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A Generalization of complex, dual and hyperbolic quaternions: hybrid quaternions

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Abstract. Hybrid numbers are a new non-commutative number system which is a generalization of the complex ($\mathbf{i}^2 = -1$), dual ($\varepsilon^2 = 0$), and hyperbolic numbers ($\mathbf{h}^2 = 1$). In this article, firstly we define a new quaternion system called hybrid quaternions by taking the coefficients of real quaternions as hybrid numbers. This new quaternion system is a combination of complex quaternions (biquaternions), hyperbolic (perplex) quaternions, and dual quaternions, and it can be viewed as a generalization of these quaternion systems. Then, we present the basic properties of hybrid quaternions including fundamental operations, conjugates, inner product, vector product, and norm. Finally, we give a schematic representation of numbers and quaternions.

1. Introduction

Quaternions, which are an extension of complex numbers, are referred to the Hamiltonian quaternions or real quaternions. They are applied to mechanics in three-dimensional space and used in many areas of science such as computer and graphics technologies. Real quaternions are defined as follows:

$$\mathbf{H} = \{q = w_0 + w_1 i + w_2 j + w_3 k : \quad w_0, w_1, w_2, w_3 \in \mathbb{R}\} \tag{1}$$

where i, j, k are the units of quaternions satisfying

$$i^2 = j^2 = k^2 = -1, \quad ijk = -1. \tag{2}$$

The set of real quaternions forms a 4-dimensional real vector space. Furthermore, $1, i, j, k$ denote the basis elements of $\mathbf{H}$ and unit real quaternions represent a rotation in Euclidean 3-space. The most striking feature of real quaternions is having non-commutativity in multiplication. Additionally, $\mathbf{H}$ is a member of a non-commutative division algebra [9, 13].

There are some significant extensions of quaternions in the literature, such as complex, dual, and hyperbolic quaternions. These quaternions are obtained by replacing the coefficient of quaternion units. That is, for a quaternion $q = w_0 + w_1 i + w_2 j + w_3 k$, if the coefficients are complex numbers ($a + \mathbf{i}b$, $\mathbf{i}^2 = -1$), dual numbers ($a + \varepsilon b$, $\varepsilon^2 = 0$), and hyperbolic numbers ($a + \mathbf{h}b$, $\mathbf{h}^2 = 1$), then it is called a complex quaternion, dual quaternion, and hyperbolic quaternion, respectively[1)].

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1) The italic “ i ” denotes the first of the three quaternion units and the bold “ $\mathbf{i}$ ” denotes complex numbers’ unit.

Complex numbers are an extension of the real numbers($\mathbb{R}$), with imaginary unit $\mathbf{i}^2 = -1$. They are described as $\mathbb{C} = \mathbb{R}[x]/\langle x^2+1\rangle$ or $\mathbb{C} \equiv \mathbb{R}[\mathbf{i}]$ in algebra [11, 30]. Complex quaternions, $\mathbf{H}_{\mathbb{C}}$, are an extension of complex numbers that uses the coefficients of quaternions as complex numbers rather than real numbers. Complex quaternions belong to the Clifford algebra and are isomorphic to the Clifford algebra $Cl_{3,0}$. They are known as biquaternions[2)] or complexified quaternions. Additionally, they have a wide range of applications. For example, the ring complex quaternions are recognized as a strong representing instrument in formulating physical laws. Moreover, they have many useful properties in the theorems of modern algebra [1, 23, 27].

Dual numbers were first defined by W.K. Clifford as a tool for his geometrical investigation. They have been efficiently developed over time. They are systematically applied to line geometry and kinematics. Dual numbers, $\mathbb{D} \equiv \mathbb{R}[\varepsilon]$, are the extension of the real numbers that include the imaginary unit ε with the property $\varepsilon^2 = 0$. Additionally, in abstract algebra, dual numbers are described as $\mathbb{D} = \mathbb{R}[x]/\langle x^2\rangle$ [2, 29]. Dual quaternions, $\mathbf{H}_{\mathbb{D}}$, are an extension of dual numbers, where the coefficients of real quaternions are dual numbers. Dual quaternions are still used by many scientists today and are applied in various areas of science. For example, they are used in animation, robotics, computer vision applications, theoretical kinematics, and as a tool for expressing and analyzing the physical properties of rigid bodies [12, 15, 24].

There is another number system in the literature which is called hyperbolic numbers [30]. They are also known as perplex numbers, duplex numbers, double numbers, or split-complex numbers. Hyperbolic numbers, $\mathbb{H} \equiv \mathbb{R}[\mathbf{h}]$, include the imaginary unit $\mathbf{h}$ with the property $\mathbf{h}^2 = 1$. Additionally, hyperbolic numbers are described as $\mathbb{H} = \mathbb{R}[x]/\langle x^2-1\rangle$ in abstract algebra [11, 16, 21, 25]. Hyperbolic quaternions, $\mathbf{H}_{\mathbb{H}}$, are an extension of hyperbolic numbers whereby the coefficients of real quaternions are hyperbolic numbers.[3)] These quaternions are also known as perplex quaternions or split biquaternions. Hyperbolic quaternions are suitable algebraic tools for expressing Lorentz space-time transformations. Further information on hyperbolic numbers and hyperbolic quaternions can be found in [3, 4, 10, 17, 19, 26, 28, 31].

The last number system we will mention here is the hybrid numbers, $\mathbb{K}$, defined by Özdemir [18]. Because of a generalization of dual, hyperbolic, and complex numbers, they are called hybrid numbers. Additionally, $\mathbf{i}, \mathbf{h}, \varepsilon$ are hybrid units and they satisfy $\mathbf{ih} = -\mathbf{hi} = \varepsilon + \mathbf{i}$.

It is known in the literature that the term norm is defined with the property of the triangle inequality $\|q+p\| \leq \|q\| + \|p\|$. But when it comes to the complex generalization of this notion, it is not possible to apply the triangle inequality to a norm with a complex value, since there is no order in complex numbers. Instead of inventing new terms, the term seminorm is then used [23]. In this work, we denote both the term norm and semi-norm with $\|q\|$. In addition, a similar situation applies to the inner product. The inner product should satisfy the properties of symmetry, bilinearity, and positive definite. Although the third feature cannot be applied to a complex-valued inner product, the same inner product term is used.

In this study, we will investigate hybrid quaternions. Firstly in Section 2, we present fundamental information and properties of both number systems and quaternions to provide the necessary background for hybrid quaternions. Then in Section 3, we introduce the set of hybrid quaternions as an extension of hybrid numbers, where the elements of the real quaternions are hybrid numbers. Additionally, after introducing hybrid quaternions, we give some definitions and properties about them, including inner product, vector product, and norm. Finally, in Section 5, number systems including hybrid quaternions are examined using Venn diagrams.

2. Number Systems

2.1. Real Quaternions

In the previous section the set of real quaternions is defined as in (1). A real quaternion can be given in the form $q = w_0 + w_1 i + w_2 j + w_3 k$ where i, j, k are units of quaternions and they provide equation (2). As

[2)] The name biquaternion was first given by Hamilton.

[3)] In the literature, there is another type of quaternions called hyperbolic quaternions with the same name. This type of hyperbolic quaternions was defined by Scottish physicist A. Macfarlane and unlike real quaternions they are obtained by taking $i^2 = j^2 = k^2 = 1$. Note that these quaternions do not have the property of associativity [6, 14].

mentioned above, the real quaternion algebra $\mathbf{H}$ is associative, i.e. $(ij)k = i(jk)$, but not commutative, that is $ij \neq ij$. For any $q \in \mathbf{H}$, scalar part and vector part of q is $S_q = w_0$ and $V_q = w_1 i + w_2 j + w_3 k$, respectively. Moreover, the addition of two quaternions is defined componentwise. For the quaternions $p = S_p + V_p$ and $q = S_q + V_q$, the addition is defined as $q + p = (S_q + S_p) + (V_q + V_p)$. Also, the quaternionic multiplication of $p = z_0 + z_1 i + z_2 j + z_3 k$ and $q = w_0 + w_1 i + w_2 j + w_3 k$ is defined as:

$$pq = S_p S_q - \langle V_p, V_q\rangle + S_p V_q + S_q Vp + V_p \wedge V_q, \tag{3}$$

where $\langle V_p, V_q\rangle = z_1 w_1 + z_2 w_2 + z_3 w_3$ and $V_p \wedge V_q = (z_2 w_3 - z_3 w_2) i - (z_1 w_3 - z_3 w_1) j + (z_1 w_2 - z_2 w_1) k$.

