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Determinants of circulant matrices with Gaussian Nickel Fibonacci numbers

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Abstract. In this study, we consider $\mathcal{K}_n := \operatorname{circ}(GN_1, GN_2, \ldots, GN_n)$ circulant matrices whose entries are the Gaussian Nickel Fibonacci numbers $GN_1, GN_2, \ldots, GN_n$. Then, we compute determinants of $\mathcal{K}_n$ by exploiting Chebyshev polynomials of the second kind. Moreover, we obtain Cassini's identity and the D'Ocagne identity for the Gaussian Nickel Fibonacci numbers.

1. Introduction

In linear algebra, a circulant matrix is a square matrix defined in [1], in which all row vectors are composed of the same elements and each row vector is rotated one element to the right relative to the preceding row vector and it is given as:

$$C_n = \begin{pmatrix} c_0 & c_1 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & \cdots & c_{n-3} & c_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_2 & c_3 & \cdots & c_0 & c_1 \\ c_1 & c_2 & \cdots & c_{n-1} & c_0 \end{pmatrix}. \tag{1}$$

It is seen that any circulant matrix is a particular kind of Toeplitz matrix. The eigenvalues of C_n are well known [2]:

$$\lambda_j = \sum_{k=0}^{n-1} c_k \omega^{jk}, \quad j = 0, 1, \ldots, n-1,$$

where $\omega = \exp(\frac{2\pi i}{n})$ and $i = \sqrt{-1}$. Therefore, we can write determinant of a non singular circulant matrix as:

$$\det C_n = \prod_{j=0}^{n-1} \left(\sum_{k=0}^{n-1} c_k \omega^{jk} \right)$$

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where $k = 0, 1, \ldots, n-1$ (see e.g. [3]).

In the literature, matrix theory has important role in mathematics and applied sciences. Moreover, there are many applications and studies of special matrices such as symmetric, orthogonal and circulant matrices. Especially, the circulant matrices have many applications in many fields of science such as statistics. The circulant matrices have many applications in many fields of science such as statistics, algebraic coding theory, acoustics, periodic stochastic process, numerical analysis, number theory, graph theory and so on, see [4–17]. Many researchers have been deal with the circulant matrices and examined their properties such as their determinants and inverses associated with some integer sequences. In 1970, Lind gave a determinant formula for $F = circ(F_r, F_{r+1}, ..., F_{r+n-1})$ $(r \geq 1)$, [18]. Then, Solak examined the matrix norms of $F = circ(F_1, F_2, ..., F_n)$ and $L = circ(L_1, L_2, ..., L_n)$, where F_s and L_s are the Fibonacci and Lucas numbers, respectively, [19]. In 2011, Shen et al. gave the following determinant formulae for circulant matrices F and L

$$\begin{aligned} \det(F) &= (1-F_{n+1})^{n-1} + F_n^{n-2}\sum_{k=1}^{n-1} F_k\left(\frac{1-F_{n+1}}{F_n}\right)^{k-1}, \\ \det(L) &= (1-L_{n+1})^{n-1} + (L_n-2)^{n-2}\sum_{k=1}^{n-1}(L_{k+2}-3L_{k+1})\left(\frac{1-L_{n+1}}{L_n-2}\right)^{k-1}, \end{aligned}$$

[19].

