Modulus of continuity of normal derivative of a harmonic functions at a boundary point

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Abstract. We give sufficient conditions which ensure that harmonic extension \( u = P[f] \) to the upper half space \( \{(x, y) \mid x \in \mathbb{R}^n, y > 0\} \) of a function \( f \in L^p(\mathbb{R}^n) \) satisfies estimate \( \frac{\partial u}{\partial y}(x, y) \leq C \omega(y)/y \) for every \( x \in E \subset \mathbb{R}^n \), where \( \omega \) is a majorant. The conditions are expressed in terms of behaviour of the Riesz transforms \( R_j f \) of \( f \) near points in \( E \). We briefly investigate related questions for the cases of harmonic and hyperbolic harmonic functions in the unit ball.

1. Introduction and preliminaries

It is easily seen that if the majorant of \( \varphi(x) \in C(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \) is \( \omega \), and if \( u(x, y) \) is the harmonic extension of \( \varphi \) to the upper half space, then for each \( y > 0 \) the function \( \varphi_y(x) = u(x, y) \) has the same majorant as \( \varphi \). This need not be true for functions \( \varphi_x(y) = u(x, y) \) defined on \( [0, +\infty) \). For example, if \( \varphi \) is Lipschitz continuous it is not necessarily true that \( u \) is also Lipschitz continuous. Therefore information on the vertical derivative of \( u \) is of interest in obtaining results on modulus of continuity of \( u(x, y) \). We point out that in that respect hyperbolic Laplacian has better properties regarding preservation of Lipschitz continuity, see [5]. In addition, information on behaviour of normal derivatives is relevant when studying mappings which are at the same time harmonic and quasiconformal, see [4] for the case when the boundary is not flat.

In the case \( n = 1 \) important role is played by the harmonic conjugate of \( \varphi \) and by the Hilbert transform of \( \varphi \). In our general case the corresponding role is played by a conjugate system of harmonic functions, see (4) and by the Riesz transforms \( R_j \), which are multi dimensional analogues of Hilbert transform.

We denote the upper half space by \( \mathbb{H}^{n+1} = \{(x, y) \mid x \in \mathbb{R}^n, y > 0\} \), the boundary of \( \mathbb{H}^{n+1} \) is identified with \( \mathbb{R}^n \). The surface measure of the unit sphere \( S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\} \) is \( n \omega_n \), where \( \omega_n \) is the volume of the unit ball \( \mathbb{B}^n \) in \( \mathbb{R}^n \). The Poisson kernel for the upper half space is

\[
P(x, y) = P_y(x) = c_n \frac{y}{(y^2 + |x|^2)^{(n+1)/2}}, \quad x \in \mathbb{R}^n, \quad y > 0
\]

where

\[
c_n = \Gamma\left(\frac{n+1}{2}\right) \pi^{-\frac{n+1}{2}}.
\]
The harmonic extension of a function \( \varphi \) on \( \mathbb{R}^n \) to \( \mathbb{H}^{n+1} \) is

\[
P[\varphi](x, y) = c_n \int_{\mathbb{R}^n} \varphi(t) \frac{y}{(y^2 + |x-t|^2)^{(n+1)/2}} \, dt
\]

Let \( R_j, 1 \leq j \leq n \), be the Riesz operators. They are defined by the following formula

\[
R_jf(x) = \lim_{\epsilon \to 0} \frac{\Gamma \left( \frac{n+1}{2} \right)}{\pi^{(n+1)/2}} \int_{|y|>\epsilon} \frac{y_j f(x-y)}{|y|^{n+1}} \, dy, \quad f \in L^p(\mathbb{R}^n), \quad 1 \leq p < \infty.
\]

These operators are bounded linear operators on \( L^p(\mathbb{R}^n) \) for \( 1 < p < \infty \).

We say that a function \( \omega : [0, +\infty) \to \mathbb{R} \) is a majorant if it is continuous, concave and increasing on \([0, +\infty)\), strictly positive on \((0, +\infty)\) and \( \omega(0) = 0 \).

