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Real hypersurfaces in $S^6(1)$ equipped with structure Jacobi operator satisfying $\mathcal{L}_X l = \nabla_X l$

Djordje Kocić[a]

[a]*Faculty of Mathematics, University of Belgrade, 11000 Belgrade, Serbia*

Abstract. The study of hypersurfaces of almost Hermitian manifolds by means of their Jacobi operators has been highly active in recent years. Specially, many recent results answer the question of the existence of hypersurfaces with a structure Jacobi operator that satisfies conditions related to their parallelism. We investigate real hypersurfaces in nearly Kähler sphere $S^6(1)$ whose Lie derivative of structure Jacobi operator coincides with the covariant derivative of it and show that such submanifolds do not exist.

1. Introduction

An almost Hermitian manifold with an almost complex structure J and the Levi-Civita connection $\tilde{\nabla}$ is respectively Kähler or nearly Kähler if the tensor field $G(X, Y) = (\tilde{\nabla}_X J)Y$ is vanishing or skew-symmetric. In [8], it was shown that an arbitrary nearly Kähler manifold can be locally decomposed into manifolds of three particular types, 6-dimensional nearly Kähler manifolds being one of those types. It is known, see [3], that there are only four homogeneous 6-dimensional strict nearly Kähler manifolds: the six-dimensional sphere $S^6(1)$, the manifold $S^3 \times S^3$, the projective space CP^3 and the flag manifold $SU(3)/U(1) \times U(1)$.

If M is a hypersurface of an almost Hermitian manifold with a unit normal vector field N, the tangent vector field $\xi = -JN$ is said to be characteristic or the Reeb vector field. The Jacobi operator with respect to a tangent vector field X on M is given by $R(\cdot, X)X$, where R is the Riemmanian curvature of M. For $X = \xi$ the Jacobi operator is called structure Jacobi operator and is denoted by $l = R(\cdot, \xi)\xi$.

We say that a hypersurface M of an almost Hermitian manifold is Hopf if ξ is principal, that is, $A\xi = \alpha\xi$ for a certain function α on M, where A is the shape operator of the hypersurface. We also note that then the function α is locally constant, see [2]. The classification of the Hopf hypersurfaces of the sphere $S^6(1)$ is well known. Such hypersurfaces are either totally geodesic spheres or tubes around almost complex curves.

The study of real hypersurfaces whose structure Jacobi operator is parallel is a problem widely investigated. In [9] the nonexistence of real hypersurfaces in nonflat complex space form with parallel structure Jacobi operator ($\nabla l = 0$) was proved. In [12] a weaker condition, $\nabla_x l = 0$, for any vector field X orthogonal to ξ, was studied and it was proved the nonexistence of such hypersurfaces in case of $\mathbb{C}P^n$ ($n \geq 3$). Nonexistence of real hypersurfaces with parallel structure Jacobi operator in $S^6(1)$ was proved in [1].

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Email address: djordje_kocic@matf.bg.ac.rs (Djordje Kocić)

Let $\mathcal{L}$ denote the Lie derivative on M, i.e. $\mathcal{L}_X Y = \nabla_X Y - \nabla_Y X$, for any X, Y tangent to M. The Lie derivative of the structure Jacobi operator, which is given by $(\mathcal{L}_X l)Y = \nabla_X(lY) + l\nabla_Y X - \nabla_{lY} X - l\nabla_X Y$, for X, Y tangent to M, is another condition that has been studied extensively, see [7, 13–15].

The parallelness of structure Jacobi operator in combination with other conditions was another problem that was studied by many others. Recently Pérez-Santos (see [16]) studied real hypersurfaces in $\mathbb{C}P^n$ for $n \geq 3$ whose structure Jacobi operator satisfies the relation $\mathcal{L}_\xi l = \nabla_\xi l$. Panagiotidou and Xenos in [11] have completed the investigation of this problem studying the case $n = 2$ in both complex projective and hyperbolic spaces. In [10] the nonexistence of real hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$, whose structure Jacobi operator satisfies the relation $\mathcal{L}_X l = \nabla_X l$, for X orthogonal to ξ, was proved.

