Filomat 37:25 (2023), 8635–8640
https://doi.org/10.2298/FIL2325635R

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Surfaces defined by bending of knots

Svetozar R. Rančić[a], Marija S. Najdanović[b], Ljubica S. Velimirović[a]

[a] *University of Niš, Faculty of Sciences and Mathematics, Serbia*
[b] *University of Priština in Kosovska Mitrovica, Faculty of Sciences and Mathematics, Kosovska Mitrovica, Serbia*

Abstract. We consider the definition of the infinitesimal bending of a curve as a vector parametric equation of a surface defined by two free variables: one of them is free variable u which define curve and another is bending parameter ϵ. In this way, while being bent curve is deformed and moved through the space forming a surface. If infinitesimal bending field is of constant intensity, deformed curves form a ruled surface that represents a ribbon. In particular, we consider surfaces obtain by bending of knots both analytically and graphically. We pay attention to the torus knot and possibility of its infinitesimal bending so that the surface determined by bending is a part of the initial torus.

1. Introduction

Let us consider continuous regular curve $C : \mathbf{r} = \mathbf{r}(u),\ u \in J \subseteq \mathbb{R}$ included in a family of the curves

$$C_\epsilon : \tilde{\mathbf{r}}(u, \epsilon) = \mathbf{r}_\epsilon(u) = \mathbf{r}(u) + \epsilon \mathbf{z}(u), \quad u \in J, \quad \epsilon \geq 0, \epsilon \to 0, \tag{1}$$

where u is a real parameter and we get C for $\epsilon = 0$ ($C = C_0$). Family of curves C_ϵ is called **infinitesimal bending of a curve** C if the difference of the squares of the line elements of the initial and deformed curves is an infinitesimal of a higher order than ϵ, i.e. if the following condition is valid

$$ds_\epsilon^2 - ds^2 = o(\epsilon). \tag{2}$$

The field $\mathbf{z} = \mathbf{z}(u)$, $\mathbf{z} \in C^1$ is corresponding **infinitesimal bending field**.

Infinitesimal bending problems are interesting not only from the aspect of differential geometry, but also in many other disciplines where they can be applied. Many papers are dedicated to the theory of infinitesimal bending. Some of them are [1]-[3], [5], [6], [8]-[16], [18]-[20].

According to [5], necessary and sufficient condition for $\mathbf{z}(u)$ to be an infinitesimal bending field of a curve C is to be

$$d\mathbf{r} \cdot d\mathbf{z} = 0, \tag{3}$$

2020 *Mathematics Subject Classification*. Primary 53A04, 53A05, Secondary 53C45, 57K10
Keywords. infinitesimal bending, curve, knot, ruled surface, ribbon.
Received: 24 February 2023; Accepted: 15 March 2023
Communicated by Zoran Rakić and Mića Stanković
Research supported by the Serbian Ministry of Education, Science and Technological Development under the research grants 451-03-47/2023-01/200123 and 451-03-47/2023-01/200124 and by the project IJ-0203 of Faculty of Sciences and Mathematics, University of Priština in Kosovska Mitrovica.
Email addresses: rancicsv@yahoo.com (Svetozar R. Rančić), marija.najdanovic@pr.ac.rs (Marija S. Najdanović), vljubica@pmf.ni.ac.rs (Ljubica S. Velimirović)

where $\cdot$ stands for the scalar product in $\mathbb{R}^3$. An explicit expression for the infinitesimal bending field is

$$\mathbf{z}(u) = \int [p(u)\mathbf{n}_1(u) + q(u)\mathbf{n}_2(u)]\,du, \tag{4}$$

where $p(u)$ and $q(u)$, are arbitrary integrable functions and vectors $\mathbf{n}_1(u)$ and $\mathbf{n}_2(u)$ are respectively unit principal normal and binormal vector fields of the curve C [19].

