The space of centered planes and generalized bilinear connection

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Abstract. We continue to study the space of centered planes in $n$-dimension projective space. We use E. Cartan’s method of external forms and the group-theoretical method of G. F. Laptev to study the space of centered planes of the same dimension. These methods are successfully applied in physics.

In a generalized bundle, a bilinear connection associated with a space is given. The connection object contains two simplest subtensors and subquasi-tensors (four simplest and three simple subquasi-tensors).

The object field of this connection defines the objects of torsion $S$, curvature-torsion $T$, and curvature $R$. The curvature tensor contains six simplest and four simple subtensors, and curvature-torsion tensor contains three simplest and two simple subtensors.

The canonical case of a generalized bilinear connection is considered.

We realize the strong Lumiste’s affine clothing (it is an analog of the strong Norde’s normalization of the space of centered planes). Covariant differentials and covariant derivatives of the clothing quasi-tensor are described. The covariant derivatives do not form a tensor. We present a geometrical characterization of the generalized bilinear connection using mappings.

1. Introduction

The theory of connections is an important area of modern differential geometry [3, 4, 13, 14, 25, 28, 29, 36]. During its centuries-old history, it has gained popularity among geometers and successfully continues to develop today (see, e.g. [10, 11, 16–18, 21, 23, 24, 37]).

In 1918, H. Weyl introduced an affine connection without a metric. The affine connection with the metric was introduced by A. Einstein. Further, the theory was developed by such mathematicians as T. Levi-Civita (1917) and C. Ehresmann (1941) and popularized by B. A. Rosenfeld in [31]. In 1923, E. Cartan gave the general concept of an affine connection from the point of view of the theory of relativity. Then this connection developed geometrically, no longer in connection with applications.

Generalized affine connections were considered in book [19], where a relation was shown between a generalized affine connection and a linear connection.

We will use the method of external forms of E. Cartan [1, 2, 5, 22, 30, 32] and the theoretical-group method of G. F. Laptev to study the space of centered planes of the same dimension.

Generalized affine connections (plane and normal) for the space of centered planes were introduced by author in 2010 (see [6, 7]) and also considered in the paper [12]. In the present paper, we introduce a new generalized bilinear connection.
2. Analytical apparatus and object of research

Projective space $P_n$ can be represented as a quotient space $L_{n+1}/\sim$ of a linear space $L_{n+1}$ with respect to equivalence (collinearity) $\sim$ of non-zero vectors, i.e.,

$$P_n = (L_{n+1} \setminus \{0\})/\sim.$$  

Projective frame in the space $P_n$ is a system formed by points $A_I$, $I = 0, \ldots, n$, and an unit point $E$. In linear space $L_{n+1}$ linearly independent vectors $e_I$ correspond to the points $A_I$, and a vector $e = \sum_{I=0}^n e_I$ corresponds to the point $E$. Moreover, these vectors are determined in the space $L_{n+1}$ with an accuracy up to a common factor. The unit point is specified together with the basic points, although you do not have to mention it every time.

We will use the method of a moving frame $\{A, A_I\}$, $I = 1, \ldots, n$, the derivation formulae of the vertices of which are (see, e.g., [9])

$$dA = \theta A + \omega_A^{I} A_I, \quad dA_I = \theta A_I + \omega_A^{J} A_J + \omega_I A, \quad (1)$$

where the form $\theta$ acts as a proportionality factor, $d$ is the symbol of ordinary differentiation in the space $P_n$, and the structure forms $\omega_A^I$, $\omega_A^J$, $\omega_I$ of the projective group $G_P(n)$, effectively acting on the space $P_n$, satisfy the following Cartan equations:

$$D\omega_A^I = \omega_A^K \wedge \omega_A^I, \quad D\omega_A^I = \omega_A^K \wedge \omega_A^I + \delta_I^J \omega_K \wedge \omega_J + \omega_I \wedge \omega_J, \quad (2)$$

where $D$ is the symbol of exterior differentiation.

In the projective space $P_n$ a space $\Pi$ of all centered $m$-dimensional planes $P_m$ is considered. Vertices $A$ and $A_a$, $a = 1, \ldots, m$, of the moving frame are placed on the centered plane, where vertex $A$ is fixed as a centre. The forms $\omega_a^\alpha$, $\omega^a$, $\omega^a_a$ ($a = m + 1, \ldots, n$) are the basic forms of the space $\Pi$.

Remark 2.1. The space $\Pi$ is a differentiable manifold whose points are $m$-dimensional centered planes.

We use the technique which is based on the Laptev – Lumiste method. This in turn requires knowledge of calculating external differential forms.

We will use the following terminology [34]:

Definition 2.2. A substructure of a structure $S$ is called simple if it is not a union of two substructures of the structure $S$.

A simple substructure is called the simplest if, in turn, does not have a substructure.