In this paper, we observe a stronger condition for the real hypersurfaces of the nearly Kähler sphere $S^6(1)$ and prove the following non-existence theorem:

Theorem 1.1. *There exist no real hypersurfaces in $S^6(1)$ equipped with structure Jacobi operator satisfying $\mathcal{L}_X l = \nabla_X l$, for all $X \in TM$.*

Note that the skew symmetry of the tensor G imposes a somewhat different approach to analizing hypersurfaces in nearly Kähler manifolds compared to the one in Kähler manifolds. As a consequence we constract a suitable moving frame along the hypersurface in order to analyse it.

2. Preliminaries

Let M be a Riemannian submanifold of the nearly Kähler sphere $S^6(1)$ with nearly Kähler structure (J, g), where J is the almost complex structure and g is the metric on $S^6(1)$. Then the $(2, 1)$-tensor field G on $S^6(1)$ defined by $G(X, Y) = (\bar{\nabla}_X J)Y$, where $\bar{\nabla}$ is the Levi-Civita connection on $S^6(1)$, is skew symmetric and also satisfies

$$G(X, JY) + JG(X, Y) = 0, \qquad g(G(X, Y), Z) + g(G(X, Z), Y) = 0.$$

Moreover, see [5], we have

$$(\bar{\nabla} G)(X, Y, Z) = g(X, Z)JY - g(X, Y)JZ - g(JY, Z)X, \tag{1}$$

for arbitrary vector fields X, Y, Z tangent to $S^6(1)$.

Let us denote by ∇ and $\nabla^\perp$ the Levi-Civita connection of M and the normal connection induced from $\bar{\nabla}$ in the normal bundle $T^\perp M$ of M in $S^6(1)$, respectively. Then the formulas of Gauss and Weingarten are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \qquad \bar{\nabla}_X N = -A_N X + \nabla^\perp_X N,$$

where X, Y are tangent, N is a normal vector field on M, h and A_N are the second fundamental form and the shape operator with respect to the section N, respectively. The second fundamental form and the shape operator are related by $g(h(X, Y), \xi) = g(A_\xi X, Y)$. Also, for tangent vector fields X, Y, Z and W, we have that the Gauss equation yields

$$R(X, Y, Z, W) = g(X, W)g(Y, Z) - g(X, Z)g(Y, W) + g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)), \tag{2}$$

where we denote by R the Riemannian curvature tensor of M.

Now, let M be a hypersurface in $S^6(1)$. Using the almost complex structure J on $S^6(1)$, for normal vector field N on M, we define corresponding Reeb vector field $\xi = -JN$ with dual 1-form $\eta(x) = g(X, \xi)$. Let $\mathcal{D} = \operatorname{Ker} \eta = \{X \in TM \mid \eta(X) = 0\}$. Then $\mathcal{D}$ is a 4-dimensional smooth distribution on M, which is J-invariant.

3. The moving frame for hypersurfaces in $S^6(1)$

Now we will present one of the convenient moving frames to work with and the relations between the connection coefficients in it, for details see [1, 4].

For each unit vector field $E_1 \in \mathcal{D}$, let $E_2 = JE_1$, $E_3 = G(E_1, \xi)$, $E_4 = JE_3$. Then the set $\{E_1, E_2, E_3, E_4, E_5 = \xi\}$ is a local orthonormal frame on M, see [4]. Moreover, the following holds.

Lemma 3.1. *([4]) For the previously defined orthonormal frame the following relations hold*

$$\begin{array}{lllll} G(E_1,E_2)=0, & G(E_1,E_3)=-\xi, & G(E_1,E_4)=N, & G(E_1,\xi)=E_3, & G(E_1,N)=-E_4,\\ G(E_2,E_3)=-N, & G(E_2,E_4)=\xi, & G(E_2,\xi)=-E_4, & G(E_2,N)=-E_3, & G(E_3,E_4)=0,\\ G(E_3,\xi)=-E_1, & G(E_3,N)=E_2, & G(E_4,\xi)=E_2, & G(E_4,N)=E_1. & \end{array} \tag{3}$$

This moving frame is not uniquely determined and depends on the choice of the vector field $E_1 \in \mathcal{D}$.

Now, we denote the coefficients of the covariant derivatives in the given frame by

$$g_{ij}^k = g(D_{E_i}E_j, E_k), \qquad h_{ij} = g(D_{E_i}E_j, N), \qquad 1 \le i, j, k \le 5, \tag{4}$$

where D is Levi-Civita connection on $\mathbb{R}^7$. The connection D is metric and the second fundamental form symmetric, which gives us $g_{ij}^k = -g_{ik}^j$, and $h_{ij} = h_{ji}$.

