{\partial L}{\partial g_{\mu\nu,\eta\xi}} g_{\mu\nu,\xi\alpha} - \delta_\alpha^\eta L\right] . \tag{40}$$

It is possible to show that, while under a general diffeomorphism transformation $x' = x'(x)$ the tensor changes as

$$\tau'^{\eta}_{\ \alpha}(x') \neq \mathrm{J}_\sigma^\eta \mathrm{J}^{-1\tau}_{\ \ \alpha} \tau_\tau^\sigma(x) \ , \tag{41}$$

with Jacobian matrix and determinant defined as

$$\mathrm{J}_\sigma^\eta = \frac{\partial x'^\eta}{\partial x^\sigma} \qquad \mathrm{J}^{-1\tau}_{\ \ \alpha} = \frac{\partial x^\tau}{\partial x'^\alpha} \qquad \det\left(\mathrm{J}_\beta^\alpha\right) = |J| = \frac{1}{\mathrm{J}^{-1}} \ , \tag{42}$$

under the following affine transformations

$$x'^\mu = \Lambda_\nu^\mu x^\nu \qquad \mathrm{J}_\nu^\mu = \Lambda_\nu^\mu \qquad |\Lambda| \neq 0 \ , \tag{43}$$

the tensor is transformed as

$$\tau'^{\eta}_{\ \alpha}(x') = \Lambda_\sigma^\eta \Lambda^{-1\tau}_{\ \ \alpha} \tau_\tau^\sigma(x) \ . \tag{44}$$

In general, the following identities occur

$$\sqrt{-g'} = \sqrt{-g} \qquad \text{where } g \text{ is a scalar density of weight } w = -2 \ ,$$
$$L' = \mathrm{J}^{-1} L \qquad \text{where } L \text{ is a scalar density of weight } w = -1 \ ,$$
$$g'_{\mu\nu,\alpha}(x') = \mathrm{J}^{-1a}_{\ \ \mu}\mathrm{J}^{-1b}_{\ \ \nu}\mathrm{J}^{-1c}_{\ \ \alpha} g_{ab,c}(x) + \partial'_\alpha\left[\mathrm{J}^{-1a}_{\ \ \mu}\mathrm{J}^{-1b}_{\ \ \nu}\right] g_{ab}(x) \ ,$$
$$\frac{\partial g_{\gamma\rho,\tau}}{\partial g'_{\mu\nu,\eta}} = \frac{1}{2}\left[\left(\delta_a^\mu\delta_b^\nu + \delta_a^\nu\delta_b^\mu\right)\delta_c^\eta\right]\mathrm{J}_\gamma^a\mathrm{J}_\rho^b\mathrm{J}_\tau^c = \mathrm{J}_\gamma^{(\mu}\mathrm{J}_\rho^{\nu)}\mathrm{J}_\tau^\eta \ ,$$
$$\frac{\partial L'}{\partial g'_{\mu\nu,\eta}} = \mathrm{J}^{-1}\mathrm{J}_\gamma^{(\mu}\mathrm{J}_\rho^{\nu)}\mathrm{J}_\tau^\eta \frac{\partial L}{\partial g_{\gamma\rho,\tau}} = \mathrm{J}^{-1}\mathrm{J}_\gamma^\mu\mathrm{J}_\rho^\nu\mathrm{J}_\tau^\eta \frac{\partial L}{\partial g_{\gamma\rho,\tau}}$$
$$\text{tensorial density (3,0) of weight } w = -1 \ ,$$
$$g'_{\mu\nu,\xi\alpha}(x') = \mathrm{J}^{-1a}_{\ \ \mu}\mathrm{J}^{-1b}_{\ \ \nu}\mathrm{J}^{-1c}_{\ \ \alpha}\mathrm{J}^{-1d}_{\ \ \xi} g_{ab,cd}(x) + \partial'^2_{\xi\alpha}\left[\mathrm{J}^{-1a}_{\ \ \mu}\mathrm{J}^{-1b}_{\ \ \nu}\right] g_{ab}(x)$$
$$+ \partial'_\alpha\left[\mathrm{J}^{-1a}_{\ \ \mu}\mathrm{J}^{-1b}_{\ \ \nu}\right]\mathrm{J}^{-1d}_{\ \ \xi} g_{ab,d}(x) + \partial'_\xi\left[\mathrm{J}^{-1a}_{\ \ \mu}\mathrm{J}^{-1b}_{\ \ \nu}\mathrm{J}^{-1c}_{\ \ \alpha}\right] g_{ab,c}(x) \ ,$$
$$\frac{\partial g_{\gamma\rho,\tau\epsilon}}{\partial g'_{\mu\nu,\eta\xi}} = \left(\delta_a^{(\mu}\delta_b^{\nu)}\delta_c^{(\eta}\delta_d^{\xi)}\right)\mathrm{J}_\gamma^a\mathrm{J}_\rho^b\mathrm{J}_\tau^c\mathrm{J}_\epsilon^d = \mathrm{J}_\gamma^{(\mu}\mathrm{J}_\rho^{\nu)}\mathrm{J}_\tau^{(\eta}\mathrm{J}_\epsilon^{\xi)} \ ,$$
$$\frac{\partial L'}{\partial g'_{\mu\nu,\eta\xi}} = \mathrm{J}^{-1}\mathrm{J}_\gamma^{(\mu}\mathrm{J}_\rho^{\nu)}\mathrm{J}_\tau^{(\eta}\mathrm{J}_\epsilon^{\xi)} \frac{\partial L}{\partial g_{\gamma\rho,\tau\epsilon}} = \mathrm{J}^{-1}\mathrm{J}_\gamma^\mu\mathrm{J}_\rho^\nu\mathrm{J}_\tau^\eta\mathrm{J}_\epsilon^\xi \frac{\partial L}{\partial g_{\gamma\rho,\tau\epsilon}}$$
$$\text{tensorial density (4,0) of weight } w = -1 \ ,$$
$$\partial'_\lambda \frac{\partial L'}{\partial g'_{\mu\nu,\eta\lambda}} = \mathrm{J}^{-1}\mathrm{J}_\gamma^\mu\mathrm{J}_\rho^\nu\mathrm{J}_\tau^\eta\mathrm{J}_\epsilon^\lambda\mathrm{J}^{-1\sigma}_{\ \ \lambda}\partial_\sigma \frac{\partial L}{\partial g_{\gamma\rho,\tau\epsilon}} + \partial'_\lambda\left[\mathrm{J}^{-1}\mathrm{J}_\gamma^\mu\mathrm{J}_\rho^\nu\mathrm{J}_\tau^\eta\mathrm{J}_\epsilon^\lambda\right]\frac{\partial L}{\partial g_{\gamma\rho,\tau\epsilon}} \ ,$$