**Lemma 1.1.** Let \( \omega \) be a majorant. Set \( M_\beta = (\beta - 1)^{-1} + (\beta - 2)^{-1} \), where \( \beta > 2 \). Then

\[
\int_1^\infty \frac{\omega(ys)}{s^\beta} \, ds \leq M_\beta \omega(y), \quad y > 0.
\]

**Proof.** Set, for \( C > 0 \) and \( t > 0 \), \( \omega_{C,t}(x) = Cx \) if \( 0 \leq x \leq t \) and \( \omega_{C,t}(x) = Ct \) if \( x > t \). Then \( \omega_{C,t} \) is a majorant and we have

\[
\int_1^\infty \frac{\omega_{C,t}(ys)}{s^\beta} \, ds = Ct \int_1^\infty \frac{ds}{s^\beta} = \frac{Ct}{\beta - 1} = \frac{1}{\beta - 1} \omega(y), \quad y \geq t.
\]

If \( 0 < y < t \), then we have

\[
\int_1^t \frac{\omega_{C,t}(ys)}{s^\beta} \, ds = \int_1^{t/y} \frac{\omega_{C,t}(ys)}{s^\beta} \, ds + \int_{t/y}^\infty \frac{\omega_{C,t}(ys)}{s^\beta} \, ds
\]

\[
= Cy \int_1^{t/y} \frac{ds}{s^{\beta-1}} + Ct \int_{t/y}^\infty \frac{ds}{s^\beta}
\]

\[
= Cy \left( \frac{1}{\beta - 2} \left( 1 - \frac{y^{\beta-2}}{\beta - 2} \right) + Ct \frac{1}{\beta - 1} \frac{y^{\beta-1}}{\beta - 2} \right)
\]

\[
\leq Cy \left( \frac{1}{\beta - 2} + Cy \frac{1}{\beta - 1} \right) \leq CyM_\beta \omega(y).
\]

We proved (1) for \( \omega = \omega_{C,t} \), clearly (1) also holds for functions of the form \( \omega_{C,t_1} + \cdots + \omega_{C,t_n} \), let us call them polygonal majorants. For arbitrary majorant \( \omega \) there is an increasing sequence \( \omega_n \) of polygonal majorants \( \omega_n \) which converges pointwise to \( \omega \). Then, by the Monotone Convergece Theorem and already proved estimate (1) for polygonal majorants, we have

\[
\int_1^\infty \frac{\omega(ys)}{s^\beta} \, ds = \lim_{n \to \infty} \int_1^\infty \frac{\omega_n(ys)}{s^\beta} \, ds \leq \lim_{n \to \infty} M_\beta \omega_n(y) = M_\beta \omega(y).
\]

\( \Box \)

The case \( \beta = 3 \) is the only one that we need below.
2. An auxiliary result

Lemma 2.1. Let \( \varphi \in L^p(\mathbb{R}^n) \) for some \( 1 \leq p \leq \infty \). Assume

\[
|\varphi(t) - \varphi(x^0)| \leq \omega(|t - x^0|), \quad t \in \mathbb{R}^n
\]

for some \( x^0 \in \mathbb{R}^n \) and some majorant \( \omega \). Then the harmonic extension \( g = P[\varphi] \) of \( \varphi \) satisfies the following estimate:

\[
\left| \frac{\partial g}{\partial x_j}(x^0, y) \right| \leq C(n)\frac{\omega(y)}{y}, \quad 0 < y < +\infty, \quad 1 \leq j \leq n.
\]

Proof. For all \((x, y) \in \mathbb{H}^{n+1}\) and all \( j = 1, \ldots, n \), we have

\[
\frac{\partial g}{\partial x_j}(x, y) = c_n \int_{\mathbb{R}^n} \varphi(t) \frac{y}{x_j - t_j} dt
\]

\[
= -(n + 1)c_n \int_{\mathbb{R}^n} \varphi(t)y \frac{x_j - t_j}{(y^2 + |x^0 - t|^2)^{n+1/2}} dt
\]

\[
= -(n + 1)c_n \int_{\mathbb{R}^n} (\varphi(t) - \varphi(x))y \frac{x_j - t_j}{(y^2 + |x^0 - t|^2)^{n+1/2}} dt.
\]

The last equality follows from the observation that \( x_j - t_j \) is an odd function of the variable \( x - t \). Therefore, using spherical coordinates centered at \( x^0 \), we obtain

\[
\left| \frac{\partial g}{\partial x_j}(x^0, y) \right| \leq (n + 1)c_n y \int_{\mathbb{R}^n} \omega(|x^0 - t|) \frac{|x^0 - t|}{(y^2 + |x^0 - t|^2)^{n+1/2}} dt
\]

\[
= n(n + 1)c_n \omega_n y \int_0^\infty \frac{\omega(r)r^n}{(y^2 + r^2)^{n+1/2}} dr
\]

\[
= n(n + 1)c_n \omega_n \frac{1}{y} \int_0^\infty \frac{\omega(ys)s^n}{(1 + s^2)^{n+1/2}} ds.
\]

