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On generalized almost para-Hermitian spaces

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Abstract. Recently, a generalized almost Hermitian metric on an almost complex manifold (M, J) is determined as a generalized Riemannian metric (i.e. an arbitrary bilinear form) $\mathcal{G}$ which satisfies $\mathcal{G}(JX, JY) = \mathcal{G}(X, Y)$, where X and Y are arbitrary vector fields on M. In the same manner we can study a generalized almost para-Hermitian metric and determine almost para-Hermitian spaces. Some properties of these spaces and special generalized almost para-Hermitian spaces including generalized para-Hermitian spaces as well as generalized nearly para-Kähler spaces are determined. Finally, a non-trivial example of generalized almost para-Hermitian space is constructed.

1. Introduction

This paper is devoted to the study of generalizations of Hermitian spaces, which generalize the well-known Kähler spaces. As is known, Kähler spaces were introduced by Kähler in 1934, but independently of him, these spaces were also studied by P.A. Shirokov, see [11, pp. 160-167]. Generalizations of these spaces in various directions can be found in research [2], papers [6, 21] and monograph [11]. Holomorphically projective mappings of Kähler spaces have been studied by Japanese mathematicians since 1950. One of the continuations is the 1971 paper [20] by M. Prvanović. Results on holomorphically projective mappings and transformations are in [10, 11]. An interesting result on holomorphically projective mappings of generalized Kähler spaces can be found in [16, 19, 22].

These spaces and mappings are generalized under the notion of F-structures, which have more general consequences, see e.g., [3, 7]. In this paper, we study generalized almost para-Hermitian spaces. Some properties of these spaces and special generalized almost para-Hermitian spaces including generalized para-Hermitian spaces as well as generalized nearly para-Kähler spaces are discussed. Finally, an example is presented in explicit form.

2. Almost Hermitian spaces and their generalizations

An almost complex structure on a real differentiable manifold M is a $(1, 1)$-tensor field J such that [23]

$$J^2 = -I,$$

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where I is the identity operator.

Let $\mathcal{X}(M)$ be the Lie algebra of smooth vector fields on M and let us assume that $X, Y \in \mathcal{X}(M)$. A real differentiable manifold M endowed with an almost complex structure J ($J^2 = -I$) is said to be an *almost complex manifold* or an *almost complex space* [23]. An almost complex space (M, J) is said to be an *almost Hermitian space* if there exists a Riemannian metric g on M such that [23]

$$g(JX, JY) = g(X, Y),$$

i.e.,

$$-g(X, JY) = g(JX, Y),$$

which evidently means that the fundamental 2-form

$$F(X, Y) := g(X, JY)$$

is skew-symmetric.

M. Prvanović in 1995 [21] considered an almost Hermitian space (M, g, J) as a particular generalized Riemannian space $(M, \mathcal{G}^{g,F} = g + F)$ in the sense of Eisenhart [5] and gave a classification of almost Hermitian spaces which heavily depends on the Einstein connection D determined by $(D_Z\mathcal{G})(X, Y) = 2\mathcal{G}(X, T(Z, Y))$, where T is the torsion tensor of D. The classification given in [21] is analogous to the classification of A. Gray and L.M. Hervella [6]. In [19] a *generalized Hermitian metric* on an almost complex manifold (M, J) is defined as a generalized Riemannian metric in the sense of Eisenhart $\mathcal{G}$ that is invariant by the almost complex structure J, i.e.,

$$\mathcal{G}(JX, JY) = \mathcal{G}(X, Y),$$

which further implies that

$$\frac{1}{2}(\mathcal{G}(JX, JY) \pm \mathcal{G}(JY, JX)) = \frac{1}{2}(\mathcal{G}(X, Y) \pm \mathcal{G}(Y, X)),$$

i.e.,

$$g(JX, JY) = g(X, Y) \quad \text{and} \quad F(JX, JY) = F(X, Y).$$

Definition 2.1. [19] *An almost complex manifold (M, J) endowed with a generalized Hermitian metric $\mathcal{G}$ is called a* generalized almost Hermitian space *and it is denoted by $(M, \mathcal{G}, J)$.*