Lemma 3.2. *([1]) For the previously defined coefficients we have*

$$\begin{array}{llllll} g_{12}^3=-g_{11}^4, & g_{12}^4=g_{11}^3, & h_{11}=-g_{12}^5, & h_{12}=g_{11}^5, & g_{22}^3=-g_{21}^4, & g_{22}^4=g_{21}^3,\\ g_{22}^5=-g_{11}^5, & h_{22}=g_{21}^5, & g_{32}^3=-g_{31}^4, & g_{32}^4=g_{31}^3, & h_{13}=1-g_{32}^5, & h_{23}=g_{31}^5,\\ g_{42}^3=-g_{41}^4, & g_{42}^4=g_{41}^3, & h_{14}=-g_{42}^5, & h_{24}=-1+g_{41}^5, & g_{52}^3=-1-g_{51}^4, & g_{52}^4=g_{51}^3,\\ h_{15}=-g_{52}^5, & h_{25}=g_{51}^5, & g_{32}^5=2+g_{14}^5, & g_{42}^5=-g_{13}^5, & g_{31}^5=-g_{24}^5, & g_{41}^5=2+g_{23}^5,\\ h_{33}=-g_{43}^5, & h_{34}=g_{33}^5, & g_{44}^5=-g_{33}^5, & h_{44}=g_{43}^5, & h_{35}=-g_{54}^5, & h_{45}=g_{53}^5. \end{array}$$

Lemma 3.3. *([1]) The differentiable functions (4) satisfy*

$$g_{52}^5 = g_{11}^2 + g_{13}^4, \quad g_{51}^5 = -g_{21}^2 - g_{23}^4, \quad g_{54}^5 = g_{31}^2 + g_{33}^4, \quad g_{53}^5 = -g_{41}^2 - g_{43}^4, \quad h_{55} = -g_{51}^2 - g_{53}^4. \tag{5}$$

Since we still have a choice for $E_1 \in \mathcal{D}$, see [1], from now on let it be parallel with the projection of $A\xi$ on $\mathcal{D}$. Then there exist differentiable functions α and β such that $A\xi = \beta E_1 + \alpha\xi$. Since the components of $A\xi$ in direction of E_2, E_3, E_4 vanish, we have

$$g_{13}^4 = -g_{11}^2 - \beta, \qquad g_{23}^4 = -g_{21}^2, \qquad g_{33}^4 = -g_{31}^2, \qquad g_{43}^4 = -g_{41}^2, \qquad g_{53}^4 = -g_{51}^2 - \alpha.$$

The Gauss equation for the different choices of the vector fields yields the following relations.

Lemma 3.4. *([1]) The functions (4), α and β satisfy*

$$(3+2g_{14}^5)g_{21}^5 - 2g_{11}^5g_{24}^5 - 2g_{24}^5g_{33}^5 + g_{34}^5 + 2g_{23}^5g_{34}^5 - 2\alpha - g_{14}^5\alpha + g_{23}^5\alpha - g_{21}^3\beta + g_{31}^2\beta = 0, \tag{6}$$

$$g_{11}^5(3 + 2g_{23}^5) - g_{33}^5 - 2g_{23}^5g_{33}^5 - 2g_{24}^5g_{43}^5 + g_{24}^5\alpha + g_{13}^5(-2g_{21}^5 + \alpha) - g_{21}^4\beta + g_{41}^2\beta = 0, \tag{7}$$

$$-4-3g_{23}^5 - g_{14}^5(3 + 2g_{23}^5) + 2g_{13}^5g_{24}^5 - 2(g_{33}^5)^2 - 2g_{34}^5g_{43}^5 + g_{34}^5 - g_{43}^5\alpha\alpha - g_{31}^4\beta + g_{41}^3\beta = 0. \tag{8}$$

4. Proof of the main theorem

Let us denote by $\eta_{ij}^k = g(l\nabla_{E_j}E_i - \nabla_{lE_j}E_i, E_k)$. The condition $\mathcal{L}_X l = \nabla_X l$ is equivalent to $\eta_{ij}^k = 0, 1 \leq i, j, k \leq 5$. From $\eta_{5i}^5 = 0, 1 \leq i \leq 4$, by using the Gauss equations, we have, respectively,

$$g_{11}^5\alpha\beta = 0, \qquad (1 + g_{21}^5\alpha)\beta = 0, \qquad g_{24}^5\alpha\beta = 0, \qquad (1 + g_{23}^5)\alpha\beta = 0. \tag{9}$$

We will now treat the cases of Hopf and non-Hopf hypersurfaces separately.