Let us denote the above integral by \( I(y) \). Then we have

\[
I(y) = \int_0^1 \frac{\omega(ys)s^n}{(1 + s^2)^{n+1/2}} ds + \int_1^\infty \frac{\omega(ys)s^n}{(1 + s^2)^{n+1/2}} ds
\]

\[
= \int_0^1 \frac{\omega(y)s^n}{(1 + s^2)^{n+1/2}} ds + \int_1^\infty \frac{\omega(y)s^n}{s^3} ds
\]

\[
\leq (1 + M_3)\omega(y)
\]

and this proves desired estimate (3). \( \square \)

The proof shows that one can take \( C(n) = 5n(n + 1)c_n \omega_n/2 \).

3. Main result

We recall that a system of harmonic functions \( u_j, 0 \leq j \leq n \), on \( \mathbb{H}^{n+1} \) is called a conjugate system if it satisfies the following system of equations

\[
\sum_{j=0}^n \frac{\partial u_j}{\partial x_j} = 0, \quad \frac{\partial u_j}{\partial x_k} = \frac{\partial u_k}{\partial x_j},
\]
where \( x_0 = y \), see \([9]\). When \( n = 2 \) this reduces to the classical Cauchy - Riemann equations. Given a function \( \varphi = \varphi(x_0, x_1, \ldots, x_n) \) harmonic in \( \mathbb{H}^{n+1} \) one gets a conjugate system by setting \( u_j = \partial \varphi / \partial x_j \), \( 0 \leq j \leq n \). Conversely, given a conjugate system \( u_j, 0 \leq j \leq n \) of harmonic functions in \( \mathbb{H}^{n+1} \), there is a (unique up to an additive constant) function \( \varphi \) on \( \mathbb{H}^{n+1} \) such that \( u_j = \partial \varphi / \partial x_j \), for \( j = 0, 1, \ldots, n \).

The above system allows one to infer estimates of \( \partial u_0 / \partial x_0 = \partial u_0 / \partial y \) from the estimates of \( \partial u_j / \partial x_j \) for \( 1 \leq j \leq n \), this is how one proves the following proposition.

**Proposition 3.1.** Let \( \omega \) be a majorant, \( E \subset \mathbb{R}^n \cong \partial \mathbb{H}^{n+1} \) and let \( f_j, 0 \leq j \leq n \), be a system of conjugate functions on \( \mathbb{H}^{n+1} \). Assume

\[
\left| \frac{\partial f_j}{\partial x_j}(x, y) \right| \leq \frac{\omega(y)}{y}, \quad x \in E, \quad y > 0, \quad 1 \leq j \leq n.
\]

Then we have

\[
\left| \frac{\partial f_0}{\partial y}(x, y) \right| \leq n \frac{\omega(y)}{y}, \quad x \in E, \quad y > 0.
\]

**Theorem 3.2.** Let \( f \in L^p(\mathbb{R}^n) \) for some \( 1 < p < \infty \) and set \( f_j = R_j f \) for \( 1 \leq j \leq n \). Let \( u = P[f] \) be the harmonic extension of \( f \) to the upper half space \( \mathbb{H}^{n+1} \). Let \( \omega \) be a majorant and let \( E \) be a subset of \( \mathbb{R}^n \cong \partial \mathbb{H}^{n+1} \). Assume

\[
|f(t) - f_j(x)| \leq \omega(|t - x|), \quad x \in E, \quad t \in \mathbb{R}^n, \quad 1 \leq j \leq n.
\]

Then there is a constant \( C = C(n) \) such that

\[
\left| \frac{\partial u_j}{\partial x_j}(x, y) \right| \leq C(n) \frac{\omega(y)}{y}, \quad y > 0, \quad x \in E.
\]

**Proof.** Let \( u_0 = u \) and \( u_j = P[f_j] \) for \( 1 \leq j \leq n \). The system \( (u_j)_{j=0}^n \) is a conjugate system of harmonic functions. By Lemma 2.1 and our assumptions we have

\[
\left| \frac{\partial u_j}{\partial x_j}(x, y) \right| \leq C(n) \frac{\omega(y)}{y}, \quad y > 0, \quad x \in E, \quad 1 \leq j \leq n.
\]

Now the above proposition gives the desired estimate. \( \square \)

The proof shows that one can take \( C = 5n^2(n + 1)c_n \omega_n / 2 \).