3. Generalized bilinear connection

Similarly to the generalized plane and normal affine connections [6, 7, 12] we consider a new generalized connection.

Definition 3.1. A smooth manifold with structural equations

$$D\omega^a = \omega^b \wedge \omega^a_b + \omega^a \wedge \omega^a_a, \quad D\omega^a = \omega^b \wedge \omega^a_b + \omega^a \wedge \omega^a_a, \quad D\omega^a_a = \omega^b \wedge (\delta^a_b \omega^a_a - \delta^a_b \omega^a_b) - \omega^a \wedge \omega_a,$$
\[ Da^a_{\beta} = a^a_{\beta} + a^a_{\alpha} \wedge a^\alpha - \delta^a_{\alpha} \wedge a^\alpha \wedge \omega^\alpha + a^a_{\beta} \wedge a^\alpha \wedge \omega^\alpha, \]

\[ Da^\alpha_{\beta} = a^\alpha_{\beta} + a^\alpha_{\alpha} \wedge a^\alpha - \delta^\alpha_{\alpha} \wedge a^\alpha \wedge \omega^\alpha + (\delta^\alpha_{\alpha} \wedge \omega^\alpha + \delta^\alpha_{\beta} \wedge \omega^\beta) \wedge \omega^\alpha, \]

is called a generalized bundle of bilinear frames \([20]\) and denoted by \(A_{n-2k+1}[-1]\), where \(k = m(n - m)\).

**Remark 3.2.** The symbol \(k\) is bracketed in the notation \(A_{n-2k+1}[-1]\) since \(k\) forms \(a^a_{\alpha}\) are the basic and fibre forms. Let’s call them basic-fibre forms (see \([35]\)).

In the generalized bundle \(A_{n-2k+1}[-1]\) we define a bilinear connection by the Laptev – Lumiste method \([15, 33]\) using the forms of planar \(\tilde{a}_{\alpha}^a\), normal linear \(\tilde{a}_{\beta}^a\) connections, and the forms \(\tilde{a}_{\alpha}^a\)

\[
\begin{align*}
\tilde{a}_{\alpha}^a &= a_{\alpha}^a - G_{a_{\beta}^a}^a \cdot \omega^a - G_{a_{\beta}^a}^b \cdot \omega^b - G_{a_{\beta}^a}^b \cdot \omega^b, \\
\tilde{a}_{\alpha}^a &= a_{\alpha}^a - G_{a_{\beta}^a}^a \cdot \omega^a - G_{a_{\beta}^a}^b \cdot \omega^b - G_{a_{\beta}^a}^b \cdot \omega^b, \\
\tilde{a}_{\alpha}^a &= a_{\alpha}^a - G_{a_{\beta}^a}^a \cdot \omega^a - G_{a_{\beta}^a}^b \cdot \omega^b - G_{a_{\beta}^a}^b \cdot \omega^b.
\end{align*}
\]

We find the exterior differentials of the connection forms and apply the Cartan – Laptev theorem \([15]\)

\[
\begin{align*}
\Delta G_{a_{\beta}^a} &= G_{a_{\beta}^a} \cdot \omega^a + (G_{a_{\beta}^a}^a \cdot \omega^a - \delta_{\alpha}^a \cdot \omega^a) \cdot a_{\beta}^a = G_{a_{\beta}^a}^a \cdot \omega^a + G_{a_{\beta}^a}^b \cdot \omega^b + G_{a_{\beta}^a}^b \cdot \omega^b, \\
\Delta G_{a_{\beta}^a} &= G_{a_{\beta}^a}^a \cdot \omega^a + G_{a_{\beta}^a}^b \cdot \omega^b + G_{a_{\beta}^a}^b \cdot \omega^b, \\
\Delta G_{a_{\beta}^a} &= G_{a_{\beta}^a}^a \cdot \omega^a + G_{a_{\beta}^a}^b \cdot \omega^b + G_{a_{\beta}^a}^b \cdot \omega^b, \\
\Delta G_{a_{\beta}^a} &= G_{a_{\beta}^a}^a \cdot \omega^a + G_{a_{\beta}^a}^b \cdot \omega^b + G_{a_{\beta}^a}^b \cdot \omega^b, \\
\Delta G_{a_{\beta}^a} &= G_{a_{\beta}^a}^a \cdot \omega^a + G_{a_{\beta}^a}^b \cdot \omega^b + G_{a_{\beta}^a}^b \cdot \omega^b, \\
\Delta G_{a_{\beta}^a} &= G_{a_{\beta}^a}^a \cdot \omega^a + G_{a_{\beta}^a}^b \cdot \omega^b + G_{a_{\beta}^a}^b \cdot \omega^b.
\end{align*}
\]

where the right-hand sides of the basic forms contain Pfaffian derivatives, and differential operator \(\Delta\) acts by the law \(\Delta G_{a_{\beta}^a} = dG_{a_{\beta}^a} + G_{a_{\beta}^a} \cdot \omega^a - G_{a_{\beta}^a}^a \cdot \omega^a - G_{a_{\beta}^a}^b \cdot \omega^b\).