Case 1:. Suppose that M is a Hopf hypersurface, i.e. $\beta = 0$, ξ is an eigenvector field for the shape operator A and α is a constant.

Note that, in this case, we still have the freedom of choosing the vector field $E_1 \in \mathcal{D}$. Let us, therefore, take E_1 to be an eigenvector field for the shape operator A. As $AE_1 = -g_{12}^5E_1 + g_{11}^5E_2 - (1 + g_{14}^5)E_3 + g_{13}^5E_4$, we get $g_{11}^5 = 0,\ g_{14}^5 = -1,\ g_{13}^5 = 0$.

From $0 = \eta_{41}^5 = 1 - g_{12}^5\alpha$, we have $g_{12}^5 \neq 0 \neq \alpha$ and

$$0 = E_2(1 - g_{12}^5\alpha) = -(g_{11}^2(g_{12}^5 + g_{21}^5) + g_{11}^4(1 + g_{23}^5) - g_{11}^3g_{24}^5)\alpha. \tag{10}$$

From $0 = \eta_{52}^2 = -g_{24}^5\alpha$, we have $g_{24}^5 = 0$ and then from $0 = \eta_{51}^3 = g_{33}^5\alpha$ we obtain $g_{33}^5 = 0$. Therefore

$$0 = E_1(g_{24}^5) = g_{11}^2 + g_{12}^5g_{21}^3 - g_{11}^4g_{21}^5 + g_{11}^2g_{23}^5 - (g_{11}^4 + g_{21}^3)g_{34}^5. \tag{11}$$

Now we have $0 = \eta_{4i}^4 = -g_{i1}^3(1 + g_{23}^5)\alpha$, $1 \leq i \leq 5$. If we assume $1 + g_{23}^5 = 0$, then from $0 = \eta_{51}^2 = -g_{12}^5(g_{12}^5 + g_{21}^5)\alpha$ and $0 = \eta_{52}^3 = -(g_{21}^5 + g_{34}^5)\alpha$ we have $g_{34}^5 = -g_{21}^5 = g_{12}^5$. Further from $0 = \eta_{52}^1 = 2(g_{12}^5 + \alpha)$ we have $g_{12}^5 = -\alpha$, so from $0 = \eta_{41}^5 = 1 + \alpha^2$ we have a contradiction. Thus $1 + g_{23}^5 \neq 0$, $g_{i1}^3 = 0$, $1 \leq i \leq 5$ and

$$0 = E_1(g_{31}^3) = -1 + 2g_{11}^4g_{31}^2 - 2g_{11}^2g_{31}^4 - g_{12}^5g_{34}^5. \tag{12}$$

Now, (10) and (11) reduce to $g_{11}^2(g_{12}^5+g_{21}^5)+g_{11}^4(1+g_{23}^5) = 0$ and $g_{11}^2(1+g_{23}^5)-g_{11}^4(g_{21}^5+g_{34}^5) = 0$, respectively, so $g_{11}^2 = 0$ if and only if $g_{11}^4 = 0$. If we assume $g_{11}^4 \neq 0$, from $0 = \eta_{21}^3 = g_{11}^4(-g_{12}^5 + g_{34}^5)\alpha$ we have $g_{34}^5 = g_{12}^5$. Then determinant of the system of equations (10) and (11) is $-(g_{12}^5 + g_{21}^5)^2 - (1 + g_{23}^5)^2 \neq 0$, so we have only trivial solution $g_{11}^2 = g_{11}^4 = 0$. Therefore $g_{11}^2 = g_{11}^4 = 0$.