and by symmetry of $B_{\alpha\beta}$, i.e., $B_{\alpha\beta} = B_{\beta\alpha}$, it follows that $A^{(\alpha\beta)}B_{\alpha\beta} = A^{\alpha\beta}B_{\alpha\beta}$. Then we have

$$\frac{\partial L'}{\partial g'_{\mu\nu,\eta}} g'_{\mu\nu,\alpha} = \mathrm{J}^{-1}\mathrm{J}_\tau^\eta\mathrm{J}^{-1\pi}_{\ \ \alpha}\frac{\partial L}{\partial g_{\gamma\rho,\tau}} g_{\gamma\rho,\pi}(x) + \frac{\partial}{\partial x'^\alpha}\left[\mathrm{J}^{-1a}_{\ \ \mu}\mathrm{J}^{-1b}_{\ \ \nu}\right] g_{ab}(x)\,\mathrm{J}^{-1}\mathrm{J}_\gamma^\mu\mathrm{J}_\rho^\nu\mathrm{J}_\tau^\eta \frac{\partial L}{\partial g_{\gamma\rho,\tau}} \ ,$$

$$\partial'_\lambda \frac{\partial L'}{\partial g'_{\mu\nu,\eta\lambda}} g'_{\mu\nu,\alpha}(x') = \mathrm{J}^{-1}\mathrm{J}^{\eta}_{\tau}\mathrm{J}^{-1c}{}_{\alpha}\partial_\sigma \frac{\partial L}{\partial g_{ab,\tau\sigma}} g_{ab,c} + \partial'_\lambda\left[\mathrm{J}^{-1}\mathrm{J}^{\mu}_{\gamma}\mathrm{J}^{\nu}_{\rho}\mathrm{J}^{\eta}_{\tau}\mathrm{J}^{\lambda}_{\epsilon}\right]\partial'_\alpha\left[\mathrm{J}^{-1a}{}_{\mu}\mathrm{J}^{-1b}{}_{\nu}\right] g_{ab}(x)\frac{\partial L}{\partial g_{\gamma\rho,\tau\epsilon}}$$
$$+\mathrm{J}^{-1}\mathrm{J}^{\mu}_{\gamma}\mathrm{J}^{\nu}_{\rho}\mathrm{J}^{\eta}_{\tau}\partial_\sigma \frac{\partial L}{\partial g_{\gamma\rho,\tau\sigma}}\partial'_\alpha\left[\mathrm{J}^{-1a}{}_{\mu}\mathrm{J}^{-1b}{}_{\nu}\right] g_{ab} + \partial'_\lambda\left[\mathrm{J}^{-1}\mathrm{J}^{\mu}_{\gamma}\mathrm{J}^{\nu}_{\rho}\mathrm{J}^{\eta}_{\tau}\mathrm{J}^{\lambda}_{\epsilon}\right]\mathrm{J}^{-1a}{}_{\mu}\mathrm{J}^{-1b}{}_{\nu}\mathrm{J}^{-1c}{}_{\alpha}\frac{\partial L}{\partial g_{\gamma\rho,\tau\epsilon}} g_{ab,c}\,,$$

$$\frac{\partial L'}{\partial g'_{\mu\nu,\eta\xi}} g'_{\mu\nu,\xi\alpha}(x') = \mathrm{J}^{-1}\mathrm{J}^{\eta}_{\tau}\mathrm{J}^{-1\omega}{}_{\alpha}\frac{\partial L}{\partial g_{\gamma\rho,\tau\epsilon}} g_{\gamma\rho,\omega\epsilon}(x) + \mathrm{J}^{-1}\partial'^2_{\xi\alpha}\left[\mathrm{J}^{-1a}{}_{\mu}\mathrm{J}^{-1b}{}_{\nu}\right] g_{ab}(x)\,\mathrm{J}^{\mu}_{\gamma}\mathrm{J}^{\nu}_{\rho}\mathrm{J}^{\eta}_{\tau}\mathrm{J}^{\xi}_{\epsilon}\frac{\partial L}{\partial g_{\gamma\rho,\tau\epsilon}}$$
$$+\mathrm{J}^{-1}\partial'_\alpha\left[\mathrm{J}^{-1a}{}_{\mu}\mathrm{J}^{-1b}{}_{\nu}\right] g_{ab,d}(x)\,\mathrm{J}^{\mu}_{\gamma}\mathrm{J}^{\nu}_{\rho}\mathrm{J}^{\eta}_{\tau}\frac{\partial L}{\partial g_{\gamma\rho,\tau d}} + \mathrm{J}^{-1}\mathrm{J}^{\mu}_{\gamma}\mathrm{J}^{\nu}_{\rho}\mathrm{J}^{\eta}_{\tau}\mathrm{J}^{\xi}_{\epsilon}\partial'_\xi\left[\mathrm{J}^{-1a}{}_{\mu}\mathrm{J}^{-1b}{}_{\nu}\mathrm{J}^{-1c}{}_{\alpha}\right] g_{ab,c}(x)\frac{\partial L}{\partial g_{\gamma\rho,\tau\epsilon}},$$

Finally, taking into account the previous relations, we get

$$\tau'^{\eta}_{\ \alpha}(x') = \mathrm{J}^{\eta}_{\sigma}\mathrm{J}^{-1\tau}{}_{\alpha}\tau^{\sigma}_{\tau}(x) + \left\{\text{terms containing } \frac{\partial^2 x}{\partial x'^2}, \frac{\partial^3 x}{\partial x'^3}\right\}\,. \tag{45}$$

