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Codimension 2 submanifolds of paracosymplectic manifolds with normal Reeb vector field

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Abstract. This study is devoted to a submanifold M of codimension 2 of an almost paracontact metric manifold $\overline{M}$, for which the Reeb vector field of the ambient manifold is normal. Some sufficient conditions for the existence of M are given. When $\overline{M}$ is paracosymplectic, then some necessary and sufficient conditions are established for M to fall in one of the following classes of almost paracontact metric manifolds according to the classification given by S. Zamkovoy and G. Nakova: normal, paracontact metric, para-Sasakian, K-paracontact, quasi-para-Sasakian, respectively. When in addition, M is para-Sasakian and $\overline{M}$ is paracosymplectic, some characterization results are obtained for M to be totally umbilical, as well as a nonexistence result for M to be totally geodesic is provided. The case when $\overline{M}$ is of a constant sectional curvature is analysed and an example is constructed.

Dedicated to the memory of Prof. Kostadin Gribachev (1938 - 2022)

1. Introduction

The notion of paracontact manifold is a corresponding notion of the contact manifold [1], which is applied in theoretical physics, mechanics, thermodynamics and so on.

In 1976, Sato [6, 7] introduced for the first time in literature, the concept of almost paracontact Riemannian manifolds, where the metric is compatible with the structure (see also a recent paper [10]), which is a different topic from the one considered by the present paper. In our paper here, we use the notion of an almost paracontact metric manifold, introduced by Kaneyuki and Kozai in 1985, which is not equal to that of Sato, since the almost paracontact structure is anti-compatible with a semi-Riemannian metric of signature $(n+1, n)$ [13]. For a more complete description of the comparison between compatible and anti-compatible metrics with the above structure, see [4], which shows that in the compatibility (resp. anti-compatibility) case, the ranks of the eigensubbundles corresponding to the eigenvalues $+1$ and -1 are arbitrary (resp. equal).

It is worthwhile to note that almost paracontact metric manifolds are the odd-dimensional version of the almost para-Hermitian manifolds, in the same way as the almost contact Riemannian manifolds are the odd-dimensional version of the almost Hermitian manifolds.

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A classification of almost paracontact metric manifolds was obtained by Zamkovoy and the second author in [15]. Among all the 12 classes of this classification, a special one is the class of paracosymplectic manifolds, studied in this paper.

In literature, the theory of hypersurfaces of manifolds endowed with certain structures is very important, but also the theory of submanifolds of codimension 2 in manifolds endowed with some structures is intensively studied. This is the reason which motivates the investigation of the present paper. There is a concern for the study of the submanifolds of codimension 2, as for instance, Kanemaki studied in [11] submanifolds of codimension 2 in quasi-Sasakian manifolds, the second author and Gribachev studied in [5] submanifolds of codimension 2 in almost contact manifolds with B-metric. When the submanifold is degenerate, Duggal and Bejancu examined in [8] lightlike submanifolds of codimension 2 and then Duggal alone or with other co-authors published several papers on this topic. Also, the first author studied in [3] lightlike submanifolds of codimension 2 in almost para-Hermitian manifolds.

The main purpose of the present paper is to study codimension 2 submanifolds of almost paracontact metric manifolds with normal Reeb vector field.

Section 2 contains some preliminaries, where the main notion of the paper, namely almost paracontact metric manifold is recalled and also some characterization results of some special classes of almost paracontact metric manifolds are cited from [15], in order to be used later on. In Section 3 an existence result for an almost paracontact metric structure on a codimension 2 submanifold of an almost paracontact metric manifold with normal Reeb vector field is obtained.

In Section 4, as a special case, some codimension 2 submanifolds of paracosymplectic manifolds with normal Reeb vector field are studied. Some necessary and sufficient conditions for such a submanifold to belong to a certain class are given. We obtain here a characterization theorem for a codimension 2 submanifold of a paracosymplectic manifold with normal Reeb vector field to fall in one of the following classes of almost paracontact metric manifolds: normal, paracontact metric, para-Sasakian, K-paracontact, quasi-para-Sasakian, respectively. In this context, some problems related to the Riemannian curvature are also studied.

The last part of the paper represents Section 5, which is devoted to the codimension 2 para-Sasakian submanifolds of paracosymplectic manifolds with normal Reeb vector field. In this context, some characterization results are given for totally umbilical submanifolds, as well as a nonexistence result for a totally geodesic submanifold is provided. Moreover, the case when the ambient manifold is of a constant sectional curvature is analysed. An example of a para-Sasakian submanifold of codimension 2 with a normal Reeb vector field in a paracosymplectic manifold of an arbitrary odd dimension is constructed. The obtained submanifold is totally umbilical, parallel and it has a constant sectional curvature -1.

2. Preliminaries

A (2n+1)-dimensional smooth manifold $\overline{M}$ has an *almost paracontact structure* $(\overline{\varphi}, \overline{\xi}, \overline{\eta})$ [12] if it admits a tensor field $\overline{\varphi}$ of type $(1,1)$, a vector field $\overline{\xi}$, called a Reeb vector field, and a 1-form $\overline{\eta}$ satisfying the following conditions:

$$\overline{\varphi}(\overline{\xi}) = 0, \quad \overline{\eta} \circ \overline{\varphi} = 0, \quad \overline{\eta}(\overline{\xi}) = 1; \tag{1}$$

$$\overline{\varphi}^2\overline{X} = \overline{X} - \overline{\eta}(\overline{X})\overline{\xi}, \quad \overline{X} \in \chi(\overline{M}); \tag{2}$$

the tensor field $\overline{\varphi}$ induces an almost paracomplex structure $\overline{\varphi}|_{\overline{\mathbb{D}}}$ on the paracontact distribution $\overline{\mathbb{D}} = \operatorname{Ker}\overline{\eta}$, that is, $\overline{\varphi}|_{\overline{\mathbb{D}}}$ is an almost product structure $(\overline{\varphi}^2|_{\overline{\mathbb{D}}} = I)$ and the eigensubbundles $\overline{\mathbb{D}}^+$and $\overline{\mathbb{D}}^-$corresponding to the eigenvalues 1 and -1 of $\overline{\varphi}|_{\overline{\mathbb{D}}}$, respectively, have the same dimension. (3)

Everywhere here we will denote by $\mathcal{F}(\overline{M})$ and $\chi(\overline{M})$ the set of all smooth real functions and vector fields on $\overline{M}$, respectively. Also, $\overline{X}, \overline{Y}, \overline{Z}, \overline{W}$ stand for vector fields belonging to $\chi(\overline{M})$.

The manifold $\overline{M}$ endowed with an almost paracontact structure $(\overline{\varphi}, \overline{\xi}, \overline{\eta})$ is called an *almost paracontact manifold*. If an almost paracontact manifold $(\overline{M}, \overline{\varphi}, \overline{\xi}, \overline{\eta})$ admits a pseudo-Riemannian metric $\overline{g}$ such that

$$\overline{g}(\overline{\varphi}\overline{X}, \overline{\varphi}\overline{Y}) = -\overline{g}(\overline{X}, \overline{Y}) + \overline{\eta}(\overline{X})\overline{\eta}(\overline{Y}), \tag{4}$$

then $(\overline{M}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ is called *an almost paracontact metric manifold* (see [14]). The metric $\overline{g}$ is called *compatible* metric and it is necessarily of signature $(n+1, n)$. Setting $\overline{Y} = \overline{\xi}$, we have $\overline{\eta}(\overline{X}) = \overline{g}(\overline{X}, \overline{\xi})$. Also, using (1) and (2) we obtain that the condition (4) is equivalent to

$$\overline{g}(\overline{\varphi}\overline{X}, \overline{Y}) = -\overline{g}(\overline{X}, \overline{\varphi}\overline{Y}). \tag{5}$$

Remark 2.1. *The restriction $\overline{g}|_{\overline{\mathbb{D}}}$ of $\overline{g}$ on the paracontact distribution $\overline{\mathbb{D}}$ is a para-Hermitian metric and $(\overline{\mathbb{D}}, \overline{\varphi}|_{\overline{\mathbb{D}}}, \overline{g}|_{\overline{\mathbb{D}}})$ is an almost para-Hermitian vector bundle. According to [13, Remark, p. 84], the eigensubbundles $\overline{\mathbb{D}}^+$ and $\overline{\mathbb{D}}^-$ have the same dimension since from* (5)*, they are maximal totally isotropic with respect to $\overline{g}|_{\overline{\mathbb{D}}}$.*