Under infinitesimal bending of curves each line element gets non-negative addition, which is the infinitesimal value of the order higher than the first with respect to ϵ [18], i. e.

$$ds_\epsilon - ds = o(\varepsilon) \geq 0. \tag{5}$$

A closed **ribbon** is a smooth mapping (or the image set) of an annulus $S^1 \times [0,1]$ into three-dimensional Euclidean space $\mathbb{R}^3$, where the sets $u \times [0,1]$ are mapped to line segments all of the same length [4]. If a knot K (i.e. a simple closed curve) is given by a regular parametrization $\mathbf{r} = \mathbf{r}(u)$, with a smooth unit vector field $\mathbf{e} = \mathbf{e}(u)$ based along K, we may define a ribbon of width R associated to the knot K and the field $\mathbf{e}$ as the set of all points

$$\mathbf{r}(u) + v\mathbf{e}(u),\ v \in [0, R]. \tag{6}$$

For large R, ribbons and their outer edge curves may have self-intersections. On the other hand, a thin ribbon does not self-intersect and the ribbon itself gives an isotopy of the two boundary curves. Thin ribbons are very important in the study of DNA [17].

It is interesting to notice: during infinitesimal bending of a knot, under the conditions $||\mathbf{z}|| = 1$ and $\epsilon \in [0, R]$, the deformed curves form a ruled surface that represents a ribbon of width R.

2. Surfaces defined by bending of curves

Let us observe the infinitesimal bending of a regular curve

$$C_\epsilon : \tilde{\mathbf{r}}(u,\epsilon) = \mathbf{r}_\epsilon(u) = \mathbf{r}(u) + \epsilon\mathbf{z}(u), \tag{7}$$

where $\mathbf{z}(u)$ is an infinitesimal bending field.

We can consider the definition of the infinitesimal bending of a curve as a vector parametric equation of a surface defined by two free variables: one of them is free variable which define curve u and another is bending parameter ϵ.

There is a connection between infinitesimal bending of curves and ruled surfaces. Namely, let us consider ruled surface

$$S: \boldsymbol{\rho}(u,v) = \mathbf{r}(u) + v\mathbf{e}(u), \quad u \in J \subseteq \mathbb{R}, \quad v \in \mathbb{R}, \quad ||\mathbf{e}(u)|| = 1,$$

with directrix $C:\ \mathbf{r} = \mathbf{r}(u)$ and generatrices in the direction of $\mathbf{e}(u)$. If the directrix C is also the striction line of ruled surface S (see [7]), then the condition

$$\dot{\mathbf{r}}(u) \cdot \dot{\mathbf{e}}(u) = 0$$

is valid, where "dot" denotes the derivative by u. Therefore, $\mathbf{e}(u)$ is infinitesimal bending field and the family of bent curves (infinitesimal bending)

$$C_\epsilon:\ \tilde{\mathbf{r}}(u,\epsilon) = \mathbf{r}_\epsilon(u) = \mathbf{r}(u) + \epsilon\mathbf{e}(u)$$

belongs to the ruled surface S. Note that each curve of the family C_ϵ is "parallel" with C, i.e. the segment of each generatrix between C and C_ϵ is of the same length:

$$||\tilde{\mathbf{r}}(u_1,\epsilon) - \mathbf{r}(u_1)|| = ||\epsilon\mathbf{e}(u_1)|| = \epsilon = ||\epsilon\mathbf{e}(u_2)|| = ||\tilde{\mathbf{r}}(u_2,\epsilon) - \mathbf{r}(u_2)||,$$

since $\|\mathbf{e}(u_1)\| = \|\mathbf{e}(u_2)\| = 1$. For $\epsilon \in [0, R]$, corresponding knot infinitesimal bending determine a ribbon of width R.

For characterization of a ruled surface obtained by bending with some examples we refer to [6]. Infinitesimal bending of curves on the ruled surfaces is discussed in [11, 13].

While being bent, curve is deformed and moved through the space. In next examples we will present some of such surfaces. We use expresion (4) for infinitesimal bending field. Bending field is defined by integral whose sub integral function includes arbitrary functions: p and q. Knot visualization and obtaining 3D model is done by using OpenGL.