The conjugate of q is defined by $\overline{q} = w_0 - w_1 i - w_2 j - w_3 k$. Furthermore, the inner(scalar) product of the real quaternions q and p is defined as

$$\langle q, p\rangle = \frac{1}{2}(q\overline{p} + p\overline{q}) = \frac{1}{2}(\overline{q}p + \overline{p}q). \tag{4}$$

Additionally, it can also be defined using a element-wise product for $q = w_0 + w_1 i + w_2 j + w_3 k$ and $p = z_0 + z_1 i + z_2 j + z_3 k$ as follows:

$$\langle q, p\rangle = w_0 z_0 + w_1 z_1 + w_2 z_2 + w_3 z_3, \quad \langle q, p\rangle \in \mathbb{R}. \tag{5}$$

Another way to define the inner product of two real quaternions is $\langle q, p\rangle = S_{q\overline{p}} = S_{p\overline{q}}$. The vector product of the real quaternions q and p is given by

$$q \times p = \frac{1}{2}(q\overline{p} - p\overline{q}) = \frac{1}{2}(\overline{q}p - \overline{p}q). \tag{6}$$

It is obvious that following equations are satisfied for real quaternions:

$$q\overline{p} + p\overline{q} \;=\; 2\langle q, p\rangle = 2S_q S_p + 2\langle V_q, V_p\rangle, \tag{7}$$

$$q\overline{p} - p\overline{q} \;= 2(q \times p) = -2S_q V_p + 2S_p V_q - 2V_q \wedge V_p, \tag{8}$$

The norm of a quaternion q is given by as follows:

$$\|q\| = \overline{q}q = q\overline{q} = w_0^2 + w_1^2 + w_2^2 + w_3^2, \quad \|q\| \in \mathbb{R}. \tag{9}$$

Non-zero quaternions have a multiplication inverse defined by $q^{-1} = \overline{q}/\|q\|$. For further details about the basics of real quaternions, see [8, 9].

2.2. *Complex Numbers and Complex Quaternions (Biquaternions)*

It is well known that complex numbers are at the form $w = a + \mathbf{i}b$, where a and b are real numbers and $\mathbf{i}$ is the complex unit satisfying $\mathbf{i}^2 = -1$. All complex numbers can be formed with the base $\{1, \mathbf{i}\}$. Thus, the set of these numbers can be defined as $\mathbb{C} = \{w = a + \mathbf{i}b : \; a, b \in \mathbb{R}, \; \mathbf{i}^2 = -1\}$. The set of complex numbers is a field and 2-dimensional vector space over the real numbers. The conjugate of a complex number w is denoted by w^* and defined by $w^* = a - \mathbf{i}b$. The modulus $|w|$ is defined by $|w| = \sqrt{ww^*} = \sqrt{a^2 + b^2}$. This modulus corresponds to the distance in Euclidean plane[4)].

The set of complex quaternions is an extension of real quaternions by complex numbers and it is defined as follows:

$$\mathbf{H}_{\mathbb{C}} = \{Q = w_0 + w_1 i + w_2 j + w_3 k : \; w_0, w_1, w_2, w_3 \in \mathbb{C}\} \tag{10}$$

[4)]Euclidean plane is defined with the Euclidean inner product: for the vectors $\vec{x} = (a_1, b_1)$, $\vec{y} = (a_2, b_2) \in \mathbb{R}^2$, $\langle \vec{x}, \vec{y}\rangle = a_1 a_2 + b_1 b_2$. Note that complex numbers ($\mathbb{C}$) are isomorphic to Euclidean plane, $\mathbb{E}^2$, [30].

where $1, i, j, k$ are the quaternion units. Complex unit $\mathbf{i}$ commutes with the quaternion units $\mathbf{i}i = i\mathbf{i}$, $\mathbf{i}j = j\mathbf{i}$, $\mathbf{i}k = k\mathbf{i}$. As a result of this commutativity, complex quaternions can be written in the following form:

$$\begin{aligned} Q =&(a_0 + \mathbf{i}b_0) + (a_1 + \mathbf{i}b_1)i + (a_2 + \mathbf{i}b_2)j + (a_3 + \mathbf{i}b_3)k = (a_0 + a_1 i + a_2 j + a_3 k) + \mathbf{i}(b_0 + b_1 i + b_2 j + b_3 k) \\ =&q_a + \mathbf{i}q_b = q_a + q_b\mathbf{i} \end{aligned}$$

for $n = 0, 1, 2, 3$ and $w_n = a_n + \mathbf{i}b_n \in \mathbb{C}$, where $q_a, q_b \in \mathbf{H}$. Therefore, complex quaternions can also be written as $Q = q_a + \mathbf{i}q_b = q_a + q_b\mathbf{i}$. As a consequence of this representation we can rewrite the set of complex quaternions as follows:

$$\mathbf{H}_{\mathbb{C}} = \{Q = q_a + \mathbf{i}q_b : \ q_a, q_b \in \mathbf{H}, \quad \mathbf{i}^2 = -1\}. \tag{11}$$

Basis elements of complex quaternions are $1, i, j, k, \mathbf{i}, i\mathbf{i}, j\mathbf{i}, k\mathbf{i}$, and $\mathbf{H}_{\mathbb{C}}$ is 8-dimensional vector space over the $\mathbb{R}$. For any $Q = w_0 + w_1 i + w_2 j + w_3 k \in \mathbf{H}_{\mathbb{C}}$, we define the complex scalar part as $S_Q = w_0$, and vector part as $V_Q = w_1 i + w_2 j + w_3 k$. Thus, the complex quaternions can be represented as $Q = S_Q + V_Q$. Also, for complex quaternions $Q = q_a + \mathbf{i}q_b$ and $P = p_a + \mathbf{i}p_b$, addition and multiplication are defined as follows

$$Q + P = q_a + p_a + \mathbf{i}(q_b + p_b), \tag{12}$$

$$QP = q_a p_a - q_b p_b + \mathbf{i}(q_a p_b + q_b p_a). \tag{13}$$

where the products between q_a and q_b are the real quaternionic products. Unlike real quaternions, there are three different conjugate definitions for complex quaternions [26]. For a complex quaternion $Q = q_a + \mathbf{i}q_b = S_Q + V_Q$, different conjugate types are defined as follows:

i) Quaternion conjugate: $\overline{Q} = \overline{q_a} + \mathbf{i}\overline{q_b} = S_Q - V_Q$,

ii) Complex conjugate: $Q^* = q_a - \mathbf{i}q_b = S_Q^* + V_Q^*$,

iii) Total conjugate: $Q^\dagger = (\overline{Q})^* = \overline{(Q^*)} = \overline{q_a} - \mathbf{i}\overline{q_b} = S_Q^* - V_Q^*$.

The inner product of complex quaternions is defined similarly to the inner product of real quaternions and the result is complex-valued. For complex quatenions $Q = q_a + \mathbf{i}q_b$ and $P = p_a + \mathbf{i}p_b$ the inner product is given by

$$\langle Q, P\rangle = \frac{1}{2}(Q\overline{P} + P\overline{Q}) = \langle q_a, p_a\rangle - \langle q_b, p_b\rangle + \mathbf{i}\Big(\langle q_a, p_b\rangle + \langle q_b, p_a\rangle\Big). \tag{14}$$

There are different definitions for the norm of complex quaternions in the literature. Classically, the term norm is real-valued and positive defined (non-negative), but when we mention the norm for complex quaternions it has a complex value. Furthermore, because there is no ordering in complex numbers, triangle inequality does not hold as mentioned in the introduction. Therefore, the term semi-norm arises. The semi-norm[5)] of the complex quaternion $Q = w_0 + w_1 i + w_2 j + w_3 k = q_a + \mathbf{i}q_b$ is defined in terms of the complex components:

$$\|Q\| = Q\overline{Q} = \overline{Q}Q = w_0^2 + w_1^2 + w_2^2 + w_3^2 = q_a\overline{q_a} - q_b\overline{q_b} + \mathbf{i}(q_a\overline{q_b} + q_b\overline{q_a}) = \big\|q_a\big\| - \big\|q_b\big\| + 2\mathbf{i}\langle q_a, q_b\rangle. \tag{15}$$

Note that even though the result is complex-valued, $\|Q\| \in \mathbb{C}$, still satisfy $Q\overline{Q} = \overline{Q}Q$. Another definition made for complex quaternions is real valued given by:

$$\|Q\| = S_{q_a\overline{(q_b)^*}} = \big\|q_a\big\| + \big\|q_b\big\| = a_0^2 + a_1^2 + a_2^2 + a_3^2 + b_0^2 + b_1^2 + b_2^2 + b_3^2. \tag{16}$$

Q is called unit complex quaternion, if $\|Q\| = 1$. The modulus of a complex quaternion Q is $|Q| = \sqrt{\|Q\|}$.

It is known that every nonzero real quaternion has a multiplicative inverse but this is not true for complex quaternions. Because, complex quaternions have zero divisors [22]. These zero divisors have no multiplicative inverse. Hence, the complex quaternion algebra $\mathbf{H}_{\mathbb{C}}$ is not a division algebra. If $Q \in \mathbf{H}_{\mathbb{C}}$ is non-zero and not a zero divisor, then the inverse Q^{-1} is a complex quaternion and satisfies $QQ^{-1} = Q^{-1}Q = 1$.

More details about complex quaternions can be found in [9, 22, 23, 27].

[5)] Some authors prefer to use the term *"norm"* instead term *"semi-norm"*.

