On generalized almost para-Hermitian spaces

Miloš Z. Petrović

Abstract. Recently, a generalized almost Hermitian metric on an almost complex manifold \((M, J)\) is determined as a generalized Riemannian metric (i.e. an arbitrary bilinear form) \(G\) which satisfies \(G(JX, JY) = G(X, Y)\), where \(X\) and \(Y\) are arbitrary vector fields on \(M\). In the same manner we can study a generalized almost para-Hermitian metric and determine almost para-Hermitian spaces. Some properties of these spaces and special generalized almost para-Hermitian spaces including generalized para-Hermitian spaces as well as generalized nearly para-Kähler spaces are determined. Finally, a non-trivial example of generalized almost para-Hermitian space is constructed.

1. Introduction

This paper is devoted to the study of generalizations of Hermitian spaces, which generalize the well-known Kähler spaces. As is known, Kähler spaces were introduced by Kähler in 1934, but independently of him, these spaces were also studied by P.A. Shirokov, see [11, pp. 160-167]. Generalizations of these spaces in various directions can be found in research [2], papers [6, 21] and monograph [11]. Holomorphically projective mappings of Kähler spaces have been studied by Japanese mathematicians since 1950. One of the continuations is the 1971 paper [20] by M. Prvanović. Results on holomorphically projective mappings and transformations are in [10, 11]. An interesting result on holomorphically projective mappings of generalized Kähler spaces can be found in [16, 19, 22].

These spaces and mappings are generalized under the notion of \(F\)-structures, which have more general consequences, see e.g., [3, 7]. In this paper, we study generalized almost para-Hermitian spaces. Some properties of these spaces and special generalized almost para-Hermitian spaces including generalized para-Hermitian spaces as well as generalized nearly para-Kähler spaces are discussed. Finally, an example is presented in explicit form.

2. Almost Hermitian spaces and their generalizations

An almost complex structure on a real differentiable manifold \(M\) is a \((1, 1)\)-tensor field \(J\) such that [23]
\[
J^2 = -I,
\]
where \( I \) is the identity operator.

Let \( \mathcal{X}(M) \) be the Lie algebra of smooth vector fields on \( M \) and let us assume that \( X, Y \in \mathcal{X}(M) \). A real differentiable manifold \( M \) endowed with an almost complex structure \( J \) \( (J^2 = -I) \) is said to be an *almost complex manifold* or an *almost complex space* [23]. An almost complex space \((M, J)\) is said to be an *almost Hermitian space* if there exists a Riemannian metric \( g \) on \( M \) such that [23]

\[
g(JX, JY) = g(X, Y),
\]

i.e.,

\[
-g(X, JY) = g(JX, Y),
\]

which evidently means that the fundamental 2-form

\[
F(X, Y) := g(X, JY)
\]

is skew-symmetric.

M. Prvanović in 1995 [21] considered an almost Hermitian space \((M, g, J)\) as a particular generalized Riemannian space \((M, G^{F, g} = g + J)\) in the sense of Eisenhart [5] and gave a classification of almost Hermitian spaces which heavily depends on the Einstein connection \( D \) determined by \((D_2G)(X, Y) = 2G(X, T(Z, Y))\), where \( T \) is the torsion tensor of \( D \). The classification given in [21] is analogous to the classification of A. Gray and L.M. Hervella [6]. In [19] a *generalized Hermitian metric* on an almost complex manifold \((M, J)\) is defined as a generalized Riemannian metric in the sense of Eisenhart \( G \) that is invariant by the almost complex structure \( J \), i.e.,

\[
G(JX, JY) = G(X, Y),
\]

which further implies that

\[
\frac{1}{2}(G(JX, JY) \pm G(JY, JX)) = \frac{1}{2}G(X, Y) \pm G(Y, X),
\]

i.e.,

\[
g(JX, JY) = g(X, Y) \quad \text{and} \quad F(JX, JY) = F(X, Y).
\]