The object of a generalized bilinear connection \(\Gamma = \{G_{a_{\beta}^a}, G_{a_{\beta}^a}, G_{a_{\beta}^a}, G_{a_{\beta}^a}, G_{a_{\beta}^a}, G_{a_{\beta}^a}, G_{a_{\beta}^a}, G_{a_{\beta}^a}, G_{a_{\beta}^a}, G_{a_{\beta}^a}\}\) associated with the space \(\Pi\) of centered planes contains

- two simplest subtensors \(G_{a_{\beta}^a}\) and \(G_{a_{\beta}^a}\) of the simple subquadra-tensor \(G_{a_{\beta}^a}^a \cdot G_{a_{\beta}^a} \cdot G_{a_{\beta}^a}\) of the connection \(\Gamma\);
- four simplest subquadra-tensors \(G_{a_{\beta}^a}, G_{a_{\beta}^a}, G_{a_{\beta}^a}, G_{a_{\beta}^a}\) and \(G_{a_{\beta}^a}, G_{a_{\beta}^a}, G_{a_{\beta}^a}, G_{a_{\beta}^a}\);
- two simple subquadra-tensors \(G_{a_{\beta}^a}, G_{a_{\beta}^a}, G_{a_{\beta}^a}\) and \(G_{a_{\beta}^a}, G_{a_{\beta}^a}, G_{a_{\beta}^a}\).

Structure equations of the basic forms can be written as

\[
\begin{align*}
D\omega^a &= \delta^a_{\beta} \wedge \omega^a + \delta^a_{\beta} \wedge \omega^a + S_{a_{\beta}^a} \cdot \omega^a \wedge \omega^a + S_{a_{\beta}^a} \cdot \omega^a \wedge \omega^a + S_{a_{\beta}^a} \cdot \omega^a \wedge \omega^a + S_{a_{\beta}^a} \cdot \omega^a \wedge \omega^a, \\
D\omega^a &= \delta^a_{\beta} \wedge \omega^a + \delta^a_{\beta} \wedge \omega^a + \delta^a_{\beta} \wedge \omega^a + \delta^a_{\beta} \wedge \omega^a + \delta^a_{\beta} \wedge \omega^a + \delta^a_{\beta} \wedge \omega^a + \delta^a_{\beta} \wedge \omega^a + \delta^a_{\beta} \wedge \omega^a + \delta^a_{\beta} \wedge \omega^a + \delta^a_{\beta} \wedge \omega^a + \delta^a_{\beta} \wedge \omega^a + \delta^a_{\beta} \wedge \omega^a + \delta^a_{\beta} \wedge \omega^a + \delta^a_{\beta} \wedge \omega^a + \delta^a_{\beta} \wedge \omega^a + \delta^a_{\beta} \wedge \omega^a.
\end{align*}
\]

where the components of torsion object \(S\) are found according to the formulae

\[
S_{a_{\beta}^a} = \Gamma_{a_{\beta}^a}, \quad S_{a_{\beta}^a} = \Gamma_{a_{\beta}^a} - G_{a_{\beta}^a}, \quad S_{a_{\beta}^a} = \Gamma_{a_{\beta}^a} - G_{a_{\beta}^a}, \quad S_{a_{\beta}^a} = G_{a_{\beta}^a} - G_{a_{\beta}^a}.
\]
where square brackets mean antisymmetrization on extreme indexes. For example, $S^{ab}_{[\alpha|\beta]} = \frac{1}{2} (S^{a}_{\alpha \beta} - S^{a}_{\beta \alpha})$.

Components of torsion object $S$ satisfy differential congruences modulo the basic forms

\[
\Delta S^a_{\beta \gamma} + S^{ab}_{[\alpha|\beta]} \omega_a - \Gamma^a_{[\alpha|\beta]} \omega_\gamma = 0,
\]

\[
\Delta S^a_{\alpha \beta} = 0,
\]

\[
\Delta S^a_{ab} = 0,
\]

\[
\Delta S^a_{ba} = 0,
\]

\[
\Delta S^a_{bc} = 0,
\]

\[
\Delta S^a_{cb} = 0.
\]

The torsion object $S$ is a geometric object (quasi-tensor) only in conjunction with the bilinear connection object $\Gamma$.