The fundamental 2-form $\overline{\phi}$ on $(\overline{M}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ is given by $\overline{\phi}(\overline{X}, \overline{Y}) = \overline{g}(\overline{\varphi}\overline{X}, \overline{Y})$ and the structure tensor field $\overline{F}$ of $\overline{M}$ is defined by

$$\overline{F}(\overline{X}, \overline{Y}, \overline{Z}) = (\overline{\nabla}_{\overline{X}}\overline{\phi})(\overline{Y}, \overline{Z}) = g((\overline{\nabla}_{\overline{X}}\overline{\varphi})\overline{Y}, \overline{Z}),$$

where $\overline{\nabla}$ is the Levi-Civita connection on $\overline{M}$. The tensor field $\overline{F}$ has the following properties:

$$\begin{aligned} &\overline{F}(\overline{X}, \overline{Y}, \overline{Z}) = -\overline{F}(\overline{X}, \overline{Z}, \overline{Y}), \\ &\overline{F}(\overline{X}, \overline{\varphi}\overline{Y}, \overline{\varphi}\overline{Z}) = \overline{F}(\overline{X}, \overline{Y}, \overline{Z}) + \overline{\eta}(\overline{Y})\overline{F}(\overline{X}, \overline{Z}, \overline{\xi}) - \overline{\eta}(\overline{Z})\overline{F}(\overline{X}, \overline{Y}, \overline{\xi}). \end{aligned} \tag{6}$$

The following 1-forms, called Lee forms, are associated with $\overline{F}$:

$$\overline{\theta}(\overline{X}) = \overline{g}^{ij}\overline{F}(\overline{e}_i, \overline{e}_j, \overline{X}), \qquad \overline{\theta}^*(\overline{X}) = \overline{g}^{ij}\overline{F}(\overline{e}_i, \overline{\varphi}\,\overline{e}_j, \overline{X}), \qquad \overline{\omega}(\overline{X}) = \overline{F}(\overline{\xi}, \overline{\xi}, \overline{X}), \tag{7}$$

where $\{\overline{e}_i, \overline{\xi}\ / i = 1, \ldots, 2n\}$ is a local basis of $T\overline{M}$, and $(\overline{g}^{ij})$ is the inverse matrix of $(\overline{g}_{ij})$, with $\overline{g}_{ij} = \overline{g}(\overline{e}_i, \overline{e}_j)$.

The tangent space $T_p\overline{M}$ at each point p in an almost paracontact metric manifold $(\overline{M}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ is the following orthogonal direct sum

$$T_p\overline{M} = \overline{\mathbb{D}_p} \oplus \mathrm{span}_{\mathbb{R}}\{\overline{\xi}(p)\}.$$

Hence, every vector $\overline{x} \in T_p\overline{M}$ can be decomposed uniquely in the following manner

$$\overline{x} = \mathrm{h}\overline{x} + \mathrm{v}\overline{x}, \ \text{ where } \ \mathrm{h}\overline{x} = \overline{\varphi}^2\overline{x} \in \overline{\mathbb{D}_p} \ \text{ and } \ \mathrm{v}\overline{x} = \overline{\eta}(\overline{x}).\overline{\xi}(p) \in \mathrm{span}_{\mathbb{R}}\{\overline{\xi}(p)\}. \tag{8}$$

Let $\mathcal{F}$ be the subspace of the space $\otimes_3^0 T_p\overline{M}$ of the tensors of type $(0,3)$ over $T_p\overline{M}$, defined by

$$\mathcal{F} = \{\overline{F} \in \otimes_3^0 T_p\overline{M} : \overline{F}(\overline{x}, \overline{y}, \overline{z}) = -\overline{F}(\overline{x}, \overline{z}, \overline{y}) = \overline{F}(\overline{x}, \overline{\varphi}\,\overline{y}, \overline{\varphi}\,\overline{z}) - \overline{\eta}(\overline{y})\overline{F}(\overline{x}, \overline{z}, \overline{\xi}) + \overline{\eta}(\overline{z})\overline{F}(\overline{x}, \overline{y}, \overline{\xi})\}.$$

In [15] the space $\mathcal{F}$ has been decomposed into 12 mutually orthogonal and invariant (under the action of the structure group $\mathbb{U}^{\pi}(n) \times \{1\}$) subspaces. Based on this decomposition, 12 basic classes of almost paracontact metric manifolds $\mathbb{G}_i$ $(i = 1, \ldots, 12)$ with respect to the tensor field $\overline{F}$ are obtained. First we recall the partial decomposition of $\mathcal{F}$ in an orthogonal direct sum of its subspaces W_i $(i = 1, 2, 3, 4)$, i.e.

$$\mathcal{F} = W_1 \oplus W_2 \oplus W_3 \oplus W_4, \tag{9}$$

where W_i $(i = 1, 2, 3, 4)$ are defined by

$$\begin{aligned} W_1 &= \{\overline{F} \in \mathcal{F} : \overline{F}(\overline{x}, \overline{y}, \overline{z}) = \overline{F}(\mathrm{h}\overline{x}, \mathrm{h}\overline{y}, \mathrm{h}\overline{z})\}, \\ W_2 &= \{\overline{F} \in \mathcal{F} : \overline{F}(\overline{x}, \overline{y}, \overline{z}) = -\overline{\eta}(\overline{y})\overline{F}(\mathrm{h}\overline{x}, \mathrm{h}\overline{z}, \overline{\xi}) + \overline{\eta}(\overline{z})\overline{F}(\mathrm{h}\overline{x}, \mathrm{h}\overline{y}, \overline{\xi})\}, \\ W_3 &= \{\overline{F} \in \mathcal{F} : \overline{F}(\overline{x}, \overline{y}, \overline{z}) = \overline{\eta}(\overline{x})\overline{F}(\overline{\xi}, \mathrm{h}\overline{y}, \mathrm{h}\overline{z}), \\ W_4 &= \{\overline{F} \in \mathcal{F} : \overline{F}(\overline{x}, \overline{y}, \overline{z}) = \overline{\eta}(\overline{x})\{\overline{\eta}(\overline{y})\overline{F}(\overline{\xi}, \overline{\xi}, \mathrm{h}\overline{z}) - \overline{\eta}(\overline{z})\overline{F}(\overline{\xi}, \overline{\xi}, \mathrm{h}\overline{y})\}, \end{aligned} \tag{10}$$

for arbitrary vectors $\overline{x}, \overline{y}, \overline{z} \in T_p\overline{M}$. The subspace W_2 is an orthogonal direct sum of the following classes of almost paracontact metric manifolds:

$$W_2 = \mathbb{G}_5 \oplus \mathbb{G}_6 \oplus \mathbb{G}_7 \oplus \mathbb{G}_8 \oplus \mathbb{G}_9 \oplus \mathbb{G}_{10} \tag{11}$$

and $W_4 = \mathbb{G}_{12}$. For later use we give the characteristic conditions of the classes $\mathbb{G}_i$ ($i = 5, \ldots, 10, 12$):

$$\mathbb{G}_5 : \overline{F}(\overline{X}, \overline{Y}, \overline{Z}) = \frac{\overline{\theta}(\overline{\xi})}{2n}\{\overline{\eta}(\overline{Y})\overline{g}(\overline{\varphi}\overline{X}, \overline{\varphi}\overline{Z}) - \overline{\eta}(\overline{Z})\overline{g}(\overline{\varphi}\overline{X}, \overline{\varphi}\overline{Y})\}; \tag{12}$$

$$\mathbb{G}_6 : \overline{F}(\overline{X}, \overline{Y}, \overline{Z}) = -\frac{\overline{\theta}^*(\overline{\xi})}{2n}\{\overline{\eta}(\overline{Y})\overline{g}(\overline{X}, \overline{\varphi}\overline{Z}) - \overline{\eta}(\overline{Z})\overline{g}(\overline{X}, \overline{\varphi}\overline{Y})\}; \tag{13}$$

$$\begin{aligned} \mathbb{G}_7 : \overline{F}(\overline{X}, \overline{Y}, \overline{Z}) &= -\overline{\eta}(\overline{Y})\overline{F}(\overline{X}, \overline{Z}, \overline{\xi}) + \overline{\eta}(\overline{Z})\overline{F}(\overline{X}, \overline{Y}, \overline{\xi}), \\ \overline{F}(\overline{X}, \overline{Y}, \overline{\xi}) &= -\overline{F}(\overline{Y}, \overline{X}, \overline{\xi}) = -\overline{F}(\overline{\varphi}\overline{X}, \overline{\varphi}\overline{Y}, \overline{\xi}), \quad \overline{\theta}^*(\overline{\xi}) = 0; \end{aligned} \tag{14}$$

$$\begin{aligned} \mathbb{G}_8 : \overline{F}(\overline{X}, \overline{Y}, \overline{Z}) &= -\overline{\eta}(\overline{Y})\overline{F}(\overline{X}, \overline{Z}, \overline{\xi}) + \overline{\eta}(\overline{Z})\overline{F}(\overline{X}, \overline{Y}, \overline{\xi}), \\ \overline{F}(\overline{X}, \overline{Y}, \overline{\xi}) &= \overline{F}(\overline{Y}, \overline{X}, \overline{\xi}) = -\overline{F}(\overline{\varphi}\overline{X}, \overline{\varphi}\overline{Y}, \overline{\xi}), \quad \overline{\theta}(\overline{\xi}) = 0; \end{aligned} \tag{15}$$

