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Weighted combinatorial Ricci flow and metrics defined by degenerate circle packings

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Abstract. Chow and Luo [1] in 2003 had shown that the combinatorial analogue of the Hamilton Ricci flow on surfaces under certain conditions converges to Thruston's circle packing metric of constant curvature. The combinatorial setting includes weights defined for edges of a triangulation. Crucial assumption in the paper [1] was that the weights are nonnegative. Recently we have shown that same statement on convergence can be proved under weaker condition: some weights can be negative and should satisfy certain inequalities, [3]. Moreover, in [6] notions of degenerate circle packing and corresponding metric were introduced. In [6] theory of combinatorial Ricci flow for such metrics was developed, which includes Chow–Luo theory as a partial case for nondegenerate circle packing and nonnegative weights on edges.

On the other hand, in [2] the combinatorial Yamabe flow was introduced and investigated. In [7, 8] we developed weighted modification of Yamabe flow.

In this paper we merge ideas from these two theories and introduce weighted combinatorial Ricci flow on metrics defined by degenerate circle packings. We prove that under certain conditions for any initial metric the flow converges to a unique metric of constant curvature.

1. Basic definition

The combinatorial Ricci flow for triangulated surfaces was introduced by Chow and Luo in [1]. They gave a complete description of the asymptotic behaviour of the solution to the combinatorial Ricci flow under certain assumptions. Both the Euclidean and the hyperbolic background geometry were considered. Brief review of their results can be found in [6].

Main results of this paper are Theorems 2.1 and 2.2. Let us describe combinatorial data used in this paper. The setting is general enough to contain the particular cases considered in [1] and in [6].

Let X be a closed surface with a triangulation T. We assume that a lift of a closed face or an edge to the universal cover $\tilde{X}$ is an embedding. Denote the sets of vertices, edges and faces of T by V, E, F correspondingly. Divide the set of vertices into a disjoint union $V = V_n \sqcup V_d$, such that there is no edge connecting two vertices from V_d. Vertices from $V_n = \{A_1, \dots, A_M\}$ are called *nondegenerate* and vertices from $V_d = \{A_{M+1}, \dots, A_N\}$ are called *degenerate*. Call a cell of T (that is edge or face) *nondegenerate* iff all its vertices are nondegenerate, and *degenerate* otherwise. Denote the set of (non)degenerate edges and faces by E_d (E_n)

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and F_d (F_n), correspondingly. Clearly, $E = E_n \sqcup E_d$ and $F = F_n \sqcup F_d$. Sometimes it is useful to denote a subset of vertices and the corresponding subset of indices by the same symbol.

Also fix a weight function (a weight) $w : E_n \to (-1, 1]$.

1.1. Euclidean background

A (degenerate) circle packing metric is defined by a collection of numbers $r = (r_1, r_2, \dots, r_N)$, where $r_j > 0$ for $1 \leqslant j \leqslant M$ and $r_j = 0$ for $M + 1 \leqslant j \leqslant N$. This definition differs from the classical circle packing metric where all r_j are positive, see [1, 5]. For the Euclidean background the length of an edge connecting two vertices A_i and A_j is defined by the formula

$$l_{ij}^2 = r_i^2 + r_j^2 + 2r_ir_jw_{ij}. \tag{1}$$

For a degenerate edge one of the numbers r_i or r_j is zero, therefore the last summand is assumed to be zero although the weight w_{ij} is not defined. To be more precise, if $r_i = 0$ then $l_{ij} = r_j$. The curvature K_i at the vertex A_i is defined as usual by formula

$$K_i = 2\pi - \sum_{\triangle A_iA_jA_k \in F} \angle A_kA_iA_j. \tag{2}$$

The curvature at a degenerate vertex does not depend on r and can be expressed in terms of the weight w. Indeed, let $A_i \in V_d$ and $\triangle A_iA_jA_k \in F$. Then $A_j, A_k \in V_n$. By the cosine law $\cos \angle A_jA_iA_k = -w_{jk}$, hence $\angle A_jA_iA_k = \pi - \arccos(w_{jk})$. Therefore,

$$K_i = 2\pi - \sum_{\triangle A_iA_jA_k \in F} (\pi - \arccos(w_{jk})), \quad i = M + 1, \dots, N. \tag{3}$$

The combinatorial Ricci flow in Euclidean background as considered in [1, 6] is the system of ODE

$$\frac{dr_i}{dt} = -K_ir_i. \tag{4}$$

Note that in degenerate vertices $r_i(t) = 0$.

