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The space of centered planes and generalized bilinear connection

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Abstract. We continue to study the space of centered planes in n-dimension projective space. We use E. Cartan's method of external forms and the group-theoretical method of G. F. Laptev to study the space of centered planes of the same dimension. These methods are successfully applied in physics.

In a generalized bundle, a bilinear connection associated with a space is given. The connection object contains two simplest subtensors and subquasi-tensors (four simplest and three simple subquasi-tensors).

The object field of this connection defines the objects of torsion S, curvature-torsion T, and curvature R. The curvature tensor contains six simplest and four simple subtensors, and curvature-torsion tensor contains three simplest and two simple subtensors.

The canonical case of a generalized bilinear connection is considered.

We realize the strong Lumiste's affine clothing (it is an analog of the strong Norde's normalization of the space of centered planes). Covariant differentials and covariant derivatives of the clothing quasi-tensor are described. The covariant derivatives do not form a tensor. We present a geometrical characterization of the generalized bilinear connection using mappings.

1. Introduction

The theory of connections is an important area of modern differential geometry [3, 4, 13, 14, 25, 28, 29, 36]. During its centuries-old history, it has gained popularity among geometers and successfully continues to develop today (see, e.g. [10, 11, 16–18, 21, 23, 24, 37]).

In 1918, H. Weyl introduced an affine connection without a metric. The affine connection with the metric was introduced by A. Einstein. Further, the theory was developed by such mathematicians as T. Levi-Civita (1917) and C. Ehresmann (1941) and popularized by B. A. Rosenfeld in [31]. In 1923, E. Cartan gave the general concept of an affine connection from the point of view of the theory of relativity. Then this connection developed geometrically, no longer in connection with applications.

Generalized affine connections were considered in book [19], where a relation was shown between a generalized affine connection and a linear connection.

We will use the method of external forms of E. Cartan [1, 2, 5, 22, 30, 32] and the theoretical-group method of G. F. Laptev to study the space of centered planes of the same dimension.

Generalized affine connections (plane and normal) for the space of centered planes were introduced by author in 2010 (see [6, 7]) and also considered in the paper [12]. In the present paper, we introduce a new generalized bilinear connection.

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2. Analytical apparatus and object of research

Projective space P_n can be represented as a quotient space $L_{n+1}/\sim$ of a linear space L_{n+1} with respect to equivalence (collinearity) $\sim$ of non-zero vectors, i.e.,

$$P_n = (L_{n+1} \setminus \{0\})/\sim .$$

Projective frame in the space P_n is a system formed by points $A_{I'}$, $\quad I' = 0, ..., n$, and an unit point E. In linear space L_{n+1} linearly independent vectors $e_{I'}$ correspond to the points $A_{I'}$, and a vector $e = \sum_{I'=0}^{n} e_{I'}$ corresponds to the point E. Moreover, these vectors are determined in the space L_{n+1} with an accuracy up to a common factor. The unit point is specified together with the basic points, although you do not have to mention it every time.

We will use the method of a moving frame $\{A, A_I\}$, $\quad I, ... = 1, ..., n$, the derivation formulae of the vertices of which are (see, e.g., [9])

$$dA = \theta A + \omega^I A_I, \qquad dA_I = \theta A_I + \omega_I^J A_J + \omega_I A, \tag{1}$$

where the form θ acts as a proportionality factor, d is the symbol of ordinary differentiation in the space P_n, and the structure forms ω^I, ω_J^I, ω_I of the projective group $GP(n)$, effectively acting on the space P_n, satisfy the following Cartan equations:

$$\begin{aligned} D\omega^I &= \omega^J \wedge \omega_J^I, \\ D\omega_J^I &= \omega_J^K \wedge \omega_K^I + \delta_J^I \omega_K \wedge \omega^K + \omega_J \wedge \omega^I, \\ D\omega_I &= \omega_I^J \wedge \omega_J, \end{aligned} \tag{2}$$

where D is the symbol of exterior differentiation.

In the projective space P_n a space Π of all centered m-dimensional planes P_m^* is considered. Vertices A and A_a, $\quad a, ... = 1, ..., m$, of the moving frame are placed on the centered plane, where vertex A is fixed as a centre. The forms ω^a, ω^α, ω_a^α $\quad (\alpha, ... = m+1, ..., n)$ are the basic forms of the space Π.

Remark 2.1. *The space Π is a differentiable manifold whose points are m-dimensional centered planes.*

We use the technique which is based on the Laptev – Lumiste method. This in turn requires knowledge of calculating external differential forms.

We will use the following terminology [34]:

Definition 2.2. *A substructure of a structure S is called simple if it is not a union of two substructures of the structure S.*

A simple substructure is called the simplest if it, in turn, does not have a substructure.

3. Generalized bilinear connection

Similarly to the generalized plane and normal affine connections [6, 7, 12] we consider a new generalized connection.

Definition 3.1. *A smooth manifold with structural equations*

$$D\omega^a = \omega^b \wedge \omega_b^a + \omega^\alpha \wedge \omega_\alpha^a,$$

$$D\omega^\alpha = \omega^\beta \wedge \omega_\beta^\alpha + \omega^a \wedge \omega_a^\alpha,$$

$$D\omega_a^\alpha = \omega_b^\beta \wedge \left(\delta_a^b \omega_\beta^\alpha - \delta_\beta^\alpha \omega_a^b\right) - \omega^\alpha \wedge \omega_a,$$

$$D\omega^a_b = \omega^c_b \wedge \omega^a_c - \omega^c \wedge \left(\delta^a_c \omega_b + \delta^a_b \omega_c\right) - \delta^a_b \omega^\alpha \wedge \omega_\alpha + \omega^\alpha_b \wedge \omega^a_\alpha,$$

$$D\omega^\alpha_\beta = \omega^\gamma_\beta \wedge \omega^\alpha_\gamma + \omega^a_\beta \wedge \omega^\alpha_a + \delta^\alpha_\beta \omega_a \wedge \omega^a + \left(\delta^\alpha_\beta \omega_\gamma + \delta^\alpha_\gamma \omega_\beta\right) \wedge \omega^\gamma$$

is called a generalized bundle of bilinear frames [20] and denoted by $A_{n^2-2k+[k]}$, where $k = m(n-m)$.

Remark 3.2. *The symbol k is bracketed in the notation $A_{n^2-2k+[k]}$ since k forms ω^α_a are the basic and fibre forms. Let's call them basic-fibre forms (see [35]).*

In the generalized bundle $A_{n^2-2k+[k]}$ we define a bilinear connection by the Laptev – Lumiste method [15, 33] using the forms of planar $\tilde{\omega}^a_b$, normal linear $\tilde{\omega}^\alpha_\beta$ connections, and the forms $\tilde{\omega}^\alpha_a$

$$\begin{aligned} \tilde{\omega}^\alpha_a &= \omega^\alpha_a - G^\alpha_{a\beta}\omega^\beta - G^\alpha_{ab}\omega^b - G^{\alpha b}_{a\beta}\omega^\beta_b, \\ \tilde{\omega}^a_b &= \omega^a_b - \Gamma^a_{b\alpha}\omega^\alpha - \Gamma^a_{bc}\omega^c - \Gamma^{ac}_{b\alpha}\omega^\alpha_c, \\ \tilde{\omega}^\alpha_\beta &= \omega^\alpha_\beta - \Gamma^\alpha_{\beta\gamma}\omega^\gamma - \Gamma^\alpha_{\beta a}\omega^a - \Gamma^{\alpha a}_{\beta\gamma}\omega^\gamma_a. \end{aligned} \tag{3}$$