2.3. Dual Numbers and Dual Quaternions

A dual number is written in the form $w = a + \varepsilon b$, where a and b are real numbers and dual unit ε satisfies $\varepsilon^2 = 0$. All dual numbers can be formed with the base $\{1, \varepsilon\}$. The set of dual numbers can be defined as $\mathbb{D} = \{w = a + \varepsilon b : a, b \in \mathbb{R}, \varepsilon^2 = 0\}$. The set of dual numbers is 2-dimensional vector space over real numbers. It is easy to see that $(\mathbb{D}, +, .)$ is a commutative ring with unity. Dual numbers without real part such as εa and εb are zero divisors. That is, $(\varepsilon a)(\varepsilon b) = \varepsilon^2(ab) = 0$. The conjugate of a dual number is defined by $w^* = a - \varepsilon b$. Moreover, the modulus $|w|$ is defined by $|w| = \sqrt{ww^*} = \sqrt{a^2} = |a|$. This modulus corresponds to the distance in 2-dimensional Galilean plane[6)].

Dual quaternions are an extention of real quaternions by dual numbers and the set of dual quaternions is defined as follows

$$\mathbf{H}_{\mathbb{D}} = \{Q = w_0 + w_1 i + w_2 j + w_3 k : \ w_0, w_1, w_2, w_3 \in \mathbb{D}\} \tag{17}$$

where $1, i, j, k$ are the quaternion units. Note that, dual unit ε commutes with the basis elements i, j, k, that is, $\varepsilon i = i\varepsilon$, $\varepsilon j = j\varepsilon$, $\varepsilon k = k\varepsilon$. As a result of this commutativity, dual quaternions can also be written as $Q = q_a + \varepsilon q_b = q_a + q_b \varepsilon$, where $q_a = a_0 + a_1 i + a_2 j + a_3 k$, $q_b = b_0 + b_1 i + b_2 j + b_3 k \in \mathbf{H}$. So, we can rewrite the set of dual quaternions as follows:

$$\mathbf{H}_{\mathbb{D}} = \{Q = q_a + \varepsilon q_b : \ q_a, q_b \in \mathbf{H}, \ \varepsilon^2 = 0\}. \tag{18}$$

Basis elements of dual quaternions are $1, i, j, k, \varepsilon, i\varepsilon, j\varepsilon, k\varepsilon$, and $\mathbf{H}_{\mathbb{D}}$ are 8-dimensional vector space over $\mathbb{R}$. For any $Q = w_0 + w_1 i + w_2 j + w_3 k \in \mathbf{H}_{\mathbb{D}}$, dual scalar part and vector part of Q is defined as $S_Q = w_0$ and $V_Q = w_1 i + w_2 j + w_3 k$, respectively. Q is called as pure dual quaternion if $S_Q = 0$. Additionally, for dual quaternions $Q = q_a + \varepsilon q_b$ and $P = p_a + \varepsilon p_b$, addition and multiplication are defined as

$$Q + P = q_a + p_a + \varepsilon(q_b + p_b), \tag{19}$$

$$QP = q_a p_a + \varepsilon(q_a p_b + q_b p_a). \tag{20}$$

Just like complex quaternions, there are three different conjugate definitions for dual quaternions. For $Q = q_a + \varepsilon q_b = S_Q + V_Q$, we have the following:

(a) Quaternion conjugate: $\overline{Q} = \overline{q_a} + \varepsilon \overline{q_b} = S_Q - V_Q$,

(b) Dual conjugate: $Q^* = q_a - \varepsilon q_b = S_Q^* + V_Q^*$,

(c) Total conjugate: $Q^\dagger = (\overline{Q})^* = \overline{(Q^*)} = \overline{q_a} - \varepsilon \overline{q_b} = S_Q^* - V_Q^*$.

The inner product of dual quaternions is defined similar to the inner product of real quaternions and the result is dual-valued. For dual quatenions $Q = q_a + \varepsilon q_b$ and $P = p_a + \varepsilon p_b$ the inner product is defined as

$$\langle Q, P \rangle = \frac{1}{2}(Q\overline{P} + P\overline{Q}) = \langle q_a, p_a \rangle + \varepsilon\Big(\langle q_a, p_c \rangle + \langle q_c, p_a \rangle\Big). \tag{21}$$

The semi norm of a dual quaternion $Q = w_0 + w_1 i + w_2 j + w_3 k = q_a + \varepsilon q_b$ can be defined as:

$$\begin{aligned} \|Q\| = Q\overline{Q} = \overline{Q}Q = w_0^2 + w_1^2 + w_2^2 + w_3^2 &= (a_0^2 + a_1^2 + a_1^2 + a_3^2) + 2\varepsilon(a_0 b_0 + a_1 b_1 + a_2 b_2 + a_3 b_3) \\ &= q_a \overline{q_a} + \varepsilon(q_a \overline{q_b} + q_b \overline{q_a}) = \big\|q_a\big\| + 2\varepsilon \langle q_a, q_b \rangle. \end{aligned} \tag{22}$$

Note that the norm is dual-valued. Furthermore, if $q_a = 0$ then $\|Q\| = 0$ and hence Q has no inverse. Thus, if $Q = q_a + \varepsilon q_b \in \mathbf{H}_{\mathbb{D}}$ is non-zero and also $q_a \neq 0$, then the inverse Q^{-1} is a dual quaternion and satisfies $QQ^{-1} = Q^{-1}Q = 1$. For more details about the basics of dual quaternions, see [12, 15, 24].

[6)] Galilean plane is defined with the Galilean inner product: for the vectors $\vec{x} = (a_1, b_1)$, $\vec{y} = (a_2, b_2) \in \mathbb{R}^2$, $\langle \vec{x}, \vec{y} \rangle = a_1 a_2$. Note that dual numbers ($\mathbb{D}$) are isomorphic to Galilean plane, $\mathbb{G}^2$, [5, 30, 31].

2.4. Hyperbolic Numbers and Hyperbolic(Perplex) Quaternions

Hyperbolic numbers are at the form $w = a + \mathbf{h}b$, together with real numbers a,b and hyperbolic unit $\mathbf{h}$ satisfies $\mathbf{h}^2 = 1$. All hyperbolic numbers can be written with the base $\{1, \mathbf{h}\}$. The set of these numbers can be defined as $\mathbb{H} = \{w = a + \mathbf{h}b : \ a, b \in \mathbb{R}, \ \mathbf{h}^2 = 1\}$. The set of hyperbolic numbers is 2-dimensional vector space over the field of real numbers. The conjugate of a hyperbolic number is defined by $w^* = a - b\mathbf{h}$. In addition, the modulus $|w|$ is defined as $|w| = \sqrt{ww^*} = \sqrt{|a^2 - b^2|}$. This modulus corresponds to the distance in the Minkowski plane[7)].

The set of hyperbolic quaternions is an extension of real quaternions by hyperbolic numbers and it is defined as

$$\mathbf{H}_{\mathbb{H}} = \{Q = w_0 + w_1 i + w_2 j + w_3 k : \ \ w_0, w_1, w_2, w_3 \in \mathbb{H}\}, \tag{23}$$

where $1, i, j, k$ are the quaternion units. Furthermore, hyperbolic unit $\mathbf{h}$ commutes with the basis elements i, j, k, namely, $\mathbf{h}i = i\mathbf{h}$, $\mathbf{h}j = j\mathbf{h}$, $\mathbf{h}k = k\mathbf{h}$. As a result of this commutativity, hyperbolic quaternions can also be written as $Q = q_a + \mathbf{h}q_b = q_a + q_b\mathbf{h}$, where $q_a = a_0 + a_1 i + a_2 j + a_3 k$, $q_b = b_0 + b_1 i + b_2 j + b_3 k \in \mathbf{H}$. So, we can rewrite the set of hyperbolic quaternions as follows:

$$\mathbf{H}_{\mathbb{H}} = \{Q = q_a + \mathbf{h}q_b : \ q_a, q_b \in \mathbf{H}, \ \mathbf{h}^2 = 1\}. \tag{24}$$

Basis elements of hyperbolic quaternions are are $1, i, j, k, \mathbf{h}, \mathbf{h}i, \mathbf{h}j, \mathbf{h}k$, and hyperbolic quaternions are form an 8-dimensional vector space over the $\mathbb{R}$. For any $Q = w_0 + w_1 i + w_2 j + w_3 k \in \mathbf{H}_{\mathbb{H}}$, the hyperbolic scalar part and vector part of Q are $S_Q = w_0$ and $V_Q = w_1 i + w_2 j + w_3 k$, respectively. If $S_Q = 0$, then Q is called a pure hyperbolic quaternion. Furthermore, for hyperbolic quaternions $Q = q_a + \mathbf{h}q_b$ and $P = p_a + \mathbf{h}p_b$, addition and multiplication are given by

$$Q + P = q_a + p_a + \mathbf{h}(q_b + p_b)\,, \tag{25}$$
$$QP = q_a p_a + q_b p_b + \mathbf{h}(q_a p_b + q_b p_a). \tag{26}$$

There are three different conjugate definitions for hyperbolic quaternions. For a hyperbolic quaternion $Q = q_a + \mathbf{h}q_b = S_Q + V_Q$, we have

(a) Quaternion conjugate: $\overline{Q} = \overline{q_a} + \mathbf{h}\overline{q_b} = S_Q - V_Q$,

(b) Hyperbolic conjugate: $Q^* = q_a - \mathbf{h}q_b = S_Q^* + V_Q^*$,

(c) Total conjugate: $Q^\dagger = (\overline{Q})^* = \overline{(Q^*)} = \overline{q_a} - \mathbf{h}\overline{q_b} = S_Q^* - V_Q^*$.