**Definition 2.1.** [19] *An almost complex manifold \((M, J)\) endowed with a generalized Hermitian metric \( G \) is called a generalized almost Hermitian space and it is denoted by \((M, G, J)\).*

In 2001, Minčić, Stanković and Velimirović [14] gave a definition of a generalized Kähler space assuming that

\[
\tilde{J}^2 = -I,
\]

\[
g(\tilde{\partial}_i, \tilde{\partial}_i) = g(\partial_i, \partial_i),
\]

\[
(\tilde{\nabla}_{\tilde{\partial}_i})\partial_j = 0 \quad \text{and} \quad (\tilde{\nabla}_{\partial_i})\partial_j = 0,
\]

where \( \tilde{\nabla} \) is a non-symmetric linear connection explicitly determined by [5]

\[
g(\tilde{\nabla}_{\partial_i}\partial_j, \partial_k) = \frac{1}{2}\left( \partial_i G(\partial_j, \partial_k) + \gamma G(\partial_j, \partial_k) + \partial_k G(\partial_i, \partial_j) \right),
\]

where \( \partial_i = \frac{\partial}{\partial x^i} \), \( \partial_j = \frac{\partial}{\partial x^j} \) and \( \partial_k = \frac{\partial}{\partial x^k} \) is standard orthonormal basis of the tangent space \( T_p(M) \) at the point \( p \) of the manifold \( M \). As is well-know a non-symmetric linear connection \( \tilde{\nabla} \) which is dual to \( \tilde{\nabla} \) is determined by

\[
\tilde{\nabla}_X Y = \tilde{\nabla}_X Y + [X, Y],
\]
or in the standard orthonormal basis as
\[ \nabla_{\partial_i} \partial_j = \nabla_{\partial_j} \partial_i. \]

Also, as is well-known the torsion-free linear connection \( ^0\nabla \) that is associated with the non-symmetric linear connections \( ^1\nabla \) and \( ^2\nabla \) is determined by
\[ ^0\nabla = \frac{1}{2} ( ^1\nabla Y + ^2\nabla Y ). \]

In [16] a more general definition of a generalized Kähler space in the sense of Eisenhart is given as in Definition 2.2.

**Definition 2.2.** [16] A generalized Riemannian space \((M, G)\) is called a generalized Kähler space in the sense of Eisenhart if there exists a \((1,1)\)-tensor field \(J\) on \(M\) such that
\[ J^2 = -I, \]
\[ g(JX, JY) = g(X, Y), \]
\[ ^0\nabla_X J_Y = 0, \]
where \( ^0\nabla_X Y = \frac{1}{2}(^1\nabla_X Y + ^2\nabla_X Y) \) is the symmetric part of the non-symmetric linear connection \( ^1\nabla \) and \( I \) is the identity operator.

**Definition 2.3.** A generalized Kähler space in the sense of Eisenhart \((M, g, J)\) is said to be a generalized Kähler space in the sense of Eisenhart with parallel torsion if the torsion tensor \( ^1T(X, Y) = ^1\nabla_X Y - ^1\nabla_Y X - [X, Y] \) satisfies
\[ ^0\nabla^1 T = 0, \]
where \( ^0\nabla_X Y = \frac{1}{2}(^1\nabla_X Y + ^2\nabla_X Y) \).

3. **Almost para-Hermitian spaces and their generalizations**

An almost product structure on a real differentiable manifold \(M\) is a \((1,1)\)-tensor field \(J\) such that [4]
\[ J^2 = I, \]
where \( I \) is the identity operator.

Let \((M, J)\) be an almost paracomplex manifold of dimension \(2n > 2\) and \(g\) be a pseudo-Riemannian metric on \(M\). The space \((M, g, J)\) is said to be an almost para-Hermitian space if the condition [4]
\[ g(JX, Y) + g(X, JY) = 0, \]
is satisfied. Almost para-Hermitian spaces were thoroughly studied for instance in [1, 4, 8].