If we add η_{41}^5 to (12) we obtain $-g_{12}^5(g_{34}^5 + \alpha) = 0$, so $g_{34}^5 = -\alpha$. Now (8) becomes $-1 - g_{23}^5 + g_{43}^5\alpha - \alpha^2 = 0$, so $g_{43}^5\alpha - g_{23}^5 = 1 + \alpha^2$, and using $0 = \eta_{41}^5 = 1 - g_{12}^5\alpha$ we obtain

$$0 = \eta_{51}^4 = -1 + 2g_{43}^5\alpha - \alpha^2 - g_{23}^5(1 + g_{12}^5\alpha) = -1 - \alpha^2 + 2(g_{43}^5\alpha - g_{23}^5) = 1 + \alpha^2,$$

which is a contradiction.

Case 2: Suppose that M is not a Hopf hypersurface, i.e. that $\beta \neq 0$. Then the second equation of (9) implies $\alpha \neq 0$, so we obtain

$$g_{11}^5 = 0, \qquad g_{21}^5\alpha = -1, \qquad g_{24}^5 = 0, \qquad g_{23}^5 = -1. \tag{13}$$

Further, from the Gauss equations and (13), we get

Lemma 4.1.

$$\begin{aligned}
0 =&\xi(g_{11}^5) = 1 + g_{21}^5g_{51}^2 + g_{13}^5g_{51}^3 + g_{51}^4 + g_{14}^5(1 + g_{51}^4) + (g_{11}^2 - \beta)\beta + g_{12}^5(g_{21}^5 + g_{51}^2 - \alpha),\\
0 =&E_1(g_{24}^5) = -g_{11}^2(1 + g_{14}^5) + (g_{12}^5 + g_{21}^5)g_{31}^2 + g_{13}^5g_{31}^3 + g_{31}^4 + g_{14}^5g_{31}^4 - g_{11}^3g_{33}^5 - g_{11}^4(g_{21}^5 + g_{34}^5) + (2 + g_{14}^5)\beta,\\
0 =&\xi(g_{24}^5) = -2g_{34}^5 + g_{14}^5(g_{21}^5 - g_{51}^2) - g_{51}^2 - g_{33}^5g_{51}^3 - (g_{21}^5 + g_{34}^5)g_{51}^4 - g_{21}^3\beta - \alpha,\\
0 =&E_1(g_{23}^5) = -g_{11}^4g_{33}^5 - (g_{12}^5 + g_{21}^5)g_{41}^2 - (1 + g_{14}^5)g_{41}^4 + g_{11}^3(-g_{21}^5 + g_{43}^5) - g_{13}^5(g_{11}^2 + g_{41}^3 - \beta),\\
0 =&\xi(g_{23}^5) = g_{13}^5(g_{21}^5 - g_{51}^2) - g_{21}^5g_{51}^3 + g_{43}^5g_{51}^3 - g_{33}^5(2 + g_{51}^4) + g_{21}^4\beta.
\end{aligned}$$

Then, from $E_2(g^5_{24}) = 0$ and $E_2(g^5_{23}) = 0$ we have

$$E_3(g^5_{21}) = -(1 + g^5_{14})g^2_{21} - g^3_{21}g^5_{33} - g^4_{21}(g^5_{21} + g^5_{34}), \qquad E_4(g^5_{21}) = g^5_{13}g^2_{21} + g^4_{21}g^5_{33} + g^3_{21}(g^5_{21} - g^5_{43}). \tag{14}$$

Now, by using the Gauss equations and (14), from $E_i(g^5_{21}\alpha) = 0$, $i = 2, 3, 4$, we obtain

$$E_2(g^5_{21}) = -(g^5_{21})^2(3g^5_{21} - g^2_{51} - \alpha)\beta, \qquad g^5_{21}g^3_{51}\beta - ((1 + g^5_{14})g^2_{21} + g^3_{21}g^5_{33} + g^4_{21}(g^5_{21} + g^5_{34}))\alpha = 0, \tag{15}$$

$$g^5_{13}g^2_{21}\alpha + g^4_{21}g^5_{33}\alpha + g^3_{21}(g^5_{21} - g^5_{43})\alpha + g^5_{21}(-2 + g^4_{51})\beta = 0. \tag{16}$$

Further, from $0 = \eta^5_{11} = -g^5_{13}\alpha$ and $0 = \eta^5_{33} = -g^5_{33}$, we have $g^5_{13} = 0$ and $g^5_{33} = 0$, and hence

$$0 = E_1(g^5_{33}) = g^4_{11}(1 + g^5_{14}) + g^2_{31} + g^5_{14}g^2_{31} - g^5_{12}g^4_{31} - g^2_{11}(g^5_{34} + g^5_{43}) - 2g^5_{34}\beta - g^5_{43}(g^4_{31} + \beta). \tag{17}$$

Note that, now, (7) has become $(-g^4_{21} + g^2_{41})\beta=0$, and therefore $g^2_{41} = g^4_{21}$.