The first example is a surface obtained by bending of the trefoil knot given by $\mathbf{r}(u) = (\sin(u) + 2\sin(2u), \cos(y) - 2\cos(2u), -\sin(3u))$, see Fig. 1. Bending field is defined by Eq. (4) for $p(u) = \cos(3u)$ and $q(u) = \sin(6u)$.

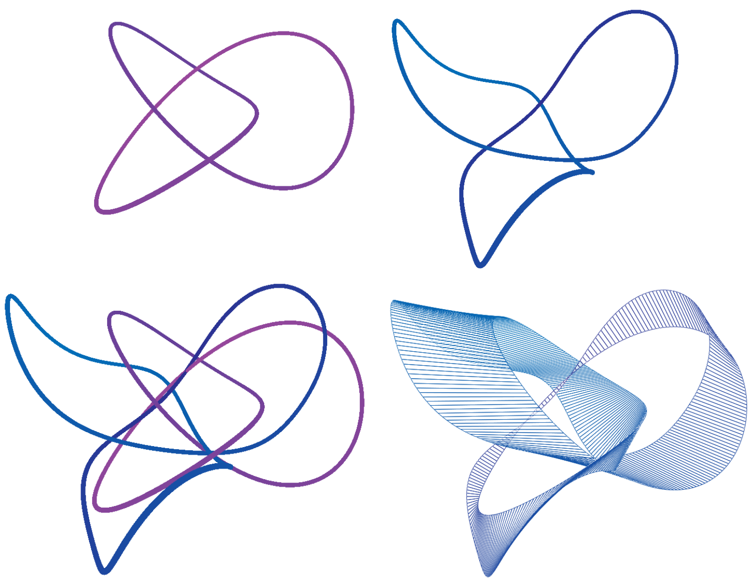

Figure 1: Surface on trefoil knot: basic and infinitesimally bent with $\epsilon = 2.0$.

The second example is a surface obtained by bending of $p3q2$ torus knot, given by scalar parametric equations: $x = (\cos(2u) + 2) \cdot \cos(3u)$, $y = (\cos(2u) + 2) \cdot \sin(3u)$, $z = -\sin(2u)$.

Bending field is defined by $p(u) = \cos(2u)$ and $q(u) = \sin(2u)$. For the bent curves see Fig. 2, obtained surface is on Fig. 3.

3. Infinitesimal bending on the torus

We posed the question whether it is possible to infinitesimally bend the torus knot so that the surface determined by bending is a part of the initial torus. Regarding that the following theorem holds.

Theorem 3.1. [10] *Let C : $\mathbf{r} : (t_1, t_2) \to R^3$ be a regular continuous curve on the torus S. There is no non-trivial vector field $\mathbf{z}(t)$ that includes the given curve under infinitesimal bending into the family of curves C_ϵ : $\mathbf{r}_\epsilon = \mathbf{r}(t) + \epsilon\mathbf{z}(t)$, $\epsilon \geq 0$, $\epsilon \to 0$, on the torus S.*

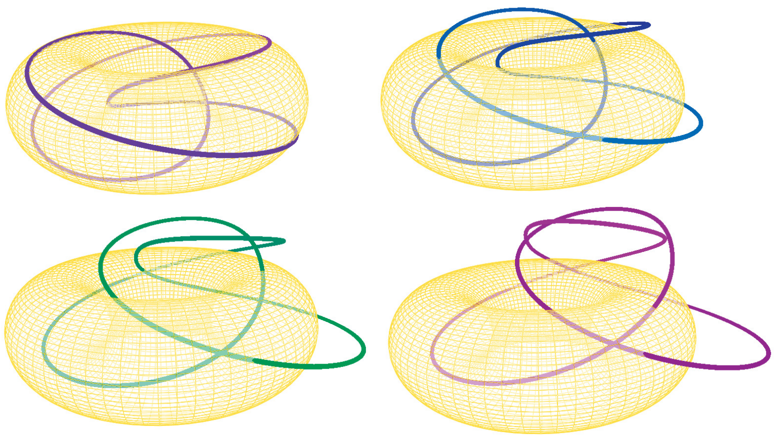

Figure 2: p3q2 knot: basic and infinitesimally bent with $\epsilon = 0.6, 1.2, 1.8$.