Taking into account the differential equations of the connection forms (3), we obtain

\[
D \tilde{\alpha}^a_b = \tilde{\alpha}^a_b \wedge \tilde{\alpha}^b_a + \tilde{\alpha}^a_b \wedge \tilde{\alpha}^b_a + T^{a}_{ab} \omega^b \wedge \omega^a + T^{a}_{ab} \omega^b \wedge \omega^a +
\]

\[
+ T^{a}_{ab} \omega^b \wedge \omega^a + T^{a}_{ab} \omega^b \wedge \omega^a + T^{a}_{ab} \omega^b \wedge \omega^a + T^{a}_{ab} \omega^b \wedge \omega^a,
\]

\[
D \tilde{\alpha}^a_b = \tilde{\alpha}^a_b \wedge \tilde{\alpha}^b_a + R^{a}_{ab} \omega^b \wedge \omega^a + R^{a}_{ab} \omega^b \wedge \omega^a +
\]

\[
+ R^{a}_{ab} \omega^b \wedge \omega^a + R^{a}_{ab} \omega^b \wedge \omega^a + R^{a}_{ab} \omega^b \wedge \omega^a + R^{a}_{ab} \omega^b \wedge \omega^a,
\]

\[
D \tilde{\alpha}^a_b = \tilde{\alpha}^a_b \wedge \tilde{\alpha}^b_a + R^{a}_{ab} \omega^b \wedge \omega^a + R^{a}_{ab} \omega^b \wedge \omega^a +
\]

\[
+ R^{a}_{ab} \omega^b \wedge \omega^a + R^{a}_{ab} \omega^b \wedge \omega^a + R^{a}_{ab} \omega^b \wedge \omega^a + R^{a}_{ab} \omega^b \wedge \omega^a.
\]

Curvature-torsion object $T$ has the following components:

\[
T^{a}_{[\alpha|\beta]} = G^{a}_{[\alpha|\beta]} - G^{a}_{[\beta|\alpha]} - X^{a}_{[\alpha|\beta]} - X^{a}_{[\beta|\alpha]},
\]

\[
T^{a}_{[\alpha|\beta]} = 2 \left( X^{a}_{[\alpha|\beta]} - X^{a}_{[\beta|\alpha]} - G^{a}_{[\alpha|\beta]} - G^{a}_{[\beta|\alpha]} \right),
\]

\[
T^{a}_{[\alpha|\beta]} = 2 \left( X^{a}_{[\alpha|\beta]} - X^{a}_{[\beta|\alpha]} - G^{a}_{[\alpha|\beta]} - G^{a}_{[\beta|\alpha]} \right),
\]

\[
T^{a}_{[\alpha|\beta]} = 2 \left( X^{a}_{[\alpha|\beta]} - X^{a}_{[\beta|\alpha]} - G^{a}_{[\alpha|\beta]} - G^{a}_{[\beta|\alpha]} \right),
\]

\[
T^{a}_{[\alpha|\beta]} = 2 \left( X^{a}_{[\alpha|\beta]} - X^{a}_{[\beta|\alpha]} - G^{a}_{[\alpha|\beta]} - G^{a}_{[\beta|\alpha]} \right),
\]

The components of the curvature object $R$ can be find

\[
R^{\alpha}_{\beta \gamma \delta} = \Gamma^{\alpha}_{[\beta|\gamma]} - \Gamma^{\alpha}_{[\gamma|\beta]} - \Gamma^{\alpha}_{[\delta|\beta]} - \Gamma^{\alpha}_{[\delta|\gamma]} - \Gamma^{\alpha}_{[\delta|\gamma]} - \Gamma^{\alpha}_{[\delta|\gamma]},
\]

\[
R^{\alpha}_{\beta \gamma \delta} = \Gamma^{\alpha}_{[\beta|\gamma]} - \Gamma^{\alpha}_{[\gamma|\beta]} - \Gamma^{\alpha}_{[\delta|\beta]} - \Gamma^{\alpha}_{[\delta|\gamma]} - \Gamma^{\alpha}_{[\delta|\gamma]} - \Gamma^{\alpha}_{[\delta|\gamma]},
\]

\[
R^{\alpha}_{\beta \gamma \delta} = \Gamma^{\alpha}_{[\beta|\gamma]} - \Gamma^{\alpha}_{[\gamma|\beta]} - \Gamma^{\alpha}_{[\delta|\beta]} - \Gamma^{\alpha}_{[\delta|\gamma]} - \Gamma^{\alpha}_{[\delta|\gamma]} - \Gamma^{\alpha}_{[\delta|\gamma]},
\]

\[
R^{\alpha}_{\beta \gamma \delta} = \Gamma^{\alpha}_{[\beta|\gamma]} - \Gamma^{\alpha}_{[\gamma|\beta]} - \Gamma^{\alpha}_{[\delta|\beta]} - \Gamma^{\alpha}_{[\delta|\gamma]} - \Gamma^{\alpha}_{[\delta|\gamma]} - \Gamma^{\alpha}_{[\delta|\gamma]},
\]