On the other hand, if one considers the edges as a kind of a reinforcing frame, then tension caused by the curvature K_i should be (equally) distributed over all edges adjacent to v_i, that is the right hand side of (4) should be divided by the degree of the vertex A_i. This (informal) idea leads to the following flow. Fix a collection of positive weights at nondegenerate vertices $\beta = \{\beta_1, \dots, \beta_M\}$. Then the *weighted combinatorial Ricci flow* in Euclidean background is the system of ODE

$$\frac{dr_i}{dt} = -\frac{K_i}{\beta_i}r_i, \quad i = 1, \dots, M. \tag{5}$$

For $i = M + 1, M + 2, \dots, N$ one has $r_i = \frac{dr_i}{dt} = 0$, hence in (5) one can assume $1 \leqslant i \leqslant N$.

For a degenerate metric define the *equilibrium curvature* K_i^0 for $i = 1, \dots, M$:

$$K_i^0 = \frac{\beta_i}{\sum\limits_{j=1}^{M} \beta_j} \left(2\pi\chi(X) - \sum_{j=M+1}^{N} K_j \right). \tag{6}$$

The *normalized combinatorial Ricci flow* is the system of ODE

$$\frac{dr_i}{dt} = -\frac{1}{\beta_i}(K_i - K_i^0)r_i, \; i = 1, \dots, M. \tag{7}$$

The normalized and non-normalized Ricci flows are in certain sense equivalent.

Lemma 1.1. *Function $r(t) = \{r_i(t)\}$ is a solution to the flow (5) iff the function $\{e^{\lambda t} r_i(t)\}$ is the solution to the flow*

$$\frac{dr_i}{dt} = -\left(\frac{K_i}{\beta_i} - \lambda\right) r_i.$$

Moreover, there is a unique $\lambda = \frac{1}{\sum_{j=1}^{M} \beta_j}\left(2\pi\chi(X) - \sum_{j=M+1}^{N} K_j\right)$ such that the last flow can have an equilibrium.
The possible equilibriums of (7) are given by such a constant functions $r_i(t)$ that $K_i(r(t)) = K_i^0$.

Proof is by straightforward computation.
The normalized combinatorial Ricci flow has the following useful property.

Lemma 1.2. *The product $\prod_{j=1}^{M} r_j(t)^{\beta_i}$ is a first integral for (7).*

Proof is by straightforward computation.

1.2. *Hyperbolic background metric*

For the hyperbolic background geometry the length of the edge e_{ij} joining vertices A_i and A_j is defined by the equation

$$\cosh l_{ij} = \cosh r_i \cosh r_j + \sinh r_i \sinh r_j w_{ij}. \tag{8}$$

As in the Euclidean case for degenerate edge one of the radii r_i or r_j is zero so the last summand is assumed to be zero though the weight w_{ij} is undefined. Clearly for $r_i = 0$ one has $l_{ij} = r_j$. The curvature K_i at the vertex A_i is defined as usual by formula (2). The curvature at a degenerate vertex A_i, $M + 1 \leq i \leq N$, is given by (3).

The *weighted combinatorial Ricci flow* in hyperbolic background is the system of ODE

$$\frac{dr_i}{dt} = -\frac{K_i}{\beta_i} \sinh r_i, \ i = 1, \dots, M. \tag{9}$$

For $i = M + 1, M + 2, \dots, N$ one has $r_i = \frac{dr_i}{dt} = 0$, hence in (9) one can assume $1 \leqslant i \leqslant N$.

We shall see that there is no need to normalize hyperbolic flow. Its equilibrium are characterized by the property $K_i(r(t)) = 0$.

1.3. *Condition (W), space of metrics, and derivative of curvatures.*

Fix a triple (X, T, w). Also fix a background (Euclidean or hyperbolic). Denote by $\mathcal{R}_w$ the set of all $r \in \mathbb{R}^M \times (0, \dots, 0) \subset \mathbb{R}^N$ such that for every face of the triangulation the triangle inequalities hold.