On this topic, Bozkurt and Tam, in [5] obtained analogues of the results with [20] for circulant matrices associated with Jacobsthal and Jacobsthal Lucas numbers. Namely, they calculated the determinant of $W = circ(W_1, W_2, ..., W_n)$,where the sequence W_n is defined by the recurrence relation $W_n = pW_{n-1} + qW_{n-2}$ $(n \geq 3)$ where $W_1 = a$ and $W_2 = b, (a, b, p, q \in \mathbb{Z})$. Then, Yazlık and Taşkara, in [10], generalized the results of Bozkurt and Tam [21] for a circulant matrix whose entries are generalized $k-$ Horadam numbers. Further, generalizing above determinant results for a sequence $\{a_k\}$ of real numbers defined by an m^{th} order linear homogeneous recurrence relation $(m \geq 1)$ in [22]. In [23], Bozkurt and Yılmaz obtained formulas for determinant and inverse of circulant matrices with Pell and Pell-Lucas numbers. Recently, Jiang, Xin and Lu [7] have studied some types of circulant matrices whose entries are Gaussian Fibonacci numbers. It is important to note that there are many number sequences in the literature such as Fibonacci, Lucas, and Leonardo numbers. They also play important roles in number theories with their applications. Researchers have studied them in different ways with different number systems, [24–29, 38].

The Nickel Fibonacci numbers create a sequence with initial values $N_0 = 1$ and $N_1 = 1$. The Nickel Fibonacci sequence is created by adding 3 times the pre-two Nickel Fibonacci number to the previous Nickel Fibonacci number. Thus, it is defined for every integer $n > 2$ as follows:

$$N_{n+1} = N_n + 3N_{n-1},$$

with initial condition $N_0 = 1$ and $N_1 = 1$. Also, this sequence

$$1, 1, 4, 7, 19, 40, 97, 217, 508, 1159, 2683, 6160, 1429, 32689, 75316, ...$$

is given in OEIS with the code A006130, [30] and it is called the second order Nickel Fibonacci sequence or $(1, 3)-$Fibonacci sequence in the literature. The characteristic equation of the second order Nickel Fibonacci sequence is

$$x^2 - x - 3 = 0.$$

The roots of this equation is found as $\alpha = \frac{1+\sqrt{13}}{2}$ and $\beta = \frac{1-\sqrt{13}}{2}$. The number α is called the Nickel Ratio (or Nickel constant), [31]. The Nickel Fibonacci numbers are studied by some researchers in different areas such as encryption / decryption algorithms and finance, (for details, see [32] and [33]). The n^{th} Gaussian Fibonacci number G_n is defined with $G_0 = i$, $G_1 = 1$ and $G_n = F_n + iF_{n-1}$ for $n \geq 2$. Similary, in [34], the n^{th} Gaussian Nickel Fibonacci number GN_n is defined with $GN_0 = \frac{i}{3}$, $GN_1 = 1$ and

$$GN_{n+1} = GN_n + 3GN_{n-1} \tag{2}$$

for $n \geq 2$. We give the first few terms of the Gaussian Nickel Fibonacci numbers in Table 1.

n	0	1	2	3	4	5
GN_n	$\frac{i}{3}$	1	$1+i$	$4+i$	$7+4i$	$19+7i$

Table 1: Some Gaussian Nickel Fibonacci Numbers

We remind that the Chebyshev polynomials of second kind satisfying $\{U_n(x)\}_{n\geqslant 0}$, where each $U_n(x)$ is of degree n, satisfy the three-term recurrence relations [35]:

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)\,, \quad \text{for all } n = 1, 2, \dots\,,$$

with initial conditions $U_0(x) = 1$ and $U_1(x) = 2x$, or, equivalently,

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}\,, \quad \text{with } x = \cos\theta \quad (0 \leqslant \theta < \pi),$$

for all $n = 0, 1, 2 \dots$. It is also standard (see e.g. [36]) that

$$\det\begin{pmatrix} a & b & & \\ c & \ddots & \ddots & \\ & \ddots & \ddots & b \\ & & c & a \end{pmatrix}_{n\times n} = \left(\sqrt{bc}\right)^n U_n\left(\frac{a}{2\sqrt{bc}}\right).$$

In [37], if

$$D_n = \begin{pmatrix} d_1 & d_2 & d_3 & \cdots & d_{n-1} & d_n \\ a & b & & & & \\ c & a & b & & & \\ & c & a & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & c & a & b \end{pmatrix} \tag{3}$$

then

$$\det D_n = \sum_{k=1}^{n} d_k b^{n-k}\left(-\sqrt{bc}\right)^{k-1} U_{k-1}\left(\frac{a}{2\sqrt{bc}}\right), \tag{4}$$

where $U_k(x)$ is the k^{th} Chebyshev polynomial of second kind, [35].