Lemma 4.2. *The coefficients (4) satisfy*

$$\begin{aligned}
&g^5_{34} = g^5_{14}g^5_{21},\\
&g^3_{21}(2 - g^5_{12}g^5_{21} + (-1 - g^5_{14} + g^5_{21}(g^5_{12} + g^5_{21}))g^2_{31} + g^5_{21}(3 + g^5_{14} + g^5_{14}g^5_{21}(-3g^5_{21} + g^2_{51} + \alpha))\beta\\
&\quad + g^5_{14}(2 + (g^5_{21})^2)) = 0,\\
&-g^4_{31} + 2g^3_{41} + g^2_{31}g^5_{43} - g^5_{21}(g^2_{31} - g^5_{12}g^4_{31} + g^5_{12}g^3_{41} - g^4_{31}g^5_{43} + g^5_{43}\beta) + g^5_{14}(-g^4_{31} + 2g^3_{41} + g^3_{21}(g^5_{21} - g^5_{43})\\
&\quad + (g^5_{21})^2(g^3_{41} + \beta)) = 0,\\
&(2 - g^5_{12}g^5_{21} + g^5_{14}(2 + (g^5_{21})^2))g^3_{51} + (g^5_{14}g^2_{21} - g^3_{11}g^5_{21} + g^3_{31})\beta = 0.
\end{aligned}$$

Proof. From $0 = \eta^3_{52} = (g^5_{14}g^5_{21} - g^5_{34})\alpha$, we get the first relation of the Lemma. Now, from $E_i(g^5_{14}g^5_{21} - g^5_{34}) = 0$, for $i = 2, 4, 5$, we have the other relations of the Lemma, respectively. □

Further,

$$\begin{aligned}
&0 = \eta^3_{31} = g^3_{11}(1 + g^5_{14})\alpha, && 0 = \eta^3_{32} = (1 + g^5_{14})g^3_{21}\alpha, && 0 = \eta^1_{31} = -(1 + g^5_{14})g^3_{31}\alpha,\\
&0 = \eta^3_{34} = (1 + g^5_{14})g^3_{41}\alpha, && 0 = \eta^3_{35} = (1 + g^5_{14})g^3_{51}\alpha, && 0 = \eta^3_{42} = (1 + g^5_{14})(g^4_{21} - g^2_{21}g^5_{21})\alpha.
\end{aligned} \tag{18}$$

At this point we can distinct two cases: $1 + g^5_{14} = 0$ or not.

Case 2.1: $g^5_{14} \neq -1$.

Then from (18) we obtain that $g^3_{11} = g^3_{21} = g^3_{31} = g^3_{41} = g^3_{51} = 0$ and $g^4_{21} = g^2_{21}g^5_{21}$, and hence

$$0 = E_2(g^3_{41}) = -1 + 2(g^4_{21})^2 - 2g^2_{21}g^4_{41} - g^5_{21}g^5_{43}. \tag{19}$$

From fifth relation of Lemma 4.1 we have $g^4_{21}\beta = 0$, and therefore $g^4_{21} = 0$. From (18) we then obtain that $g^2_{21} = 0$. From fourth relation of Lemma 4.1 it follows that $-(1 + g^5_{14})g^4_{41} = 0$, so $g^4_{41} = 0$. From (16) and (19) we have $g^5_{21}(-2 + g^4_{51})\beta = 0$ and $-1 - g^5_{21}g^5_{43} = 0$. Therefore $g^4_{51} = 2$, and (9) implies that $g^5_{43} = \alpha$. Further, from $0 = \eta^4_{15} = -2g^4_{31}\beta$ we have $g^4_{31} = 0$. Finally, (7) becomes $-1 - \alpha^2 = 0$, which is a contradiction.

Case 2.2: $g^5_{14} = -1$.