$$\begin{aligned} \mathbb{G}_9 : \overline{F}(\overline{X}, \overline{Y}, \overline{Z}) &= -\overline{\eta}(\overline{Y})\overline{F}(\overline{X}, \overline{Z}, \overline{\xi}) + \overline{\eta}(\overline{Z})\overline{F}(\overline{X}, \overline{Y}, \overline{\xi}), \\ \overline{F}(\overline{X}, \overline{Y}, \overline{\xi}) &= -\overline{F}(\overline{Y}, \overline{X}, \overline{\xi}) = \overline{F}(\overline{\varphi}\overline{X}, \overline{\varphi}\overline{Y}, \overline{\xi}); \end{aligned} \tag{16}$$

$$\begin{aligned} \mathbb{G}_{10} : \overline{F}(\overline{X}, \overline{Y}, \overline{Z}) &= -\overline{\eta}(\overline{Y})\overline{F}(\overline{X}, \overline{Z}, \overline{\xi}) + \overline{\eta}(\overline{Z})\overline{F}(\overline{X}, \overline{Y}, \overline{\xi}), \\ \overline{F}(\overline{X}, \overline{Y}, \overline{\xi}) &= \overline{F}(\overline{Y}, \overline{X}, \overline{\xi}) = \overline{F}(\overline{\varphi}\overline{X}, \overline{\varphi}\overline{Y}, \overline{\xi}); \end{aligned} \tag{17}$$

$$\mathbb{G}_{12} : \overline{F}(\overline{X}, \overline{Y}, \overline{Z}) = \overline{\eta}(\overline{X})\left\{\overline{\eta}(\overline{Y})\overline{F}(\overline{\xi}, \overline{\xi}, \overline{Z}) - \overline{\eta}(\overline{Z})\overline{F}(\overline{\xi}, \overline{\xi}, \overline{Y})\right\}. \tag{18}$$

Definition 2.2. *[9] An almost paracontact metric manifold $(\overline{M}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ is said to be paracosymplectic if* $\overline{\nabla}\,\overline{\phi} = \overline{\nabla}\overline{\eta} = 0$.

Remark 2.3. *The above Definition 2.2 is introduced in [9], but the condition* $\overline{\nabla}\,\overline{\phi} = 0$ *is enough since the condition* $\overline{\nabla}\overline{\eta} = 0$ *is a consequence of the previous one.*

It is clear that paracosymplectic manifolds constitute the special class $\mathbb{G}_0$, determined by the condition $\overline{F}(\overline{X}, \overline{Y}, \overline{Z}) = 0$. This class is the intersection of the basic twelve classes. Hence, $\mathbb{G}_0$ is the class of the almost paracontact metric manifolds with parallel structures, i.e.

$$\overline{\nabla}\,\overline{\phi} = \overline{\nabla}\overline{\varphi} = \overline{\nabla}\,\overline{\xi} = \overline{\nabla}\overline{\eta} = \overline{\nabla}\overline{g} = 0.$$

An almost paracontact metric manifold is called (see [12, 14])

- *normal* if $\overline{N}(\overline{X}, \overline{Y}) - 2\mathrm{d}\overline{\eta}(\overline{X}, \overline{Y})\overline{\xi} = 0$, where $\overline{N}(\overline{X}, \overline{Y}) = \overline{\varphi}^2[\overline{X}, \overline{Y}] + [\overline{\varphi}\overline{X}, \overline{\varphi}\overline{Y}] - \overline{\varphi}[\overline{\varphi}\overline{X}, \overline{Y}] - \overline{\varphi}[\overline{X}, \overline{\varphi}\overline{Y}]$ is the Nijenhuis torsion tensor of $\overline{\varphi}$;
- *paracontact metric* if $\overline{\phi} = \mathrm{d}\overline{\eta}$;
- *para-Sasakian* if it is normal and paracontact metric;
- *K-paracontact* if it is paracontact and $\overline{\xi}$ is a Killing vector field;
- *quasi-para-Sasakian* if it is normal and $\mathrm{d}\overline{\phi} = 0$.

The classes of normal, paracontact metric, para-Sasakian, K-paracontact and quasi-para-Sasakian manifolds are determined in [15] . Here, we recall some results which we need.

Further we consider the subclass of the class $\mathbb{G}_5$, which consists of all $(2n+1)$-dimensional almost paracontact metric manifolds belonging to $\mathbb{G}_5$, such that $\overline{\theta}(\overline{\xi}) = 2n$ (see [15]). Here we denote this subclass by $\mathbb{G}_5^*$. The characteristic condition of $\mathbb{G}_5^*$ is:

$$\mathbb{G}_5^* : \overline{F}(\overline{X}, \overline{Y}, \overline{Z}) = \overline{\eta}(\overline{Y})\overline{g}(\overline{\varphi}\overline{X}, \overline{\varphi}\overline{Z}) - \overline{\eta}(\overline{Z})\overline{g}(\overline{\varphi}\overline{X}, \overline{\varphi}\overline{Y}). \tag{19}$$

Theorem 2.4. *[15] A $(2n+1)$-dimensional almost paracontact metric manifold $(\overline{M}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ is:*

(i) normal if and only if $\overline{M}$ belongs to one of the classes $\mathbb{G}_1$, $\mathbb{G}_2$, $\mathbb{G}_5$, $\mathbb{G}_6$, $\mathbb{G}_7$, $\mathbb{G}_8$ or to the classes which are their direct sums;

(ii) paracontact metric if and only if $\overline{M}$ belongs to the class $\mathbb{G}_5^*$ or to the classes which are direct sums of $\mathbb{G}_5^*$ with $\mathbb{G}_4$ or $\mathbb{G}_{10}$;

(iii) para-Sasakian if and only if $\overline{M}$ belongs to the class $\mathbb{G}_5^*$;

(iv) K-paracontact metric if and only if $\overline{M}$ belongs to the classes $\mathbb{G}_5^*$ or $\mathbb{G}_5^* \oplus \mathbb{G}_4$;

(v) quasi-para-Sasakian if and only if $\overline{M}$ belongs to the classes $\mathbb{G}_5$, $\mathbb{G}_8$ or $\mathbb{G}_5 \oplus \mathbb{G}_8$.

3. Codimension 2 submanifolds of almost paracontact metric manifolds with normal Reeb vector field

Let $(\overline{M}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ be a $(2n+3)$-dimensional almost paracontact metric manifold and let M be a submanifold of codimension 2 embedded in $\overline{M}$ such that the normal vector fields $\overline{N}_1$ and $\overline{N}_2$ to M satisfy the conditions

$$\overline{g}(\overline{N}_1, \overline{N}_1) = -\overline{g}(\overline{N}_2, \overline{N}_2) = 1, \quad \overline{g}(\overline{N}_1, \overline{N}_2) = 0. \tag{20}$$

We assume that $\overline{\xi}$ is a normal vector field to M. Then for $\overline{\xi}$ we have

$$\overline{\xi} = a\overline{N}_1 + b\overline{N}_2, \tag{21}$$

where a and b are functions on M.

In what follows, we use the notation $\chi(M)$ for the set of all vector fields on M and X, Y, Z, W for arbitrary vector fields belonging to $\chi(M)$.

From (21) we obtain $\overline{\eta}(X) = 0, \quad a = \overline{\eta}(\overline{N}_1), \quad b = -\overline{\eta}(\overline{N}_2)$ and

$$a^2 - b^2 = 1. \tag{22}$$

The equality (22) implies that $a \neq 0$. Now, by using $\overline{\varphi}\overline{\xi} = 0$ and (21), we get $\overline{\varphi}\overline{N}_1 = -\frac{b}{a}\overline{\varphi}\overline{N}_2$. The vector field $\overline{\varphi}\overline{N}_2 \in \chi(\overline{M})$ has the following unique decomposition

$$\overline{\varphi}\overline{N}_2 = \zeta + c\overline{N}_1 + d\overline{N}_2,$$

where: $\zeta \in \chi(M)$ is the tangent part of $\overline{\varphi}\overline{N}_2$; $c\overline{N}_1 + d\overline{N}_2$ is the normal part of $\overline{\varphi}\overline{N}_2$; c, d are functions on M. From the latter equality, taking into account $\overline{g}(\overline{\varphi}\overline{N}_2, \overline{N}_2) = \overline{g}(\overline{\varphi}\overline{N}_2, \overline{\xi}) = 0$, (21) and $a \neq 0$, we obtain $c = d = 0$. Hence, for $\overline{\varphi}\overline{N}_1$ and $\overline{\varphi}\overline{N}_2$ the following equalities hold good

$$\overline{\varphi}\overline{N}_1 = -\frac{b}{a}\zeta, \quad \overline{\varphi}\overline{N}_2 = \zeta, \tag{23}$$

which means that both $\overline{\varphi}\overline{N}_1$ and $\overline{\varphi}\overline{N}_2$ are vector fields on M. The transform vector field $\overline{\varphi}X$ has the unique decomposition

$$\overline{\varphi}X = \varphi X + \alpha(X)\overline{N}_1 + \beta(X)\overline{N}_2,$$

where φ is a $(1,1)$-tensor field on M and α, β are 1-forms. From the above equality using (23) we get

$$\alpha(X) = \frac{b}{a}\beta(X), \qquad \beta(X) = \overline{g}(X,\zeta). \tag{24}$$