In this paper, firstly, we deal with circulant matrices associated with Gaussian Nickel Fibonacci numbers. Then we give well-known Cassini's and D'Ocagne identities for these numbers.

2. Main results

In this section, we consider the n-square circulant matrix

$$\mathcal{K}_n := \mathrm{circ}\,(GN_1, GN_2, \dots, GN_n)$$

where GN_n is the n^{th} Gaussian Nickel Fibonacci number. Then, we obtain determinant formula for the matrix $\mathcal{K}_n$ by exploiting spectacular properties of Chebyshev polynomials of the second kind.

Let us define n-square matrices L_n and M_n as below:

$$L_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ -3 & 0 & 0 & \cdots & 0 & 1 & -1 \\ 0 & 0 & 0 & \iddots & \iddots & \iddots & -3 \\ \vdots & \iddots & \iddots & \iddots & \iddots & \iddots & 0 \\ \vdots & \iddots & \iddots & \iddots & \iddots & \iddots & \vdots \\ 0 & 1 & -1 & -3 & 0 & \cdots & 0 \end{pmatrix}$$

and

$$M_n = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & & & 0 \\ \vdots & \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & 1 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \end{pmatrix}. \tag{5}$$

Then, we have the following property.

Lemma 2.1.

$$\det(L_n) = \det(M_n) = \begin{cases} -1, & \begin{matrix} n \equiv 3(mod\ 4) \\ n \equiv 0(mod\ 4) \end{matrix} \\ & \\ 1, & \begin{matrix} n \equiv 1(mod\ 4) \\ n \equiv 2(mod\ 4) \end{matrix} \end{cases}.$$

Proof. By using Laplace expansion on the first row, the proof can be seen, clearly. □

Theorem 2.2. *For $n \geq 3$; we have*

$$\begin{aligned} det(\mathcal{K}_n) &= -GN_3 \sum_{k=1}^{n-1} (GN_{k+2} + GN_{k+1}) Z^{k-1} \left(-\sqrt{ZX}\right)^{n-k-1} U_{n-k-1}\left(\frac{Y}{2\sqrt{ZX}}\right) \\ &\quad + GN_2 \sum_{k=2}^{n-1} (GN_{k+2} + GN_{k+1}) Z^{k-2} \left(-\sqrt{ZX}\right)^{n-k} U_{n-k}\left(\frac{Y}{2\sqrt{ZX}}\right) \\ &\quad + \left(GN_2^2 + GN_2 GN_{n+1}\right) Z^{n-2} U_0 \left(\frac{Y}{2\sqrt{ZX}}\right) \end{aligned}$$

where $X = (GN_{n+2} - GN_{n+3}) + (GN_3 - GN_2)$, $Y = (GN_3 - GN_{n+3})$, $Z = (GN_2 - GN_{n+2})$.

Proof. For $n \geq 3$; let us multiply the matrices $L_n, \mathcal{K}_n, M_n$ which is given above, as below:

$$H_n = L_n \mathcal{K}_n M_n$$

Then, we have the following matrix

$$H_n = \begin{bmatrix} GN_2 & GN_{n+1}+GN_n & GN_n+GN_{n-1} & \cdots & GN_4+GN_3 & GN_3 \\ GN_3 & GN_2+GN_{n+1} & GN_{n+1}+GN_n & \cdots & GN_4+GN_5 & GN_4 \\ 0 & Y & Z & \cdots & 0 & 0 \\ 0 & X & Y & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & X & Y & Z \end{bmatrix} \tag{6}$$

where $Y = (GN_3 - GN_{n+3})$, $Z = (GN_2 - GN_{n+2})$ and $X = Y - Z$. By adding the first column to the n^{th} column, we have

$$H_n = \begin{bmatrix} GN_2 & GN_{n+1}+GN_n & GN_n+GN_{n-1} & \cdots & GN_4+GN_3 & GN_3+GN_2 \\ GN_3 & GN_2+GN_{n+1} & GN_{n+1}+GN_n & \cdots & GN_4+GN_5 & GN_4+GN_3 \\ 0 & Y & Z & \cdots & 0 & 0 \\ 0 & X & Y & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & X & Y & Z \end{bmatrix}. \tag{7}$$