Extra terms that include derivatives of order equal to or greater than two vanish for each non-singular affine transformation but not for generic diffeomorphisms. This proves that gravitational stress–energy tensor is a non-covariant but an affine object, that is, it is invariant under affine transformations due to non-covariance of the derivatives of the metric tensor $g_{\mu\nu}$, that make it at least affine. Generalizing to n-th order Lagrangian, the metric tensor derivatives change as

$$g'_{\mu\nu,i_1\cdots i_m\alpha}(x') = \mathrm{J}^{-1\alpha}{}_{\mu}\mathrm{J}^{-1\beta}{}_{\nu}\mathrm{J}^{-1j_1}{}_{i_1}\cdots\mathrm{J}^{-1j_m}{}_{i_m}\mathrm{J}^{-1\tau}{}_{\alpha} g_{\alpha\beta,j_1\cdots j_m\tau}(x)$$
$$+\left\{\text{containing terms } \frac{\partial^2 x}{\partial x'^2},\cdots,\frac{\partial^{m+2}x}{\partial x'^{m+2}}\right\}, \tag{46}$$

and the Lagrangian derivatives as

$$\frac{\partial L'}{\partial g'_{\mu\nu,\eta i_0\cdots i_m}} = \mathrm{J}^{-1}\mathrm{J}^{\mu}_{\gamma}\mathrm{J}^{\nu}_{\rho}\mathrm{J}^{\eta}_{\tau}\mathrm{J}^{i_1}_{j_1}\cdots\mathrm{J}^{i_m}_{j_m}\frac{\partial L}{\partial g_{\gamma\rho,\tau j_1\cdots j_m}} \quad \text{tensorial density (m+3,0) of weight } w = -1\,,$$

so that the non-covariance of tensor τ^{η}_{α} is clear. On the other hand, we obtain, for affine transformations,

$$\frac{\partial^2 x}{\partial x'^2} = \cdots = \frac{\partial^{m+2}x}{\partial x'^{m+2}} = 0\,,$$
$$\tau'^{\eta}_{\ \alpha}(x') = \Lambda^{\eta}_{\sigma}\Lambda^{-1\tau}{}_{\alpha}\tau^{\sigma}_{\tau}(x)\ ,$$

that is, the energy–momentum tensor of gravitational field is a pseudo-tensor. This result generalizes the result in [1] to Extended Theories of Gravity . The affine character of the stress–energy tensor τ^{η}_{α} exhibits the non-localizability of gravitational energy density. Specifically, the gravitational energy in a finite-dimensional space, at a given time, depends on the choice of coordinate system [25, 28]. It is worth highlighting that the existence of particular Lagrangians for which extra terms in Eq. (46) vanish cannot be excluded a priori. This is because terms depending on derivatives in the bracket (45), such as $\frac{\partial^2 x}{\partial x'^2},\cdots,\frac{\partial^{m+2}x}{\partial x'^{m+2}}$, can cancel each other out. Consequently, the energy–momentum pseudo-tensor τ^{η}_{α} becomes a covariant tensor. However, due to the structure of (45), in general, τ^{η}_{α} is a pseudo-tensor.

3.3. The gravitational energy–momentum pseudo-tensor of higher-order gravity

Let us now investigate theories of gravity of order higher than fourth considering $\Box^k$ operators. Hence, we introduce the linear and quadratic part of the Ricci scalar R, the first $\overline{R}$ depends only on first derivative of metric tensor $g_{\mu\nu}$ and the second $R^\star$ depends linearly on second derivative of metric tensor, as follows [1, 22, 27]

$$R = R^\star + \overline{R}\,, \tag{47}$$

$$R^\star = g^{\mu\nu}\left(\Gamma^\rho_{\mu\nu,\rho} - \Gamma^\rho_{\mu\rho,\nu}\right)\,, \tag{48}$$

$$\overline{R} = g^{\mu\nu}\left(\Gamma^\sigma_{\mu\nu}\Gamma^\rho_{\sigma\rho} - \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\rho}\right)\,. \tag{49}$$

Hence, we want to derive the energy–momentum pseudo-tensor τ^η_α for a gravitational Lagrangian given by

$$L_g = (\overline{R} + a_0R^2 + \sum_{k=1}^{p} a_kR\Box^kR)\sqrt{-g}\,. \tag{50}$$

Therefore, with the purpose to derive the pseudo-tensor τ^η_α, we have to calculate derivatives present into Eq. (23), namely

$$\frac{\partial L}{\partial g_{\mu\nu,\eta}} = \sqrt{-g}\left[\frac{\partial\overline{R}}{\partial g_{\mu\nu,\eta}} + \left(2a_0R + \sum_{k=1}^{p} a_k\Box^kR\right)\frac{\partial R}{\partial g_{\mu\nu,\eta}} + \sum_{k=1}^{p} a_kR\frac{\partial\Box^kR}{\partial g_{\mu\nu,\eta}}\right]\,, \tag{51}$$

$$-\partial_\lambda\left(\frac{\partial L}{\partial g_{\mu\nu,\eta\lambda}}\right) = -\partial_\lambda\left(\sqrt{-g}\left[\left(2a_0R + \sum_{k=1}^{p} a_k\Box^kR\right)\frac{\partial R}{\partial g_{\mu\nu,\eta\lambda}} + \sum_{k=1}^{p} a_kR\frac{\partial\Box^kR}{\partial g_{\mu\nu,\eta\lambda}}\right]\right)\,, \tag{52}$$

$$\begin{aligned}\sum_{m=2}^{n-1}(-1)^m\left(\frac{\partial L}{\partial g_{\mu\nu,\eta i_0\cdots i_m}}\right)_{,i_0\cdots i_m} &= \sum_{m=2}^{n-1}\sum_{k=1}^{p}(-1)^m\,\partial_{i_0\cdots i_m}\left[\sqrt{-g}a_kR\frac{\partial\Box^kR}{\partial g_{\mu\nu,\eta i_0\cdots i_m}}\right]\\ &= \sum_{k=1}^{p}\sum_{m=2}^{2p+3}(-1)^m\,\partial_{i_0\cdots i_m}\left[\sqrt{-g}a_kR\frac{\partial\Box^kR}{\partial g_{\mu\nu,\eta i_0\cdots i_m}}\right]\\ &= \sum_{k=1}^{p}\sum_{m=2}^{2k+1}(-1)^m\,\partial_{i_0\cdots i_m}\left[\sqrt{-g}a_kR\frac{\partial\Box^kR}{\partial g_{\mu\nu,\eta i_0\cdots i_m}}\right]\,,\end{aligned} \tag{53}$$

where $\lambda = i_1$, $n = 2p+4$ and

$$\frac{\partial\Box^kR}{\partial g_{\mu\nu,\eta i_0\cdots i_m}} = 0 \qquad \text{if} \quad m > 2k+1\,. \tag{54}$$

Then, after algebraic manipulations, one have

$$\begin{aligned}&\sum_{j=0}^{n-2}\sum_{m=j+1}^{n-1}(-1)^j\left(\frac{\partial L}{\partial g_{\mu\nu,\eta i_0\cdots i_m}}\right)_{,i_0\cdots i_j}\\ &= \sum_{h=1}^{p}\sum_{j=0}^{2p+2}\sum_{m=j+1}^{2p+3}(-1)^j\left(\sqrt{-g}a_hR\frac{\partial\Box^hR}{\partial g_{\mu\nu,\eta i_0\cdots i_m}}\right)_{,i_0\cdots i_j}\,.\end{aligned} \tag{55}$$