In 2001, Minčić, Stanković and Velimirović [14] gave a definition of a generalized Kähler space assuming that

$$\begin{aligned} J^2 &= -I, \\ g(J\partial_i, J\partial_j) &= g(\partial_i, \partial_j), \\ (\overset{1}{\nabla}_{\partial_i} J)\partial_j &= 0 \text{ and } (\overset{2}{\nabla}_{\partial_i} J)\partial_j = 0, \end{aligned}$$

where $\overset{1}{\nabla}$ is a non-symmetric linear connection explicitly determined by [5]

$$g(\overset{1}{\nabla}_{\partial_i}\partial_j, \partial_k) = \frac{1}{2}\left(\partial_i\mathcal{G}(\partial_j, \partial_k) + Y\mathcal{G}(\partial_i, \partial_k) - \partial_k\mathcal{G}(\partial_j, \partial_i)\right),$$

where $\partial_i = \frac{\partial}{\partial x^i}$, $\partial_j = \frac{\partial}{\partial x^j}$ and $\partial_k = \frac{\partial}{\partial x^k}$ is standard orthonormal basis of the tangent space $T_p(M)$ at the point p of the manifold M. As is well-know a non-symmetric linear connection $\overset{2}{\nabla}$ which is dual to $\overset{1}{\nabla}$ is determined by

$$\overset{2}{\nabla}_X Y = \overset{1}{\nabla}_Y X + [X, Y],$$

or in the standard orthonormal basis as

$$\overset{2}{\nabla}_{\partial_i}\partial_j = \overset{1}{\nabla}_{\partial_j}\partial_i.$$

Also, as is well-know the torsion-free linear connection $\overset{0}{\nabla}$ that is associated with the non-symmetric linear connections $\overset{1}{\nabla}$ and $\overset{2}{\nabla}$ is determined by

$$\overset{0}{\nabla} = \frac{1}{2}\left(\overset{1}{\nabla}_X Y + \overset{2}{\nabla}_X Y\right).$$

In [16] a more general definition of a generalized Kähler space in the sense of Eisenhart is given as in Definition 2.2.

Definition 2.2. [16] *A generalized Riemannian space* $(M, \mathcal{G})$ *is called a* generalized Kähler space in the sense of Eisenhart *if there exists a* $(1,1)$*-tensor field* J *on* M *such that*

$$\begin{aligned} J^2 &= -I, \\ g(JX, JY) &= g(X, Y), \\ (\overset{0}{\nabla}_X J)Y &= 0, \end{aligned}$$

where $\overset{0}{\nabla}_X Y = \frac{1}{2}(\overset{1}{\nabla}_X Y + \overset{2}{\nabla}_X Y)$ *is the symmetric part of the non-symmetric linear connection* $\overset{1}{\nabla}$ *and* I *is the identity operator.*

Definition 2.3. *A generalized Kähler space in the sense of Eisenhart* (M, g, J) *is said to be a* generalized Kähler space in the sense of Eisenhart with parallel torsion *if the torsion tensor* $\overset{1}{T}(X, Y) = \overset{1}{\nabla}_X Y - \overset{1}{\nabla}_Y X - [X, Y]$ *satisfies*

$$\overset{0}{\nabla}\overset{1}{T} = 0, \quad \textit{where} \quad \overset{0}{\nabla}_X Y = \frac{1}{2}(\overset{1}{\nabla}_X Y + \overset{2}{\nabla}_X Y).$$

3. Almost para-Hermitian spaces and their generalizations

An almost product structure on a real differentiable manifold M is a $(1,1)$-tensor field J such that [4]

$$J^2 = I,$$

where I is the identity operator.