Lemma 1.3 (see [6]). *Suppose any face of the triangulation satisfies one of the following conditions:*
(a) *the face is nondegenerate and all the weights of its edges are nonnegative;*
(b) *the face is nondegenerate, exactly one weight w_{ij} of its edges is negative, two others weights w_{ik}, w_{jk} are positive, and $w_{ij} + w_{ik}w_{jk} \geqslant 0$;*
(c) *the face is degenerate and the weight of the nondegenerate edge of the face is not equal to 1.*
Then for both the Euclidean and the hyperbolic background geometry one has $\mathcal{R}_w = \mathbb{R}_+^M$.

We refer to the conditions of Lemma 1.3 as to the **conditions (W)**.

We need equations for time derivatives of curvatures K_i provided r_i satisfy one of the flows (5), (7) or (9).

Proposition 1.4 (see [6]). *Let (X, T, w) be a closed surface X with a triangulation T and a weight function w. Assume w satisfies the condition (W).*

Suppose $r_i(t), i = 1, \dots, M$, satisfy the equations $\frac{dr_i}{dt} = -L_i(r_1, \dots, r_M)s(r_i)$, where $s(r_i) = r_i$ in the Euclidean case and $s(r_i) = \sinh r_i$ in the hyperbolic case.

Then for the time derivatives of the curvatures K_i at the nondegenerate vertices ($1 \le i \le M$) one has

$$\frac{dK_i}{dt} = \sum_{i\sim j, j\leqslant M} C_{ij}(L_j - L_i) + \lambda C_i L_i,$$

where $C_{ij} = C_{ji}$ and C_i are positive elementary functions in $r_1, \dots, r_M$, and summation is over all nondegenerate vertices A_j adjacent to A_i. Here $\lambda = 0$ for the Euclidean background and $\lambda = -1$ for the hyperbolic background.

2. Main results

Assume $I \subset V_n$ is a proper subset of nondegenerate vertices. Let F_I be the subcomplex formed by simplices with vertices from I. Denote by D_I the set of all degenerate vertices adjacent to a vertex from I. Also let $Lk(I)$ be the set of pairs (v, e), $v \in I$, $e \in E$, such that both end points of e are not in I and v, e form a triangle.

For the Euclidean background we have the following statement.

Theorem 2.1. *Suppose X is a closed surface with a triangulation T and weights w and β. Assume the condition (W) is satisfied.*

The solution to the normalized Ricci flow (7) converges for any initial metric iff for any proper subset $I \in V_n$,

$$\sum_{i\in I} K_i^0 + \sum_{j\in D_I} K_j > -\sum_{(e,v)\in Lk(I\cup D_I)} (\pi - \arccos w(e)) + 2\pi\chi(F_{I\cup D_I}). \tag{10}$$

Furthermore, if the solution converges, then it converges exponentially fast to the metric with $K_i = K_i^0$, $i = 1, \dots, M$.

Similar statement for hyperbolic background also holds provided X is a surface of negative Euler characteristic.

Theorem 2.2. *Suppose X is a closed surface of negative Euler characteristic with a triangulation T and weights w and β. Assume the condition (W) is satisfied.*

The solution to the hyperbolic Ricci (9) flow converges for any initial metric iff for any subset $I \in V_n$,

$$\sum_{j\in D_I} K_j > -\sum_{(e,v)\in Lk(I\cup D_I)} (\pi - \arccos w(e)) + 2\pi\chi(F_{I\cup D_I}). \tag{11}$$

Furthermore, if the solution converges, then it converges exponentially fast to the metric with $K_i = 0$, $i = 1, \dots, M$.

Both theorems contains main results from [1, 6] as particular cases when all $\beta_i = 1$.

3. Simple properties of solutions for the weighted combinatorial Ricci flows

Results of this section are proved by arguments close to those from [6]. Therefore, we give only statements since they have certain changes.