We find the exterior differentials of the connection forms and apply the Cartan – Laptev theorem [15]

$$\begin{aligned} \Delta G^\alpha_{a\beta} - G^\alpha_{ab}\omega^b_\beta + (G^{\alpha b}_{a\beta} - \delta^\alpha_\beta\delta^b_a)\omega_b &= G^\alpha_{a\beta,\gamma}\omega^\gamma + G^\alpha_{a\beta,b}\omega^b + G^{\alpha,b}_{a\beta,\gamma}\omega^\gamma_b, \\ \Delta G^\alpha_{ab} &= G^\alpha_{ab,\beta}\omega^\beta + G^\alpha_{ab,c}\omega^c + G^{\alpha,c}_{ab,\beta}\omega^\beta_c, \\ \Delta G^{\alpha b}_{a\beta} &= G^{\alpha b}_{a\beta,\gamma}\omega^\gamma + G^{\alpha b}_{a\beta,c}\omega^c + G^{\alpha b,c}_{a\beta,\gamma}\omega^\gamma_c, \\ \Delta\Gamma^a_{bc} - \delta^a_b\omega_c - \delta^a_c\omega_b &= \Gamma^a_{bc,\alpha}\omega^\alpha + \Gamma^a_{bc,e}\omega^e + \Gamma^{a,e}_{bc,\alpha}\omega^\alpha_e, \\ \Delta\Gamma^a_{b\alpha} - \Gamma^a_{bc}\omega^c_\alpha + \Gamma^{ac}_{b\alpha}\omega_c - \delta^a_b\omega_\alpha &= \Gamma^a_{b\alpha,\beta}\omega^\beta + \Gamma^a_{b\alpha,c}\omega^c + \Gamma^{a,c}_{b\alpha,\beta}\omega^\beta_c, \\ \Delta\Gamma^{ac}_{b\alpha} + \delta^c_b\omega^a_\alpha &= \Gamma^{ac}_{b\alpha,\beta}\omega^\beta + \Gamma^{ac}_{b\alpha,e}\omega^e + \Gamma^{ac,e}_{b\alpha,\beta}\omega^\beta_e, \\ \Delta\Gamma^\alpha_{\beta a} - \delta^\alpha_\beta\omega_a &= \Gamma^\alpha_{\beta a,\gamma}\omega^\gamma + \Gamma^\alpha_{\beta a,b}\omega^b + \Gamma^{\alpha,b}_{\beta a,\gamma}\omega^\gamma_b, \\ \Delta\Gamma^\alpha_{\beta\gamma} - \Gamma^\alpha_{\beta a}\omega^a_\gamma + \Gamma^{\alpha a}_{\beta\gamma}\omega_a - \delta^\alpha_\beta\omega_\gamma - \delta^\alpha_\gamma\omega_\beta &= \Gamma^\alpha_{\beta\gamma,\mu}\omega^\mu + \Gamma^\alpha_{\beta\gamma,a}\omega^a + \Gamma^{\alpha,a}_{\beta\gamma,\mu}\omega^\mu_a, \\ \Delta\Gamma^{\alpha a}_{\beta\gamma} - \delta^\alpha_\gamma\omega^a_\beta &= \Gamma^{\alpha a}_{\beta\gamma,\mu}\omega^\mu + \Gamma^{\alpha a}_{\beta\gamma,b}\omega^b + \Gamma^{\alpha a,b}_{\beta\gamma,\mu}\omega^\mu_b, \end{aligned} \tag{4}$$

where the right-hand sides of the basic forms contain Pfaffian derivatives, and differential operator Δ acts by the law $\Delta G^\alpha_{a\beta} = dG^\alpha_{a\beta} + G^\gamma_{a\beta}\omega^\alpha_\gamma - G^\alpha_{b\beta}\omega^b_a - G^\alpha_{a\gamma}\omega^\gamma_\beta$.

The object of a generalized bilinear connection $\overset{B}{\Gamma} = \{G^\alpha_{a\beta},\ G^\alpha_{ab},\ G^{\alpha b}_{a\beta},\ \Gamma^a_{b\alpha},\ \Gamma^a_{bc},\ \Gamma^{ac}_{b\alpha},\ \Gamma^\alpha_{\beta\gamma},\ \Gamma^\alpha_{\beta a},\ \Gamma^{\alpha a}_{\beta\gamma}\}$ associated with the space Π of centered planes contains

- two simplest subtensors G^α_{ab} and $G^{\alpha b}_{a\beta}$ of the simple subquasi-tensor $\{G^\alpha_{a\beta},\ G^\alpha_{ab},\ G^{\alpha b}_{a\beta}\}$ of the connection $\overset{B}{\Gamma}$;
- four simplest subquasi-tensors $\Gamma^a_{bc},\ \Gamma^{ac}_{b\alpha},\ \Gamma^\alpha_{\beta a},\ \Gamma^{\alpha a}_{\beta\gamma}$;
- two simple subquasi-tensors $\{\Gamma^a_{bc},\ \Gamma^a_{b\alpha},\ \Gamma^{ac}_{b\alpha}\}$ and $\{\Gamma^\alpha_{\beta a},\ \Gamma^\alpha_{\beta\gamma},\ \Gamma^{\alpha a}_{\beta\gamma}\}$.

Structure equations of the basic forms can be written as

$$D\omega^\alpha = \omega^a \wedge \tilde{\omega}^\alpha_a + \omega^\beta \wedge \tilde{\omega}^\alpha_\beta + S^\alpha_{\beta\gamma}\omega^\beta \wedge \omega^\gamma + S^\alpha_{\beta a}\omega^\beta \wedge \omega^a + S^{\alpha a}_{\beta\gamma}\omega^\beta \wedge \omega^\gamma_a + S^\alpha_{ab}\omega^a \wedge \omega^b + S^{\alpha b}_{a\beta}\omega^a \wedge \omega^\beta_b,$$

$$D\omega^a = \omega^b \wedge \tilde{\omega}^a_b + \omega^\alpha \wedge \omega^a_\alpha + S^a_{b\alpha}\omega^b \wedge \omega^\alpha + S^a_{bc}\omega^b \wedge \omega^c + S^{ac}_{b\alpha}\omega^b \wedge \omega^\alpha_c,$$

where the components of torsion object S are found according to the formulae

$$S^\alpha_{\beta\gamma} = \Gamma^\alpha_{[\beta\gamma]}, \qquad S^\alpha_{\beta a} = \Gamma^\alpha_{\beta a} - G^\alpha_{a\beta}, \qquad S^{\alpha a}_{\beta\gamma} = \Gamma^{\alpha a}_{\beta\gamma}, \qquad S^\alpha_{ab} = G^\alpha_{[ab]},$$

$$S^{\alpha b}_{a\beta} = G^{\alpha b}_{a\beta}, \qquad S^{a}_{b\alpha} = \Gamma^{a}_{b\alpha}, \qquad S^{a}_{bc} = \Gamma^{a}_{[bc]}, \qquad S^{ac}_{b\alpha} = \Gamma^{ac}_{b\alpha},$$

where square brackets mean antisymmetrization on extreme indexes. For example, $\Gamma^{\alpha}_{[\beta\gamma]} = \dfrac{1}{2}\left(\Gamma^{\alpha}_{\beta\gamma} - \Gamma^{\alpha}_{\gamma\beta}\right)$.