### 4. The unit ball setting: further results and remarks

Let \( P_B \) denote harmonic Poisson kernel for the unit ball \( \mathbb{B}^n \) and also the corresponding extension operator with the same kernel. Let \( \Lambda_\alpha \) denote the class of H"older continuous function with exponent \( 0 < \alpha \leq 1 \) and set \( \text{Lip} = \Lambda_1 \). It is known that \( P_B \) maps \( \Lambda_\alpha(\mathbb{S}^{n-1}) \) into \( \Lambda_\alpha(\mathbb{B}^n) \) whenever \( 0 < \alpha < 1 \). However if \( f \in \text{Lip}(\mathbb{S}^{n-1}) \), then in general \( P[f] \) is not in \( \text{Lip}(\mathbb{B}^n) \).

It is natural to consider the corresponding question for the hyperbolic Poisson kernel \( P_B \) for the unit ball and the corresponding extension operator.

**Problem 4.1.** Are the partial derivatives of \( P_B[f] \) bounded for every \( f \) in \( \text{Lip}(\mathbb{S}^{n-1}) \)?

The answer is positive; see \([1, 5]\). More precisely, if \( f \in \text{Lip}(\mathbb{S}^{n-1}) \), then \( P_B[f] \) is in \( \text{Lip}(\mathbb{B}^n) \). This is not true in the standard (euclidean) harmonic case.

Let us introduce needed terminology and notation. If \( x_0 \in G \subset \mathbb{R}^n \) is not an isolated point of \( G \) and \( 0 < \alpha \leq 1 \) we introduce, for \( f : G \to \mathbb{R}^m \),

\[
H_\alpha f(x_0) = \limsup_{G \ni x \to x_0} \frac{|f(x) - f(x_0)|}{|x - x_0|^\alpha}
\]

and write \( Lf(x_0) \) instead of \( H_1 f(x_0) \). We say that \( f : G \to \mathbb{R}^m \) is locally H"older (\( \alpha \)-H"older) continuous at \( x_0 \in G \) if \( H_\alpha f(x_0) < \infty \), for \( \alpha = 1 \) we use term locally Lipschitz continuous at \( x_0 \).

The following results appears in \([7]\):
Let us review some results from [7]. In the proof of the next theorem we use Poisson integral representation of harmonic functions (see formula (12) below) by the Poisson kernel on the unit ball \( B \) which is given by

\[
P_B(x, \eta) = \frac{1 - |x|^2}{n |x - \eta|^n}, \quad x \in B^n, \quad \eta \in S^{n-1}.
\]

Let \( d\sigma \) denote positive Borel measure on \( S^{n-1} \) invariant with respect to orthogonal group \( O(n) \) normalized such that \( \sigma(S^{n-1}) = 1 \).

For convenience of the reader we prove the following proposition which appeared in [7].

**Proposition 4.3 ([7])**. Suppose that \( 0 < \alpha < 1 \) and \( x = re_n, 0 < r < 1 \). Then

\[
I_\alpha(re_n) =: \int_{S^{n-1}} \frac{|e_n - t|^\alpha}{|x - t|^n} d\sigma(t) \leq \frac{c_{n,\alpha}}{(1 - r)^{1 - \alpha}}.
\]

**Proof.** Since the integral is a continuous function of \( 0 \leq r < 1 \), it suffices to prove the estimate under additional assumption \( 1/2 \leq r < 1 \). The integrand depends only on the angle \( \theta = \angle(t, e_n) \) so we can use integration in polar coordinates on the sphere \( S^{n-1} \). This gives

\[
I_\alpha(re_n) \leq c_n \int_0^{\pi} \frac{\theta^{n-2}}{(1 - r)^2 + \frac{4r}{n} \theta^2} \frac{d\theta}{\theta^{n/2}} < c_n \int_0^{\pi} \frac{\theta^{n-2}}{(1 - r)^2 + \frac{4r}{n} \theta^2} \frac{d\theta}{\theta^{n/2}}.
\]

Next using \( (1 + \frac{4r}{n} u^2)^{-1} \leq C(1 + u^2)^{-1} \) for \( \frac{1}{2} \leq r < 1 \) and a change of variable \( \theta = (1 - r)u \), we find

\[
I_\alpha(re_n) \leq C(1 - r)^{n-1} \int_0^{\pi} \frac{u^{n-2}}{(1 + u^2)^{n/2}} du.
\]