Thereby, after observing that $j+1 \le m \le 2h+1 \to j \le 2h$, we finally get

$$\sum_{j=0}^{n-2}\sum_{m=j+1}^{n-1}(-1)^j\left(\frac{\partial L}{\partial g_{\mu\nu,\eta i_0\cdots i_m}}\right)_{,i_0\cdots i_j} = \sum_{h=1}^{p}\sum_{j=0}^{2h}\sum_{m=j+1}^{2h+1}(-1)^j\left(\sqrt{-g}a_hR\frac{\partial\Box^hR}{\partial g_{\mu\nu,\eta i_0\cdots i_m}}\right)_{,i_0\cdots i_j}\,.$$

By inserting these expressions into (23), we obtain the gravitational energy–momentum pseudo-tensor for the Lagrangian (50)

$$\begin{aligned}\tau^{\eta}_{\alpha} = \tau^{\eta}_{\alpha|GR} + \frac{1}{2\chi\sqrt{-g}}\Bigg\{ & \sqrt{-g}\left(2a_0R + \sum_{k=1}^{p} a_k \Box^k R\right)\left[\frac{\partial R}{\partial g_{\mu\nu,\eta}} g_{\mu\nu,\alpha} + \frac{\partial R}{\partial g_{\mu\nu,\eta\lambda}} g_{\mu\nu,\lambda\alpha}\right] \\ & - \partial_\lambda \left[\sqrt{-g}\left(2a_0R + \sum_{k=1}^{p} a_k \Box^k R\right)\frac{\partial R}{\partial g_{\mu\nu,\eta\lambda}}\right] g_{\mu\nu,\alpha} \\ & + \Theta_{[1,+\infty[}(p) \sum_{h=1}^{p}\Bigg\{\sum_{q=0}^{2h+1} (-1)^q \,\partial_{i_0\cdots i_q}\left[\sqrt{-g}a_hR\frac{\partial \Box^h R}{\partial g_{\mu\nu,\eta i_0\cdots i_q}}\right] g_{\mu\nu,\alpha} \\ & + \sum_{j=0}^{2h}\sum_{m=j+1}^{2h+1} (-1)^j \,\partial_{i_0\cdots i_j}\left[\sqrt{-g}a_hR\frac{\partial \Box^h R}{\partial g_{\mu\nu,\eta i_0\cdots i_m}}\right] g_{\mu\nu,i_{j+1}\cdots i_m\alpha}\Bigg\} \\ & - \delta^{\eta}_{\alpha}\left(a_0R^2 + \sum_{k=1}^{p} a_k R\Box^k R\right)\sqrt{-g}\Bigg\}\end{aligned} \quad , \tag{56}$$

where the notation $\partial_{i_0} = \mathbb{I}$ is the identity operator and $\tau^{\eta}_{\alpha|GR}$ indicates the energy–momentum pseudo-tensor of general relativity [28] defined as

$$\tau^{\eta}_{\alpha|GR} = \frac{1}{2\chi}\left(\frac{\partial \overline{R}}{\partial g_{\mu\nu,\eta}} g_{\mu\nu,\alpha} - \delta^{\eta}_{\alpha}\overline{R}\right) . \tag{57}$$

Given that only $\overline{R}$ contributes to the field equations, we can replace the scalar density $\sqrt{-g}R$ with $\sqrt{-g}\overline{R}$,which is not a scalar density.

4. The weak-field limit of energy–momentum pseudo-tensor

The low energy pseudo-tensor (56) related to Lagrangian (50) can be obtained by weakly perturbing spacetime metric $g_{\mu\nu}$ around the Minkowski metric $\eta_{\mu\nu}$ as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad \text{being} \quad |h_{\mu\nu}| \ll 1 , \tag{58}$$

where $h = \eta^{\mu\nu}h_{\mu\nu}$ is the trace of perturbation. Thus, we expand the energy–momentum pseudo-tensor to lower order in h, namely, retaining terms up to h^2. Let us see what becomes the weakly perturbed pseudo-tensor (57) in harmonic coordinates where $g^{\mu\nu}\Gamma^{\sigma}_{\mu\nu} = 0$. The quadratic part of the Ricci scalar $\overline{R}$ yields

$$\overline{R} = -g^{\mu\nu}\left(\Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho}\right) , \tag{59}$$

that is

$$\overline{R} = -\frac{1}{4}g^{\mu\nu}g^{\sigma\lambda}g^{\rho\epsilon}\left(g_{\epsilon\mu,\sigma} + g_{\epsilon\sigma,\mu} - g_{\mu\sigma,\epsilon}\right)\left(g_{\lambda\nu,\rho} + g_{\lambda\rho,\nu} - g_{\nu\rho,\lambda}\right) . \tag{60}$$

Keeping terms up to second order in h^2, we have

$$\left(\frac{\partial \overline{R}}{\partial g_{\alpha\beta,\gamma}}\right)^{(1)}\left(g_{\alpha\beta,\delta}\right)^{(1)} \overset{h^2}{=} \left(\frac{1}{2}h^{\alpha\beta}{}_{,}{}^{\gamma}h_{\alpha\beta,\delta} - h^{\gamma\alpha}{}_{,}{}^{\beta}h_{\alpha\beta,\delta}\right) , \tag{61}$$

according to

$$\frac{\partial \overline{R}}{\partial g_{\alpha\beta,\gamma}} g_{\alpha\beta,\delta} = -\frac{1}{4}\Big\{\left(g^{\mu\beta}g^{\sigma\alpha}g^{\epsilon\gamma} + g^{\mu\gamma}g^{\sigma\alpha}g^{\beta\epsilon} - g^{\mu\alpha}g^{\sigma\gamma}g^{\beta\epsilon}\right)\left(g_{\epsilon\mu,\sigma} + g_{\epsilon\sigma,\mu} - g_{\sigma\mu,\epsilon}\right) + \left(g^{\beta\nu}g^{\gamma\lambda}g^{\rho\alpha} + g^{\gamma\nu}g^{\beta\lambda}g^{\rho\alpha} - g^{\alpha\lambda}g^{\beta\nu}g^{\rho\gamma}\right)\left(g_{\lambda\nu,\rho} + g_{\lambda\rho,\nu} - g_{\nu\rho,\lambda}\right)\Big\} g_{\alpha\beta,\delta} \ , \quad (62)$$

and also

$$\overline{R}^{(2)} = -\frac{1}{4}\left(h^{\sigma\lambda}{}_{,\rho}h_{\lambda\sigma,}{}^{\rho} - 2h^{\sigma\lambda}{}_{,\rho}h^{\rho}{}_{\lambda,\sigma}\right) \ . \quad (63)$$