The inner product of hyperbolic quaternions is defined similar to the inner product of real quaternions and the result is hyperbolic-valued. For hyperbolic quatenions $Q = q_a + \mathbf{h}q_b$ and $P = p_a + \mathbf{h}p_b$ the inner product is defined as

$$\langle Q, P\rangle = \frac{1}{2}(Q\overline{P} + P\overline{Q}) = \langle q_a, p_a\rangle + \langle q_d, p_d\rangle + \mathbf{h}\Big(\langle q_a, p_d\rangle + \langle q_d, p_a\rangle\Big). \tag{27}$$

The semi norm of a hyperbolic quaternion $Q = w_0 + w_1 i + w_2 j + w_3 k = q_a + \mathbf{h}q_b$ can be defined as:

$$\begin{aligned} ||Q|| = Q\overline{Q} = \overline{Q}Q &= w_0^2 + w_1^2 + w_2^2 + w_3^2, \quad ||Q|| \in \mathbb{H} \\ &= (a_0^2 + a_1^2 + a_1^2 + a_3^2) + (b_0^2 + b_1^2 + b_1^2 + b_3^2) + 2\mathbf{h}(a_0 b_0 + a_1 b_1 + a_2 b_2 + a_3 b_3) \\ &= q_a\overline{q_a} + q_b\overline{q_b} + \mathbf{h}(q_a\overline{q_b} + q_b\overline{q_a}) = \left\|q_a\right\| + \left\|q_b\right\| + 2\mathbf{h}\langle q_a, q_b\rangle \end{aligned} \tag{28}$$

For further details, see [3, 4, 10, 19, 28, 31].

[7)] In the literature, Minkowski plane is also known as Lorentzian plane. Minkowski plane is defined with the Minkowski inner product: for the vectors $\vec{x} = (a_1, b_1)$, $\vec{y} = (a_2, b_2) \in \mathbb{R}^2$, $\langle \vec{x}, \vec{y}\rangle = a_1 a_2 - b_1 b_2$. Note that hyperbolic numbers ($\mathbb{H}$) are isomorphic to Minkowski plane, $\mathbb{R}^2_1$, [17, 20].

2.5. Hybrid Numbers

A general element of hybrid numbers has the form $w = a + \mathbf{i}b + \varepsilon c + \mathbf{h}d$, where a, b, c, d are real numbers, and $\{1, \mathbf{i}, \varepsilon, \mathbf{h}\}$ are the standart basis elements. The set of hybrid numbers is defined by:

$$\mathbb{K} = \left\{ w = a + \mathbf{i}b + \varepsilon c + \mathbf{h}d : \ a, b, c, d \in \mathbb{R}, \mathbf{h}^2 = 1, \mathbf{i}^2 = -1, \varepsilon^2 = 0, \varepsilon + \mathbf{i} = \mathbf{ih} = -\mathbf{hi} \right\}. \tag{29}$$

The addition of hybrid numbers is done component-wise and satisfies the commutative property and associativity. Also, the additive inverse of any hybrid number is $-w$. Multiplication of hybrid numbers is easily found by using the properties of hybrid units. In addition, it is clear that the multiplication operation is not commutative but associative. For a hybrid number $w = a + \mathbf{i}b + \varepsilon c + \mathbf{h}d$, the conjugate of w is defined by $w^* = a - \mathbf{i}b - \varepsilon c - \mathbf{h}d$. Additionally, according to the hybridian product it is easy to see that $w\overline{w} = \overline{w}w$. The character of a hybrid number w is defined as $C(w) = ww^* = w^*w = a^2 + (b - c)^2 - c^2 - d^2$ where $C(w)$ is a real number. Moreover, the modulus of a hybrid number $|w|$ is defined by $|w| = \sqrt{C(w)}$. This definition of the modulus is a generalized modulus of complex, dual and hyperbolic numbers. That is, if $c = d = 0$, then $|w| = \sqrt{a^2 + b^2}$; if $b = d = 0$, then $|w| = \sqrt{a^2}$; if $b = c = 0$, then $|w| = \sqrt{|a^2 - d^2|}$. For more details, see [7, 18].

3. Hybrid Quaternions

The set of hybrid quaternions that we will describe in this section is a generalization of the complex, dual, and hyperbolic quaternions found in the literature. Therefore, it includes these three quaternion systems and provides their properties completely in special cases.

Hybrid quaternions are an extension of real quaternions by hybrid numbers ($\mathbb{K}$). The set of hybrid quaternions is denoted by $\mathbf{H}_{\mathbb{K}}$ and defined as

$$\mathbf{H}_{\mathbb{K}} = \{Q = w_0 + w_1 i + w_2 j + w_3 k : w_0, w_1, w_2, w_3 \in \mathbb{K}\} \tag{30}$$

where i, j and k are the quaternion units and they satisfy $i^2 = j^2 = k^2 = -1, ijk = -1$. Additionally, for hybrid coefficients w_0, w_1, w_2, w_3 the hybrid units $\mathbf{i}, \varepsilon, \mathbf{h}$ satisfy $\varepsilon^2 = 0, \mathbf{i}^2 = -1$, $\mathbf{h}^2 = 1$, $\varepsilon + \mathbf{i} = \mathbf{ih} = -\mathbf{hi}$ and they commute with the quaternion units:

$$\mathbf{i}i = i\mathbf{i}, \mathbf{i}j = j\mathbf{i}, \mathbf{i}k = k\mathbf{i}, \varepsilon i = i\varepsilon, \varepsilon j = j\varepsilon, \varepsilon k = k\varepsilon, \mathbf{h}i = i\mathbf{h}, \mathbf{h}j = j\mathbf{h}, \mathbf{h}k = k\mathbf{h}.$$

For the coefficients $w_n = a_n + \mathbf{i}b_n + \varepsilon c_n + \mathbf{h}d_n \in \mathbb{K}$, $n = 0, 1, 2, 3$, any hybrid quaternion $Q = w_0 + w_1 i + w_2 j + w_3 k$ can be written as

$$\begin{aligned} Q =& (a_0 + \mathbf{i}b_0 + \varepsilon c_0 + \mathbf{h}d_0) + (a_1 + \mathbf{i}b_1 + \varepsilon c_1 + \mathbf{h}d_1)i + (a_2 + \mathbf{i}b_2 + \varepsilon c_2 + \mathbf{h}d_2)j + (a_3 + \mathbf{i}b_3 + \varepsilon c_3 + \mathbf{h}d_3)k \\ =& q_a + \mathbf{i}q_b + \varepsilon q_c + \mathbf{h}q_d = q_a + q_b\mathbf{i} + q_c\varepsilon + q_d\mathbf{h} \end{aligned}$$

where $q_a, q_b, q_c, q_d \in \mathbf{H}$ and $a_n, b_n, c_n, d_n \in \mathbb{R}$. As a consequence of this representation we can rewrite the set of hybrid quaternions as follows:

$$\mathbf{H}_{\mathbb{K}} = \left\{ Q = q_a + \mathbf{i}q_b + \varepsilon q_c + \mathbf{h}q_d : \ q_a, q_b, q_c, q_d \in \mathbf{H}, \varepsilon^2 = 0, \mathbf{i}^2 = -1, \mathbf{h}^2 = 1, \mathbf{ih} = -\mathbf{hi} = \varepsilon + \mathbf{i} \right\}.$$

Hybrid quaternions form a 16-dimensional real vector space and the basis elements are $1, i, j, k, \mathbf{i}, \mathbf{i}i, \mathbf{i}j, \mathbf{i}k, \varepsilon, \varepsilon i, \varepsilon j, \varepsilon k, \mathbf{h}, \mathbf{h}i, \mathbf{h}j, \mathbf{h}k$. With the help of multiplication rules of the hybrid numbers, multiplication table for the basis elements of hybrid quaternions can be established easily.

Definition 3.1. *For any $Q = w_0 + w_1 i + w_2 j + w_3 k \in \boldsymbol{H}_{\mathbb{K}}$, we define the hybrid scalar part of Q as $S_Q = w_0$, hybrid vector part of Q as $V_Q = w_1 i + w_2 j + w_3 k$. If $S_Q = 0$, then Q is called 1^{st} type pure hybrid quaternion. Here, note that the hybrid vector part consists of hybrid coefficients.*

Definition 3.2. *For any $Q = q_a + \boldsymbol{i}q_b + \boldsymbol{\varepsilon}q_c + \boldsymbol{h}q_d \in \boldsymbol{H}_{\mathbb{K}}$, scalar part and quaternion vector part of Q are defined by $\tilde{\mathcal{S}}_Q = q_a$ and $\tilde{\mathcal{V}}_Q = \boldsymbol{i}q_b + \boldsymbol{\varepsilon}q_c + \boldsymbol{h}q_d$, respectively. Q is called 2^{nd} type pure hybrid quaternion, if $\tilde{\mathcal{S}}_Q = 0$. Here the coefficients$(q_a, q_b, q_c, q_d)$ are real quaternions. Additionally, two hybrid quaternions are equal if all of their components are sequentially equal.*

Remark 3.3. *Let us have the matrices $X = \begin{bmatrix} 1 & i & j & k \end{bmatrix}^T$ and $Y = \begin{bmatrix} 1 & \boldsymbol{i} & \boldsymbol{\varepsilon} & \boldsymbol{h} \end{bmatrix}^T$ which are formed by the quaternion units and the hybrid units, respectively. Additionally, for the 4×4 matrix M which consists the coefficients of a hybrid quaternion Q as*

$$M = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ c_0 & c_1 & c_2 & c_3 \\ d_0 & d_1 & d_2 & d_3 \end{bmatrix} \tag{31}$$

then, the equations $Q = X^T MY$ and $Q = Y^T M^T X$ are hold. Note that $Q = Q^T$ and $(X^T MY)^T = Y^T M^T X$.