Then $0 = E_5(g^5_{14}) = -(g^5_{12} + g^5_{21})g^3_{51} - g^3_{11}\beta$, and from (15) we have $g^5_{21}g^3_{51}\beta = 0$, so $g^3_{51} = 0$ and therefore $g^3_{11} = 0$. Also, we have

$$0 = \eta^5_{41} = 1 - g^5_{12}\alpha - \beta^2. \tag{20}$$

Now, from fifth relation of Lemma 4.1 and fourth relation from Lemma 4.2 we have, respectively, $g_{21}^4\beta = 0$ and $(-g_{21}^2 + g_{31}^3)\beta = 0$, so $g_{21}^4 = 0$ and $g_{31}^3 = g_{21}^2$. If we compare $(8) - \eta_{51}^4 - \eta_{41}^5 = -1 - g_{43}^5\alpha = 0$ with (13), we obtain $g_{43}^5 = g_{21}^5$. Now, using (2), from $0 = \xi(g_{21}^5 - g_{43}^5) = (g_{21}^2 - g_{41}^4)\beta$, we have $g_{41}^4 = g_{21}^2$ and then from $0 = E_2(g_{21}^5 - g_{43}^5) = -(g_{21}^5)^2(3g_{21}^5 - g_{51}^2 - \alpha)\beta$ we obtain $g_{51}^2 = 3g_{21}^5 - \alpha$. From (16) we have $g_{21}^5(-2 + g_{51}^4)\beta = 0$, hence $g_{51}^4 = 2$ and second relation of Lemma 4.1 become $(g_{12}^5 + g_{21}^5)g_{31}2 + \beta = 0$, so $\beta = -(g_{12}^5 + g_{21}^5)g_{31}^2$ and special $g_{21}^5 \neq -g_{12}^5$. Second relation from Lema 4.2 gives $-g_{21}^5(g_{12}^5 + g_{21}^5)(g_{21}^3 + g_{31}^2) = 0$ hence $g_{31}^2 = -g_{21}^3$. From (17) we have $-g_{12}^5 g_{31}^4 + g_{21}^5(-g_{31}^4 + \beta) = 0$. If we multiply it with α, using (13), we obtain $\beta = -(g_{12}^5 + g_{21}^5)g_{31}^4\alpha$, hence $g_{31}^4 \neq 0$ and $g_{31}^2 = g_{31}^4\alpha$. Further, from (6) and $\eta_{11}^2 = 0$, we have, respectively $2(g_{21}^5 - \alpha - (g_{12}^5 + g_{21}^5)(g_{31}^4)^2\alpha^2) = 0$ and $g_{11}^2(g_{12}^5 + g_{21}^5)(\alpha + (g_{12}^5 + g_{21}^5)(g_{31}^4)^2\alpha^2) = 0$. From the first relation we have $(g_{12}^5 + g_{21}^5)(g_{31}^4)^2\alpha^2 = g_{21}^5 - \alpha$, and then the second relation reduce to $g_{11}^2 g_{21}^5(g_{12}^5 + g_{21}^5) = 0$, so $g_{11}^2 = 0$. Multiplying third relation of Lemma 4.1 with $g_{12}^5 + g_{21}^5$ and subtracting from first relation we obtain $3g_{12}^5 g_{21}^5 + 2(g_{21}^5)^2 - g_{12}^5\alpha = 0$. If we multiply this with α, using $g_{21}^5\alpha = -1$, we have $-2g_{21}^5 - g_{12}^5(3 + \alpha^2) = 0$, hence $g_{21}^5 = -\frac{1}{2}g_{12}^5(3 + \alpha^2)$. From $g_{21}^5\alpha = -1$ we obtain $g_{12}^5 = 2/(3\alpha + \alpha^3)$. Further, (6) became

$$\frac{2(1 + \alpha^2)(-3 + ((g_{31}^4)^2 - 1)\alpha^2)}{\alpha(3 + \alpha^2)} = 0, \text{ hence } -3 + ((g_{31}^4)^2 - 1)\alpha^2 = 0.$$

Finally, from (20) we have

$$0 = \eta_{41}^5 = -\frac{(1 + \alpha^2)[-3 + (g_{31}^4)^2 + ((g_{31}^4)^2 - 1)\alpha^2]}{(3 + \alpha^2)^2} = -\frac{(1 + \alpha^2)(g_{31}^4)^2}{(3 + \alpha^2)^2}.$$

Since $g_{31}^4 \neq 0$, we obtain a contradiction.

This completes the proof.

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