Let (M, J) be an almost paracomplex manifold of dimension $2n > 2$ and g be a pseudo-Riemannian metric on M. The space (M, g, J) is said to be an *almost para-Hermitian space* if the condition [4]

$$g(JX, Y) + g(X, JY) = 0,$$

is satisfied. Almost para-Hermitian spaces were thoroughly studied for instance in [1, 4, 8].

In the same way as M. Prvanović did in [21] in the case of almost Hermitian manifolds and similar approach was also used in [9] we can use the following 2-form

$$F(X, Y) := g(JX, Y) = -g(X, JY) = -g(JY, X) = -F(Y, X)$$

and consider the bilinear form

$$\mathcal{G}^{g,F}(X, Y) := g(X, Y) + F(X, Y),$$

which is neither symmetric nor skew-symmetric.

Let us consider a $2n$-dimensional smooth manifold M endowed with an almost para-complex structure J and a bilinear form $\mathcal{G}$ which satisfies

$$\mathcal{G}(JX, JY) = -\mathcal{G}(X, Y),$$

or equivalently

$$\mathcal{G}(JX, Y) + \mathcal{G}(X, JY) = 0.$$

The bilinear form $\mathcal{G}$, which is neither symmetric nor skew-symmetric, can be described via its symmetric part g and skew-symmetric part ω as follows

$$\mathcal{G}(X, Y) = g(X, Y) + \omega(X, Y).$$

It is not difficult to conclude that the metric g and 2-form ω satisfy

$$g(JX, JY) = -g(X, Y) \quad \text{and} \quad \omega(JX, JY) = -\omega(X, Y),$$

Therefore,

$$g(JX, Y) + g(X, JY) = 0 \quad \text{and} \quad \omega(JX, Y) + \omega(X, JY) = 0.$$

Obviously, the bilinear form $\mathcal{G}$ is different than $\mathcal{G}^{g,F}$.

Let (M, J) be an almost paracomplex manifold and $\mathcal{G}$ be a generalized pseudo-Riemannian metric on M. If the equality

$$\mathcal{G}(JX, Y) + \mathcal{G}(X, JY) = 0,$$

holds, then the metric $\mathcal{G}$ is said to be a *generalized almost para-Hermitian metric* and consequently the space $(M, \mathcal{G} = g + \omega, J)$ is called a *generalized almost para-Hermitian space*.

Definition 3.1 (Generalized para-Hermitian space). *A generalized almost para-Hermitian space (M, g, J), where J is an integrable almost para-Hermitian structure, is called a* generalized para-Hermitian space.

It is well-known that the almost paracomplex structure J is integrable if and only if the Nijenhuis tensor identically vanishes, i.e., [23]

$$N(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] + [X, Y] = 0.$$

As a particular case of generalized almost para-Hermitian space we can consider a generalized nearly para-Kähler space.

Definition 3.2 (Generalized nearly para-Kähler space). *A generalized almost para-Hermitian space $(M, \mathcal{G} = g + \omega, J)$ is said to be a* generalized nearly para-Kähler space *if*

$$(\nabla^g_X J)X = 0,$$

where ∇^g is the Levi-Civita connection of metric g.

The definition of generalized hyperbolic Kähler spaces introduced in [15] was extended in [**?**] and some other types of generalized para Kähler spaces were described in [18].

Definition 3.3 (Generalized para-Kähler space). [**?**] *A generalized pseudo-Riemannian space $(M, \mathcal{G} = g + \omega)$ of dimension $2n \geq 4$ is called a* generalized para-Kähler space *if there exists a $(1, 1)$-tensor field J on M such that*

$$\begin{aligned} J^2 &= I, \\ g(JX, JY) &= -g(X, Y), \\ (\nabla^g_X J)Y &= 0, \end{aligned}$$

where ∇^g is the Levi-Civita connection of metric g and I is the identity operator.

In the same manner as *Example 3.1* in [16] here we construct *Example 3.1*.