Then $\overline{\varphi}X$ becomes

$$\overline{\varphi}X = \varphi X + \frac{b}{a}\beta(X)\overline{N}_1 + \beta(X)\overline{N}_2. \tag{25}$$

Further, by straightforward computations, we obtain that the induced objects φ, β, ζ on the submanifold M of $\overline{M}$ satisfy the following conditions:

- from $\overline{\varphi}^2X = X$, (25) and (22) we derive

$$\varphi^2X = X - \frac{1}{a^2}\beta(X)\zeta, \qquad \beta(\varphi X) = 0; \tag{26}$$

- the equalities (2), (21), (23), (25) and (22) imply

$$\varphi\zeta = 0, \qquad \beta(\zeta) = a^2; \tag{27}$$

- by using (4) and (25) we get

$$\overline{g}(\varphi X, \varphi Y) = -\overline{g}(X,Y) + \frac{1}{a^2}\beta(X)\beta(Y). \tag{28}$$

Now, we define a vector field ξ and a 1-form η on M by

$$\xi = \frac{1}{a}\zeta, \qquad \eta(X) = \frac{1}{a}\beta(X). \tag{29}$$

Taking into account (24), (26), (27) and (29), we verify that the following equalities hold good on M:

$$\eta(\xi) = 1, \qquad \eta(X) = \overline{g}(X,\xi), \qquad \varphi\xi = 0, \qquad \eta(\varphi X) = 0, \qquad \varphi^2X = X - \eta(X)\xi. \tag{30}$$

We denote by g the restriction of the metric $\overline{g}$ on M. For g, by virtue of (28) and (29), we have

$$g(\varphi X, \varphi Y) = -g(X,Y) + \eta(X)\eta(Y). \tag{31}$$

Theorem 3.1. *Let $(\overline{M}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ be a $(2n+3)$-dimensional almost paracontact metric manifold and let M be a $(2n+1)$-dimensional submanifold of $\overline{M}$ such that $\overline{\xi}$ is a normal vector field to M satisfying* (21). *Then $(M, \varphi, \xi, \eta, g)$ is an almost paracontact metric manifold with an almost paracontact structure (φ, ξ, η), where ξ and η are defined by* (29),

$$\varphi X = \overline{\varphi}X - \eta(X)\left(b\overline{N}_1 + a\overline{N}_2\right) \tag{32}$$

and g is the restriction of the metric $\overline{g}$ on M.

Proof. According to (30) and (31), for the structure (φ, ξ, η) the conditions (1), (2) are fulfilled and g is a compatible metric on M.

We denote by $\mathbb{D} = \operatorname{Ker}\eta$ and $\varphi|_{\mathbb{D}}$ the paracontact distribution of M and the restriction of φ on $\mathbb{D}$, respectively. Now, we show that the eigensubbundles $\mathbb{D}^+$ and $\mathbb{D}^-$ corresponding to the eigenvalues 1 and -1 of the almost product structure $\varphi|_{\mathbb{D}}$, respectively, have equal dimension n. From $\overline{\eta}(X) = 0$ it follows that the $(2n+2)$-dimensional paracontact distribution $\overline{\mathbb{D}}$ of $\overline{M}$ contains the $(2n+1)$-dimensional tangent bundle TM of M, i.e. $TM \subset \overline{\mathbb{D}}$. By using (30) and (32), we get $\overline{\varphi}\xi = b\overline{N}_1 + a\overline{N}_2$. Hence, for any $X \in TM$ we have

$\overline{g}(X, \overline{\varphi}\xi) = 0$. The latter and $\overline{\varphi}\xi \in \overline{\mathbb{D}}$ imply that the vector bundle $\{\overline{\varphi}\xi\}$ spanned by $\overline{\varphi}\xi$ is the orthogonal complement of TM in $\overline{\mathbb{D}}$, i.e. $\overline{\mathbb{D}} = TM \oplus span\{\overline{\varphi}\xi\}$. Since $TM = \mathbb{D} \oplus span\{\xi\}$ (the sum $\oplus$ is orthogonal), we obtain that $\overline{\mathbb{D}}$ is the following orthogonal direct sum

$$\overline{\mathbb{D}} = \mathbb{D} \oplus \mathbb{W},$$

where $\mathbb{W} = \text{span}\{\xi, \overline{\varphi}\xi\}$. By virtue of (32) for any $X \in \mathbb{D}$ we get $\varphi X = \overline{\varphi}X$, which means that $\varphi|_{\mathbb{D}} = \overline{\varphi}|_{\mathbb{D}}$ and the subbundle $\mathbb{D}$ of $\overline{\mathbb{D}}$ is invariant under the action of $\overline{\varphi}$. By using $\overline{\eta}(\xi) = 0$ it is easy to check that the subbundle $\mathbb{W}$ of $\overline{\mathbb{D}}$ is also $\overline{\varphi}$-invariant.

Let P be the matrix of $\overline{\varphi}|_{\mathbb{D}}$ in the basis $\{e_1, \dots, e_{2n}\}$ of $\mathbb{D}$ and let Q be the matrix of $\overline{\varphi}|_{\mathbb{W}}$ in the basis $\{e_{2n+1}, e_{2n+2}\}$ of $\mathbb{W}$, where both bases consist of eigenvectors of $\overline{\varphi}|_{\mathbb{D}}$ and $\overline{\varphi}|_{\mathbb{W}}$, respectively. From a well known algebraic result it follows that the matrix L of $\overline{\varphi}|_{\overline{\mathbb{D}}}$ in the basis $\{e_1, \dots, e_{2n}, e_{2n+1}, e_{2n+2}\}$ of $\overline{\mathbb{D}}$ has the form $L = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}$ (P, Q are matrices of type $(2n \times 2n)$ and (2×2), respectively). Since for $\overline{\varphi}|_{\overline{\mathbb{D}}}$ the condition (3) is fulfilled, i.e. $\dim\overline{\mathbb{D}}^{+} = \dim\overline{\mathbb{D}}^{-} = n + 1$, L is a diagonal matrix and its main diagonal consists of 1 and -1, the number of which is the same, equal to $(n + 1)$. Taking into account that $Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\varphi|_{\mathbb{D}} = \overline{\varphi}|_{\mathbb{D}}$, we conclude that $\dim\mathbb{D}^{+} = \dim\mathbb{D}^{-} = n$, which completes the proof. $\square$

Let $\overline{\nabla}$ and ∇ be the Levi-Civita connections of the metrics $\overline{g}$ and g on $\overline{M}$ and M, respectively. Then the Gauss-Weingarten formulas are:

$$\begin{aligned} &\overline{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \\ &\overline{\nabla}_X \overline{N}_1 = -A_{\overline{N}_1} X + D_X \overline{N}_1, \\ &\overline{\nabla}_X \overline{N}_2 = -A_{\overline{N}_2} X + D_X \overline{N}_2. \end{aligned} \tag{33}$$

Here, σ is the second fundamental form, $A_{\overline{N}_i}$ is the shape operator with respect to $\overline{N}_i$ $(i = 1, 2)$ and D is the normal connection on the normal bundle $TM^{\perp}$. For σ, $A_{\overline{N}_i}$ $(i = 1, 2)$ and D we obtain

$$\begin{aligned} &\sigma(X, Y) = g(A_{\overline{N}_1} X, Y)\overline{N}_1 - g(A_{\overline{N}_2} X, Y)\overline{N}_2 = g(X, A_{\overline{N}_1} Y)\overline{N}_1 - g(X, A_{\overline{N}_2} Y)\overline{N}_2, \\ &D_X \overline{N}_1 = \gamma(X)\overline{N}_2, \quad D_X \overline{N}_2 = \gamma(X)\overline{N}_1, \end{aligned} \tag{34}$$

where γ is a 1-form on M.

4. Codimension 2 submanifolds of paracosymplectic manifolds with normal Reeb vector field

From now on, $(\overline{M}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ is a $(2n + 3)$-dimensional paracosymplectic manifold and $(M, \varphi, \xi, \eta, g)$ is the submanifold of codimension 2 of $\overline{M}$ which is considered in Section 3. From $\overline{\nabla}\,\overline{\xi} = 0$, (21), (33), (34) we derive $A_{\overline{N}_1} X = -\dfrac{b}{a} A_{\overline{N}_2} X$ and

$$\gamma(X) = -\frac{1}{a}(Xb). \tag{35}$$

Then the Gauss-Weingarten formulas become

$$\begin{aligned} &\overline{\nabla}_X Y = \nabla_X Y - g(AX, Y)\left(\frac{b}{a}\overline{N}_1 + \overline{N}_2\right), \\ &\overline{\nabla}_X \overline{N}_1 = \frac{b}{a} AX + \gamma(X)\overline{N}_2, \\ &\overline{\nabla}_X \overline{N}_2 = -AX + \gamma(X)\overline{N}_1, \end{aligned} \tag{36}$$

where we have denoted the shape operator $A_{\overline{N}_2}$ by A for brevity. By using $\overline{\nabla}\,\overline{\varphi} = 0$, (32) and (36), we obtain

$$(\nabla_X\varphi)Y = \frac{1}{a}\left\{\eta(Y)AX - g(AX, Y)\xi\right\}. \tag{37}$$

According to Theorem 3.1, $(M, \varphi, \xi, \eta, g)$ is an almost paracontact metric manifold whoose fundamental 2-form we denote by ϕ, given by $\phi(X, Y) = g(\varphi X, Y)$. Then the structure tensor field F of M is defined by

$$F(X, Y, Z) = (\nabla_X\phi)(Y, Z) = g((\nabla_X\varphi)Y, Z). \tag{38}$$

By virtue of (37) and (38), for F we get

$$F(X, Y, Z) = \frac{1}{a}\left\{\eta(Y)g(AX, Z) - \eta(Z)g(AX, Y)\right\}. \tag{39}$$

Further, depending on the properties of the shape operator A, we determine the classes to which M belongs according to the classification of almost paracontact metric manifolds given in [15].