The next step is to weaken the condition from the previous theorem. Namely, let

$$C: \begin{array}{l} x(u) = (\cos(qu) + a) \cdot \cos(pu) \\ y(u) = (\cos(qu) + a) \cdot \sin(pu) \\ z(u) = -\sin(qu) \end{array} \tag{8}$$

be (p, q) torus knot, $0 \leq u \leq 2\pi$, which lies on the torus

$$S: \ (a - \sqrt{x^2 + y^2})^2 + z^2 = 1^2.$$

Let

$$C_\epsilon: \begin{array}{l} x_\epsilon(u) = (\cos(qu) + a) \cdot \cos(pu) + \epsilon z_1(u) \\ y_\epsilon(u) = (\cos(qu) + a) \cdot \sin(pu) + \epsilon z_2(u) \\ z_\epsilon(u) = -\sin(qu) + \epsilon z_3(u) \end{array} \tag{9}$$

be infinitesimal bending of C, where $\mathbf{z}(u) = (z_1(u), z_2(u), z_3(u))$ is infinitesimal bending field. If we put

$$F(x, y, z) = (a - \sqrt{x^2 + y^2})^2 + z^2 - 1^2,$$

then the torus knot satisfies the implicit equation of the torus S, $F(x(u), y(u), z(u)) = 0$. We request that the infinitesimal bending is approximately (with a given precision) on the torus S, i.e. we set the condition

$$F(x_\epsilon(u), y_\epsilon(u), z_\epsilon(u)) = o(\epsilon), \tag{10}$$

where $o(\epsilon)$ is an infinitesimal of higher order with respect to ϵ. That means it must be valid

$$(a - \sqrt{((\cos(qu) + a) \cdot \cos(pu) + \epsilon z_1(u))^2 + ((\cos(qu) + a) \cdot \sin(pu) + \epsilon z_2(u))^2})^2 + (-\sin(qu) + \epsilon z_3(u))^2 - 1 = 0,$$

wherefrom, after some calculations and using Maclaurin series, one obtains

$$\cos(pu)\cos(qu)z_1(u) + \sin(pu)\cos(qu)z_2(u) - \sin(qu)z_3(u) = 0. \tag{11}$$

Thus we proved the following theorem.

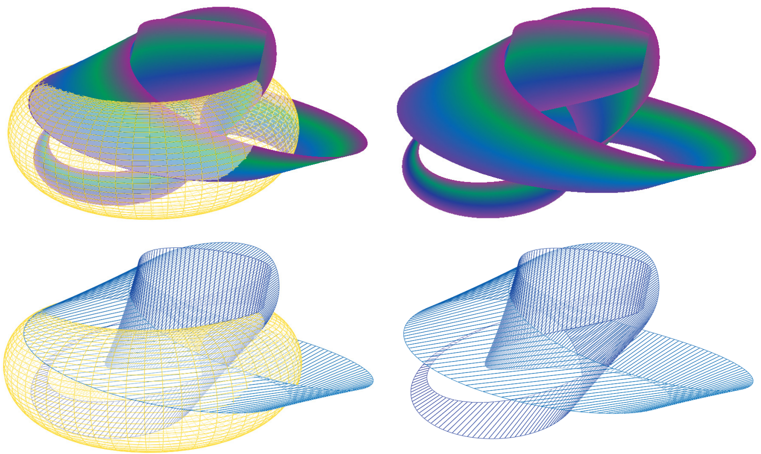

Figure 3: Surface on p3q2 knot: basic and infinitesimally bent with $\epsilon = 0.6, 1.2, 1.8$.