Definition 3.4. *The sum of two hybrid quaternions is defined by summing their components. Let $Q = q_a + \boldsymbol{i}q_b + \boldsymbol{\varepsilon}q_c + \boldsymbol{h}q_d$ and $P = p_a + \boldsymbol{i}p_b + \boldsymbol{\varepsilon}p_c + \boldsymbol{h}p_d$ be any elements of $\boldsymbol{H}_{\mathbb{K}}$, the sum of Q and P is*

$$Q + P = (q_a + \boldsymbol{i}q_b + \boldsymbol{\varepsilon}q_c + \boldsymbol{h}q_d) + (p_a + \boldsymbol{i}p_b + \boldsymbol{\varepsilon}p_c + \boldsymbol{h}p_d) = (q_a + p_a) + \boldsymbol{i}(q_b + p_b) + \boldsymbol{\varepsilon}(q_c + p_c) + \boldsymbol{h}(q_d + p_d).$$

Addition of hybrid quaternions is both commutative and associative. Zero is the null element and the inverse element of Q is $-Q$. This properties signifies that $(\mathbf{H}_{\mathbb{K}}, +)$ is an Abelian group.

Definition 3.5. *The multiplication of hybrid quaternions $Q = q_a + \boldsymbol{i}q_b + \boldsymbol{\varepsilon}q_c + \boldsymbol{h}q_d$ and $P = p_a + \boldsymbol{i}p_b + \boldsymbol{\varepsilon}p_c + \boldsymbol{h}p_d$ can be given as follows*

$$\begin{aligned} QP = \; & (q_a p_a - q_b p_b + q_b p_c + q_c p_b + q_d p_d) + \boldsymbol{i}(q_a p_b + q_b p_a + q_b p_d - q_d p_b) \\ & + \boldsymbol{\varepsilon}(q_a p_c + q_c p_a + q_b p_d - q_d p_b + q_d p_c - q_c p_d) + \boldsymbol{h}(q_a p_d + q_d p_a + q_c p_b - q_b p_c). \end{aligned} \tag{32}$$

It is obvious that the multiplication of hybrid quaternions is not commutative. But it satisfies the property of associativity. The set of hybrid quaternions, $\mathbf{H}_{\mathbb{K}}$, constitutes a non-commutative ring with addition, multiplication, and the identity element. Furthermore, hybrid quaternions form a 16-dimensional non-commutative associative algebra over $\mathbb{R}$.

Remark 3.6. *For $Q = q_a + \boldsymbol{i}q_b + \boldsymbol{\varepsilon}q_c + \boldsymbol{h}q_d$, $P = p_a + \boldsymbol{i}p_b + \boldsymbol{\varepsilon}p_c + \boldsymbol{h}p_d \in \boldsymbol{H}_{\mathbb{K}}$, the multiplication QP, which is obtained above, is a generalization of multiplication of complex, dual, and hyperbolic quaternions. Thus, under the special cases, we can obtain the multiplication of complex, dual, and hyperbolic quaternions from the equation* (32), *as follows:*

i) If $q_c = q_d = p_c = p_d = 0$, then $QP = q_a p_a - q_b p_b + \boldsymbol{i}(q_a p_b + q_b p_a)$,

ii) If $q_b = q_d = p_b = p_d = 0$, then $QP = q_a p_a + \boldsymbol{\varepsilon}(q_a p_c + q_c p_a)$,

iii) If $q_b = q_c = p_b = p_c = 0$, then $QP = q_a p_a + q_d p_d + \boldsymbol{h}(q_a p_d + q_d p_a)$.

Here, the results obtained from the first, second, and third cases exactly correspond to the equations achieved by the multiplication of the two complex, dual, and hyperbolic quaternions in (13),(20), *and* (26), *respectively. In other cases, it can be found the multiplication of a complex and a dual quaternion or a complex and a hyperbolic quaternion or a dual and a hyperbolic quaternion.*

Remark 3.7. *Since the multiplication of the two general Hybrid quaternions means the product of two sixteen-component numbers, it is not easy to deal with this multiplication manually and it is very likely to make mistakes. We overcame this difficulty by writing a software program in C++ to multiply the hybrid quaternions.*

Definition 3.8. *Three different conjugates can be defined for a hybrid quaternion. For a hybrid quaternion $Q = q_a + \boldsymbol{i}q_b + \boldsymbol{\varepsilon}q_c + \boldsymbol{h}q_d$, following definitions for conjugates can be given:*

(a) *Quaternion conjugate:* $\overline{Q} = \overline{q_a} + \boldsymbol{i}\overline{q_b} + \boldsymbol{\varepsilon}\overline{q_c} + \boldsymbol{h}\overline{q_d}$.

(b) *Hybrid conjugate:* $Q^* = q_a - \boldsymbol{i}q_b - \boldsymbol{\varepsilon}q_c - \boldsymbol{h}q_d$

(c) *Total conjugate:* $Q^\dagger = (\overline{Q})^* = \overline{(Q^*)} = \overline{q_a} - \boldsymbol{i}\overline{q_b} - \boldsymbol{\varepsilon}\overline{q_c} - \boldsymbol{h}\overline{q_d}$.

Theorem 3.9. *For any* $Q = q_a + \boldsymbol{i}q_b + \boldsymbol{\varepsilon}q_c + \boldsymbol{h}q_d$, $P = p_a + \boldsymbol{i}p_b + \boldsymbol{\varepsilon}p_c + \boldsymbol{h}p_d \in \boldsymbol{H}_{\mathbb{K}}$ *the following properties are satisfied:*

(1) $\overline{(\overline{Q})} = (Q^\dagger)^\dagger = (Q^*)^* = Q$,

(2) $\overline{Q+P} = \overline{Q} + \overline{P}$, $(Q+P)^* = Q^* + P^*$, $(Q+P)^\dagger = Q^\dagger + P^\dagger$

(3) $\overline{Q} = Q$ *iff* $Q \in \mathbb{K}$, $Q^* = Q$ *iff* $Q \in \boldsymbol{H}$, $Q^\dagger = Q$ *iff* $Q \in \mathbb{R}$.

Proof. Straightforward. □

Some important properties are resulted from the non-commutativity of hybrid quaternions. Before giving a corollary covering these essential properties, we observe them in the following example.

Example 3.10. *Let* $Q = (2-i) + (1-j)\boldsymbol{i} + (3-k)\boldsymbol{\varepsilon}$ *and* $P = (1-k)\boldsymbol{i} + (2-3i)\boldsymbol{\varepsilon} + (1+2k)\boldsymbol{h}$ *be two hybrid quaternions. Then,*

1) $\overline{Q} = (2+i) + (1+j)\boldsymbol{i} + (3+k)\boldsymbol{\varepsilon}$

2) $Q^* = (2-i) - (1-j)\boldsymbol{i} - (3-k)\boldsymbol{\varepsilon}$

3) $Q^\dagger = (2+i) - (1+j)\boldsymbol{i} - (3+k)\boldsymbol{\varepsilon}$

4) $\overline{P} = (1+k)\boldsymbol{i} + (2+3i)\boldsymbol{\varepsilon} + (1-2k)\boldsymbol{h}$

5) $P^* = -(1-k)\boldsymbol{i} - (2-3i)\boldsymbol{\varepsilon} - (1+2k)\boldsymbol{h}$

6) $P^\dagger = -(1+k)\boldsymbol{i} - (2+3i)\boldsymbol{\varepsilon} - (1-2k)\boldsymbol{h}$

7) $Q\overline{Q} = 9 + 4\boldsymbol{i} + 12\boldsymbol{\varepsilon} + (2i + 6j - 2k)\boldsymbol{h}$

8) $\overline{Q}Q = 9 + 4\boldsymbol{i} + 12\boldsymbol{\varepsilon} + (2i - 6j + 2k)\boldsymbol{h}$

9) $QQ^* = (-3 - 4i - 4j + 2k) + (-2k)\boldsymbol{i} + (2j)\boldsymbol{\varepsilon} + (2i)\boldsymbol{h}$

10) $Q^*Q = (-3 - 4i - 4j + 2k) + (2k)\boldsymbol{i} + (-2j)\boldsymbol{\varepsilon} + (2i)\boldsymbol{h}$

11) $QQ^\dagger = 1 + (2i - 4j + 2k)\boldsymbol{i} + (6i - 2j - 4k)\boldsymbol{\varepsilon} + (-2i - 6j + 2k)\boldsymbol{h}$