In [15] it is shown that the decomposition (9) implies that the tensor F, given by (38), has a unique representation in the form $F(X, Y, Z) = \sum_{j=1}^{4} F^{W_j}(X, Y, Z)$, where F^{W_j} are the projections of F in the subspaces W_j $(j = 1, 2, 3, 4)$. Taking into account (6), (8) and (10), the projections F^{W_j} $(j = 1, 2, 3, 4)$ of F can be written as follows:

$$\begin{aligned}
&F^{W_1}(X, Y, Z) = F(\varphi^2X, \varphi^2Y, \varphi^2Z),\\
&F^{W_2}(X, Y, Z) = -\eta(Y)F(\varphi^2X, \varphi^2Z, \xi) + \eta(Z)F(\varphi^2X, \varphi^2Y, \xi),\\
&F^{W_3}(X, Y, Z) = \eta(X)F(\xi, \varphi Y, \varphi Z),\\
&F^{W_4}(X, Y, Z) = \eta(X)\{\eta(Y)F(\xi, \xi, Z) - \eta(Z)F(\xi, \xi, Y)\}.
\end{aligned}$$

By using (39) for the projections of the structure tensor F of M, we obtain $F^{W_1} = F^{W_3} = 0$,

$$F^{W_2}(X, Y, Z) = \frac{1}{a}\left\{\eta(Y)g(A(\varphi^2X), \varphi^2Z) - \eta(Z)g(A(\varphi^2X), \varphi^2Y)\right\}, \tag{40}$$

$$F^{W_4}(X, Y, Z) = \frac{1}{a}\eta(X)\left\{\eta(Y)g(A\xi, \varphi^2Z) - \eta(Z)g(A\xi, \varphi^2Y)\right\}. \tag{41}$$

Hence, for F we have

$$F(X, Y, Z) = F^{W_2}(X, Y, Z) + F^{W_4}(X, Y, Z). \tag{42}$$

Thus, we proved the following proposition:

Proposition 4.1. *The submanifold* $(M, \varphi, \xi, \eta, g)$ *of* $(\overline{M}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ *belongs to the direct sum* $W_2 \oplus W_4$.

By virtue of (12), (15) and (17), we obtain:

Lemma 4.2. *The characteristic conditions of the classes* $\mathbb{G}_5 \oplus \mathbb{G}_8 \oplus \mathbb{G}_{10}$ *and* $\mathbb{G}_5 \oplus \mathbb{G}_8$ *of almost paracontact metric manifolds with respect to F are as follows:*

(*i*) $\mathbb{G}_5 \oplus \mathbb{G}_8 \oplus \mathbb{G}_{10}$:

$$F(X, Y, Z) = -\eta(Y)F(X, Z, \xi) + \eta(Z)F(X, Y, \xi), \tag{43}$$

$$F(X, Y, \xi) = F(Y, X, \xi); \tag{44}$$

(*ii*) $\mathbb{G}_5 \oplus \mathbb{G}_8$: *the conditions* (43), (44) *and*

$$F(X, Y, \xi) = -F(\varphi X, \varphi Y, \xi). \tag{45}$$

Theorem 4.3. *For the submanifold $(M, \varphi, \xi, \eta, g)$ of $(\overline{M}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$, the following assertions are equivalent:*
(i) *$M \in W_2$;*
(ii) *F is given by*

$$F(X, Y, Z) = \frac{1}{a}\left\{\eta(Y)g(A(\varphi^2X), \varphi^2Z) - \eta(Z)g(A(\varphi^2X), \varphi^2Y)\right\} \tag{46}$$

and (46) *can be written in the following equivalent form*

$$F(X, Y, Z) = -\eta(Y)F(X, Z, \xi) + \eta(Z)F(X, Y, \xi), \tag{47}$$

where

$$F(X, Y, \xi) = -\frac{1}{a}g(A(\varphi^2X), \varphi^2Y); \tag{48}$$

(iii) *A satisfies*

$$A\xi = \eta(A\xi)\xi; \tag{49}$$

(iv) *$M \in \mathbb{G}_5 \oplus \mathbb{G}_8 \oplus \mathbb{G}_{10}$.*

Proof. $(i) \Rightarrow (ii)$ If $M \in W_2$, then $F = F^{W_2}$. Having in mind (40), we see that (46) is fulfilled. Substituting Z with ξ in (46), we get (48). Then (46) and (48) yield (47).

$(ii) \Rightarrow (i)$ This implication is obvious.

$(ii) \Rightarrow (iii)$ If (46) holds, then $F = F^{W_2}$. Now, from (42) it follows that $F^{W_4} = 0$. Substituting X and Z with ξ in (41), we derive $g(A\xi, \varphi^2Y) = 0$. From the latter equality, by using (2) and (5), we obtain (49).

$(iii) \Rightarrow (ii)$ Conversely, let us assume that A satisfies (49). Thus, from (41) we get $F^{W_4} = 0$, which together with (42) implies $F = F^{W_2}$, i.e. (46) is valid.

$(i) \Rightarrow (iv)$ According to the proved assertion $(i) \Rightarrow (ii)$, the tensor F satisfies (47) and (48). Since A is self-adjoint with respect to g, from (48) it follows that $F(X, Y, \xi) = F(Y, X, \xi)$. Now, applying Lemma 4.2, we conclude that $M \in \mathbb{G}_5 \oplus \mathbb{G}_8 \oplus \mathbb{G}_{10}$.

$(iv) \Rightarrow (i)$ This implication is an immediate consequence from (11). □

Theorem 4.4. *For the submanifold $(M, \varphi, \xi, \eta, g)$ of $(\overline{M}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$, the following assertions hold:*
(i) *$M \in \mathbb{G}_5 \oplus \mathbb{G}_8$ if and only if $A \circ \varphi = \varphi \circ A$. Moreover, if $M \in \mathbb{G}_5 \oplus \mathbb{G}_8$, then*

$$\theta(\xi) = \frac{1}{a}\{\eta(A\xi) - \mathrm{tr}(A)\}; \tag{50}$$

(ii) *$M \in \mathbb{G}_{10}$ if and only if $A \circ \varphi = -\varphi \circ A$;*
(iii) *$M \in \mathbb{G}_5$ if and only if*

$$AX = \frac{1}{2n}\{[\mathrm{tr}(A) - \eta(A\xi)]X + [(2n+1)\eta(A\xi) - \mathrm{tr}(A)]\eta(X)\xi\}, \quad \mathrm{tr}(A) \neq \eta(A\xi); \tag{51}$$

(iv) *$M \in \mathbb{G}_5^*$ if and only if*

$$AX = -aX + [a + \eta(A\xi)]\eta(X)\xi; \tag{52}$$

(v) *$M \in \mathbb{G}_8$ if and only if $A \circ \varphi = \varphi \circ A$ and $\mathrm{tr}(A) = \eta(A\xi)$;*
(vi) *$M \in W_4 = \mathbb{G}_{12}$ if and only if*

$$AX = \eta(X)A\xi + \{\eta(AX) - \eta(X)\eta(A\xi)\}\xi; \tag{53}$$

(vii) M is a paracosymplectic manifold if and only if $AX = \eta(X)\eta(A\xi)\xi$.

Proof. First, we note that the direct sum $\mathbb{G}_5 \oplus \mathbb{G}_8 \oplus \mathbb{G}_{10}$ contains the classes $\mathbb{G}_5 \oplus \mathbb{G}_8$, $\mathbb{G}_{10}$, $\mathbb{G}_5$, $\mathbb{G}_5^*$ and $\mathbb{G}_8$. Therefore, if M belongs to some of these classes, then the assertions $(i) \div (iii)$ from Theorem 4.3 are valid.

(i) Let us assume that $M \in \mathbb{G}_5 \oplus \mathbb{G}_8$. Then (45) and (48) imply $-\frac{1}{a}g(A(\varphi^2X), \varphi^2Y) = \frac{1}{a}g(A(\varphi X), \varphi Y)$. Employing the latter equality, with the help of (1), (2), (5) and (49), we obtain $A \circ \varphi = \varphi \circ A$.