Then, we have

$$det(H_n) = det(L_n\mathcal{K}_nM_n) = det(L_n)det(\mathcal{K}_n)det(M_n).$$

By Lemma 2.1, it is seen that

$$det(H_n) = det(\mathcal{K}_n).$$

Then, by Laplace expansion on the first column we have

$$\begin{aligned} \det(\mathcal{K}_n) &= -GN_3\sum_{k=1}^{n-1}(GN_{k+2}+GN_{k+1})Z^{k-1}\left(-\sqrt{ZX}\right)^{n-k-1}U_{n-k-1}\left(\frac{Y}{2\sqrt{ZX}}\right) \\ &\quad +GN_2\sum_{k=2}^{n-1}(GN_{k+2}+GN_{k+1})Z^{k-2}\left(-\sqrt{ZX}\right)^{n-k}U_{n-k}\left(\frac{Y}{2\sqrt{ZX}}\right) \\ &\quad +\left(GN_2^2+GN_2GN_{n+1}\right)Z^{n-2}U_0\left(\frac{Y}{2\sqrt{ZX}}\right) \end{aligned}$$

where $U_k(x)$ is the k^{th} Chebyshev polynomial of second kind, the proof can be seen easily. □

Example 2.3. *For $n = 5$,*

$$\begin{aligned} \det(\mathcal{K}_5) &= -GN_3\sum_{k=1}^{4}(GN_{k+2}+GN_{k+1})Z^{k-1}\left(-\sqrt{ZX}\right)^{4-k}U_{4-k}\left(\frac{Y}{2\sqrt{ZX}}\right) \\ &\quad +GN_2\sum_{k=2}^{4}(GN_{k+2}+GN_{k+1})Z^{k-2}\left(-\sqrt{ZX}\right)^{5-k}U_{5-k}\left(\frac{Y}{2\sqrt{ZX}}\right) \\ &\quad +\left(GN_2^2+GN_2GN_6\right)Z^3U_0\left(\frac{Y}{2\sqrt{ZX}}\right) \\ &= -9.2836\times10^7+1.31492\times10^8 i = \det(H_5). \end{aligned}$$

Note that the determinant of the matrix $\mathcal{K}_n$ can be obtained in another way as it presented in the following theorem.

Theorem 2.4. *For $n \geq 3$;*

$$det(\mathcal{K}_n) = \left(\left(GN_2 + \sum_{k=1}^{n-2} GN_{k+3}\left(\frac{-X}{Z}\right)^{n-(k+1)}\right)GN_2 - \left(\sum_{k=1}^{n-1} GN_{k+2}\left(\frac{-X}{Z}\right)^{n-(k+1)}\right)GN_3\right)Z^{n-2}$$

where $Y = (GN_3 - GN_{n+3})$, $Z = (GN_2 - GN_{n+2})$ and $X = Y - Z$.