12) $Q^\dagger Q = 1 + (2i - 4j - 2k)\boldsymbol{i} + (6i + 2j - 4k)\boldsymbol{\varepsilon} + (-2i + 6j - 2k)\boldsymbol{h}$

13) $QP = (3 - 4i - j - 6k) + (3 - 3i - 2j)\boldsymbol{i} + (-3 - 10i - j - 3k)\boldsymbol{\varepsilon} + (2 + 2i + 4j + 3k)\boldsymbol{h}$

14) $PQ = (3 - 2i - j) + (1 - 3i + 2j - 4k)\boldsymbol{i} + (5 - 10i + j + 3k)\boldsymbol{\varepsilon} + (2 - 4i - 4j + 11k)\boldsymbol{h}$

15) $\overline{QP} = (3 + 4i + j + 6k) + (3 + 3i + 2j)\boldsymbol{i} + (-3 + 10i + j + 3k)\boldsymbol{\varepsilon} + (2 - 2i - 4j - 3k)\boldsymbol{h}$

16) $\overline{P}\,\overline{Q} = (3 + 4i + j + 6k) + (1 - i + 4k)\boldsymbol{i} + (5 + 6i - j - 3k)\boldsymbol{\varepsilon} + (2 + 4i - 5k)\boldsymbol{h}$

17) $(QP)^* = (3 - 4i - j - 6k) + (-3 + 3i + 2j)\boldsymbol{i} + (3 + 10i + j + 3k)\boldsymbol{\varepsilon} - (2 + 2i + 4j + 3k)\boldsymbol{h}$

18) $P^*Q^* = (3 - 2i - j) + (-3 - i)\boldsymbol{i} + (3 + 6i + j + 3k)\boldsymbol{\varepsilon} + (-2 - 2i + 3k)\boldsymbol{h}$

19) $(QP)^\dagger = (3 + 4i + j + 6k) + (-3 - 3i - 2j)\boldsymbol{i} + (3 - 10i - j - 3k)\boldsymbol{\varepsilon} + (-2 + 2i + 4j + 3k)\boldsymbol{h}$

20) $P^{\dagger}Q^{\dagger} = (3+4i+j+6k)+(1-i+4k)\boldsymbol{i}+(5+6i-j-3k)\boldsymbol{\varepsilon}+(2+4i-5k)\boldsymbol{h}$

With these examples we can easily obtain the following corollary for hybrid quaternions.

Corollary 3.11. *Let Q and P be two hybrid quaternions. Then the followings are satisfied in general:*

(a) $Q\overline{Q} \neq \overline{Q}Q$, $QQ^{\dagger} \neq Q^{\dagger}Q$, $QQ^{*} \neq Q^{*}Q$,

(b) $\overline{QP} \neq \overline{P}\,\overline{Q}$, $(QP)^{\dagger} \neq P^{\dagger}Q^{\dagger}$, $(QP)^{*} \neq P^{*}Q^{*}$.

Definition 3.12. *The inner product of hybrid quaternions is defined as*

$$\langle Q,P\rangle = S_{\frac{1}{2}(Q\overline{P}+P\overline{Q})}, \quad \langle Q,P\rangle \in \mathbb{K}. \tag{33}$$

where $Q,P \in \boldsymbol{H}_{\mathbb{K}}$ *and* $S_{\frac{1}{2}(Q\overline{P}+P\overline{Q})}$ *is the hybrid scalar part of the hybrid quaternion* $\frac{1}{2}(Q\overline{P}+P\overline{Q})$*. This inner product fulfill all the usual properties of an inner product except positive definite.*

Remark 3.13. *Note that we select to define a different inner product than that defined for real quaternions. Certainly alternative definitions can be given for the inner product.*

Theorem 3.14. *Let* $Q = q_a + \boldsymbol{i}q_b + \boldsymbol{\varepsilon}q_c + \boldsymbol{h}q_d$ *and* $P = p_a + \boldsymbol{i}p_b + \boldsymbol{\varepsilon}p_c + \boldsymbol{h}p_d$ *be two hybrid quaternions. The inner product of Q and P:*

$$\begin{aligned} S_{\frac{1}{2}(Q\overline{P}+P\overline{Q})} &= \langle q_a,p_a\rangle - \langle q_b,p_b\rangle + \langle q_d,p_d\rangle + \langle q_c,p_b\rangle + \langle q_b,p_c\rangle \\ &\quad + \boldsymbol{i}\Big(\langle q_a,p_b\rangle + \langle q_b,p_a\rangle\Big) + \boldsymbol{\varepsilon}\Big(\langle q_a,p_c\rangle + \langle q_c,p_a\rangle\Big) + \boldsymbol{h}\Big(\langle q_a,p_d\rangle + \langle q_d,p_a\rangle\Big). \end{aligned}$$

Proof. With the help of multiplication of two quaternions (32), we can calculate $Q\overline{P}$ and $P\overline{Q}$:

$$\begin{aligned} Q\overline{P} &= (q_a + \mathbf{i}q_b + \boldsymbol{\varepsilon}q_c + \mathbf{h}q_d)(\overline{p_a} + \mathbf{i}\overline{p_b} + \boldsymbol{\varepsilon}\overline{p_c} + \mathbf{h}\overline{p_d}) \\ &= (q_a\overline{p_a} - q_b\overline{p_b} + q_b\overline{p_c} + q_c\overline{p_b} + q_d\overline{p_d}) + \mathbf{i}(q_a\overline{p_b} + q_b\overline{p_a} + q_b\overline{p_d} - q_d\overline{p_b}) \\ &\quad + \boldsymbol{\varepsilon}(q_a\overline{p_c} + q_c\overline{p_a} + q_b\overline{p_d} - q_d\overline{p_b} + q_d\overline{p_c} - q_c\overline{p_d}) + \mathbf{h}(q_a\overline{p_d} + q_d\overline{p_a} + q_c\overline{p_b} - q_b\overline{p_c}). \end{aligned} \tag{34}$$

$$\begin{aligned} P\overline{Q} &= (p_a + \mathbf{i}p_b + \boldsymbol{\varepsilon}p_c + \mathbf{h}p_d)(\overline{q_a} + \mathbf{i}\overline{q_b} + \boldsymbol{\varepsilon}\overline{q_c} + \mathbf{h}\overline{q_d}) \\ &= (p_a\overline{q_a} - p_b\overline{q_b} + p_b\overline{q_c} + p_c\overline{q_b} + p_d\overline{q_d}) + \mathbf{i}(p_a\overline{q_b} + p_b\overline{q_a} + p_b\overline{q_d} - p_d\overline{q_b}) \\ &\quad + \boldsymbol{\varepsilon}(p_a\overline{q_c} + p_c\overline{q_a} + p_b\overline{q_d} - p_d\overline{q_b} + p_d\overline{q_c} - p_c\overline{q_d}) + \mathbf{h}(p_a\overline{q_d} + p_d\overline{q_a} + p_c\overline{q_b} - p_b\overline{q_c}). \end{aligned} \tag{35}$$

By using equations (7) and (8) we can easily get

$$\begin{aligned} \frac{1}{2}\Big(Q\overline{P} + P\overline{Q}\Big) &= \langle q_a,p_a\rangle - \langle q_b,p_b\rangle + \langle q_d,p_d\rangle + \langle q_c,p_b\rangle + \langle q_b,p_c\rangle \\ &\quad + \mathbf{i}\Big(\langle q_a,p_b\rangle + \langle q_b,p_a\rangle + q_b \times p_d + p_b \times q_d\Big) \\ &\quad + \boldsymbol{\varepsilon}\Big(\langle q_a,p_c\rangle + \langle q_c,p_a\rangle + q_b \times p_d + p_b \times q_d + q_d \times p_c + p_d \times q_c\Big) \\ &\quad + \mathbf{h}\Big(\langle q_a,p_d\rangle + \langle q_d,p_a\rangle + q_c \times p_b + p_c \times q_d\Big). \end{aligned}$$

Then, the hybrid scalar part of $\frac{1}{2}\Big(Q\overline{P} + P\overline{Q}\Big)$ is obtained as

$$\begin{aligned} S_{\frac{1}{2}(Q\overline{P}+P\overline{Q})} &= \langle q_a,p_a\rangle - \langle q_b,p_b\rangle + \langle q_d,p_d\rangle + \langle q_c,p_b\rangle + \langle q_b,p_c\rangle + \mathbf{i}\Big(\langle q_a,p_b\rangle + \langle q_b,p_a\rangle\Big) \\ &\quad + \boldsymbol{\varepsilon}\Big(\langle q_a,p_c\rangle + \langle q_c,p_a\rangle\Big) + \mathbf{h}\Big(\langle q_a,p_d\rangle + \langle q_d,p_a\rangle\Big). \end{aligned} \tag{36}$$

□

Corollary 3.15. *The inner product of two hybrid quaternions which is defined above is a generalized inner product for the complex quaternions (biquaternions), dual quaternions and hyperbolic(perplex) quaternions. In other words, for* $Q = q_a + \boldsymbol{i}q_b + \boldsymbol{\varepsilon}q_c + \boldsymbol{h}q_d$, $P = p_a + \boldsymbol{i}p_b + \boldsymbol{\varepsilon}p_c + \boldsymbol{h}p_d \in \boldsymbol{H}_{\mathbb{K}}$, *from the inner product* $\langle Q, P\rangle$ *we can get the inner product of complex, dual, and hyperbolic quaternions in special cases.*

(i) If $q_c = q_d = p_c = p_d = 0$, *then we get* $S_{\frac{1}{2}(Q\overline{P}+P\overline{Q})} = \langle q_a, p_a\rangle - \langle q_b, p_b\rangle + \boldsymbol{i}\Big(\langle q_a, p_b\rangle + \langle q_b, p_a\rangle\Big)$

(ii) If $q_b = q_d = p_b = p_d = 0$, *then we get* $S_{\frac{1}{2}(Q\overline{P}+P\overline{Q})} = \langle q_a, p_a\rangle + \boldsymbol{\varepsilon}\Big(\langle q_a, p_c\rangle + \langle q_c, p_a\rangle\Big)$.