Theorem 3.2. *The necessary and sufficient condition for the field* $\mathbf{z}(u) = (z_1(u), z_2(u), z_3(u))$ *to include* (p, q) *torus knot (8) into the family of deformed curves on the torus with a given precision is that Eq. (11) is satisfied.*

From Eq. (11) we obtain

$$z_3(u) = [\cos(pu)z_1(u) + \sin(pu)z_2(u)]\cot(qu). \tag{12}$$

Therefore, we are looking for the field

$$\mathbf{z}(u) = (z_1(u), z_2(u), [\cos(pu)z_1(u) + \sin(pu)z_2(u)]\cot(qu))$$

which satisfies the necessary and sufficient condition of infinitesimal bending, i.e.

$$\dot{\mathbf{r}}(u) \cdot \dot{\mathbf{z}}(u) = 0.$$

Since

$$\dot{\mathbf{r}}(u) = (-q\sin(qu)\cos(pu) - p(\cos(qu) + a)\sin(pu), -q\sin(qu)\sin(pu) + p(\cos(qu) + a)\cos(pu), -q\cos(qu))$$

$$\begin{aligned}\dot{\mathbf{z}}(u) = (&\dot{z}_1(u), \dot{z}_2(u), (-p\sin(pu)z_1(u) + \cos(pu)\dot{z}_1(u) + p\cos(pu)z_2(u) + \sin(pu)\dot{z}_2(u))\cot(qu)\\ &+ (\cos(pu)z_1(u) - q\sin(pu)z_2(u))\frac{1}{\sin^2(qu)})\end{aligned}$$

using one of the functions $z_1(u)$ and $z_2(u)$ arbitrarily, for instance, $z_1(u)$, we get the other by solving the following linear differential equation:

$$A(u)\dot{z}_2(u) + B(u)z_2(u) + C(u) = 0, \tag{13}$$

where

$$A(u) = -q\sin(qu)\sin(pu) + p(\cos(qu) + a)\cos(pu) - q\cos(qu)\cot(qu)\sin(pu), \tag{14}$$

$$B(u) = -pq\cos(pu)\cos(qu)\cot(qu) + q^2\frac{\cos(qu)}{\sin^2(qu)}\sin pu, \tag{15}$$

$$\begin{aligned}C(u) = &[-q\sin(qu)\cos(pu) - p(\cos(qu) + a)\sin(pu)]\dot{z}_1(u)\\ &- q\cos(qu)[-p\sin(pu)z_1(u) + \cos(pu)\dot{z}_1(u)]\cot(qu) + q^2\frac{\cos(qu)}{\sin^2(qu)}\cos(pu)z_1(u).\end{aligned} \tag{16}$$

Therefore, the following theorem holds.

Theorem 3.3. *Vector field*

$$\mathbf{z}(u) = (z_1(u), z_2(u), [\cos(pu)z_1(u) + \sin(pu)z_2(u)]\cot(qu)), \tag{17}$$

where $z_1(u)$ is arbitrary real continuous differentiable function and $z_2(u)$ is given by

$$z_2(u) = e^{-\int \frac{B(u)}{A(u)}\,du}\Big[c - \int \frac{C(u)}{A(u)} e^{\int \frac{B(u)}{A(u)}\,du}\,du\Big], \tag{18}$$

c is a constant, $A(u)$, $B(u)$ and $C(u)$ are given in Eqs. (14), (15) and (16), respectively, is infinitesimal bending field of (p, q) torus knot under which all bent curves are on the initial torus with a given precision.

Example 3.4. *In particular, let $z_3(u) = 0$. Then the condition (11) reduces to* $\cos(pu)z_1(u) + \sin(pu)z_2(u) = 0$, *wherefrom we have $z_1(u) = -\tan(pu)z_2(u)$ and*

$$\mathbf{z}(u) = (-\tan(pu)z_2(u), z_2(u), 0). \tag{19}$$

From the condition $\dot{\mathbf{r}}(u) \cdot \dot{\mathbf{z}}(u) = 0$, we obtain homogenous linear differential equation

$$\dot{z}_2(u) + \Big(\frac{q\sin(qu)}{\cos(qu) + a} + \frac{p\sin(pu)}{\cos(pu)}\Big)z_2(u) = 0$$

whose solution is

$$z_2(u) = c(\cos(qu) + a)\cos(pu), \tag{20}$$

c is a constant. Finally

$$\mathbf{z}(u) = (-c\tan(pu)(\cos(qu) + a)\cos(pu), c(\cos(qu) + a)\cos(pu), 0). \tag{21}$$

It is easy to check that this vector field satisfies the following conditions: $\dot{\mathbf{r}} \cdot \dot{\mathbf{z}} = 0$ and $F(x_\epsilon(u), y_\epsilon(u), z_\epsilon(u)) = o(\epsilon)$.