Hence, when we put these terms into (57), the stress–energy pseudo-tensor, in general relativity, up to order h^2 takes the form

$$\tau^{\eta}_{\alpha|GR} = \frac{1}{2\chi}\left[\frac{1}{2}h^{\mu\nu,\eta}h_{\mu\nu,\alpha} - h^{\eta\mu,\nu}h_{\mu\nu,\alpha} - \frac{1}{4}\delta^{\eta}_{\alpha}\left(h^{\sigma\lambda}{}_{,\rho}h_{\lambda\sigma}{}^{,\rho} - 2h^{\sigma\lambda}{}_{,\rho}h^{\rho}{}_{\lambda,\sigma}\right)\right] \ . \quad (64)$$

Therefore, the corrections of the pseudo-tensor (56), due to extended gravity terms, are obtained expanding it to the second order in $h_{\mu\nu}$. We get the following expansions to lower order in h

$$\left(\frac{\partial R}{\partial g_{\mu\nu,\eta\lambda}}\right)^{(0)} = \frac{1}{2}\left(g^{\mu\eta}g^{\nu\lambda} + g^{\mu\lambda}g^{\nu\eta} - 2g^{\mu\nu}g^{\eta\lambda}\right)^{(0)} = \frac{1}{2}\left(\eta^{\mu\eta}\eta^{\nu\lambda} + \eta^{\mu\lambda}\eta^{\nu\eta} - 2\eta^{\mu\nu}\eta^{\eta\lambda}\right) \ , \quad (65)$$

$$\left(\frac{\partial R}{\partial g_{\mu\nu,\eta\lambda}}\right)^{(0)}\left(g_{\mu\nu,\lambda\alpha}\right)^{(1)} = \left(h^{\lambda\eta}{}_{,\lambda\alpha} - h^{,\eta}{}_{\alpha}\right) = \left(h^{\lambda\eta} - \eta^{\eta\lambda}h\right)_{,\lambda\alpha} \overset{\text{h.g.}}{=} -\frac{1}{2}h^{,\eta}{}_{\alpha} \ , \quad (66)$$

$$\left(\frac{\partial R}{\partial g_{\mu\nu,\eta\lambda}}\right)^{(0)}\left(g_{\mu\nu,\alpha}\right)^{(1)} = \left(h^{\lambda\eta} - \eta^{\eta\lambda}h\right)_{,\alpha} \ , \quad (67)$$

$$\left(\frac{\partial \Box^h R}{\partial g_{\mu\nu,\eta i_0\cdots i_m}}\right)^{(0)} = \left(\frac{\partial \Box^h R}{\partial g_{\mu\nu,\eta i_0\cdots i_q}}\right)^{(0)} = \left(\frac{\partial \Box^h R}{\partial g_{\mu\nu,\eta i_0\cdots i_{2h+1}}}\right)^{(0)} = \eta^{i_2 i_3}\cdots\eta^{i_{2h}i_{2h+1}}\left(\eta^{\mu i_1}\eta^{\nu\eta} - \eta^{\mu\nu}\eta^{\eta i_1}\right) + \cdots \ . \quad (68)$$

Hence, retaining only terms up to h^2 in harmonic gauge, one gets

$$\left(2a_0R + \sum_{k=1}^{p} a_k\Box^k R\right)\frac{\partial R}{g_{\mu\nu,\eta\lambda}} g_{\mu\nu,\lambda\alpha} \overset{\substack{h^2\\ \text{h.g.}}}{=} \frac{1}{4}\left(\sum_{k=0}^{p} a_k\Box^{k+1}h\right)h^{,\eta}{}_{\alpha} + \frac{1}{4}a_0h^{,\eta}{}_{\alpha}\Box h \ , \quad (69)$$

$$-\partial_\lambda\left[\sqrt{-g}\left(2a_0R + \sum_{k=1}^{p} a_k\Box^k R\right)\frac{\partial R}{\partial g_{\mu\nu,\eta\lambda}}\right] g_{\mu\nu,\alpha} \overset{\substack{h^2\\ \text{h.g.}}}{=} a_0\Box h_{,\lambda}\left(h^{\lambda\eta} - \eta^{\eta\lambda}h\right)_{,\alpha} + \frac{1}{2}\sum_{k=1}^{p} a_k\Box^{k+1}h_{,\lambda}\left(h^{\lambda\eta} - \eta^{\lambda\eta}h\right)_{,\alpha} \ , \quad (70)$$

$$\sum_{h=1}^{p}\sum_{q=0}^{2h+1}(-1)^q\,\partial_{i_0\cdots i_q}\left[\sqrt{-g}a_hR\frac{\partial \Box^h R}{\partial g_{\mu\nu,\eta i_0\cdots i_q}}\right] g_{\mu\nu,\alpha} \overset{\substack{h^2\\ \text{h.g.}}}{=} \frac{1}{2}\sum_{h=1}^{p} a_h\Box^{h+1}h_{,\lambda}\left(h^{\eta\lambda} - \eta^{\eta\lambda}h\right)_{,\alpha} + \left(A_p\right)^{\eta}_{\alpha} \ ,$$

(71)

$$\sum_{h=1}^{p}\sum_{j=0}^{2h}\sum_{m=j+1}^{2h+1}(-1)^j\,\partial_{i_0\cdots i_j}\left[\sqrt{-g}a_hR\frac{\partial\Box^hR}{\partial g_{\mu\nu,\eta i_0\cdots i_m}}\right]g_{\mu\nu,i_{j+1}\cdots i_m\alpha}\overset{h^2}{\underset{\text{h.g.}}{=}}\frac{1}{4}\sum_{h=1}^{p}a_h\Box h\Box^h h'^{\eta}{}_{\alpha}$$

$$+\frac{1}{2}\sum_{h=0}^{1}\sum_{j=h}^{p-1+h}\sum_{m=j+1-h}^{p}(-1)^h\,a_m\Box^{m-j}\left(h^{\eta\lambda}-\eta^{\eta\lambda}h\right)_{,i_h\alpha}\Box^{j+1-h}h_{,\lambda}{}^{i_h}+\left(B_p\right)^{\eta}_{\alpha}\ . \quad (72)$$