Conversely, by using $A \circ \varphi = \varphi \circ A$ and $\varphi\xi = 0$, we get (49). According to Theorem 4.3, from (49) it follows that $M \in \mathbb{G}_5 \oplus \mathbb{G}_8 \oplus \mathbb{G}_{10}$ and (48) holds. Now, by using (48) we check that $F(X, Y, \xi) = -F(\varphi X, \varphi Y, \xi)$. Thus, in view of Lemma 4.2, we obtain that $M \in \mathbb{G}_5 \oplus \mathbb{G}_8$.

It is known [14] that there exists a local orthonormal basis (called a φ-basis) $\{e_1, \ldots, e_n, \varphi e_1, \ldots, \varphi e_n, \xi\}$ on M, such that $g(e_i, e_i) = -g(\varphi e_i, \varphi e_i) = 1$ $(i = 1, \ldots, n)$. Let us suppose that $M \in \mathbb{G}_5 \oplus \mathbb{G}_8$. Hence, the equality $A \circ \varphi = \varphi \circ A$ is fulfilled. Then, by virtue of (4), (7) and (48), we find

$$\theta(\xi) = \sum_{i=1}^{n} \{F(e_i, e_i, \xi) - F(\varphi e_i, \varphi e_i, \xi)\} = \frac{1}{a}\sum_{i=1}^{n}\left\{-g(A(\varphi^2 e_i), \varphi^2 e_i) + g(A(\varphi e_i), \varphi e_i)\right\} = -\frac{2}{a}\sum_{i=1}^{n} g(Ae_i, e_i),$$

$$\mathrm{tr}(A) = \sum_{i=1}^{n}\{g(Ae_i, e_i) - g(A(\varphi e_i), \varphi e_i)\} + g(A\xi, \xi) = 2\sum_{i=1}^{n} g(Ae_i, e_i) + \eta(A\xi).$$

Thus, from the expressions for $\theta(\xi)$ and $\mathrm{tr}(A)$ we obtain (50).

(ii) We omit the proof of (ii), because by using (17), we can prove it in a similar way as (i). Note that if $M \in \mathbb{G}_{10}$, then $\theta(\xi) = 0$ and $\mathrm{tr}(A) = \eta(A\xi)$.

(iii) If $M \in \mathbb{G}_5$, then from (i) it follows that $A \circ \varphi = \varphi \circ A$. By using the latter, (4), (48) and (49), we get

$$F(X, Y, \xi) = \frac{1}{a}\{-g(AX, Y) + \eta(A\xi)\eta(X)\eta(Y)\}. \tag{54}$$

On the other hand, from (12) we have

$$F(X, Y, \xi) = -\frac{\theta(\xi)}{2n}g(\varphi X, \varphi Y), \tag{55}$$

where $\theta(\xi)$ is given by (50). Equating the right sides of (54) and (55) we obtain (51).

Conversely, let the shape operator A satisfies (51). Then we have $A \circ \varphi = \varphi \circ A$. Hence $M \in \mathbb{G}_5 \oplus \mathbb{G}_8$, which implies that (47), (48) and (50) hold good. Since $\mathrm{tr}(A) \neq \eta(A\xi)$, from (50) it follows that $\theta(\xi) \neq 0$. Substituting (51) in (48) and having in mind (47) we obtain (12), which completes the proof of (iii).

(iv) If $M \in \mathbb{G}_5^*$, then from the definition of $\mathbb{G}_5^*$ it follows that M is a manifold from the class $\mathbb{G}_5$ for which $\theta(\xi) = 2n$. This enables us to apply (iii). Hence, (50) and (51) are fulfilled and from (50) we get $\mathrm{tr}(A) = -2na + \eta(A\xi)$. Substituting the latter in (51) we derive (52).

Conversely, suppose (52) is satisfied. Then we have $A \circ \varphi = \varphi \circ A$, which implies that $M \in \mathbb{G}_5 \oplus \mathbb{G}_8$. Hence, (47) and (48) are valid. By virtue of (48) and (52) we obtain (19), i.e. $M \in \mathbb{G}_5^*$.

(v) From (15) and Lemma 4.2 it follows that $M \in \mathbb{G}_8$ if and only if $M \in \mathbb{G}_5 \oplus \mathbb{G}_8$ and $\theta(\xi) = 0$. Therefore, the truth of (v) one can easily establish by using (i).

(vi) $M \in W_4$ if and only if $F = F^{W_4}$. From (42) it follows that $M \in W_4$ if and only if $F^{W_2} = 0$. Having in mind (40), the vanishing of the projection F^{W_2} is equivalent to $g(A(\varphi^2X), \varphi^2Z) = 0$. After standard computations, by using (2) and (5), we obtain that the latter condition is equivalent to (53).

(vii) M is paracosymplectic if and only if $F = 0$, which is equivalent to $F^{W_2} = F^{W_4} = 0$. Considering the proofs of $(ii) \Rightarrow (iii)$ of Theorem 4.3 and (vi) of Theorem 4.4, we infer that $F = 0$ if and only if both (49) and (53) hold. Now, the equivalence of the condition $AX = \eta(X)\eta(A\xi)\xi$ with (49) and (53) is easily seen. □

As an immediate consequence of Theorem 2.4 and Theorem 4.4, we state

Theorem 4.5. *For the submanifold* $(M, \varphi, \xi, \eta, g)$ *of* $(\overline{M}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ *the following statements are equivalent:*

(i) M is normal;

(ii) M is quasi-para-Sasakian;

(iii) one of the following conditions is fulfilled: (51), $A \circ \varphi = \varphi \circ A$.

Furthermore, the statements given below are also equivalent:
(iv) M is paracontact metric;
(v) M is para-Sasakian;
(vi) M is K-paracontact metric;
(vii) A satisfies (52).

Let $\overline{R}$ and R be the Riemannian curvature tensor fields of $\overline{M}$ and M given by $\overline{R}(\overline{X},\overline{Y})\overline{Z} = \overline{\nabla}_{\overline{X}}\overline{\nabla}_{\overline{Y}}\overline{Z} - \overline{\nabla}_{\overline{Y}}\overline{\nabla}_{\overline{X}}\overline{Z} - \overline{\nabla}_{[\overline{X},\overline{Y}]}\overline{Z}$ and $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$, respectively. The corresponding to $\overline{R}$ and R tensor fields of type $(0,4)$ are defined by

$$\overline{R}(\overline{X},\overline{Y},\overline{Z},\overline{W}) = \overline{g}(\overline{R}(\overline{X},\overline{Y})\overline{Z},\overline{W}) \quad \text{and} \quad R(X,Y,Z,W) = g(R(X,Y)Z,W),$$

respectively. From the conditions $\overline{\nabla}\overline{\varphi} = \overline{\nabla}\,\overline{\xi} = 0$ and (4) it follows that $\overline{R}$ has the property

$$\overline{R}(\overline{X},\overline{Y},\overline{\varphi}\overline{Z},\overline{\varphi}\overline{W}) = -\overline{R}(\overline{X},\overline{Y},\overline{Z},\overline{W}). \tag{56}$$

By using (36), for the second fundamental form σ of M and $(\nabla_X\sigma)Y$ we have

$$\begin{aligned} &\sigma(X,Y) = -g(AX,Y)\left(\frac{b}{a}\overline{N}_1 + \overline{N}_2\right), \\ &(\nabla_X\sigma)(Y,Z) = D_X\sigma(Y,Z) - \sigma(\nabla_X Y,Z) - \sigma(Y,\nabla_X Z) \\ &\qquad = \left\{-g((\nabla_X A)Y,Z) - \frac{b}{a}\gamma(X)g(AY,Z)\right\}\left\{\frac{b}{a}\overline{N}_1 + \overline{N}_2\right\}. \end{aligned} \tag{57}$$

Then we obtain the equations of Gauss and Codazzi

$$\begin{aligned} \overline{R}(X,Y,Z,W) &= R(X,Y,Z,W) + \overline{g}(\sigma(X,Z),\sigma(Y,W)) - \overline{g}(\sigma(Y,Z),\sigma(X,W)) \\ &= R(X,Y,Z,W) + \frac{1}{a^2}\pi_1(AX,AY,Z,W), \end{aligned} \tag{58}$$

where the tensor π_1 is defined by $\pi_1(X,Y,Z,W) = g(Y,Z)g(X,W) - g(X,Z)g(Y,W)$ and

$$\begin{aligned} (\overline{R}(X,Y)Z)^{\perp} &= (\nabla_X\sigma)(Y,Z) - (\nabla_Y\sigma)(X,Z) \\ &= \left\{g((\nabla_Y A)X,Z) - g((\nabla_X A)Y,Z) + \frac{b}{a}[\gamma(Y)g(AX,Z) - \gamma(X)g(AY,Z)]\right\}\left\{\frac{b}{a}\overline{N}_1 + \overline{N}_2\right\}, \end{aligned} \tag{59}$$

respectively.