Since the above improper integral is convergent the proof is completed. \( \square \)

If \( \omega \) is a majorant satisfying the following condition

\[
\int_0^{\pi} \frac{u^{n-2}}{(1 + u^2)^{n/2}} \omega(\delta u) du \leq C_\alpha(\delta), \quad 0 < \delta \leq 1,
\]

then one proves, by a similar argument, the following proposition.
Proposition 4.4.
\[ I_{\omega}(r) = \int_{S^{n-1}} \frac{\omega(|e_n - t|)}{|x - t^n|} d\sigma(t) \leq c \frac{\omega(1 - r)}{1 - r}, \quad 0 \leq r < 1. \]

Theorem 4.5. Let \( \omega \) be a majorant satisfying condition (10). Assume \( h \) is continuous on \( B^n \), harmonic on \( B^n \) and let \( x_0 \) be a point in \( S^{n-1} \). In addition suppose the following estimate is valid:
\[ |h(x) - h(x_0)| \leq C_\omega(|x - x_0|), \quad x \in S^{n-1}. \tag{11} \]

Then there is a constant \( M = M_{\alpha, \omega} \) such that
\[ |\partial_r h(x_0)| \leq MC_\omega(1 - r) \frac{1}{1 - r}, \quad 0 \leq r < 1. \]

Proof. Since \( h \) is harmonic on \( B^n \) and continuous on \( \overline{B^n} \) we have
\[ h(x) = \int_{S^{n-1}} P_B(x, \eta)h(\eta)d\sigma(\eta), \quad x \in B^n. \tag{12} \]

Set \( d := 1 - |x|^2 \). By computation \( \partial_\sigma P_B(x, t) = -\left(\frac{2}{1 - r^2} + d \frac{\sin^2 \theta}{|x - t^n|^2}\right) \). Since \( d \leq 2(1 - |x|) \leq 2|t - x| \) for all \( t \in S^{n-1} \) we obtain
\[ |\partial_\sigma P_B(x, t)| \leq \frac{c_n}{|x - t^n|} \quad x \in B^n, \quad t \in S^{n-1}. \tag{13} \]

We can assume \( x_0 = e_n \). Let \( x = re_n \) and let \( \theta \) be the angle between \( t \) and \( e_n \). Note that \( s := |x - t|^2 = 1 - 2r \cos \theta + r^2 \) depends only on \( \theta \) for fixed \( x \). Next, since \( \int_{S^{n-1}} \partial_\sigma P_B(x, t)h(e_n)d\sigma(t) = 0 \), we find
\[ \partial_\sigma h(x) = \int_{S^{n-1}} \partial_\theta P_B(x, t)(h(t) - h(e_n))d\sigma(t). \tag{14} \]

Hence by (13) and the hypothesis (11) we get
\[ |\partial_\sigma h(x)| \leq c_n C \int_{S^{n-1}} \frac{\omega(|e_n - t|)}{|x - t^n|} d\sigma(t) \tag{15} \]
and the result follows from Proposition 4.4. \( \square \)

Remark 4.6. It is convenient to denote expressions that appear in formulae (7) and (8) without constants by \( A(r, \alpha) \) and \( B(r, \alpha) \) respectively. Note that \( A(r, \alpha) \) is finite for \( 0 \leq r < 1 \) and \( 0 < \alpha \leq 1 \) and that \( B(r, 1) = +\infty \). In order to estimate \( A(r, \alpha) \) we used a change of variable \( \theta = (1 - r)u \) and transformed the integral over \([0, \pi]\) into integral over \([0, a(r)]\) with respect to \( u \), where \( a(r) = \pi(1 - r)^{-1} \). Since \( a(r) \to \infty \) as \( r \to 1 \), it is convenient to estimate integral \( A(r, \alpha) \) by integral \( B(r, \alpha) \) over interval \([0, \infty]\).

Remark 4.7. The above proof breaks down for \( \alpha = 1 \) because \( B(r, 1) = +\infty \). Moreover, for each \( n = 2 \), there is a Lipschitz continuous map \( f : S^{n-1} \to B^n \) such that \( u = P_B[f] \) is not Lipschitz continuous. In the planar case, consider \( f = u + iv \) such that \( zf' = -\log(1 - z) \). \( u' \) is bounded while its harmonic conjugate \( ru' \) is not bounded. In the spatial case, consider \( U(x_1, x_2, \ldots, x_n) = u(x_1 + ix_2, x_3, \ldots, x_n) \).
References