In Eqs. (71), (72) and (68), we have disregarded the index permutations $(\mu\nu)$ and $(\eta i_1\cdots i_{2h+1})$ because $\left(A_p\right)^{\eta}_{\alpha}$ and $\left(B_p\right)^{\eta}_{\alpha}$ terms vanish if averaged on a suitable spacetime region, according to the appendices in the papers [14, 15]. Then, inserting the equalities (69), (70), (71) and (72) into (56), we find the extra term of pseudo-tensor τ^{η}_{α} at second order, that we call $\tilde{\tau}^{\eta}_{\alpha}$, i.e.

$$\tilde{\tau}^{\eta}_{\alpha}\overset{h^2}{=}\frac{1}{2\chi}\Bigg\{\frac{1}{4}\left(\sum_{k=0}^{p}a_k\Box^{k+1}h\right)h'^{\eta}{}_{\alpha}+\frac{1}{2}\sum_{t=0}^{p}a_t\Box^{t+1}h_{,\lambda}\left(h^{\eta\lambda}-\eta^{\eta\lambda}h\right)_{,\alpha}$$

$$+\frac{1}{2}\sum_{h=0}^{1}\sum_{j=h}^{p}\sum_{m=j}^{p}(-1)^h\,a_m\Box^{m-j}\left(h^{\eta\lambda}-\eta^{\eta\lambda}h\right)_{,\alpha i_h}\Box^{j+1-h}h_{,\lambda}{}^{i_h}$$

$$+\frac{1}{4}\sum_{l=0}^{p}a_l\Box^l\left(h'^{\eta}{}_{\alpha}-\Box h\delta^{\eta}_{\alpha}\right)\Box h+\Theta_{[1,+\infty[}(p)\left[\left(A_p\right)^{\eta}_{\alpha}+\left(B_p\right)^{\eta}_{\alpha}\right]\Bigg\}\ , \quad (73)$$

where conventions used are

$$()_{,\alpha i_0}=()_{,\alpha}\qquad h_{,\lambda}{}^{i_0}=h_{,\lambda}\ .$$

Summing up, we can split the gravitational energy–momentum pseudo-tensor in general relativity and extended gravity part, i.e.

$$\tau^{\eta}_{\alpha}\overset{h^2}{=}\tau^{\eta}_{\alpha|GR}+\tilde{\tau}^{\eta}_{\alpha}\ . \quad (74)$$

For p equal to 0 and 1, we can derive the simplest corrections to the pseudo-tensor $\tilde{\tau}^{\eta}_{\alpha}$. For $p=0$, i.e. $L_g=\left(\overline{R}+a_0R^2\right)\sqrt{-g}$, we get

$$\tau^{\eta}_{\alpha}\overset{h^2}{=}\tau^{\eta}_{\alpha|GR}+\tilde{\tau}^{\eta}_{\alpha}\ ,$$

with

$$\tilde{\tau}^{\eta}_{\alpha}\overset{h^2}{=}\frac{a_0}{2\chi}\left(\frac{1}{2}h'^{\eta}{}_{\alpha}\Box h+h^{\eta}{}_{\lambda,\alpha}\Box h^{\lambda}-h_{,\alpha}\Box h^{\eta}-\frac{1}{4}\left(\Box h\right)^2\delta^{\eta}_{\alpha}\right)\ . \quad (75)$$

While for $p=1$, i.e. $L_g=\left(\overline{R}+a_0R^2+a_1R\Box R\right)\sqrt{-g}$, we find

$$\tau^{\eta}_{\alpha}\overset{h^2}{=}\tau^{\eta}_{\alpha|GR}+\tilde{\tau}^{\eta}_{\alpha}\ ,$$

where corrections are

$$\tilde{\tau}^{\eta}_{\alpha}\overset{h^2}{=}\frac{1}{2\chi}\Bigg\{\frac{1}{4}\left(2a_0\Box h+a_1\Box^2h\right)h'^{\eta}{}_{\alpha}+\frac{1}{2}\left(2a_0\Box h_{,\lambda}+a_1\Box^2h_{,\lambda}\right)\left(h^{\eta\lambda}-\eta^{\eta\lambda}h\right)_{,\alpha}$$

$$+\frac{1}{2}a_1\Box\left(h^{\eta\lambda}-\eta^{\eta\lambda}h\right)_{,\alpha}\Box h_{,\lambda}+\frac{1}{2}a_1\left(h^{\eta\lambda}-\eta^{\eta\lambda}h\right)_{,\alpha}\Box^2 h_{,\lambda}-\frac{1}{2}a_1\left(h^{\eta\lambda}-\eta^{\eta\lambda}h\right)_{,\sigma\alpha}\Box h_{,\lambda}^{\ \ \sigma}$$
$$+\frac{1}{4}a_1\Box h^{,\eta}_{\ \ \alpha}\Box h-\frac{1}{4}\delta^{\eta}_{\alpha}\left[a_0\left(\Box h\right)+a_1\left(\Box^2 h\right)\right]\Box h+\left(A_1\right)^{\eta}_{\alpha}+\left(B_1\right)^{\eta}_{\alpha}\Bigg\}\,. \quad (76)$$

Possible applications of the above results are the following.

5. Emitted power carried away by gravitational waves

In order to calculate the power emitted by an isolated self-gravitating system, we have to average the pseudo-tensor over a suitable space-time domain considering its local conservation (28). The wave solutions of the linearized field equations in vacuum, associated to the Lagrangian (50), can be expressed as [33]

$$h_{\mu\nu}\left(x\right)=\sum_{m=1}^{p+2}\int_{\Omega}\frac{d^3\mathbf{k}}{\left(2\pi\right)^3}\left(B_m\right)_{\mu\nu}\left(\mathbf{k}\right)e^{i\left(k_m\right)_\alpha x^\alpha}+c.c.\,, \quad (77)$$

where c.c. stands for the complex conjugate and

$$\left(B_m\right)_{\mu\nu}\left(\mathbf{k}\right)=\begin{cases}C_{\mu\nu}\left(\mathbf{k}\right) & \text{for} \quad m=1\\ \frac{1}{3}\left[\frac{\eta_{\mu\nu}}{2}+\frac{\left(k_m\right)_\mu\left(k_m\right)_\nu}{k^2_{(m)}}\right]\mathrm{A}_m\left(\mathbf{k}\right) & \text{for} \quad m\geq 2\end{cases}, \quad (78)$$

with $C_{\mu\nu}\left(\mathbf{k}\right)$ the polarization tensor in TT-gauge, $\mathrm{A}_m\left(\mathbf{k}\right)$ the amplitude of the wave at a fixed value $\mathbf{k}$ and $k^\mu_m=\left(\omega_m,\mathbf{k}\right)$ the wave vector with $k^2_m=\omega^2_m-|\mathbf{k}|^2=\mathrm{M}^2$ where $k^2_1=0$ and $k^2_m\neq 0$ for $m\geq 2$.Now, let us calculate the trace of tensor (78). We obtain

$$\left(B_m\right)^\lambda_\lambda\left(\mathbf{k}\right)=\begin{cases}C^\lambda_\lambda\left(\mathbf{k}\right) & \text{for} \quad m=1\\ \mathrm{A}_m\left(\mathbf{k}\right) & \text{for} \quad m\geq 2\end{cases}, \quad (79)$$