(iii) If $q_b = q_c = p_b = p_c = 0$, *then we get* $S_{\frac{1}{2}(Q\overline{P}+P\overline{Q})} = \langle q_a, p_a\rangle + \langle q_d, p_d\rangle + \boldsymbol{h}\Big(\langle q_a, p_d\rangle + \langle q_d, p_a\rangle\Big)$.

Here, the results obtained under the first, second, and third cases exactly correspond to the equations achieved by the inner product of complex, dual, and hyperbolic quaternions in (14), (21), *and* (27), *respectively.*

(iv) If $q_c = q_d = p_b = p_d = 0$, *the inner product of a complex quaternion and a dual quaternion can be obtained as* $S_{\frac{1}{2}(Q\overline{P}+P\overline{Q})} = \langle q_a, p_a\rangle + \langle q_b, p_c\rangle + \boldsymbol{i}\langle q_b, p_a\rangle + \boldsymbol{\varepsilon}\langle q_a, p_c\rangle$.

(v) If $q_c = q_d = p_b = p_c = 0$, *the inner product of a complex quaternion and a hyperbolic quaternion can be obtained as* $S_{\frac{1}{2}(Q\overline{P}+P\overline{Q})} = \langle q_a, p_a\rangle + \boldsymbol{i}\langle q_b, p_a\rangle + \boldsymbol{h}\langle q_a, p_d\rangle$.

(vi) If $q_b = q_d = p_b = p_c = 0$, *the inner product of a dual quaternion and a hyperbolic quaternion can be obtained as* $S_{\frac{1}{2}(Q\overline{P}+P\overline{Q})} = \langle q_a, p_a\rangle + \boldsymbol{\varepsilon}\langle q_c, p_a\rangle + \boldsymbol{h}\langle q_a, p_d\rangle$.

All these special cases show that hybrid quaternions are a powerful generalization of the complex quaternions (biquaternions), dual quaternions, and hyperbolic(perplex) quaternions.

Definition 3.16. *For* $Q, P \in \boldsymbol{H}_{\mathbb{K}}$, *vector product of hybrid quaternions is defined as*

$$Q \times P = V_{\frac{1}{2}(Q\overline{P}-P\overline{Q})}, \quad Q \times P \in \boldsymbol{H}_{\mathbb{K}}. \tag{37}$$

Here, $V_{\frac{1}{2}(Q\overline{P}-P\overline{Q})}$ *is the hybrid vector part of the hybrid quaternion* $\frac{1}{2}(Q\overline{P} - P\overline{Q})$.

Theorem 3.17. *Let* $Q = q_a + \boldsymbol{i}q_b + \boldsymbol{\varepsilon}q_c + \boldsymbol{h}q_d$ *and* $P = p_a + \boldsymbol{i}p_b + \boldsymbol{\varepsilon}p_c + \boldsymbol{h}p_d$ *be two hybrid quaternions. The vector product of Q and P is given by*

$$\begin{aligned} V_{\frac{1}{2}(Q\overline{P}-P\overline{Q})} &= q_a \times p_a - q_b \times p_b + q_d \times p_d + q_c \times p_b + q_b \times p_c + \boldsymbol{i}\Big(q_a \times p_b + q_b \times p_a\Big) \\ &\quad + \boldsymbol{\varepsilon}\Big(q_a \times p_c + q_c \times p_a\Big) + \boldsymbol{h}\Big(q_a \times p_d + q_d \times p_a\Big). \end{aligned} \tag{38}$$

Proof. Previously $Q\overline{P}$ and $P\overline{Q}$ were calculated as in (34), (35) and also by using equations (7) and (8) we can easily obtain $\frac{1}{2}\Big(Q\overline{P} - P\overline{Q}\Big)$ as follows:

$$\begin{aligned} \frac{1}{2}\Big(Q\overline{P} - P\overline{Q}\Big) &= q_a \times p_a - q_b \times p_b + q_d \times p_d + q_c \times p_b + q_b \times p_c + \mathbf{i}\Big(q_a \times p_b + q_b \times p_a + \langle q_b, p_d\rangle + \langle p_b, q_d\rangle\Big) \\ &\quad + \boldsymbol{\varepsilon}\Big(q_a \times p_c + q_c \times p_a + \langle q_b, p_d\rangle + \langle p_b, q_d\rangle + \langle q_d, p_c\rangle + \langle p_d, q_c\rangle\Big) \\ &\quad + \mathbf{h}\Big(q_a \times p_d + q_d \times p_a + \langle q_c, p_b\rangle + \langle p_c, q_b\rangle\Big). \end{aligned}$$

Then, the hybrid vector part is found as follows:

$$\begin{aligned} V_{\frac{1}{2}(Q\overline{P}-P\overline{Q})} &= q_a \times p_a - q_b \times p_b + q_d \times p_d + q_c \times p_b + q_b \times p_c \\ &\quad + \mathbf{i}\Big(q_a \times p_b + q_b \times p_a\Big) + \boldsymbol{\varepsilon}\Big(q_a \times p_c + q_c \times p_a\Big) + \mathbf{h}\Big(q_a \times p_d + q_d \times p_a\Big). \end{aligned}$$

□

Corollary 3.18. *The vector product of two hybrid quaternions which is defined as in* (37), *is a generalized vector product for the complex quaternions (biquaternions), dual quaternions, and hyperbolic(perplex) quaternions. That is, for $Q, P \in \boldsymbol{H}_{\mathbb{K}}$, under the special cases we can get the vector product of complex, dual, and hyperbolic quaternions from the vector product $Q \times P$.*

Example 3.19. *Let's find the inner product and vector product of hybrid quaternions $Q = (1+i+j)+\boldsymbol{i}(1-k)+\boldsymbol{\varepsilon}(i+j)$ and $P = \boldsymbol{i}(i+k)+\boldsymbol{h}(1+k)$. Using the definition of inner product* (3.12) *and the definition of vector product* (3.16) *we get the results:*

$$\langle Q, P\rangle = S_{\frac{1}{2}(Q\overline{P}+P\overline{Q})} = 2 + \boldsymbol{i} + 2\boldsymbol{\varepsilon} + \boldsymbol{h},$$

$$Q \times P = V_{\frac{1}{2}(Q\overline{P}-P\overline{Q})} = 2k + \boldsymbol{i}(-2i + j) + \boldsymbol{\varepsilon}(i + j) + \boldsymbol{h}(2j - k).$$

Definition 3.20. *Let $Q = w_0 + w_1 i + w_2 j + w_3 k$ be a hybrid quaternion. The norm of a hybrid quaternion is a kind of semi-norm which can be defined as in terms of the inner product of a hybrid quaternion with itself, that is $\|Q\| = \langle Q, Q\rangle$. Also, it can be directly defined as:*

$$\|Q\| = |Q|^2 = {w_0}^2 + {w_1}^2 + {w_2}^2 + {w_3}^2, \tag{39}$$

where, this semi-norm is hybrid-valued. Moreover, with the definition of inner product of two hybrid quaternions, $\|Q\|$ can be written as:

$$\|Q\| = \langle Q, Q\rangle = S_{\frac{1}{2}(Q\overline{Q}+Q\overline{Q})} = S_{(Q\overline{Q})}, \quad \|Q\| \in \boldsymbol{H}_{\mathbb{K}}. \tag{40}$$

Theorem 3.21. *Let $Q = q_a + \boldsymbol{i}q_b + \boldsymbol{\varepsilon}q_c + \boldsymbol{h}q_d$ be a non-zero hybrid quaternion. The norm $\|Q\|$ is*

$$\|Q\| = S_{(Q\overline{Q})} = \left\|q_a\right\| - \left\|q_b - q_c\right\| + \left\|q_c\right\| + \left\|q_d\right\| + 2\boldsymbol{i}\langle q_a, q_b\rangle + 2\boldsymbol{\varepsilon}\langle q_a, q_c\rangle + 2\boldsymbol{h}\langle q_a, q_d\rangle.$$