In the same way as M. Prvanović did in [21] in the case of almost Hermitian manifolds and similar approach was also used in [9] we can use the following 2-form
\[ F(X, Y) := g(JX, Y) = -g(X, JY) = -g(JY, X) = -F(Y, X) \]
and consider the bilinear form
\[ G^{ef}(X, Y) := g(X, Y) + F(X, Y), \]
which is neither symmetric nor skew-symmetric.

Let us consider a $2n$-dimensional smooth manifold $M$ endowed with an almost para-complex structure $J$ and a bilinear form $G$ which satisfies

$$G(JX, JY) = -G(X, Y),$$

or equivalently

$$G(JX, JY) + G(X, Y) = 0.$$ 

The bilinear form $G$, which is neither symmetric nor skew-symmetric, can be described via its symmetric part $g$ and skew-symmetric part $\omega$ as follows

$$G(X, Y) = g(X, Y) + \omega(X, Y).$$

It is not difficult to conclude that the metric $g$ and 2-form $\omega$ satisfy

$$g(JX, JY) = -g(X, Y) \quad \text{and} \quad \omega(JX, JY) = -\omega(X, Y),$$

Therefore,

$$g(JX, Y) + g(X, JY) = 0 \quad \text{and} \quad \omega(JX, Y) + \omega(X, JY) = 0.$$ 

Obviously, the bilinear form $G$ is different than $G^{\pm \mathcal{F}}$.

Let $(M, J)$ be an almost paracomplex manifold and $G$ be a generalized pseudo-Riemannian metric on $M$. If the equality

$$G(JX, Y) + G(X, JY) = 0,$$

holds, then the metric $G$ is said to be a generalized almost para-Hermitian metric and consequently the space $(M, G = g + \omega, J)$ is called a generalized almost para-Hermitian space.

**Definition 3.1 (Generalized para-Hermitian space).** A generalized almost para-Hermitian space $(M, g, J)$, where $J$ is an integrable almost para-Hermitian structure, is called a generalized para-Hermitian space.

It is well-known that the almost paracomplex structure $J$ is integrable if and only if the Nijenhuis tensor identically vanishes, i.e., [23]

$$N(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] + [X, Y] = 0.$$ 

As a particular case of generalized almost para-Hermitian space we can consider a generalized nearly para-Kähler space.

**Definition 3.2 (Generalized nearly para-Kähler space).** A generalized almost para-Hermitian space $(M, G = g + \omega, J)$ is said to be a generalized nearly para-Kähler space if

$$(\nabla^g_X J)X = 0,$$

where $\nabla^g$ is the Levi-Civita connection of metric $g$.

The definition of generalized hyperbolic Kähler spaces introduced in [15] was extended in [? ] and some other types of generalized para Kähler spaces were described in [18].

**Definition 3.3 (Generalized para-Kähler space).** A generalized pseudo-Riemannian space $(M, G = g + \omega)$ of dimension $2n \geq 4$ is called a generalized para-Kähler space if there exists a $(1,1)$-tensor field $J$ on $M$ such that

$$J^2 = I,$$

$$g(JX, JY) = -g(X, Y),$$

$$(\nabla^g_X J)Y = 0,$$

where $\nabla^g$ is the Levi-Civita connection of metric $g$ and $I$ is the identity operator.
In the same manner as Example 3.1 in [16] here we construct Example 3.1.

**Example 3.1.** Let us consider a space \((M, \mathcal{G} = g + \omega, f)\) of real dimension \(n = 4\), where the components of the bilinear form \(\mathcal{G} = g + \omega\) and the almost product structure \(f\) are, respectively, given by

\[
(G_{ij}) = \begin{pmatrix}
    e^{2(t+r)} & q \cos^2 \theta & 0 & 0 \\
    -q \cos^2 \theta & -e^{2(t+r)} & 0 & 0 \\
    0 & 0 & q^2 \sin^2 \theta & -q(t+r)^2 \\
    0 & 0 & q(t+r)^2 & -q^2 \sin^2 \theta
\end{pmatrix}
\]

and

\[
(f^i_j) = \begin{pmatrix}
    0 & -1 & 0 & 0 \\
    -1 & 0 & 0 & 0 \\
    0 & 0 & 0 & 1 \\
    0 & 0 & 1 & 0
\end{pmatrix}
\]

where \(t, r, q \neq 0\) and \(\theta \neq k\pi, k \in \mathbb{Z}\).