Let $\overline{M}(t) = \max(K_1(t)/\beta_1, \dots, K_M(t)/\beta_M)$ and $\underline{M}(t) = \min(K_1(t)/\beta_1, \dots, K_M(t)/\beta_M)$. The solution to the weighted combinatorial Ricci flow with any given initial metric exists for all $t \in [0, +\infty)$. This can be proved by the same argument as in Proposition 3.4 of [1] using the following maximum principle.

Proposition 3.1. *Let $r(t) = (r_1(t), \dots, r_M(t))$ be a solution to the weighted Ricci flow (7) or (9) on an interval. Then*

(1) for the Euclidean geometry the function $\overline{M}(t)$ is non-increasing and the function $\underline{M}(t)$ is non-decreasing;

(2) for the hyperboilc geometry the function $\max(0, \overline{M}(t))$ is non-increasing and the function $\min(0, \underline{M}(t))$ is non-decreasing.

Proposition 3.2. *Let $r(t) = (r_1(t), \dots, r_M(t))$ is a solution to the normalized weighted Ricci flow (7). Suppose the curve $r(t)$ is contained in a compact subset of $\mathbb{R}^M_+$. Then $r(t)$ converges to a point in $\mathbb{R}^M_+$ such that the corresponding curvatures at nondegenerate vertices are equal to $K^0_i = \frac{\beta_i}{\sum\limits_j b_j}(2\pi\chi(X) - \sum\limits_{j\geqslant M+1} K_j)$. The convergence is exponentially fast.*

The proof goes along lines of the proof of Proposition 4 from [6] using function

$$g(t) = \sum_{j=1}^{M} \frac{1}{\beta_j}(K_j(t) - K_j^0)^2$$

and its derivative

$$g'(t) = -2 \sum_{i \sim j \leqslant M} C_{ij}(K_i/\beta_i - K_j/\beta_j)^2.$$

Proposition 3.3. *Let $r(t) = (r_1(t), ..., r_M(t))$ be a solution to the Ricci flow (9). Suppose curve $r(t)$ is contained in a compact subset $\mathbb{R}^M_+$. Then $r(t)$ converges exponentially fast to a point $(r_1, \dots, r_M) \in \mathbb{R}^M_+$ such that the corresponding curvatures $K_1, \dots, K_M$ vanish.*

The proof is similar to the proof of Proposition 3.7 from [1].

Thus, convergence of the solution to the weighted combinatorial Ricci flow is reduced to certain compactness property of the solution.

4. Ricci flow as a skew negative gradient flow

Consider the following change of variables. For the Euclidean background geometry define $u_j = \ln r_j$, and for the hyperbolic background geometry define $u_j = \ln \tanh r_j/2$. Then both Ricci flows (5) and (9) take the form

$$\frac{du_j}{dt} = -\frac{K_j}{\beta_j}, \ j = 1, \dots, M. \tag{12}$$

Under assumption (W) for the Euclidean background $u = (u_1, \dots, u_M)$ belongs to $\mathcal{U} = R^M$, and for the hyperbolic background u belongs to $\mathcal{U} = (-\infty, 0)^M \subset \mathbb{R}^M$. In [1] very important equality was proved:

$$\frac{\partial K_i}{\partial u_j} = \frac{\partial K_j}{\partial u_i}\ , \ i, j = 1, \dots M.$$

Thus, the 1-form $\Omega = \sum\limits_{j=1}^{M} K_j du_j$ is closed. Since in both cases $\mathcal{U}$ is simply connected, there exists a function $F(u_1, \dots, u_M) : U \to \mathbb{R}$ such that $dF = \Omega$.

Proposition 4.1. *Assume the weight function satisfied the condition (W). Then*

(a) for hyperbolic background the function $F(u_1, \dots, u_M)$ is strictly convex;

(b) for the Euclidean background the function $F(u_1, \dots, u_M)$ is strictly convex on any plane $\sum\limits_{j=1}^{M} \beta_j u_j = \text{const}$.

Proof is the same as the proof of the Proposition 3.9 from [1], suitable changes are necessary for (b).

From this Proposition it easily follows that in the hyperbolic background the metric is determined by its curvatures, and that in the Euclidian background the metric is determined by its curvatures up to a scalar multiple. This gives us rigidity of circle packings with degenerations.