Components of torsion object S satisfy differential congruences modulo the basic forms

$$\Delta S^{\alpha}_{\beta\gamma} + S^{\alpha a}_{[\beta\gamma]}\omega_a - \Gamma^{\alpha}_{[\beta a}\omega^{a}_{\gamma]} \equiv 0, \qquad \Delta S^{\alpha}_{\beta a} - S^{\alpha b}_{a\beta}\omega_b + G^{\alpha}_{ab}\omega^{b}_{\beta} \equiv 0,$$

$$\Delta S^{\alpha a}_{\beta\gamma} - \delta^{\alpha}_{\gamma}\omega^{a}_{\beta} \equiv 0, \qquad \Delta S^{\alpha}_{ab} \equiv 0, \qquad \Delta S^{\alpha b}_{a\beta} \equiv 0,$$

$$\Delta S^{a}_{b\alpha} + S^{ac}_{b\alpha}\omega_c - \Gamma^{a}_{bc}\omega^{c}_{\alpha} - \delta^{a}_{b}\omega_{\alpha} \equiv 0, \qquad \Delta S^{a}_{bc} \equiv 0,$$

$$\Delta S^{ac}_{b\alpha} + \delta^{c}_{b}\omega^{a}_{\alpha} \equiv 0.$$

The torsion object S is a geometric object (quasi-tensor) only in conjunction with the bilinear connection object $\overset{B}{\Gamma}$.

Taking into account the differential equations (4) in the structural equations of the connection forms (3), we obtain

$$D\tilde{\omega}^{\alpha}_{a} = \tilde{\omega}^{b}_{a} \wedge \tilde{\omega}^{\alpha}_{b} + \tilde{\omega}^{\beta}_{a} \wedge \tilde{\omega}^{\alpha}_{\beta} + T^{\alpha}_{a\beta\gamma}\omega^{\beta} \wedge \omega^{\gamma} + T^{\alpha}_{a\beta b}\omega^{\beta} \wedge \omega^{b} +$$

$$+T^{\alpha b}_{a\beta\gamma}\omega^{\beta} \wedge \omega^{\gamma}_{b} + T^{\alpha}_{abc}\omega^{b} \wedge \omega^{c} + T^{\alpha c}_{ab\beta}\omega^{b} \wedge \omega^{\beta}_{c} + T^{\alpha bc}_{a\beta\gamma}\omega^{\beta}_{b} \wedge \omega^{\gamma}_{c},$$

$$D\tilde{\omega}^{a}_{b} = \tilde{\omega}^{c}_{b} \wedge \tilde{\omega}^{a}_{c} + R^{a}_{b\beta\gamma}\omega^{\beta} \wedge \omega^{\gamma} + R^{a}_{bc\alpha}\omega^{c} \wedge \omega^{\alpha} +$$

$$+R^{ac}_{b\alpha\beta}\omega^{\alpha} \wedge \omega^{\beta}_{c} + R^{a}_{bce}\omega^{c} \wedge \omega^{e} + R^{ae}_{bc\alpha}\omega^{c} \wedge \omega^{\alpha}_{e} + R^{\alpha ce}_{b\alpha\beta}\omega^{\alpha}_{c} \wedge \omega^{\beta}_{e},$$

$$D\tilde{\omega}^{\alpha}_{\beta} = \tilde{\omega}^{\gamma}_{\beta} \wedge \tilde{\omega}^{\alpha}_{\gamma} + R^{\alpha}_{\beta\gamma\mu}\omega^{\gamma} \wedge \omega^{\mu} + R^{\alpha}_{\beta\gamma a}\omega^{\gamma} \wedge \omega^{a} +$$

$$+R^{\alpha a}_{\beta\gamma\mu}\omega^{\gamma} \wedge \omega^{\mu}_{a} + R^{\alpha}_{\beta ab}\omega^{a} \wedge \omega^{b} + R^{\alpha b}_{\beta a\gamma}\omega^{a} \wedge \omega^{\gamma}_{b} + R^{\alpha ab}_{\beta\gamma\mu}\omega^{\gamma}_{a} \wedge \omega^{\mu}_{b}.$$

Curvature-torsion object T has the following components:

$$T^{\alpha}_{a\beta\gamma} = G^{\alpha}_{a[\beta,\gamma]} - \Gamma^{b}_{a[\beta}G^{\alpha}_{b\gamma]} - G^{\mu}_{a[\beta}\Gamma^{\alpha}_{\mu\gamma]},$$

$$T^{\alpha}_{a\beta b} = 2\left(G^{\alpha}_{a[\beta,b]} - \Gamma^{c}_{a[\beta}G^{\alpha}_{cb]} - G^{\gamma}_{a[\beta}\Gamma^{\alpha}_{\gamma b]}\right),$$

$$T^{\alpha b}_{a\beta\gamma} = G^{\alpha,b}_{a\beta,\gamma} - G^{\alpha b}_{a\gamma,\beta} - \Gamma^{c}_{a\beta}G^{\alpha b}_{c\gamma} + \Gamma^{cb}_{a\gamma}G^{\alpha}_{c\beta} + \delta^{\alpha}_{\gamma}\Gamma^{b}_{a\beta} - \delta^{b}_{a}\Gamma^{\alpha}_{\gamma\beta} - G^{\mu}_{a\beta}\Gamma^{\alpha b}_{\mu\gamma} + G^{\mu b}_{a\gamma}\Gamma^{\alpha}_{\mu\beta},$$

$$T^{\alpha}_{abc} = G^{\alpha}_{a[b,c]} - \Gamma^{e}_{a[b}G^{\alpha}_{ec]} - G^{\beta}_{a[b}\Gamma^{\alpha}_{\beta c]},$$

$$T^{\alpha c}_{ab\beta} = 2G^{\alpha,c}_{a[b,\beta]} + \Gamma^{ec}_{a\beta}G^{\alpha}_{eb} - \Gamma^{e}_{ab}G^{\alpha c}_{e\beta} + G^{\gamma c}_{a\beta}\Gamma^{\alpha}_{\gamma b} - G^{\gamma}_{ab}\Gamma^{\alpha c}_{\gamma\beta} + \delta^{\alpha}_{\beta}\Gamma^{c}_{ab} - \delta^{c}_{a}\Gamma^{\alpha}_{\beta b} - \delta^{c}_{b}G^{\alpha}_{a\beta},$$

$$T^{\alpha bc}_{a\beta\gamma} = G^{\alpha}_{a}[{}^{b,c}_{\beta,\gamma}] - \Gamma^{e}_{a}[{}^{b}_{\beta}G^{\alpha c}_{e\gamma}] - G^{\mu}_{a}[{}^{b}_{\beta}\Gamma^{\alpha c}_{\mu\gamma}] - \delta^{\alpha}_{\lfloor\beta}\Gamma^{\lceil bc}_{a\gamma}] + \delta^{\lceil b}_{a}\Gamma^{\alpha c}_{\lfloor\beta\gamma}].$$