Proof. Clearly, $det(\mathcal{K}_3) = 54 + 432i$. For $n > 3$, if we multiply $\mathcal{K}_n$ with Q on the right and P on the left, we obtain a special Hessenberg matrix that have nonzero entries only on first two rows, main diagonal and subdiagonal. In other words:

$$P = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ -3 & 0 & 0 & \cdots & 0 & 1 & -1 \\ 0 & 0 & 0 & \iddots & \iddots & \iddots & -3 \\ \vdots & \iddots & \iddots & \iddots & \iddots & \iddots & 0 \\ \vdots & \iddots & \iddots & \iddots & \iddots & \iddots & \vdots \\ 0 & 1 & -1 & -3 & 0 & \cdots & 0 \end{bmatrix} \tag{8}$$

and

$$Q = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \left(\frac{-X}{Z}\right)^{n-2} & 0 & \cdots & 0 & 0 & 1 \\ 0 & \left(\frac{-X}{Z}\right)^{n-3} & 0 & \cdots & 0 & 1 & 0 \\ 0 & \left(\frac{-X}{Z}\right)^{n-4} & 0 & \cdots & \iddots & 0 & 0 \\ \vdots & \vdots & \vdots & & \iddots & \iddots & \vdots \\ 0 & \left(\frac{-X}{Z}\right) & 1 & \iddots & \iddots & \iddots & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{9}$$

Notice that we obtain the following equality:

$$\begin{aligned} H_n &= P\mathcal{K}_nQ \\ &= \begin{bmatrix} GN_2 & J'_n & GN_n & GN_{n-1} & GN_{n-2} & \cdots & GN_3 \\ GN_3 & J_n & GN_{n+1} & GN_n & GN_{n-1} & \cdots & GN_4 \\ 0 & 0 & Z & 0 & 0 & \cdots & 0 \\ 0 & 0 & X & Z & 0 & & 0 \\ \vdots & \vdots & \ddots & & \ddots & \ddots & \vdots \\ & & & & \ddots & & 0 \\ 0 & 0 & \cdots & & 0 & X & Z \end{bmatrix} \end{aligned}$$

where

$$\begin{aligned} J'_n &= \sum_{k=1}^{n-1} GN_{k+2}\left(\frac{-X}{Z}\right)^{n-(k+1)}, \\ J_n &= GN_2 + \sum_{k=1}^{n-2} GN_{k+3}\left(\frac{-X}{Z}\right)^{n-(k+1)}. \end{aligned}$$

We obtain

$$\begin{aligned} det(P)det(H_n)det(Q) &= det(PH_nQ) \\ &= (J_nGN_2 - J'_nGN_3)Z^{n-2} \\ &= \left(\left(GN_2 + \sum_{k=1}^{n-2} GN_{k+3}\left(\frac{-X}{Z}\right)^{n-(k+1)}\right)GN_2\right. \\ &\quad \left. - \left(\sum_{k=1}^{n-1} GN_{k+2}\left(\frac{-X}{Z}\right)^{n-(k+1)}\right)GN_3\right)Z^{n-2}. \end{aligned}$$

Thus, the proof is complete. □

Example 2.5. *For n=5,*

$$\begin{aligned} \det(\mathcal{K}_5) &= \left(\left(GN_2 + \sum_{k=1}^{3} GN_{k+3}\left(\frac{-X}{Z}\right)^{4-k}\right)GN_2\right. \\ &\quad \left. - \left(\sum_{k=1}^{4} GN_{k+2}\left(\frac{-X}{Z}\right)^{4-k}\right)GN_3\right)Z^3 \\ &= \left(\left(GN_2 + GN_4\left(-\frac{X}{Z}\right)^3 + GN_5\left(-\frac{X}{Z}\right)^2 + GN_6\left(-\frac{X}{Z}\right)\right)GN_2\right. \\ &\quad \left. - \left(GN_3\left(-\frac{X}{Z}\right)^3 + GN_4\left(-\frac{X}{Z}\right)^2 + GN_5\left(-\frac{X}{Z}\right) + GN_6\right)GN_3\right)Z^3 \\ &= -9.2836 \times 10^7 + 1.31492 \times 10^8 i = \det(H_5). \end{aligned}$$

Note that the results of Example 2.3 and 2.5 are the same as we expected.