\[
R^{\alpha}_{\beta \gamma \delta} = \Gamma^{\alpha}_{[\beta|\gamma]} - \Gamma^{\alpha}_{[\gamma|\beta]} - \Gamma^{\alpha}_{[\delta|\beta]} - \Gamma^{\alpha}_{[\delta|\gamma]} - \Gamma^{\alpha}_{[\delta|\gamma]} - \Gamma^{\alpha}_{[\delta|\gamma]},
\]

\[
R^{\alpha}_{\beta \gamma \delta} = \Gamma^{\alpha}_{[\beta|\gamma]} - \Gamma^{\alpha}_{[\gamma|\beta]} - \Gamma^{\alpha}_{[\delta|\beta]} - \Gamma^{\alpha}_{[\delta|\gamma]} - \Gamma^{\alpha}_{[\delta|\gamma]} - \Gamma^{\alpha}_{[\delta|\gamma]},
\]

\[
R^{\alpha}_{\beta \gamma \delta} = \Gamma^{\alpha}_{[\beta|\gamma]} - \Gamma^{\alpha}_{[\gamma|\beta]} - \Gamma^{\alpha}_{[\delta|\beta]} - \Gamma^{\alpha}_{[\delta|\gamma]} - \Gamma^{\alpha}_{[\delta|\gamma]} - \Gamma^{\alpha}_{[\delta|\gamma]},
\]

\[
R^{\alpha}_{\beta \gamma \delta} = \Gamma^{\alpha}_{[\beta|\gamma]} - \Gamma^{\alpha}_{[\gamma|\beta]} - \Gamma^{\alpha}_{[\delta|\beta]} - \Gamma^{\alpha}_{[\delta|\gamma]} - \Gamma^{\alpha}_{[\delta|\gamma]} - \Gamma^{\alpha}_{[\delta|\gamma]},
\]

\[
R^{\alpha}_{\beta \gamma \delta} = \Gamma^{\alpha}_{[\beta|\gamma]} - \Gamma^{\alpha}_{[\gamma|\beta]} - \Gamma^{\alpha}_{[\delta|\beta]} - \Gamma^{\alpha}_{[\delta|\gamma]} - \Gamma^{\alpha}_{[\delta|\gamma]} - \Gamma^{\alpha}_{[\delta|\gamma]},
\]
The components of the curvature-torsion object $T$ satisfy the following differential congruences modulo basic forms:
\[
\begin{align*}
T^a_{\beta\gamma} + \delta^a_{\gamma} T^a_{\beta\gamma} &\equiv 0, \\
T^a_{\alpha\beta} + 2T^a_{\alpha\beta} &\equiv 0, \\
T^a_{\beta\gamma} - T^a_{\alpha\gamma} &\equiv 0, \\
T^a_{\alpha\beta} - T^a_{\beta\alpha} &\equiv 0, \\
T^a_{\beta\gamma} - T^a_{\gamma\beta} &\equiv 0, \\
T^a_{\alpha\gamma} + T^a_{\alpha\gamma} &\equiv 0.
\end{align*}
\]
These imply the following Theorem.
Theorem 3.3. The curvature-torsion object of the generalized bilinear connection is a tensor containing three simplest subtensors $T^a_{abc}$, $T^ac_{ab\gamma}$, $T^abc_{ab\gamma}$, and two simple subtensors $\{ T^a_{ab\gamma}, T^c_{ab\gamma}, T^b_{abc}\}$.

4. The canonical case

Let’s consider the case when $G^a_{ab} = 0$ and $G^ab_{ab} = 0$. From these conditions we have $\delta^a_{\gamma} = \omega^a_{\alpha} - G^a_{ab}a^b$ and the left-hand sides of the 2nd and the 3d equations (4) are identically vanishing, then the 1st equations (4) will be simplified

$$\Delta T^a_{\alpha\beta\gamma} - \delta^a_{\gamma} \omega^\alpha = G^a_{ab\gamma} + G^b_{\alpha\beta\gamma} + G^c_{\alpha\beta\gamma}.$$

In the canonical case, the quasi-tensor $G$ of the generalized bilinear connection is reduced to the quasi-tensor $G^a_{ab\gamma}$, while the connection object is simplified $\Gamma = \{G^a_{ab\gamma}, \ 0, \ 0, \ \Gamma^a_{bc\gamma}, \ \Gamma^\alpha_{bac}, \ \Gamma^\beta_{bc\gamma}, \ \Gamma^\gamma_{bc\gamma} \}.$

Substituting $G^a_{ab\gamma} = 0$, $G^ab_{ab\gamma} = 0$ into the expressions for the components of the curvature-torsion tensor they will take the form

$$0 T^a_{ab\gamma} = G^a_{ab\gamma} - \Gamma^b_{ab\gamma} - \Gamma^0_{ab\gamma},$$

$$0 T^a_{ab\gamma} = \Gamma^b_{ab\gamma} + \Gamma^0_{ab\gamma},$$

$$0 T^a_{ab\gamma} = -\delta^a_{\gamma} \Gamma^\gamma_{ab} - \delta^0_{\gamma} \Gamma^\gamma_{ab},$$

$$0 T^a_{ab\gamma} = \delta^a_{\gamma} \Gamma^\gamma_{ab} + \delta^0_{\gamma} \Gamma^\gamma_{ab}.$$

The curvature-torsion tensor in the canonical case is not equal to zero, but it contains zero components $0 T^a_{abc}$. 