The components of the curvature object R can be find

$$R^{a}_{bce} = \Gamma^{a}_{b[c,e]} - \Gamma^{d}_{b[c}\Gamma^{a}_{de]},$$

$$R^{a}_{bc\alpha} = \Gamma^{a}_{bc,\alpha} - \Gamma^{a}_{b\alpha,c} + \Gamma^{e}_{b\alpha}\Gamma^{a}_{ec} - \Gamma^{e}_{bc}\Gamma^{a}_{e\alpha},$$

$$R^{ad}_{bc\alpha} = \Gamma^{a,d}_{bc,\alpha} - \Gamma^{ad}_{b\alpha,c} + \Gamma^{ed}_{b\alpha}\Gamma^{a}_{ec} - \Gamma^{e}_{bc}\Gamma^{ad}_{e\alpha} - \delta^{d}_{c}\Gamma^{a}_{b\alpha},$$

$$R^{a}_{b\alpha\beta} = \Gamma^{a}_{b[\alpha,\beta]} - \Gamma^{c}_{b[\alpha}\Gamma^{a}_{c\beta]},$$

$$R^{ac}_{b\alpha\beta} = \Gamma^{c,b}_{b\alpha,\beta} - \Gamma^{ac}_{b\beta,\alpha} + \Gamma^{dc}_{b\beta}\Gamma^{a}_{d\alpha} - \Gamma^{d}_{b\alpha}\Gamma^{ac}_{d\beta},$$

$$R^{acd}_{b\alpha\beta} = \Gamma^{a}_{b}[{}^{c,d}_{\alpha,\beta}] - \Gamma^{e}_{b}[{}^{c}_{\alpha}\Gamma^{ad}_{e\beta}], \qquad R^{\alpha}_{\beta ab} = \Gamma^{\alpha}_{\beta[a,b]} + \Gamma^{\alpha}_{\gamma[a}\Gamma^{\gamma}_{\beta b]},$$

$$R^{\alpha}_{\beta\gamma a} = \Gamma^{\alpha}_{\beta\gamma,a} - \Gamma^{\alpha}_{\beta a,\gamma} + \Gamma^{\alpha}_{\mu\gamma}\Gamma^{\mu}_{\beta a} - \Gamma^{\alpha}_{\mu a}\Gamma^{\mu}_{\beta\gamma},$$
$$R^{\alpha b}_{\beta a\gamma} = \Gamma^{\alpha,b}_{\beta a,\gamma} - \Gamma^{\alpha b}_{\beta\gamma,a} + \Gamma^{\alpha}_{\mu a}\Gamma^{\mu b}_{\beta\gamma} - \Gamma^{\alpha b}_{\mu\gamma}\Gamma^{\mu}_{\beta a} - \delta^{b}_{a}\Gamma^{\alpha}_{\beta\gamma},$$
$$R^{\alpha}_{\beta\gamma\mu} = \Gamma^{\alpha}_{\beta[\gamma,\mu]} + \Gamma^{\alpha}_{\eta[\gamma}\Gamma^{\eta}_{\beta\mu]},$$
$$R^{\alpha a}_{\beta\gamma\mu} = \Gamma^{\alpha,a}_{\beta[\gamma,\mu]} - \Gamma^{\alpha a}_{\beta\gamma,\mu} + \Gamma^{\alpha}_{\eta\gamma}\Gamma^{\eta a}_{\beta\mu} - \Gamma^{\alpha a}_{\eta\mu}\Gamma^{\eta}_{\beta\gamma},$$
$$R^{\alpha ab}_{\beta\gamma\mu} = \Gamma^{\alpha}_{\beta}[^{a,b}_{\gamma,\mu}] + \Gamma^{\alpha}_{\eta}[^{a}_{\gamma}\Gamma^{\eta b}_{\beta\mu}].$$

Differential congruences for R have the form

$$\Delta R^{a}_{bce} \equiv 0, \qquad \Delta R^{a}_{bc\alpha} - 2R^{a}_{bcd}\omega^{d}_{\alpha} + R^{ad}_{bc\alpha}\omega_{d} \equiv 0, \qquad \Delta R^{ad}_{bc\alpha} \equiv 0,$$
$$\Delta R^{a}_{b\alpha\beta} + R^{a}_{bc[\alpha}\omega^{c}_{\beta]} + R^{ac}_{b[\alpha\beta]}\omega_{c} \equiv 0,$$
$$\Delta R^{ac}_{b\alpha\beta} - 2R^{acd}_{b\alpha\beta}\omega_{d} - R^{ac}_{bd\beta}\omega^{d}_{\alpha} \equiv 0,$$
$$\Delta R^{acd}_{b\alpha\beta} \equiv 0, \qquad \Delta R^{\alpha}_{\beta ab} \equiv 0,$$
$$\Delta R^{\alpha}_{\beta\gamma a} + 2R^{\alpha}_{\beta ab}\omega^{b}_{\gamma} - R^{\alpha b}_{\beta a\gamma}\omega_{b} \equiv 0, \qquad \Delta R^{\alpha b}_{\beta a\gamma} \equiv 0,$$
$$\Delta R^{\alpha}_{\beta\gamma\mu} - R^{\alpha}_{\beta[\gamma a}\omega^{a}_{\mu]} + R^{\alpha a}_{\beta[\gamma\mu]}\omega_{a} \equiv 0,$$
$$\Delta R^{\alpha a}_{\beta\gamma\mu} - 2R^{\alpha ab}_{\beta\mu\gamma}\omega_{b} - R^{\alpha a}_{\beta b\mu}\omega^{b}_{\gamma} \equiv 0, \qquad \Delta R^{\alpha ab}_{\beta\gamma\mu} \equiv 0.$$

From the previous congruences it can be seen that the curvature object R is a tensor containing

- six simplest subtensors R^{a}_{bce}, $R^{ad}_{bc\alpha}$, $R^{acd}_{b\alpha\beta}$, $R^{\alpha}_{\beta ab}$, $R^{\alpha b}_{\beta a\gamma}$, $R^{\alpha ab}_{\beta\gamma\mu}$;
- four simple subtensors $\left\{R^{a}_{bc\alpha}, R^{a}_{bcd}, R^{ad}_{bc\alpha}\right\}$, $\left\{R^{ac}_{b\alpha\beta}, R^{acd}_{b\alpha\beta}, R^{ac}_{bd\beta}\right\}$, $\left\{R^{\alpha}_{\beta\gamma a}, R^{\alpha}_{\beta ab}, R^{\alpha b}_{\beta a\gamma}\right\}$, $\left\{R^{\alpha a}_{\beta\gamma\mu}, R^{\alpha ab}_{\beta\mu\gamma}, R^{\alpha a}_{\beta b\mu}\right\}$.