The bilinear form \(\mathcal{G} = g + \omega\) has non-trivial components of the symmetric part \(g\) and the skew-symmetric part \(\omega\) that are respectively given by

\[
(g_{ij}) = \begin{pmatrix}
    e^{2(t+r)} & 0 & 0 & 0 \\
    0 & -e^{2(t+r)} & 0 & 0 \\
    0 & 0 & \frac{1}{q^2 \sin^2 \theta} & 0 \\
    0 & 0 & 0 & -\frac{1}{q^2 \sin^2 \theta}
\end{pmatrix}
\]

and

\[
(\omega_{ij}) = \begin{pmatrix}
    \frac{1}{q^2 \sin^2 \theta} & q \cos^2 \theta & 0 & 0 \\
    -q \cos^2 \theta & \frac{1}{q^2 \sin^2 \theta} & 0 & 0 \\
    0 & 0 & q^2 \sin^2 \theta & -q(t+r)^2 \\
    0 & 0 & q(t+r)^2 & -q^2 \sin^2 \theta
\end{pmatrix}
\]

Obviously, the metric \(g\) is indefinite. Moreover, \(\det(g_{ij}) = e^{4(t+r)}q^4 \sin^4 \theta \neq 0\), which means that the metric \(g\) is regular. The components of the inverse metric \(g^{-1}\) of the metric \(g\) are given by

\[
(g^{ij}) = \begin{pmatrix}
    e^{-2(t+r)} & 0 & 0 & 0 \\
    0 & -e^{-2(t+r)} & 0 & 0 \\
    0 & 0 & \frac{1}{q^2 \sin^2 \theta} & 0 \\
    0 & 0 & 0 & -\frac{1}{q^2 \sin^2 \theta}
\end{pmatrix}
\]

It is not difficult to check that \(G^i_{mn}f^m_j = -G_{ij}\), i.e., \(f^i_jG^i_{mn}f^m_j = -G_{ij}\):

\[
\begin{pmatrix}
    0 & -1 & 0 & 0 \\
    -1 & 0 & 0 & 0 \\
    0 & 0 & 0 & 1 \\
    0 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
    e^{2(t+r)} & q \cos^2 \theta & 0 & 0 \\
    -q \cos^2 \theta & -e^{2(t+r)} & 0 & 0 \\
    0 & 0 & q^2 \sin^2 \theta & -q(t+r)^2 \\
    0 & 0 & q(t+r)^2 & -q^2 \sin^2 \theta
\end{pmatrix}
= \begin{pmatrix}
    -q \cos^2 \theta & -e^{2(t+r)} & 0 & 0 \\
    e^{2(t+r)} & q \cos^2 \theta & 0 & 0 \\
    0 & 0 & q^2 \sin^2 \theta & -q(t+r)^2 \\
    0 & 0 & q(t+r)^2 & -q^2 \sin^2 \theta
\end{pmatrix}
\]

We can conclude that the space \((M, \mathcal{G} = g + \omega, f)\) is a generalized almost para-Hermitian space. The non-zero components of the Riemannian curvature tensor \(R^i_{jk}\) that corresponds to the pseudo-Riemannian metric \(g\) are given by

\[
R^4_{343} = -\frac{\cos^2 \theta}{\sin^2 \theta} + \frac{1}{q^2} - 1,
\]

\[
R^3_{443} = -\frac{(q^2 - 1) \sin^2 \theta + q^2 \cos^2 \theta}{q^2 \sin^2 \theta}.
\]

4. Acknowledgements

The research was partially supported by the Ministry of Science, Technological Development and Innovation, Republic of Serbia (Contract reg. no. 451-03-47/2023-01/200383).
References