Proposition 4.6. *For the curvature tensor R of (M,φ,ξ,η,g) the following equality holds*

$$\begin{aligned} R(X,Y,\varphi Z,\varphi W) = {}& -R(X,Y,Z,W) - \frac{1}{a^2}\pi_1(AX,AY,Z,W) - \frac{1}{a^2}\pi_1(AX,AY,\varphi Z,\varphi W) \\ & + \frac{1}{a}\eta(W)\left\{g((\nabla_Y A)X,\varphi Z) - g((\nabla_X A)Y,\varphi Z) + \frac{b}{a}[\gamma(Y)g(AX,\varphi Z) - \gamma(X)g(AY,\varphi Z)]\right\} \\ & - \frac{1}{a}\eta(Z)\left\{g((\nabla_Y A)X,\varphi W) - g((\nabla_X A)Y,\varphi W) + \frac{b}{a}[\gamma(Y)g(AX,\varphi W) - \gamma(X)g(AY,\varphi W)]\right\}. \end{aligned} \tag{60}$$

Proof. With the help of (32) we get

$$\begin{aligned} \overline{R}(X,Y,\overline{\varphi}Z,\overline{\varphi}W) = {}& \overline{R}(X,Y,\varphi Z,\varphi W) + \eta(W)[b\overline{R}(X,Y,\varphi Z,\overline{N}_1) + a\overline{R}(X,Y,\varphi Z,\overline{N}_2)] \\ & + \eta(Z)[b\overline{R}(X,Y,\overline{N}_1,\varphi W) + a\overline{R}(X,Y,\overline{N}_2,\varphi W)]. \end{aligned} \tag{61}$$

From $\overline{R}(X, Y, \overline{\xi}) = 0$ and (21) we deduce $\overline{R}(X, Y, Z, \overline{N}_1) = -\frac{b}{a}\overline{R}(X, Y, Z, \overline{N}_2)$. Hence, (61) becomes

$$\overline{R}(X, Y, \overline{\varphi}Z, \overline{\varphi}W) = \overline{R}(X, Y, \varphi Z, \varphi W) + \frac{1}{a}\eta(W)\overline{R}(X, Y, \varphi Z, \overline{N}_2) - \frac{1}{a}\eta(Z)\overline{R}(X, Y, \varphi W, \overline{N}_2). \tag{62}$$

On the other hand, $\overline{R}$ satisfies (56). Equating the right sides of (62) and (56), we derive

$$\overline{R}(X, Y, \varphi Z, \varphi W) + \frac{1}{a}\eta(W)\overline{R}(X, Y, \varphi Z, \overline{N}_2) - \frac{1}{a}\eta(Z)\overline{R}(X, Y, \varphi W, \overline{N}_2) = -\overline{R}(X, Y, Z, W). \tag{63}$$

By virtue of (59) we have

$$\overline{R}(X, Y, \varphi Z, \overline{N}_2) = \left\{ g((\nabla_Y A)X, \varphi Z) - g((\nabla_X A)Y, \varphi Z) + \frac{b}{a}[\gamma(Y)g(AX, \varphi Z) - \gamma(X)g(AY, \varphi Z)] \right\}. \tag{64}$$

Finally, by using (58), (63) and (64), we obtain (60). □

Definition 4.7. *Let S be a submanifold in a (semi-) Riemannian manifold (N, h). The normal connection D on the normal bundle $TS^\perp$ is called flat if $R^\perp(X, Y)V = 0$, where the curvature tensor field $R^\perp$ is defined by*

$$R^\perp(X, Y)V = D_X D_Y V - D_Y D_X V - D_{[X,Y]}V,$$

for any $X, Y \in \chi(S)$ and any $V \in TS^\perp$.

Proposition 4.8. *The normal connection D of the submanifold $(M, \varphi, \xi, \eta, g)$ of $(\overline{M}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ is flat if and only if the 1-form γ is closed.*

Proof. To prove $R^\perp(X, Y)V = 0$ for any $V \in TM^\perp$, it is sufficient to prove that $R^\perp(X, Y)\overline{N}_1 = R^\perp(X, Y)\overline{N}_2 = 0$. Employing (36), we get $R^\perp(X, Y)\overline{N}_1 = \mathrm{d}\gamma(X, Y)\overline{N}_2$ and $R^\perp(X, Y)\overline{N}_2 = \mathrm{d}\gamma(X, Y)\overline{N}_1$. The vanishing of $R^\perp(X, Y)\overline{N}_1$ and $R^\perp(X, Y)\overline{N}_2$ is equivalent to $\mathrm{d}\gamma(X, Y) = 0$, which completes the proof. □

5. Codimension 2 para-Sasakian submanifolds of paracosymplectic manifolds with normal Reeb vector field

In this section we deal with curvature properties of para-Sasakian submanifolds of paracosymplectic manifolds. Also, we construct an example of a para-Sasakian submanifold of the considered type in the present paper.

Definition 5.1. *[2] A submanifold S in a (semi-) Riemannian manifold (N, h) is said to be:*
(*i*) *totally geodesic if its shape operator vanishes identically, that is, $A = 0$ or equivalently the second fundamental form σ vanishes identically;*
(*ii*) *umbilical with respect to the normal vector field V to S if $A_V = fI$ (I is the identity transformation) for some function f;*
(*iii*) *totally umbilical if S is umbilical with respect to every normal vector field to S;*
(*iv*) *parallel if $\nabla\sigma = 0$.*

Remark 5.2. *We note that if $(M, \varphi, \xi, \eta, g)$ is a para-Sasakian submanifold of $(\overline{M}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$, then from Theorem 4.5 it follows that A satisfies* (52). *Therefore, we have* $\mathrm{tr}(A) = -2na + \eta(A\xi)$.

Proposition 5.3. *There exist no totally geodesic para-Sasakian submanifolds of paracosymplectic manifolds.*

Proof. Let us assume that there exists a totally geodesic para-Sasakian submanifold $(M, \varphi, \xi, \eta, g)$ of a paracosymplectic manifold $(\overline{M}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$. Taking into account Definition 5.1, we have $AX = 0$ for any $X \in \chi(M)$. Then from (52) we get $X = \frac{[a + \eta(A\xi)]}{a}\eta(X)\xi$. Since $\dim M = 2n + 1 \geq 3$, the latter equality leads to a contradiction. □

Theorem 5.4. *Let $(M, \varphi, \xi, \eta, g)$ be a para-Sasakian submanifold of $(\overline{M}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$. Then the following assertions are equivalent:*
(i) M is parallel;
(ii) $\eta(A\xi) = -a$;
(iii) M is totally umbilical.

Proof. $(i) \Rightarrow (ii)$ Let M be parallel. According to Definition 5.1, $(\nabla_X \sigma)(Y, Z) = 0$ for any $X, Y, Z \in \chi(M)$. By virtue of (52) and (57), we obtain

$$(\nabla_X \sigma)(Y, Z)$$

$$= \left\{ \left[-\frac{b}{a}\gamma(X)\eta(A\xi) - (X\eta(A\xi)) \right] \eta(Y)\eta(Z) - [a + \eta(A\xi)][\eta(Y)g(\varphi X, Z) + \eta(Z)g(\varphi X, Y)] \right\} \left\{ \frac{b}{a}\overline{N}_1 + \overline{N}_2 \right\} = 0.$$

In the above equality we replace Y and Z with φY and ξ, respectively. Thus, we get

$$[a + \eta(A\xi)]g(\varphi X, \varphi Y) \left\{ \frac{b}{a}\overline{N}_1 + \overline{N}_2 \right\} = 0.$$

Now, the linear independence of $\overline{N}_1$ and $\overline{N}_2$ implies $[a + \eta(A\xi)]g(\varphi X, \varphi Y) = 0$ for any $X, Y \in \chi(M)$. Hence, $\eta(A\xi) = -a$ holds good.

$(ii) \Rightarrow (i)$ Let us assume that $\eta(A\xi) = -a$. By using the obtained expression for $(\nabla_X \sigma)(Y, Z)$ above we directly verify that $(\nabla_X \sigma)(Y, Z) = 0$, i.e. M is parallel.

$(ii) \Rightarrow (iii)$ If $\eta(A\xi) = -a$, then from (52) we have $AX = -aX$, which means that M is umbilical with respect to $\overline{N}_2$. From $A_{\overline{N}_1}X = -\frac{b}{a}A_{\overline{N}_2}X = -\frac{b}{a}AX$ it follows that M is also umbilical with respect to $\overline{N}_1$. Therefore, M is totally umbilical.