The result of a prolongation the differential equations (4)

$$\Delta G^{\alpha}_{a\beta,\gamma} - \left(\delta^{\mu}_{\gamma}G^{\alpha}_{a\beta,b} + \delta^{\mu}_{\beta}G^{\alpha}_{ab,\gamma}\right)\omega^{b}_{\mu} + \left(\delta^{\mu}_{\beta}G^{\alpha}_{a\gamma} + \delta^{\mu}_{\gamma}G^{\alpha}_{a\beta} - \delta^{\alpha}_{\gamma}G^{\mu}_{a\beta}\right)\omega_{\mu} + \left(G^{\alpha,b}_{a\beta,\gamma} + G^{\alpha b}_{a\beta,\gamma}\right)\omega_{b} \equiv 0,$$
$$\Delta G^{\alpha}_{a\beta,b} - G^{\alpha}_{ac,b}\omega^{c}_{\beta} + G^{\alpha}_{ab}\omega_{\beta} + \left(G^{\alpha c}_{a\beta,b} + \delta^{c}_{a}G^{\alpha}_{b\beta} + \delta^{c}_{b}G^{\alpha}_{a\beta}\right)\omega_{c} \equiv 0,$$
$$\Delta G^{\alpha,b}_{a\beta,\gamma} - \left(\delta^{\alpha}_{\gamma}\delta^{b}_{c}G^{\mu}_{a\beta} + \delta^{\mu}_{\gamma}\delta^{b}_{a}G^{\alpha}_{c\beta} + \delta^{\mu}_{\beta}G^{\alpha,b}_{ac,\gamma} - \delta^{\mu}_{\beta}\delta^{b}_{c}G^{\alpha}_{a\gamma}\right)\omega^{c}_{\mu} + \left(G^{\alpha b}_{a\beta} - \delta^{\alpha}_{\beta}\delta^{b}_{a}\right)\omega_{\gamma} + G^{\alpha c,b}_{a\beta,\gamma}\omega_{c} \equiv 0,$$
$$\Delta G^{\alpha}_{ab,\beta} - G^{\alpha}_{ab,c}\omega^{c}_{\beta} + \left(\delta^{\gamma}_{\beta}G^{\alpha}_{ab} - \delta^{\alpha}_{\beta}G^{\gamma}_{ab}\right)\omega_{\gamma} + G^{\alpha,c}_{ab,\beta}\omega_{c} \equiv 0,$$
$$\Delta G^{\alpha}_{ab,c} + \left(\delta^{e}_{a}G^{\alpha}_{cb} + \delta^{e}_{b}G^{\alpha}_{ac} + \delta^{e}_{c}G^{\alpha}_{ab}\right)\omega_{e} \equiv 0,$$
$$\Delta G^{\alpha,c}_{ab,\beta} - \left(\delta^{\alpha}_{\beta}\delta^{e}_{c}G^{\gamma}_{ab} + \delta^{\gamma}_{\beta}\delta^{c}_{a}G^{\alpha}_{eb} + \delta^{\gamma}_{\beta}\delta^{c}_{b}G^{\alpha}_{ae}\right)\omega^{e}_{\gamma} \equiv 0,$$
$$\Delta G^{\alpha b}_{a\beta,\gamma} - G^{\alpha b}_{a\beta,c}\omega^{c}_{\gamma} + \left(\delta^{\mu}_{\beta}G^{\alpha b}_{a\gamma} - \delta^{\alpha}_{\gamma}G^{\mu b}_{a\beta}\right)\omega_{\mu} + G^{\alpha b,c}_{a\beta,\gamma}\omega_{c} \equiv 0,$$
$$\Delta G^{\alpha b}_{a\beta,c} + \left(\delta^{e}_{a}G^{\alpha b}_{c\beta} - \delta^{b}_{c}G^{\alpha e}_{a\beta}\right)\omega_{e} \equiv 0,$$
$$\Delta G^{\alpha b,c}_{a\beta,\gamma\mu} + \left(\delta^{\mu}_{\gamma}\delta^{b}_{e}G^{\alpha c}_{a\beta} + \delta^{\mu}_{\beta}\delta^{c}_{e}G^{\alpha b}_{a\gamma} - \delta^{\alpha}_{\gamma}\delta^{c}_{e}G^{\mu b}_{a\beta} - \delta^{\mu}_{\gamma}\delta^{c}_{a}G^{\alpha b}_{e\beta}\right)\omega^{e}_{\mu} \equiv 0.$$

The components of the curvature-torsion object T satisfy the following differential congruences modulo basic forms:

$$\Delta T^{\alpha}_{a\beta\gamma} + \delta^{\mu}_{[\beta}T^{\alpha}_{a\gamma]b}\omega^{b}_{\mu} + T^{\alpha b}_{a[\beta\gamma]}\omega_{b} \equiv 0,$$
$$\Delta T^{\alpha}_{a\beta b} + 2T^{\alpha}_{abc}\omega^{c}_{\beta} - T^{\alpha c}_{ab\beta}\omega_{c} \equiv 0,$$
$$\Delta T^{\alpha b}_{a\beta\gamma} - T^{\alpha b}_{ac\gamma}\omega^{c}_{\beta} - T^{\alpha bc}_{a\gamma\beta}\omega_{c} \equiv 0,$$
$$\Delta T^{\alpha}_{abc} \equiv 0, \ \Delta T^{\alpha c}_{ab\beta} \equiv 0, \ \Delta T^{\alpha bc}_{a\beta\gamma} \equiv 0.$$

This imply the following Theorem.

Theorem 3.3. *The curvature-torsion object of the generalized bilinear connection is a tensor containing three simplest subtensors* T^{α}_{abc}, $T^{\alpha c}_{ab\beta}$, $T^{\alpha bc}_{a\beta\gamma}$ *and two simple subtensors* $\left\{T^{\alpha}_{abc},\ T^{\alpha c}_{ab\beta},\ T^{\alpha}_{a\beta b}\right\}$, $\left\{T^{\alpha c}_{ab\beta},\ T^{\alpha bc}_{a\beta\gamma},\ T^{\alpha b}_{a\beta\gamma}\right\}$.

4. The canonical case

Let's consider the case when $G^{\alpha}_{ab} = 0$ and $G^{\alpha b}_{a\beta} = 0$. From these conditions we have $\tilde{\omega}^{\alpha}_{a} = \omega^{\alpha}_{a} - G^{\alpha}_{a\beta}\omega^{\beta}$ and the left-hand sides of the 2nd and the 3d equations (4) are identically vanishing, then the 1st equations (4) will be simplified

$$\Delta\overset{0}{G}{}^{\alpha}_{a\beta} - \delta^{\alpha}_{\beta}\omega_a = \overset{0}{G}{}^{\alpha}_{a\beta,\gamma}\omega^{\gamma} + \overset{0}{G}{}^{\alpha}_{a\beta,b}\omega^{b} + \overset{0}{G}{}^{\alpha b}_{a\beta,\gamma}\omega^{\gamma}_{b}.$$

In the canonical case, the quasi-tensor G of the generalized bilinear connection is reduced to the quasi-tensor $\overset{0}{G}{}^{\alpha}_{a\beta}$, while the connection object is simplified $\overset{B0}{\Gamma} = \{\overset{0}{G}{}^{\alpha}_{a\beta},\ 0,\ 0,\ \Gamma^{a}_{b\alpha},\ \Gamma^{a}_{bc},\ \Gamma^{ac}_{b\alpha},\ \Gamma^{\alpha}_{\beta\gamma},\ \Gamma^{\alpha}_{\beta a},\ \Gamma^{\alpha a}_{\beta\gamma}\}$.