If we keep $\mathbf{k}$ fixed, we find the following relations

$$h_{,\alpha}^{\ \ \eta}=2Re\left\{\sum_{j=1}^{p+2}\left(-1\right)\left(k_j\right)_\alpha\left(k_j\right)^\eta A_j e^{ik_jx}\right\}\,, \quad (80)$$

$$\Box^m h_{,\lambda}=2Re\left\{\left(-1\right)^m i\sum_{j=1}^{p+2}\left(k_j\right)_\lambda\left(k^2_j\right)^m A_j e^{ik_jx}\right\}\,, \quad (81)$$

$$\Box^q\left(h^{\eta\lambda}-\eta^{\eta\lambda}h\right)_{,\alpha}=2Re\left\{\left(-1\right)^q i\sum_{l=1}^{p+2}\left(k_l\right)_\alpha\left(k^2_l\right)^q\left[\left(B_l\right)^{\eta\lambda}-\eta^{\eta\lambda}\left(B_l\right)^\rho_\rho\right]e^{ik_lx}\right\}\,, \quad (82)$$

$$\Box^m h_{,\lambda}^{\ \ \sigma}=2Re\left\{\left(-1\right)^{m+1}\sum_{j=1}^{p+2}\left(k_j\right)_\lambda\left(k_j\right)^\sigma\left(k^2_j\right)^m A_j e^{ik_jx}\right\}\,, \quad (83)$$

$$\Box^q\left(h^{\eta\lambda}-\eta^{\eta\lambda}h\right)_{,\sigma\alpha}=2Re\left\{\left(-1\right)^{q+1}\sum_{l=1}^{p+2}\left(k_l\right)_\sigma\left(k_l\right)_\alpha\left(k^2_l\right)^q\left[\left(B_l\right)^{\eta\lambda}-\eta^{\eta\lambda}\left(B_l\right)^\rho_\rho\right]e^{ik_lx}\right\}\,, \quad (84)$$

$$\Box^n h=2Re\left\{\left(-1\right)^n\sum_{r=2}^{p+2}\left(k^2_r\right)^n A_r e^{ik_rx}\right\}\,. \quad (85)$$

Let us choose a spacetime domain Ω such that $|\Omega| \gg \frac{1}{|k|}$ (see [22]). We can average the gravitational energy–momentum pseudo-tensor τ^{η}_{α} over our region and all integrals, including terms such as $e^{i\left(k_i-k_j\right)_\alpha x^\alpha}$, vanish thanks to the following identities

$$Re\{f\}Re\{g\} = \frac{1}{2}Re\{fg\} + \frac{1}{2}Re\{f\bar{g}\}\ , \tag{86}$$

$$(k_l)_\lambda \left[(B_l)^{\eta\lambda} - \eta^{\eta\lambda}(B_l)^{\rho}_{\rho}\right] = -\frac{(k_l)^\eta}{2}A_l\ . \tag{87}$$

In the harmonic gauge, after averaging and performing some algebraic manipulations, we get (see Ref. [14])

$$\begin{aligned}
\left\langle \Box^m h_{,\lambda} \Box^q \left(h^{\eta\lambda} - \eta^{\eta\lambda} h\right)_{,\alpha}\right\rangle &= (-1)^{m+q+1} \sum_{l=2}^{p+2} (k_l)_\alpha (k_l)^\eta \left(k_l^2\right)^{(m+q)} |A_l|^2\ ,\\
\left\langle \Box^m h^{\ \sigma}_{,\lambda} \Box^q \left(h^{\eta\lambda} - \eta^{\eta\lambda} h\right)_{,\sigma\alpha}\right\rangle &= (-1)^{m+q+1} \sum_{l=2}^{p+2} (k_l)_\alpha (k_l)^\eta \left(k_l^2\right)^{(m+q)+1} |A_l|^2\ ,\\
\left\langle \Box^q h^{,\eta}_{\ \alpha} \Box^m h\right\rangle &= 2\,(-1)^{m+q+1} \sum_{r=2}^{p+2} (k_r)_\alpha (k_r)^\eta \left(k_r^2\right)^{(m+q)} |A_r|^2\ ,\\
\langle \Box^m h \Box h\rangle &= 2\,(-1)^{m+1} \sum_{j=2}^{p+2} \left(k_j^2\right)^{m+1} |A_j|^2\ ,\\
\langle \left(A_p\right)^{\eta}_{\alpha}\rangle &= \langle \left(B_p\right)^{\eta}_{\alpha}\rangle = 0\ .
\end{aligned} \tag{88}$$

A basis of polarization tensors is explicitly shown in Ref. [33]. Hence, we calculate the average value of the energy–momentum pseudo-tensor taking into account the equalities (88)

$$\begin{aligned}
\left\langle \tau^{\eta}_{\alpha}\right\rangle &= \frac{1}{2\chi}\left[(k_1)^\eta (k_1)_\alpha \left(C^{\mu\nu}C^*_{\mu\nu} - \frac{1}{2}|C^{\lambda}_{\lambda}|^2\right)\right]\\
&+ \frac{1}{2\chi}\left[\left(-\frac{1}{6}\right)\sum_{j=2}^{p+2}\left(\left(k_j\right)^\eta \left(k_j\right)_\alpha - \frac{1}{2}k_j^2 \delta^\eta_\alpha\right)|A_j|^2\right]\\
&+ \frac{1}{2\chi}\left\{\left[\sum_{l=0}^{p}(l+2)(-1)^l a_l \sum_{j=2}^{p+2}\left(k_j\right)^\eta \left(k_j\right)_\alpha \left(k_j^2\right)^{l+1}|A_j|^2\right]\right.\\
&\left. - \frac{1}{2}\sum_{l=0}^{p}(-1)^l a_l \sum_{j=2}^{p+2}\left(k_j^2\right)^{l+2}|A_j|^2 \delta^\eta_\alpha\right\}.
\end{aligned} \tag{89}$$