$(iii) \Rightarrow (ii)$ Let us suppose that M is totally umbilical. Then $AX = fX$, where $f \in \mathcal{F}(M)$. According to Remark 5.2, we have $2nf + \eta(A\xi) = -2na + \eta(A\xi)$. Hence, we get $f = -a$. Thus, the left-hand side of (52) becomes $-aX$, which implies $\eta(A\xi) = -a$. □

Theorem 5.5. *Let $(\overline{M}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ be of a constant sectional curvature $\overline{K}$ and let $(M, \varphi, \xi, \eta, g)$ be a para-Sasakian submanifold of $\overline{M}$. Then M is of a constant sectional curvature $(\overline{K} - 1)$ if and only if M is totally umbilical.*

Proof. With the help of (52) and (58) we obtain

$$\begin{aligned} \overline{R}(X, Y, Z, W) &= R(X, Y, Z, W) + \pi_1(X, Y, Z, W) \\ &+ \frac{[a + \eta(A\xi)]}{a} \{g(X, Z)\eta(Y)\eta(W) + g(Y, W)\eta(X)\eta(Z) - g(Y, Z)\eta(X)\eta(W) - g(X, W)\eta(Y)\eta(Z)\}, \end{aligned} \tag{65}$$

which is the equation of Gauss for a para-Sasakian submanifold M of $\overline{M}$. Since $\overline{M}$ is of a constant sectional curvature, the curvature tensor $\overline{R}$ of $\overline{M}$ has the form $\overline{R}(X, Y, Z, W) = \overline{K}\overline{\pi}_1(X, Y, Z, W)$. Taking into account that the restriction of $\overline{g}$ on M is g, we have $\overline{\pi}_1(X, Y, Z, W) = \overline{g}(Y, Z)\overline{g}(X, W) - \overline{g}(X, Z)\overline{g}(Y, W) = \pi_1(X, Y, Z, W)$. Hence, (65) becomes

$$\begin{aligned} \overline{K}\pi_1(X, Y, Z, W) &= R(X, Y, Z, W) + \pi_1(X, Y, Z, W) \\ &+ \frac{[a + \eta(A\xi)]}{a} \{g(X, Z)\eta(Y)\eta(W) + g(Y, W)\eta(X)\eta(Z) - g(Y, Z)\eta(X)\eta(W) - g(X, W)\eta(Y)\eta(Z)\}. \end{aligned} \tag{66}$$

Now, let us suppose that M is of a constant sectional curvature $(\overline{K} - 1)$. Substituting $R(X, Y, Z, W)$ with $(\overline{K} - 1)\pi_1(X, Y, Z, W)$ in (66), we get

$$\frac{[a + \eta(A\xi)]}{a} \{g(X, Z)\eta(Y)\eta(W) + g(Y, W)\eta(X)\eta(Z) - g(Y, Z)\eta(X)\eta(W) - g(X, W)\eta(Y)\eta(Z)\} = 0$$

for any $X, Y, Z, W \in \chi(M)$. From the latter equality we derive $\eta(A\xi) = -a$. Applying Theorem 5.4 we conclude that M is totally umbilical.

Conversely, let M be totally umbilical. Then the condition $\eta(A\xi) = -a$ and (66) imply $(\overline{K}-1)\pi_1(X, Y, Z, W) = R(X, Y, Z, W)$. Thus, for the sectional curvature K of a non-degenerate section $\alpha = \operatorname{span}\{X, Y\}$ in M we obtain

$$K = \frac{R(X, Y, Y, X)}{\pi_1(X, Y, Y, X)} = \overline{K} - 1.$$

□

Example 5.6. *Let $\overline{M} = \mathbb{R}^{2n+3}_{n+1} = \{u = (x^1, \dots, x^{n+1}, y^1, \dots, y^{n+1}, t) \mid x^i, y^i, t \in \mathbb{R}\}$. We define an almost paracontact metric structure $(\overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ on $\mathbb{R}^{2n+3}_{n+1}$ in the following way:*

$$\overline{\varphi}\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}, \qquad \overline{\varphi}\left(\frac{\partial}{\partial y^i}\right) = \frac{\partial}{\partial x^i}, \qquad \overline{\varphi}\left(\frac{\partial}{\partial t}\right) = 0, \quad i = 1, \dots, n+1; \qquad \overline{\xi} = \frac{\partial}{\partial t}; \qquad \overline{\eta} = \mathrm{d}t;$$

and $\overline{g}$ is a pseudo-Euclidean scalar product, given by

$$\langle u, u\rangle = \sum_{i=1}^{n+1} \left\{(x^i)^2 - (y^i)^2\right\} + t^2.$$

Since the components $\overline{g}_{ij}$ of the matrix of $\overline{g}$ with respect to the local basis $\left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial t}\right\}$, $(i = 1, \dots, n+1)$ are constants, the Levi-Civita connection $\overline{\nabla}$ of $\overline{g}$ is flat. Now, it is easy to see that $\overline{\nabla}\,\overline{\varphi} = 0$. Hence, $\overline{M} = (\mathbb{R}^{2n+3}_{n+1}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ is a paracosymplectic manifold.

Let us consider the submanifold $M = \{u = (x^1, \dots, x^{n+1}, y^1, \dots, y^{n+1}, 0) \in \overline{M} \mid \langle u, u\rangle = -1\}$ of $\overline{M}$, which is given locally by the following immersion:

$$\begin{aligned} &i\left(x^1, \dots, x^{n+1}, y^1, \dots, y^n\right) \\ &\quad = \left(x^1, \dots, x^{n+1}, y^1, \dots, y^n, y^{n+1} = \sqrt{(x^1)^2 + \dots + (x^{n+1})^2 - (y^1)^2 - \dots - (y^n)^2 + 1}, 0\right). \end{aligned} \tag{67}$$

Identifying a point p in M with its position vector Z, we obtain

$$TM = \operatorname{span}\left\{\frac{\partial Z}{\partial x^i} = \frac{\partial}{\partial x^i} + \frac{\partial y^{n+1}}{\partial x^i}, \frac{\partial Z}{\partial y^j} = \frac{\partial}{\partial y^j} + \frac{\partial y^{n+1}}{\partial y^j}\right\}, \ i = 1, \dots, n+1; \quad j = 1, \dots, n.$$

The vector fields $\overline{\xi}$ and Z are normal to M and $\overline{g}(Z, Z) = -1$. From $\overline{g}(\overline{\xi}, \overline{\varphi}Z) = \overline{g}(Z, \overline{\varphi}Z) = 0$ it follows that $\overline{\varphi}Z \in \chi(M)$. Moreover, we have $\overline{g}(\overline{\varphi}Z, \overline{\varphi}Z) = 1$.

Now, we determine an almost paracontact metric structure (φ, ξ, η, g) on M by

$$\xi = \overline{\varphi}Z, \quad \varphi X = \overline{\varphi}X - \eta(X)Z, \ \ X \in \chi(M), \tag{68}$$

where g is the restriction of $\overline{g}$ on M and $\eta(X) = g(X, \xi)$. Hence, $(M, \varphi, \xi, \eta, g)$ is an almost paracontact metric manifold, which is a submanifold of codimension 2 of $\overline{M}$ such that $\overline{\xi}$ is normal to M and satisfies (21). *We note that we put $\overline{N}_1 = \overline{\xi}$, $\overline{N}_2 = Z$, for which the conditions in* (20) *hold. Then $a = 1$ and $b = 0$ in* (21). *Since $\overline{\nabla}$ is flat, we have $\overline{\nabla}_X Z = X$ for any $X \in \chi(M)$, which means that $A = -I$. Thus, the Gauss-Weingarten formulas are*

$$\begin{aligned} &\overline{\nabla}_X Y = \nabla_X Y + g(X, Y)\overline{N}_2, \\ &\overline{\nabla}_X N_2 = X. \end{aligned} \tag{69}$$

By using (68) *and* (69), *we find $(\nabla_X\varphi)Y = g(X, Y)\xi - \eta(Y)X$. From the latter equality we obtain*

$$F(X, Y, Z) = \eta(Y)g(\varphi X, \varphi Z) - \eta(Z)g(\varphi X, \varphi Y). \tag{70}$$

Let $\overline{R}$ and R be the curvature tensors of $\overline{M}$ and M, respectively. By virtue of (69) *and $\overline{R} = 0$, we get*

$$R(X, Y, Z, W) = -\pi_1(X, Y, Z, W). \tag{71}$$

Theorem 5.7. *Let $(M, \varphi, \xi, \eta, g)$ be the submanifold of $\overline{M} = (\mathbb{R}^{2n+3}_{n+1}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$, defined by* (67)*. Then we have:*

(i) M is a para-Sasakian manifold;

(ii) M is parallel;

(iii) M is totally umbilical;

(iv) M is of constant sectional curvature -1.

Proof. (i) *Taking into account* (70), (19) *and Theorem 2.4, we conclude that M is para-Sasakian.*

(ii) *By direct calculations, using* (69)*, we obtain* $(\nabla_X \sigma)(Y, Z) = 0$ *for any $X, Y, Z \in \chi(M)$. Then, according to Definition 5.1, M is parallel.*

(iii) *The truth of this assertion follows from $A = -I$ and Definition 5.1.*

(iv) *It is an immediate consequence of* (71). □

In conclusion, we remark that for the shape operator A of M we have $A\xi = -\xi$, which implies $\eta(A\xi) = -1$. This means that the condition $\eta(A\xi) = -a$ holds. Therefore, the constructed example confirms the results obtained in Theorem 4.5, Theorem 5.4 and Theorem 5.5.

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