Substituting $G^{\alpha}_{ab} = 0$, $\quad G^{\alpha b}_{a\beta} = 0$ into the expressions for the components of the curvature-torsion tensor they will take the form

$$\overset{0}{T}{}^{\alpha}_{a\beta\gamma} = \overset{0}{G}{}^{\alpha}_{a[\beta,\gamma]} - \Gamma^{b}_{a[\beta}\overset{0}{G}{}^{\alpha}_{b\gamma]} - \overset{0}{G}{}^{\mu}_{a[\beta}\Gamma^{\alpha}_{\mu\gamma]},$$

$$\overset{0}{T}{}^{\alpha}_{a\beta b} = \overset{0}{G}{}^{\alpha}_{a\beta,b} + \Gamma^{c}_{ab}\overset{0}{G}{}^{\alpha}_{c\beta} - \overset{0}{G}{}^{\gamma}_{a\beta}\Gamma^{\alpha}_{\gamma b},$$

$$\overset{0}{T}{}^{\alpha b}_{a\beta\gamma} = -\overset{0}{G}{}^{\alpha b}_{a\gamma,\beta} + \Gamma^{cb}_{a\gamma}\overset{0}{G}{}^{\alpha}_{c\beta} + \delta^{\alpha}_{\gamma}\Gamma^{b}_{a\beta} - \delta^{b}_{a}\Gamma^{\alpha}_{\gamma\beta} - \overset{0}{G}{}^{\mu}_{a\beta}\Gamma^{\alpha b}_{\mu\gamma},$$

$$\overset{0}{T}{}^{\alpha}_{abc} = 0,$$

$$\overset{0}{T}{}^{\alpha c}_{ab\beta} = \delta^{\alpha}_{\beta}\Gamma^{c}_{ab} - \delta^{c}_{a}\Gamma^{\alpha}_{\beta b} - \delta^{c}_{b}\overset{0}{G}{}^{\alpha}_{a\beta},$$

$$\overset{0}{T}{}^{\alpha bc}_{a\beta\gamma} = -\delta^{\alpha}_{\lfloor\beta}\Gamma^{\lceil bc}_{a\gamma\rceil} + \delta^{\lceil b}_{a}\Gamma^{\alpha c\rceil}_{\lfloor\beta\gamma\rfloor}.$$

The curvature-torsion tensor in the canonical case is not equal to zero, but it contains zero components $\overset{0}{T}{}^{\alpha}_{abc}$.

Theorem 4.1. *The canonical generalized bilinear connection without curvature-torsion is characterized by the following properties:*

1) alternating bilinear Pfaffian derivatives $\overset{0}{G}{}^{\alpha}_{a[\beta,\gamma]}$ *of the connection quasi-tensor* $\overset{0}{G}{}^{\alpha}_{a\beta}$ *are formed by alternations of convolutions of the quasi-tensor* $\overset{0}{G}{}^{\alpha}_{b\gamma}$ *and subobjects* $\Gamma^{a}_{b\alpha}$, $\Gamma^{\alpha}_{\beta\gamma}$ *of the quasi-tensors* $\{\Gamma^{a}_{bc},\ \Gamma^{a}_{b\alpha},\ \Gamma^{ac}_{b\alpha}\}$ *and* $\{\Gamma^{\alpha}_{\beta a},\ \Gamma^{\alpha}_{\beta\gamma},\ \Gamma^{\alpha a}_{\beta\gamma}\}$ *of the bilinear connection;*

2) the Pfaffian derivatives $\overset{0}{G}{}^{\alpha}_{a\beta,b}$ *of the connection quasi-tensor* $\overset{0}{G}{}^{\alpha}_{a\beta}$ *are formed by convolutions of the quasi-tensor* $\overset{0}{G}{}^{\alpha}_{a\beta}$ *and the components of the simplest quasi-tensors* Γ^{a}_{bc} *and* $\Gamma^{\alpha}_{\beta a}$*;*

3) the Pfaffian derivatives $\overset{0}{G}{}^{\alpha b}_{a\gamma,\beta}$ *of the connection quasi-tensor* $\overset{0}{G}{}^{\alpha}_{a\beta}$ *are the algebraic sum of convolutions of the quasi-tensor* $\overset{0}{G}{}^{\alpha}_{a\beta}$ *itself with the simplest quasi-tensors* $\Gamma^{ac}_{b\alpha}$ *and* $\Gamma^{\alpha a}_{\beta\gamma}$ *and the components* $\Gamma^{b}_{a\beta}$ *and* $\Gamma^{\alpha}_{\gamma\beta}$.

Proof. When the curvature-torsion tensor T vanishes we have

$$\overset{0}{G}{}^{\alpha}_{a[\beta,\gamma]} = \Gamma^{b}_{a[\beta}\overset{0}{G}{}^{\alpha}_{b\gamma]} + \overset{0}{G}{}^{\mu}_{a[\beta}\Gamma^{\alpha}_{\mu\gamma]},$$

$$\overset{0}{G}{}^{\alpha}_{a\beta,b} = -\overset{0}{G}{}^{\alpha}_{c\beta}\Gamma^{c}_{ab} + \overset{0}{G}{}^{\gamma}_{a\beta}\Gamma^{\alpha}_{\gamma b},$$

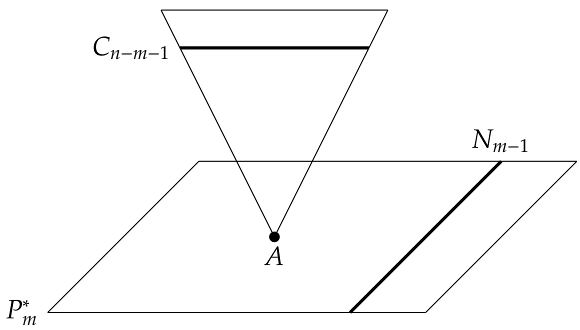
$$\overset{0}{G}{}^{\alpha b}_{a\gamma,\beta} = \overset{0}{G}{}^{\alpha}_{c\beta}\Gamma^{cb}_{a\gamma} - \overset{0}{G}{}^{\mu}_{a\beta}\Gamma^{\alpha b}_{\mu\gamma} + \delta^{\alpha}_{\gamma}\Gamma^{b}_{a\beta} - \delta^{b}_{a}\Gamma^{\alpha}_{\gamma\beta}.$$

□

5. The analog of Norden's strong normalization

Let us perform the analog of Norden's strong normalization [8, 26] of the space of centered planes under consideration by fields of the following geometric patterns:

1) an $(n-m-1)$-plane C_{n-m-1} having no common points with the plane P^*_m;
2) an $(m-1)$-plane N_{m-1} contained in the plane P^*_m and not passing through its center A



An analog of the Cartan's plane C_{n-m-1} can be defined by the points $B_\alpha = A_\alpha + \lambda^a_\alpha A_a + \lambda_\alpha A$; and the normal of the second kind of Norden N_{m-1} can be defined by $B_a = A_a + \lambda_a A$.