Theorem 4.1. The canonical generalized bilinear connection without curvature-torsion is characterized by the following properties:

1) alternating Pfaffian derivatives $0 G^a_{ab\gamma}$ of the connection quasi-tensor $0 G^a_{ab\gamma}$ are formed by alternations of convolutions of the quasi-tensor $0 \Gamma^a_{bc\gamma}, 0 \Gamma^\gamma_{bc\gamma}$ of the quasi-tensors $\{ \Gamma^a_{bc\gamma}, \ \Gamma^\gamma_{bc\gamma} \}$ and the components of the simplest quasi-tensors $\Gamma^a_{bc\gamma}$ and $\Gamma^\gamma_{bc\gamma}$.

2) the Pfaffian derivatives $0 G^a_{ab\gamma}$ of the connection quasi-tensor $0 G^a_{ab\gamma}$ are formed by convolutions of the quasi-tensor $0 G^a_{ab\gamma}$ and the components of the simplest quasi-tensors $\Gamma^a_{bc\gamma}$ and $\Gamma^\gamma_{bc\gamma}$.

3) the Pfaffian derivatives $0 G^a_{ab\gamma}$ of the connection quasi-tensor $0 G^a_{ab\gamma}$ are the algebraic sum of convolutions of the quasi-tensor $0 G^a_{ab\gamma}$ itself with the simplest quasi-tensors $\Gamma^a_{bc\gamma}$ and $\Gamma^\gamma_{bc\gamma}$ and the components $\Gamma^a_{bc\gamma}$ and $\Gamma^\gamma_{bc\gamma}$.

Proof. When the curvature-torsion tensor $T$ vanishes we have

$$0 G^a_{ab\gamma} = \Gamma^a_{ab\gamma} G^a_{ab\gamma} + G^b_{ab\gamma} \Gamma^\gamma_{ab\gamma},$$

$$0 G^a_{ab\gamma} = \delta^a_{\gamma} \Gamma^\gamma_{ab} + \delta^0_{\gamma} \Gamma^\gamma_{ab}. $$
Theorem 5.1. The clothing geometrical object in the associated fibering.

Proof.

\[
\begin{align*}
\delta_{\alpha \beta} = 0 &= \frac{\partial}{\partial r} \Gamma_{\alpha \beta}^r - \frac{\partial}{\partial \mu} \Gamma_{\alpha \beta}^\mu + \frac{\partial}{\partial \lambda} \Gamma_{\alpha \beta}^\lambda - \frac{\partial}{\partial \alpha} \Gamma_{\lambda \
\mu}^\lambda_{\alpha \beta}. \\
\end{align*}
\]

\[\Box\]

5. The analog of Norden’s strong normalization

Let us perform the analog of Norden’s strong normalization [8, 26] of the space of centered planes under consideration by fields of the following geometric patterns:

1) an \((n - m - 1)\)-plane \(C_{n-m-1}\) having no common points with the plane \(P_m\),
2) an \((m-1)\)-plane \(N_{m-1}\) contained in the plane \(P_m\) and not passing through its center \(A\)

\[
\begin{align*}
\text{An analog of the Cartan’s plane } C_{n-m-1} \text{ can be defined by the points } B_a &= A_a + \lambda_a^x A_x + \lambda_a A; \text{ and the normal of the second kind of Norden } N_{m-1} \text{ can be defined by } B_a = A_a + \lambda_a A. \\
dB_a &= \theta B_a + (\alpha_a^e - a_b^e \lambda_b \omega^a + \alpha_a \beta^a + a_b^a \omega^a)B_b + (\beta_a \omega^a + \omega_a^b)B_a + (d \lambda_a - \lambda_a^b \omega^b + \lambda_a \omega^a - \lambda_a \lambda_a \beta^a + \mu_a \omega^a)A, \\
\text{where } \mu_a &= \lambda_a \lambda_a^a - \lambda_a; \\
dB_a &= \theta B_a + (\alpha_a^e - \lambda_a \beta^a + a_b^a \omega^a)B_b + (d \lambda_a - \lambda_a \beta^a + \alpha_a \omega^a - \lambda_a^b \omega^b + \alpha_b \omega^a - \lambda_a \omega^a + \lambda_a \lambda_a \beta^a + \mu_a \omega^a)B_a + \\
( - \lambda_b (d \lambda_a^a + \lambda_a^b \alpha_a^b - \alpha_a^b \lambda_a^b) + d \lambda_a - \lambda_b \beta^a + \lambda_a^b \alpha_a^b + \alpha_b + \lambda_a \mu_b \beta^a - \lambda_a \lambda_a \beta^a - \lambda_a \mu_b \omega^a + \lambda_a^b \mu_b \omega^a)A. \\
\Delta \lambda_a + \alpha_a &= 0, \\
\Delta \lambda_a + \lambda_a^b \alpha_a + \lambda_a &= 0. \\
\end{align*}
\]