Proof. From the equation (34) we can easily obtain $Q\overline{Q}$ as

$$\begin{aligned} Q\overline{Q} = {} & (q_a\overline{q_a} - q_b\overline{q_b} + q_b\overline{q_c} + q_c\overline{q_b} + q_d\overline{q_d}) + \mathbf{i}(q_a\overline{q_b} + q_b\overline{q_a} + q_b\overline{q_d} - q_d\overline{q_b}) \\ & + \boldsymbol{\varepsilon}(q_a\overline{q_c} + q_c\overline{q_a} + q_b\overline{q_d} - q_d\overline{q_b} + q_d\overline{q_c} - q_c\overline{q_d}) + \mathbf{h}(q_a\overline{q_d} + q_d\overline{q_a} + q_c\overline{q_b} - q_b\overline{q_c}). \end{aligned} \tag{41}$$

Then, the scalar part is

$$S_{Q\overline{Q}} = (q_a\overline{q_a} - q_b\overline{q_b} + q_b\overline{q_c} + q_c\overline{q_b} + q_d\overline{q_d}) + \mathbf{i}(q_a\overline{q_b} + q_b\overline{q_a}) + \boldsymbol{\varepsilon}(q_a\overline{q_c} + q_c\overline{q_a}) + \mathbf{h}(q_a\overline{q_d} + q_d\overline{q_a}). \tag{42}$$

Using the properties of real quaternions we get

$$S_{(Q\overline{Q})} = \left\|q_a\right\| - \left\|q_b - q_c\right\| + \left\|q_c\right\| + \left\|q_d\right\| + 2\mathbf{i}\langle q_a, q_b\rangle + 2\boldsymbol{\varepsilon}\langle q_a, q_c\rangle + 2\mathbf{h}\langle q_a, q_d\rangle. \tag{43}$$

□

Corollary 3.22. *The norm of a hybrid quaternion which is defined above is a generalized norm for the complex quaternions (biquaternions), dual quaternions and hyperbolic(perplex) quaternions. In other words, for $Q = q_a + \boldsymbol{i}q_b + \boldsymbol{\varepsilon}q_c + \boldsymbol{h}q_d \in \boldsymbol{H}_{\mathbb{K}}$, under special cases norm of complex, dual, and hyperbolic quaternions can be obtained as following:*

(i) If $q_c = q_d = 0$, then we get $\|Q\| = S_{(Q\overline{Q})} = \left\|q_a\right\| - \left\|q_b\right\| + 2\boldsymbol{i}\langle q_a, q_b\rangle$.

(ii) If $q_b = q_d = 0$, then we get $\|Q\| = S_{(Q\overline{Q})} = \left\|q_a\right\| + 2\boldsymbol{\varepsilon}\langle q_a, q_c\rangle$.

(iii) If $q_b = q_c = 0$, then we get $\|Q\| = S_{(Q\overline{Q})} = \left\|q_a\right\| + \left\|q_d\right\| + 2\boldsymbol{h}\langle q_a, q_d\rangle$.

Here, the results obtained under the first, second, and third cases exactly corresponds to the equations achieved by the norm of a complex quaternion, dual quaternion, and hyperbolic quaternion in (15), (22), *and* (28), *respectively. The semi norm which is defined for hybrid quaternions above is invariant under quaternion conjugation,* $\|Q\| = \left\|\overline{Q}\right\|$.

It is known that every non-zero real quaternions have a multiplicative inverse but as mentioned previously this does not true for complex quaternions and dual quaternions. Because, zero divisors in complex quaternions and dual quaternions have no inverse. As hybrid quaternions are a generalization of complex and dual quaternions, every element in this set does not have an inverse. So, the hybrid quaternion algebra $\mathbf{H}_{\mathbb{K}}$ is not a division algebra. For proper $Q \in \mathbf{H}_{\mathbb{K}}$, the inverse Q^{-1} exists and it is given by the relation $QQ^{-1} = Q^{-1}Q = 1$.

Example 3.23. *For the hybrid quaternions* $Q_1 = (i+j) + (1+j)\boldsymbol{i} + (1+k)\boldsymbol{\varepsilon}$ *and* $Q_2 = (1+i) + (j)\boldsymbol{i} + (i)\boldsymbol{\varepsilon} + (k)\boldsymbol{h}$ *the inverses* Q_1^{-1} *and* Q_2^{-1} *are given by*

$$Q_1^{-1} = \frac{1}{6}\Big((-i+k) + (1+i-2k)\boldsymbol{i} + (2+i-j-2k)\boldsymbol{\varepsilon} - (1+i+j+k)\boldsymbol{h}\Big),$$

$$Q_2^{-1} = \frac{1}{2}\Big((-i+j) + \boldsymbol{i} + (i)\boldsymbol{\varepsilon} + (-k)\boldsymbol{h}\Big).$$

It can be seen that these hybrid quaternions satisfy $QQ^{-1} = Q^{-1}Q = 1$.

4. Venn Diagrams of Number Systems

In the previous sections, we have explained and presented the fundamental informations about number systems. In this section, we will illustrate and give the relationships between these number systems with a Venn diagram.

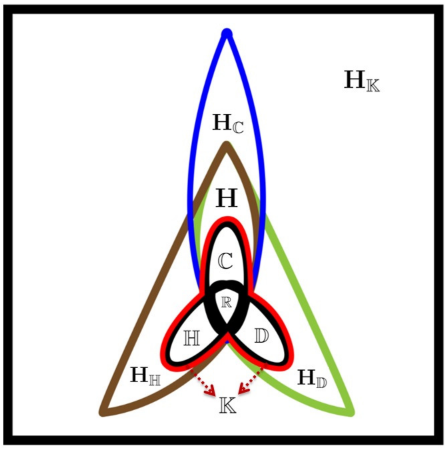

Figure 1: Venn diagram of number systems and quaternions

The relationships between sets of numbers and the sets of quaternions can be listed as follows:

- $\mathbb{C} \cup \mathbb{D} \cup \mathbb{H} \subseteq \mathbb{K}, \quad \mathbb{R} \subseteq \mathbb{C} \subseteq \mathbb{K}, \quad \mathbb{R} \subseteq \mathbb{D} \subseteq \mathbb{K}, \quad \mathbb{R} \subseteq \mathbb{H} \subseteq \mathbb{K}$
- $\mathbb{C} \cap \mathbb{D} = \mathbb{D} \cap \mathbb{H} = \mathbb{H} \cap \mathbb{C} = \mathbb{C} \cap \mathbb{D} \cap \mathbb{H} = \mathbb{R}$
- $\mathbf{H}_{\mathbb{C}} \cup \mathbf{H}_{\mathbb{D}} \cup \mathbf{H}_{\mathbb{H}} \subseteq \mathbf{H}_{\mathbb{K}}, \mathbf{H} \subseteq \mathbf{H}_{\mathbb{C}} \subseteq \mathbf{H}_{\mathbb{K}}, \mathbf{H} \subseteq \mathbf{H}_{\mathbb{D}} \subseteq \mathbf{H}_{\mathbb{K}}, \mathbf{H} \subseteq \mathbf{H}_{\mathbb{H}} \subseteq \mathbf{H}_{\mathbb{K}}$
- $\mathbf{H}_{\mathbb{C}} \cap \mathbf{H}_{\mathbb{D}} = \mathbf{H}_{\mathbb{C}} \cap \mathbf{H}_{\mathbb{H}} = \mathbf{H}_{\mathbb{D}} \cap \mathbf{H}_{\mathbb{H}} = \mathbf{H}_{\mathbb{C}} \cap \mathbf{H}_{\mathbb{D}} \cap \mathbf{H}_{\mathbb{H}} = \mathbf{H}$.
- $\mathbb{R} \subseteq \mathbb{C} \subseteq \mathbf{H} \subseteq \mathbf{H}_{\mathbb{C}} \subseteq \mathbf{H}_{\mathbb{K}}, \mathbb{R} \subseteq \mathbb{D} \subseteq \mathbf{H}_{\mathbb{D}} \subseteq \mathbf{H}_{\mathbb{K}},$
- $\mathbb{R} \subseteq \mathbb{H} \subseteq \mathbf{H}_{\mathbb{H}} \subseteq \mathbf{H}_{\mathbb{K}}, \mathbb{K} \subseteq \mathbf{H} \cup \mathbf{H}_{\mathbb{C}} \cup \mathbf{H}_{\mathbb{D}} \cup \mathbf{H}_{\mathbb{H}} \subseteq \mathbf{H}_{\mathbb{K}}$.

5. Conclusion

In this study, we have systematically reviewed, and summarized number systems ($\mathbb{C}, \mathbb{D}, \mathbb{H}, \mathbb{K}$) and quaternions ($\mathbf{H}_{\mathbb{C}}, \mathbf{H}_{\mathbb{D}}, \mathbf{H}_{\mathbb{H}}$) built on these numbers. Based on these, we have introduced a new quaternion system called hybrid quaternions ($\mathbf{H}_{\mathbb{K}}$) as a combination of complex quaternions(biquaternions), dual quaternions, and hyperbolic(perplex) quaternions. We then defined basic operations, inner product, norm, and vector product on this new system. With these inclusive definitions, hybrid quaternions take a generalized form of the other three quaternion types. This feature makes hybrid quaternions unique since each quaternion system is a special case of this new quaternion system. Finally, we were able to diagram all number systems and quaternion systems as a result of the broad consideration.

Examination of hybrid quaternions as a geometric algebra, matrices of hybrid quaternions, classifications of hybrid quaternions, polar representations, De Moivre formulas, and applications of hybrid quaternions can be worked as further studies.

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