$$dB_a = \theta B_a + (\omega^b_a - \lambda^b_\alpha\lambda_a\omega^\alpha + \lambda_a\omega^b - \lambda^b_\alpha\omega^\alpha_a)B_b + (\lambda_a\omega^\alpha + \omega^\alpha_a)B_\alpha + (d\lambda_a - \lambda_b\omega^b_a + \omega_a + \lambda_a\mu_\alpha\omega^\alpha - \lambda_a\lambda_b\omega^b + \mu_\alpha\omega^\alpha_a)A, \quad (5)$$

where $\mu_\alpha = \lambda_a\lambda^a_\alpha - \lambda_\alpha$;

$$\begin{aligned} dB_\alpha = \theta B_\alpha + (\omega^\beta_\alpha + \lambda_\alpha\omega^\beta + \lambda^a_\alpha\omega^\beta_a)B_\beta + (d\lambda^a_\alpha + \lambda^b_\alpha\omega^a_b - \lambda^a_\beta\omega^\beta_\alpha + \omega^a_\alpha - \lambda^a_\beta\lambda_\alpha\omega^\beta + \lambda_\alpha\omega^a - \lambda^a_\beta\lambda^b_\alpha\omega^\beta_b)B_a + \\ \Big(- \lambda_a(d\lambda^a_\alpha + \lambda^b_\alpha\omega^a_b - \lambda^a_\beta\omega^\beta_\alpha + \omega^a_\alpha) + d\lambda_\alpha - \lambda_\beta\omega^\beta_\alpha + \lambda^a_\alpha\omega_a + \omega_\alpha + \lambda_\alpha\mu_\beta\omega^\beta - \lambda_a\lambda_\alpha\omega^a + \lambda^a_\alpha\mu_\beta\omega^\beta_a\Big)A. \end{aligned} \quad (6)$$

$$\begin{gathered} \Delta\lambda_a + \omega_a \equiv 0, \\ \Delta\lambda^a_\alpha + \omega^a_\alpha \equiv 0, \qquad \Delta\lambda_\alpha + \lambda^a_\alpha\omega_a + \omega_\alpha \equiv 0. \end{gathered} \quad (7)$$

Theorem 5.1. *The clothing geometrical object λ is a quasi-tensor.*

Theorem 5.2. *The clothing of the space of centered planes by fields of equipment planes allows to define the connection in the associated fibering.*

Proof.

$$\overset{01}{\Gamma}{}^{a}_{bc} = -\delta^a_b\lambda_c - \delta^a_c\lambda_b, \qquad \overset{01}{\Gamma}{}^{a}_{b\alpha} = \delta^a_b\mu_\alpha + \lambda^a_\alpha\lambda_b, \qquad \overset{01}{\Gamma}{}^{ac}_{b\alpha} = \delta^c_b\lambda^a_\alpha, \quad (8)$$

$$\overset{01}{\Gamma}{}^{\alpha}_{\beta a} = -\delta^\alpha_\beta\lambda_a, \qquad \overset{01}{\Gamma}{}^{\alpha}_{\beta\gamma} = \delta^\alpha_\beta\mu_\gamma - \delta^\alpha_\gamma\lambda_\beta, \qquad \overset{01}{\Gamma}{}^{\alpha a}_{\beta\gamma} = -\delta^\alpha_\gamma\lambda^a_\beta, \quad (9)$$

$$\overset{01}{G}{}^{\alpha}_{a\beta} = -\delta^\alpha_\beta\lambda_a, \qquad \overset{01}{G}{}^{\alpha}_{ab} = 0, \qquad \overset{01}{G}{}^{\alpha b}_{a\beta} = 0. \quad (10)$$

□

Remark 5.3. *For the plane and normal generalized connections [12] the clothing of the space of centered planes by fields of equipment planes allows to define the connection*

$$\overset{01}{C}{}^{a}_{b} = 0, \quad \overset{01}{C}{}^{a}_{\alpha} = \lambda^{a}_{\alpha}, \quad \overset{01}{C}{}^{ab}_{\alpha} = 0, \quad \overset{01}{\Gamma}{}^{a}_{bc} = -\delta^{a}_{b}\lambda_{c} - \delta^{a}_{c}\lambda_{b}, \quad \overset{01}{\Gamma}{}^{a}_{b\alpha} = \delta^{a}_{b}\mu_{\alpha} + \lambda^{a}_{\alpha}\lambda_{b}, \quad \overset{01}{\Gamma}{}^{ac}_{b\alpha} = \delta^{c}_{b}\lambda^{a}_{\alpha}; \tag{11}$$

$$\overset{01}{L}{}^{\alpha}_{a} = 0, \quad \overset{01}{L}{}^{\alpha}_{\beta} = 0, \quad \overset{01}{L}{}^{\alpha a}_{\beta} = 0, \quad \overset{01}{\Gamma}{}^{\alpha}_{\beta a} = -\delta^{\alpha}_{\beta}\lambda_{a}, \quad \overset{01}{\Gamma}{}^{\alpha}_{\beta\gamma} = \delta^{\alpha}_{\beta}\mu_{\gamma} - \delta^{\alpha}_{\gamma}\lambda_{\beta}, \quad \overset{01}{\Gamma}{}^{\alpha a}_{\beta\gamma} = -\delta^{\alpha}_{\gamma}\lambda^{a}_{\beta}. \tag{12}$$

6. Covariant differentials and derivatives

The differential congruences of the object λ can be represented as equations

$$\begin{aligned}
\Delta\lambda_a + \omega_a &= \lambda_{a,\alpha}\omega^{\alpha} + \lambda_{a,b}\omega^{b} + \lambda^{,b}_{a,\alpha}\omega^{\alpha}_{b}, \\
\Delta\lambda^{a}_{\alpha} + \omega^{a}_{\alpha} &= \lambda^{a}_{\alpha,\beta}\omega^{\beta} + \lambda^{a}_{\alpha,b}\omega^{b} + \lambda^{a,b}_{\alpha,\beta}\omega^{\beta}_{b}, \\
\Delta\lambda_{\alpha} + \lambda^{a}_{\alpha}\omega_a + \omega_{\alpha} &= \lambda_{\alpha,\beta}\omega^{\beta} + \lambda_{\alpha,a}\omega^{a} + \lambda^{,a}_{\alpha,\beta}\omega^{\beta}_{a}.
\end{aligned} \tag{13}$$

We have the following congruences modulo basic forms

$$\Delta\lambda_{a,\alpha} - \lambda_{a,b}\omega^{b}_{\alpha} + \lambda^{,b}_{a,\alpha}\omega_b + \lambda_a\omega_{\alpha} \equiv 0, \qquad \Delta\lambda_{a,b} + \lambda_a\omega_b + \lambda_b\omega_a \equiv 0, \qquad \Delta\lambda^{,b}_{a,\alpha} - \delta^{b}_{a}\lambda_c\omega^{c}_{\alpha} + \delta^{b}_{a}\omega_{\alpha} \equiv 0,$$

$$\Delta\lambda^{a}_{\alpha,\beta} - \lambda^{a}_{\alpha,b}\omega^{b}_{\beta} + \lambda^{a,b}_{\alpha,\beta}\omega^{b} + \lambda^{a}_{\beta}\omega_{\alpha} \equiv 0, \qquad \Delta\lambda^{a}_{\alpha,b} - \delta^{a}_{b}\lambda^{c}_{\alpha}\omega_c - \delta^{a}_{b}\omega_{\alpha} \equiv 0, \qquad \Delta\lambda^{a,b}_{\alpha,\beta} + \lambda^{b}_{\alpha}\omega^{a}_{\beta} + \lambda^{a}_{\beta}\omega^{b}_{\beta} \equiv 0,$$

$$\Delta\lambda_{\alpha,\beta} - \lambda_{\alpha,a}\omega^{a}_{\beta} + (\lambda^{,a}_{\alpha,\beta} + \lambda^{a}_{\alpha,\beta})\omega_a + \lambda_{\alpha}\omega_{\beta} + \lambda_{\beta}\omega_{\alpha} \equiv 0, \qquad \Delta\lambda_{\alpha,a} + \lambda^{b}_{\alpha,a}\omega_b + \lambda_{\alpha}\omega_a \equiv 0,$$

$$\Delta\lambda^{,a}_{\alpha,\beta} + \lambda_{\beta}\omega^{a}_{\alpha} + \lambda^{a}_{\alpha}\omega_{\beta} + \lambda^{b,a}_{\alpha,\beta}\omega_b \equiv 0.$$