In the momentum space, the first mode k_1 and the residual modes k_m are express in TT-gauge and harmonic gauge respectively, i.e.,

$$\begin{cases} (k_1)_\mu C^{\mu\nu} = 0 \quad \wedge \quad C^{\lambda}_{\lambda} = 0 & \text{if} \quad m = 1 \\ (k_m)_\mu (B_m)^{\mu\nu} = \frac{1}{2}(B_m)^{\lambda}_{\lambda} k^\nu & \text{if} \quad m \geq 2 \end{cases}\ . \tag{90}$$

Let us now consider a gravitational wave propagating along $+z$-direction at $\mathbf{k}$ fixed, with 4-wave vector given by $k^\mu = (\omega, 0, 0, k_z)$, where $\omega_1^2 = k_z^2$ if $k_1^2 = 0$, and $k_m^2 = m^2 = \omega_m^2 - k_z^2$ otherwise for $k_z > 0$. Consequently, the averaged time-space tensorial component ,which can be seen as the flux of gravitational energy along the z axis through the surface delimiting the Ω domain, reads

$$\left\langle \tau^3_0 \right\rangle = \frac{c^4}{8\pi G}\omega_1^2\left(C_{11}^2 + C_{12}^2\right) + \frac{c^4}{16\pi G}\left[\left(-\frac{1}{6}\right)\sum_{j=2}^{p+2}\omega_j k_z |A_j|^2 + \sum_{l=0}^{p}(l+2)(-1)^l a_l \sum_{j=2}^{p+2}\omega_j k_z m_j^{2(l+1)}|A_j|^2\right]. \quad (91)$$

Finally, we can calculate the emitted power per unit solid angle Ω, radiated by the localized sources, in a direction $\hat{x}$ at fixed $\mathbf{k}$. Choosing a suitable gauge, for the local conservation of the energy–momentum pseudo-tensor (28), the power can be written as

$$\frac{dP}{d\Omega} = r^2 \hat{x}^i \left\langle \tau^i_0 \right\rangle . \quad (92)$$

By ranging the index p of the pseudo-tensor (91) over $\{0, 1, 2\}$, we obtain the following three cases for p=0

$$\left\langle \tau^3_0 \right\rangle = \frac{c^4\omega_1^2}{8\pi G}\left[C_{11}^2 + C_{12}^2\right] + \frac{c^4}{16\pi G}\left\{\left(-\frac{1}{6}\right)\omega_2|A_2|^2 k_z + 2a_0\omega_2 m_2^2|A_2|^2 k_z\right\}, \quad (93)$$

for p=1

$$\left\langle \tau^3_0 \right\rangle = \frac{c^4\omega_1^2}{8\pi G}\left[C_{11}^2 + C_{12}^2\right] + \frac{c^4}{16\pi G}\left\{\left(-\frac{1}{6}\right)\left(\omega_2|A_2|^2 + \omega_3|A_3|^3\right)k_z + 2a_0\left[\left(\omega_2 m_2^2|A_2|^2 + \omega_3 m_3^2|A_3|^2|^2\right)k_z\right] - 3a_1\left[\left(\omega_2 m_2^4|A_2|^2 + \omega_3 m_3^4|A_3|^2\right)k_z\right]\right\}, \quad (94)$$

and for p=2

$$\begin{aligned}\left\langle \tau^3_0 \right\rangle = \frac{c^4\omega_1^2}{8\pi G}\left[C_{11}^2 + C_{12}^2\right] + \frac{c^4}{16\pi G}\Big\{\left(-\frac{1}{6}\right)\left(\omega_2|A_2|^2 + \omega_3|A_3|^3 + \omega_4|A_4|^2\right)k_z \\ + 2a_0\left[\left(\omega_2 m_2^2|A_2|^2 + \omega_3 m_3^2|A_3|^2 + \omega_4 m_4^2|A_4|^2\right)k_z\right] \\ - 3a_1\left[\left(\omega_2 m_2^4|A_2|^2 + \omega_3 m_3^4|A_3|^2 + \omega_4 m_4^4|A_4|^2\right)k_z\right] \\ + 4a_2\left[\left(\omega_2 m_2^6|A_2|^2 + \omega_3 m_3^6|A_3|^2 + \omega_4 m_4^6|A_4|^2\right)\right]\Big\}.\end{aligned} \quad (95)$$

By Eqs. (93), (94) and (95), the first term returns the result of general relativity while the corrections strongly depends on the value p. In any context where corrections to general relativity can be studied, this approach could constitute a paradigm for searching for higher-order effects.

6. Conclusions

In view of solving incongruences of general relativity at ultraviolet and infrared scales (e.g. quantum gravity, dark matter and dark energy issues, etc.), many proposals have been formulated to extend or modify the Einstein theory by improving the geometric structure by curvature, torsion and non-metricity invariants [34–43]. In this context, the present paper is devoted to the generalization of the gravitational energy–momentum pseudo-tensor τ^η_α to higher-order theories where, in particular, terms like $\Box^k$are present into the gravitational Lagrangian, such as $L_g = (\overline{R} + a_0R^2 + \sum_{k=1}^{p} a_k R\Box^k R)\sqrt{-g}$. We have found that, in the framework of these theories, the local conservation of energy–momentum complex holds. It has been shown that τ^η_α is an affine and non-covariant object because it evolves like a tensor only under linear transformations but not under general coordinate transformations. The second-order perturbed pseudo-tensor, in h^2, of higher-order gravity has been obtained and, thanks to its average over a four-dimensional domain, under a given gauge, and its local conservation, the power emitted by a radiant gravitational source can be calculated. Hence, from the modified gravitational wave (77) it is possible to express the average emitted power in terms of the amplitudes $A_m(\mathbf{k})$, $C_{11}(\mathbf{k})$ and $C_{22}(\mathbf{k})$, and the free parameters a_m. In the cases discussed here, i.e. $p = 0, 1, 2$, clearly the corrections to the general relativity are evident.

In conclusion, the gravitational energy–momentum pseudo-tensor is a fundamental tool to seek for further gravitational wave polarizations and corrections to the quadrupole formula. In this context, it could be useful to fix the features of theories of gravity eventually extending or modifying general relativity [44]. Forthcoming astrophysical observations, in particular the so called multimessenger astronomy, could be extremely relevant in this debate.

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