References

[1] A. D. Aleksandrov, O beskonechno malyh izgibaniyah neregulyarnyh poverhnostei, Matem. sbornik 1(43), 3 (1936) 307-321.
[2] V. Alexandrov, On the total mean curvature of a nonrigid surface, Siberian Mathematical Journal **50**, 5, (2009) 757–759.
[3] O. Belova, J. Mikeš, M. Sherkuziyev, An Analytical Inflexibility of Surfaces Attached Along a Curve to a Surface Regarding a Point and Plane, Results Math. 76(56) (2021).
[4] S. C. Brooks, O. Durumeric, J. Simon, Knots connected by wide ribbons, Journal of Knots Theory and Its Ramifications Vol. 28, No. 12 (2019), 1950071 (22 pages).
[5] N. Efimov, Kachestvennye voprosy teorii deformacii poverhnostei, UMN 3.2 (1948) 47–158.
[6] U. Gözütok, ; H. A. Çoban, Y. Sağiroğlu, Ruled surfaces obtained by bending of curves,Turkish Journal of Mathematics **139**, 44 (2020) 300–306.
[7] A. Gray, Modern differential geometry of curves and surfaces with Mathematica, CRC Press, Boca Raton, 1998.
[8] I. Hinterleitner, J. Mikeš, J Stránská, Infinitesimal f-planar transformations, J. Russ. Math. **52(4)**,(2008) 13–18.
[9] I. Ivanova-Karatopraklieva, I. Sabitov, Bending of surfaces II, J. Math. Sci., New York 74(3) (1995) 997–1043.
[10] M. D. Maksimović, S. R. Rančić, M. S. Najdanović, Lj. S. Velimirović, E. S. Ljajko, On the torsional energy of torus knots under infinitesimal bending, Analele Stiintifice ale Universitatii Ovidius Constanta, Seria Matematica, **31(1)**, (2023) 181–197.
[11] M. Najdanović, Lj. Velimirović, Infinitesimal bending of curves on the ruled surfaces, University thought - Publication in Natural Sciences, Vol.8, No.1, (2018), 46-51,
[12] M. S. Najdanović, S. R. Rančić, L. Kauffman, L., Lj.S. Velimirović,The total curvature of knots under second-order infinitesimal bending, Journal of Knot Theory and Its Ramifications **28**, 01, (2019) 1950005.
[13] M. Najdanović, M. Maksimović, Lj. Velimirović, Curves on ruled surfaces under infinitesimal bending, Bulletin of Natural Sciences Research, Vol.11, No.1, (2021), 38-43.
[14] M. S. Najdanović, Lj. S. Velimirović, S. R. Rančić, The total torsion of knots under second order infinitesimal bending, Appl. Anal. Discrete Math. **15**, (2021) 283-294.
[15] S. Rančić, M. Najdanović, Lj. Velimirović, Total normalcy of knots, Filomat **33**, 4, (2019) 1259–1266.
[16] L. Rýparová, J. Mikeš, Infinitesimal rotary transformation, Filomat **33(4)**, (2019) 1153-1157.
[17] D. W. L. Sumners, Knot Theory and DNA, New Scientific Applications of Geometry and Topology, Proc. Symp. Applied Mathematics, ed. D. W. L. Sumners, American Mathematical Society, Vol. 45 (AMS, Providence, RI, 1992), 39–72.
[18] I. Vekua, Obobschennye analiticheskie funkcii, Moskva, 1959.
[19] Lj. Velimirović, Change of geometric magnitudes under infinitesimal bending, Facta Universitates, 3(11) (2001)135–148.
[20] Lj. Velimirović, S. Minčić, M. Stanković, Infinitesimal rigidity and flexibility of a non-symmetric affine connection space, Eur. J. Comb. 31(4) (2010) 1148-1159.