5. Existence of an equilibrium and convergence of the flows

In this section we outline the proofs for Theorems 2.1 and 2.2.

First of all we need a condition for existence of an equilibrium.

Proposition 5.1 (see [6]). *Suppose X is a closed surface with a triangulation T and a weight w, satisfying the condition (W). Let I be a proper subset of V_n. Consider a sequence of metrics $r^{(n)} = (r_i^{(n)} : i = 1, \dots, M)$ in the Euclidean or in the hyperbolic backgound geometry such that $\lim_{n\to\infty} r_i^{(n)} = 0$ for $i \in I$ and $\lim_{n\to\infty} r_i^{(n)} > 0$ for $i \in \{1, \dots, M\} \setminus I$. Then*

$$\lim_{n\to\infty} \sum_{i\in I} K_i(r^{(n)}) + \sum_{j\in D_I} K_j = -\sum_{(e,v)\in Lk(I\cup D_I)} (\pi - \arccos w(e)) + 2\pi\chi(F_{I\cup D_I}). \tag{13}$$

Moreover, for any metric r in the Euclidean or in the hyperbolic backgound geometry and any proper subset $I \in V_n$ we have

$$\sum_{i\in I} K_i(r) + \sum_{j\in D_I} K_j > -\sum_{(e,v)\in Lk(I\cup D_I)} (\pi - \arccos w(e)) + 2\pi\chi(F_{I\cup D_I}). \tag{14}$$

For Euclidean case consider the set metrics

$$M_a = \{(r_1, \dots, r_M) \mid r_i > 0 \text{ for all } i = 1, \dots, M \text{ and } \prod_{i=1}^{M} r_i^{\beta_i} = a\},$$

where $a > 0$. Also consider the curvature map

$$\Xi : M_a \to \mathbb{R}^M, \ \ \Xi(r) = (K_1(r), \dots, K_M(r)).$$

By Proposition 4.1 the map Ξ is injective. Its image is contained in the hyperplane

$$\Pi = \{(K_1, \dots, K_M) \in \mathbb{R}^M \mid \sum_{i=1}^{M} K_i = 2\pi\xi(X) - \sum_{j=M+1}^{N} K_j\}.$$

Consider the convex open polytope $P_K \subset \Pi$, defined by the inequalities

$$\sum_{i\in I} K_i > -\sum_{(e,v)\in Lk(I\cup D_I)} (\pi - \arccos w(e)) + 2\pi\chi(F_{I\cup D_I}) - \sum_{j\in D_I} K_j,$$

where I runs through all proper subsets $I \subset V_n$. Then map $\Xi : M_a \to P_K$ is anjective, while both M_a and P_K are homeomorphic to $\mathbb{R}^{M-1}$. By the invariance of domain theorem the map Ξ is a homeomorphism of M_a onto im Ξ. Applying Proposition 5.1 we see that im $\Xi = P_K$. Under assumptions of Theorem 2.1 $(K_1^0, \dots, K_M^0) \in P_K$. Thus, there exists a unique $r^{(0)} = (r_1^{(0)}, \dots, r_m^{(0)}) \in M_a$ such that $\Xi(r^{(0)}) = (K_1^0, \dots, K_M^0) \in P_K$, that is the equilibrium of (7) and is unique up to scalar multiple.

Now we discuss convergence of the normalized Ricci flow

$$\frac{du_i}{dt} = -\frac{1}{\beta_i}(K_i(u) - K^{av}). \tag{15}$$

Arguing as above, we see that there exists a function G such that $\frac{\partial G}{\partial u_i} = K_i(u) - K_i^0$. Hence, (15) is a kind of negative skew gradient flow of G. Fix $a = 1$. The restriction of G on the hyperplane $U_0 = \{u \in \mathbb{R}^M \mid \sum_{i=1}^{M} u_i = 0\}$ is strictly convex. By the argument in the beginning of the proof G has a unique critical point $u^{(0)} \in U_0$. Therefore, this point is a minimum of G. Applying arguments from the end of the proof of Theorem 3 in [6], we see that integral curves of the negative skew gradient field of G converge to $u^{(0)} \in U_0$.

The hyperbolic case is considered in similar way as in [6] and we omit the details.

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