Theorem 5.1. The clothing geometrical object \(\lambda\) is a quasi-tensor.

Theorem 5.2. The clothing of the space of centered planes by fields of equipment planes allows to define the connection in the associated fibering.

Proof.

\[
\begin{align*}
01_{\lambda k} &= -\delta^a_{b, \lambda} \alpha_c - \delta^c_{b, \lambda} \alpha_b, \\
01_{\mu k} &= \delta^a_{b, \mu} + \lambda_a^b \lambda_b, \\
01_{\nu k} &= \delta^c_{b, \nu} \alpha_b. \\
\end{align*}
\]

\[
\begin{align*}
01_{\alpha \beta} &= -\delta^a_{b, \alpha} \lambda_a - \delta^c_{b, \alpha} \lambda_b, \\
01_{\beta \gamma} &= \delta^a_{b, \beta} \mu_a - \delta^c_{b, \beta} \lambda, \\
01_{\nu \gamma} &= -\delta^a_{b, \nu} \lambda, \\
01_{\alpha} &= -\delta^a_{b, \alpha} \lambda, \\
01_{\nu} &= 0, \\
01_{\mu} &= 0. \\
\end{align*}
\]

\[\Box\]
Remark 5.3. For the plane and normal generalized connections [12] the clothing of the space of centered planes by fields of equipment planes allows to define the connection

\[
\begin{align*}
0^1_{\alpha} & = 0, & 0^1_{\beta} & = 0, & 0^1_{\alpha} & = \lambda^\alpha_{\alpha}, & 0^1_{\beta} & = \lambda^\beta_{\beta}, & 0^1_{\alpha} & = -\delta^\alpha_\beta \lambda_{\beta} - \delta^\beta_\lambda \lambda_{\alpha}, & 0^1_{\beta} & = \delta^\alpha_\beta \lambda_{\alpha} + \lambda^\alpha_{\alpha} \lambda_{\beta}, & 0^1_{\beta} & = \delta^\alpha_\beta \lambda_{\alpha}, \quad (11)
\end{align*}
\]

\[
\begin{align*}
0^1_{\alpha} & = 0, & 0^1_{\beta} & = 0, & 0^1_{\alpha} & = \lambda^\alpha_{\alpha}, & 0^1_{\beta} & = \lambda^\beta_{\beta}, & 0^1_{\alpha} & = -\delta^\alpha_\beta \lambda_{\beta} - \delta^\beta_\lambda \lambda_{\alpha}, & 0^1_{\beta} & = \delta^\alpha_\beta \lambda_{\alpha} + \lambda^\alpha_{\alpha} \lambda_{\beta}, & 0^1_{\beta} & = -\delta^\alpha_\beta \lambda_{\alpha}. \quad (12)
\end{align*}
\]

6. Covariant differentials and derivatives

The differential congruences of the object \( \lambda \) can be represented as equations

\[
\begin{align*}
\Delta \lambda_\alpha + \omega_\alpha &= \lambda_{\alpha, \gamma} \omega^\gamma + \lambda_{\alpha, \delta} \omega^\delta + \lambda_{\alpha, \beta} \omega^\beta, \\
\Delta \lambda^\alpha_\alpha + \omega^\alpha_\alpha &= \lambda^\alpha_{\alpha, \beta} \omega^\beta + \lambda_{\alpha, \beta} \omega^\beta + \lambda_{\alpha, \beta} \omega^\beta, \\
\Delta \lambda^\alpha_\beta + \lambda^\alpha_{\alpha, \beta} \omega^\beta &= \lambda_{\alpha, \beta} \omega^\beta + \lambda_{\alpha, \beta} \omega^\beta + \lambda_{\alpha, \beta} \omega^\beta.
\end{align*}
\]