Proposition 2.6. *(Cassini's identity) For $n > 0$, the following identity holds*

$$GN_{n-1}GN_{n+1} - (GN_n)^2 = (-1)^n(4-i)3^{n-2}. \tag{10}$$

Proof. Consider the matrix

$$V_1 = \begin{pmatrix} GN_0 & GN_1 \\ GN_1 & GN_2 \end{pmatrix} = \begin{pmatrix} \frac{i}{3} & 1 \\ 1 & 1+i \end{pmatrix}.$$

We construct V_2 from V_1 by adding the second row of V_1 to the first row of V_2 and then interchanging the two rows. Continuing this process for n times, we have

$$V_n = \begin{pmatrix} GN_{n-1} & GN_n \\ GN_n & GN_{n+1} \end{pmatrix}, \tag{11}$$

which can be easily proved by induction. The first one of above elementary matrix row operation does not affect the determinant and the second changes only the sign. Therefore, $\det V_n = (-1)^{n-1} \det V_1$. Thus, we have

$$GN_{n-1}GN_{n+1} - (GN_n)^2 = (-1)^n(4-i)3^{n-2}. \tag{12}$$

We call the last equality Cassini's identity for Gaussian Nickel Fibonacci numbers. □

Example 2.7. *For n=3,*

$$\begin{aligned} GN_2GN_4-(GN_3)^2 &= (1+i)(7+4i)-(4+i)^2 \\ &= (-1)^3(4-i)3 \\ &= 3(i-4) \\ &= 3i-12. \end{aligned}$$

Proposition 2.8. *(D'Ocagne's identity) For $n \geq m \geq 0$, the following property is true*

$$GN_mGN_{n+1}-GN_nGN_{m+1}=(4-i)N_{n-m} \tag{13}$$

where N_n is n^{th} Nickel Fibonacci number.

Proof. Let us consider the following matrix:

$$M_0=\begin{pmatrix} GN_n & GN_n \\ GN_{n+1} & GN_{n+1} \end{pmatrix}.$$

Then, by multiplying the first column of V_n by 3, given by (11), and by adding the first column of it to M_0, obviously, we get

$$M_1=\begin{pmatrix} GN_{n+1} & GN_n \\ GN_{n+2} & GN_{n+1} \end{pmatrix}.$$

Then, it is clear that $M_1=\det V_n=(-1)^n(4-i)$. Then, by multiplying the first column of M_0 by 3, adding the first column of M_0 to that of M_1 gives us

$$M_2=\begin{pmatrix} GN_{n+2} & GN_n \\ GN_{n+3} & GN_{n+1} \end{pmatrix}.$$

By induction, we can see that

$$M_s=\begin{pmatrix} GN_{n+s} & GN_n \\ GN_{n+s+1} & GN_{n+1} \end{pmatrix}.$$

Using the sum property of determinant, we have $\det M_s=\det M_{s-1}+\det M_{s-2}$ which shows that $\{\det M_s\}$ is a generalized Nickel sequence. Therefore, we have

$$\det M_s=N_{s-1}\det M_0+N_s\det M_1.$$

Since, $\det M_0=0$ and

$$\det M_s=N_s\det M_1=(4-i)N_s$$

we get

$$GN_{n+s}GN_{n+1}-GN_{n+s+1}GN_n=(4-i)N_s.$$

Substituting $m=n+s$ gives us

$$GN_mGN_{n+1}-GN_{m+1}GN_n=(4-i)N_{n-m}.$$

So is the proof completed. □

Example 2.9. *For $n=5$, $m=1$,*

$$\begin{aligned} GN_1GN_6-GN_2GN_5 &= 1(40+19i)-(1+i)(19+7i) \\ &= (4-i)N_4=(4-i)7 \\ &= 28-7i. \end{aligned}$$

3. Conclusion

In this paper, we examine circulant matrices with Gaussian Nickel Fibonacci number entries. We give a formula for calculating the determinant of these matrices by exploiting the Chebyshev polynomials of the second kind. Also, we give Cassini's and D'Ocagne's identities for Gaussian Nickel Fibonacci numbers.

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