Example 3.1. *Let us consider a space $(M, \mathcal{G} = g + \omega, J)$ of real dimension $n = 4$, where the components of the bilinear form $\mathcal{G} = g + \omega$ and the almost product structure J are, respectively, given by*

$$(\mathcal{G}_{ij}) = \begin{pmatrix} e^{2(t+r)} & \varphi\cos^2\theta & 0 & 0 \\ -\varphi\cos^2\theta & -e^{2(t+r)} & 0 & 0 \\ 0 & 0 & \varphi^2\sin^2\theta & -\varphi(t+r)^2 \\ 0 & 0 & \varphi(t+r)^2 & -\varphi^2\sin^2\theta \end{pmatrix} \quad \textit{and} \quad (J^h_i) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

where $t, r, \varphi \neq 0$ and $\theta \neq k\pi$, $k \in \mathbb{Z}$.

The bilinear form $\mathcal{G} = g + \omega$ has non-trivial components of the symmetric part g and the skew-symmetric part ω that are respectively given by

$$(g_{ij}) = \begin{pmatrix} e^{2(t+r)} & 0 & 0 & 0 \\ 0 & -e^{2(t+r)} & 0 & 0 \\ 0 & 0 & \varphi^2\sin^2\theta & 0 \\ 0 & 0 & 0 & -\varphi^2\sin^2\theta \end{pmatrix} \quad \textit{and} \quad (\omega_{ij}) = \begin{pmatrix} 0 & \varphi\cos^2\theta & 0 & 0 \\ -\varphi\cos^2\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & -\varphi(t+r)^2 \\ 0 & 0 & \varphi(t+r)^2 & 0 \end{pmatrix}.$$

Obviously, the metric g is indefinite. Moreover, $\det(g_{ij}) = e^{4(t+r)}\varphi^4\sin^4\theta \neq 0$, which means that the metric g is regular. The components of the inverse metric g^{-1} of the metric g are given by

$$(g^{ij}) = \begin{pmatrix} e^{-2(t+r)} & 0 & 0 & 0 \\ 0 & -e^{-2(t+r)} & 0 & 0 \\ 0 & 0 & \frac{1}{\varphi^2\sin^2\theta} & 0 \\ 0 & 0 & 0 & -\frac{1}{\varphi^2\sin^2\theta} \end{pmatrix}.$$

It is not difficult to check that $\mathcal{G}_{pq}J^p_iJ^q_j = -\mathcal{G}_{ij}$, i.e., $J^p_i\mathcal{G}_{pq}J^q_j = -\mathcal{G}_{ij}$:

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} e^{2(t+r)} & \varphi\cos^2\theta & 0 & 0 \\ -\varphi\cos^2\theta & -e^{2(t+r)} & 0 & 0 \\ 0 & 0 & \varphi^2\sin^2\theta & -\varphi(t+r)^2 \\ 0 & 0 & \varphi(t+r)^2 & -\varphi^2\sin^2\theta \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$
$$= -\begin{pmatrix} e^{2(t+r)} & \varphi\cos^2\theta & 0 & 0 \\ -\varphi\cos^2\theta & -e^{2(t+r)} & 0 & 0 \\ 0 & 0 & \varphi^2\sin^2\theta & -\varphi(t+r)^2 \\ 0 & 0 & \varphi(t+r)^2 & -\varphi^2\sin^2\theta \end{pmatrix}.$$

We can conclude that the space $(M, \mathcal{G} = g + \omega, J)$ is a generalized almost para-Hermitian space. The non-zero components of the Riemannian curvature tensor R^h_{ijk} that corresponds to the pseudo-Riemannian metric g are given by

$$R^4_{343} = -\frac{\cos^2\theta}{\sin^2\theta} + \frac{1}{\varphi^2} - 1,$$
$$R^3_{443} = -\frac{(\varphi^2-1)\sin^2\theta + \varphi^2\cos^2\theta}{\varphi^2\sin^2\theta}.$$

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