We have the following congruences modulo basic forms

\[
\begin{align*}
\Delta \lambda_{\alpha, \beta} - \lambda_{\alpha, \beta} \omega^\beta + \lambda_{\alpha, \beta} \omega^\beta + \lambda_{\alpha, \beta} \omega^\beta &\equiv 0, \\
\Delta \lambda_{\alpha, \beta} + \lambda_{\alpha, \beta} \omega^\beta + \lambda_{\alpha, \beta} \omega^\beta + \lambda_{\alpha, \beta} \omega^\beta &\equiv 0, \\
\Delta \lambda_{\alpha, \beta} - \lambda_{\alpha, \beta} \omega^\beta + \lambda_{\alpha, \beta} \omega^\beta + \lambda_{\alpha, \beta} \omega^\beta &\equiv 0, \\
\Delta \lambda_{\alpha, \beta} - \lambda_{\alpha, \beta} \omega^\beta + \lambda_{\alpha, \beta} \omega^\beta + \lambda_{\alpha, \beta} \omega^\beta &\equiv 0, \\
\Delta \lambda_{\alpha, \beta} - \lambda_{\alpha, \beta} \omega^\beta + \lambda_{\alpha, \beta} \omega^\beta + \lambda_{\alpha, \beta} \omega^\beta &\equiv 0.
\end{align*}
\]

Covariant differentials

\[
\nabla \lambda_\alpha = d \lambda_\alpha - \lambda_{\alpha, \beta} \omega^\beta + \omega_\alpha, \quad \nabla \lambda^\alpha_\beta = d \lambda^\alpha_\beta + \lambda_{\alpha, \beta} \omega^\beta - \lambda_{\beta, \alpha} \omega^\alpha + \omega^\alpha_\beta.
\]

Covariant derivatives

\[
\begin{align*}
\nabla^\alpha \lambda_\alpha &= d \lambda_\alpha - \lambda_{\alpha, \beta} \omega^\beta + \omega_\alpha, \\
\nabla^\alpha \lambda^\alpha_\beta &= d \lambda^\alpha_\beta + \lambda_{\alpha, \beta} \omega^\beta - \lambda_{\beta, \alpha} \omega^\alpha + \omega^\alpha_\beta
\end{align*}
\]

Since covariant derivatives are not tensors, it is impossible to consider parallel displacements (cf. [27]) of the equipping planes \( C_{n-1} \) and \( N_{m-1} \).
7. The central projections

We give an geometrical interpretation of the generalized bilinear connection using central projections.

Theorem 7.1. The simple subobject \( \Gamma_1 = \{ \Gamma^a_{a'c}, \Gamma^a_{b'a}, \Gamma^a_{c'b} \} \) of connection object \( \Gamma \) is characterized by the central projection of the plane \( N_{m-1} + dN_{m-1} \) adjacent to the equipping plane \( N_{m-1} \) onto the initial plane \( N_{m-1} \) from the first-kind normal \( N_{m-1} = [C_{m-1}, A] \) (the center of projection), i.e.,

\[
\Gamma_1 : \quad N_{m-1} + dN_{m-1} \xrightarrow{N_{m-1}} N_{m-1}
\]

Proof. The plane \( N_{m-1} \) is determined by the points \( B_a = A_a + \lambda_a A \), whose displacements are determined by the expression

\[
dB_a = \theta B_a + \omega^\alpha_1 B_\alpha + (\lambda_a a^\alpha + a_\alpha^a)B_a + [\nabla \lambda_a - \lambda_a \lambda_b \lambda_c^a a^\alpha + \lambda_a \omega_\alpha^a + (\delta^\alpha_\beta \mu_a - \lambda_a \lambda_\beta^a a_\beta^a)]A
\]

where \( \theta = \theta + \mu_a a^\alpha - \lambda_a a^\alpha \).

This means that the projection \( N_{m-1} + dN_{m-1} \xrightarrow{N_{m-1}} N_{m-1} \) is performed. \( \square \)

Theorem 7.2. The simple subobject \( \Gamma_2 = \{ \Gamma^a_{c'a'b}, \Gamma^a_{b'a'}, \Gamma^a_{c'b'} \} \) of connection object \( \Gamma \) is characterized by the central projection of the plane \( C_{m-1} + dC_{m-1} \) adjacent to the equipping plane \( C_{m-1} \) onto the initial plane \( C_{m-1} \) from the generating plane \( P_m \) (the center of projection), i.e.,

\[
\Gamma_2 : \quad C_{m-1} + dC_{m-1} \xrightarrow{P_m} C_{m-1}
\]

Proof. The plane \( C_{m-1} \) is determined by the points \( B_a = A_a + \lambda_a^a A + \lambda_a A \), whose displacements are determined by the expression

\[
dB_a = \theta B_a + \omega^\alpha_1 B_\alpha + (\nabla \lambda_a^\alpha + \lambda_\beta^\alpha a_\beta^a + \lambda_a \omega_\alpha^a + (\delta^\alpha_\beta \mu_a - \lambda_a \lambda_\beta^a a_\beta^a)]A
\]

This means that the projection \( C_{m-1} + dC_{m-1} \xrightarrow{P_m} C_{m-1} \) is performed. \( \square \)

References