Covariant differentials

$$\nabla\lambda_a = d\lambda_a - \lambda_b\tilde{\omega}^{b}_{a} + \omega_a, \qquad \nabla\lambda^{a}_{\alpha} = d\lambda^{a}_{\alpha} + \lambda^{b}_{\alpha}\tilde{\omega}^{a}_{b} - \lambda^{a}_{\beta}\tilde{\omega}^{\beta}_{\alpha} + \omega^{a}_{\alpha},$$

$$\nabla\lambda_{\alpha} = d\lambda_{\alpha} - \lambda_{\beta}\tilde{\omega}^{\beta}_{\alpha} + \lambda^{a}_{\alpha}\omega_a + \omega_{\alpha}.$$

Covariant derivatives

$$\begin{aligned}
&\nabla_b\lambda_a = \lambda_{a,b} + \lambda_c\Gamma^{c}_{ab}, \qquad \nabla_{\alpha}\lambda_a = \lambda_{a,\alpha} + \lambda_b\Gamma^{b}_{a\alpha}, \qquad \nabla^{b}_{\alpha}\lambda_a = \lambda^{,b}_{a,\alpha} + \lambda_c\Gamma^{cb}_{a\alpha}, \\
&\nabla_b\lambda^{a}_{\alpha} = \lambda^{a}_{\alpha,b} - \lambda^{c}_{\alpha}\Gamma^{a}_{cb} + \lambda^{a}_{\beta}\Gamma^{\beta}_{\alpha b}, \qquad \nabla_{\beta}\lambda^{a}_{\alpha} = \lambda^{a}_{\alpha,\beta} - \lambda^{b}_{\alpha}\Gamma^{a}_{b\beta} + \lambda^{a}_{\gamma}\Gamma^{\gamma}_{\alpha\beta}, \qquad \nabla^{b}_{\beta}\lambda^{a}_{\alpha} = \lambda^{a,b}_{\alpha,\beta} - \lambda^{c}_{\alpha}\Gamma^{ab}_{c\beta} + \lambda^{a}_{\gamma}\Gamma^{\gamma b}_{\alpha\beta}, \\
&\nabla_a\lambda_{\alpha} = \lambda_{\alpha,a} + \lambda_{\beta}\Gamma^{\beta}_{\alpha a}, \qquad \nabla_{\beta}\lambda_{\alpha} = \lambda_{\alpha,\beta} + \lambda_{\gamma}\Gamma^{\gamma}_{\alpha\beta}, \qquad \nabla^{a}_{\beta}\lambda_{\alpha} = \lambda^{,a}_{\alpha,\beta} + \lambda_{\gamma}\Gamma^{\gamma a}_{\alpha\beta}, \\
&\Delta\nabla_{\alpha}\lambda_a - \nabla_b\lambda_a\omega^{b}_{\alpha} + (\nabla^{b}_{\alpha}\lambda_a + \Gamma^{b}_{a\alpha})\omega_b \equiv 0, \qquad \Delta\nabla_b\lambda_a + \Gamma^{c}_{ab}\omega_c \equiv 0, \qquad \Delta\nabla^{b}_{\alpha}\lambda_a + \delta^{b}_{a}\omega_{\alpha} + \Gamma^{cb}_{a\alpha}\omega_c \equiv 0, \\
&\Delta\nabla_b\lambda^{a}_{\alpha} - \Gamma^{a}_{cb}\omega^{c}_{\alpha} + \Gamma^{\beta}_{\alpha b}\omega^{a}_{\beta} - \delta^{a}_{b}\omega_{\alpha} \equiv 0, \qquad \Delta\nabla_{\beta}\lambda^{a}_{\alpha} - \nabla_b\lambda^{a}_{\alpha}\omega^{b}_{\beta} + \nabla^{b}_{\beta}\lambda^{a}_{\alpha}\omega_b - \Gamma^{a}_{b\beta}\omega^{b}_{\alpha} + \Gamma^{\gamma}_{\alpha\beta}\omega^{a}_{\gamma} \equiv 0, \\
&\Delta\nabla^{b}_{\beta}\lambda^{a}_{\alpha} - \Gamma^{ab}_{c\beta}\omega^{c}_{\alpha} + \Gamma^{\gamma b}_{\alpha\beta}\omega^{a}_{\gamma} \equiv 0, \\
&\Delta\nabla_a\lambda_{\alpha} + (\nabla_a\lambda^{b}_{\alpha} + \lambda^{c}_{\alpha}\Gamma^{b}_{ca})\omega_b + \Gamma^{\beta}_{\alpha a}\omega_{\beta} \equiv 0, \qquad \Delta\nabla_{\beta}\lambda_{\alpha} - \nabla_a\lambda_{\alpha}\omega^{a}_{\beta} + (\nabla^{a}_{\beta}\lambda_{\alpha} + \nabla_{\beta}\lambda^{a}_{\alpha} + \lambda^{b}_{\alpha}\Gamma^{a}_{b\beta})\omega_a + \Gamma^{\gamma}_{\alpha\beta}\omega_{\gamma} \equiv 0, \\
&\Delta\nabla^{a}_{\beta}\lambda_{\alpha} + (\nabla^{a}_{\beta}\lambda^{b}_{\alpha} + \lambda^{c}_{\alpha}\Gamma^{ba}_{c\beta})\omega_b + \lambda^{a}_{\alpha}\omega_{\beta} + \Gamma^{\gamma a}_{\alpha\beta}\omega_{\gamma} \equiv 0.
\end{aligned} \tag{14}$$

Since covariant derivatives are not tensors, it is impossible to consider parallel displacements (cf. [27]) of the equipping planes C_{n-m-1} and N_{m-1}.

7. The central projections

We give an geometrical interpretation of the generalized bilinear connection using central projections.

Theorem 7.1. *The simple subobject* $\overset{B01}{\Gamma}_1 = \{\overset{01}{\Gamma}{}^a_{b\alpha}, \overset{01}{\Gamma}{}^a_{bc}, \overset{01}{\Gamma}{}^{ac}_{b\alpha}\}$ *of connection object* $\overset{B01}{\Gamma}$ *is characterized by the central projection of the plane* $N_{m-1} + dN_{m-1}$ *adjacent to the equipping plane* N_{m-1} *onto the initial plane* N_{m-1} *from the first-kind normal* $N_{n-m} = [C_{n-m-1}, A]$ *(the center of projection), i.e.,*

$$\overset{B01}{\Gamma}_1 : \qquad N_{m-1} + dN_{m-1} \xrightarrow{N_{n-m}} N_{m-1} \tag{15}$$

Proof. The plane N_{m-1} is determined by the points $B_a = A_a + \lambda_a A$, whose displacements are determined by the expression

$$dB_a = \vartheta B_a + \overset{01}{\tilde{\omega}}{}^b_a B_b + (\lambda_a \omega^\alpha + \omega^\alpha_a) B_\alpha + [\overset{01}{\nabla} \lambda_a - \lambda_a \lambda_b \lambda^b_\alpha \omega^\alpha + \lambda_a \lambda_b \omega^b + (\delta^b_a \mu_\alpha - \lambda_a \lambda^b_\alpha) \omega^\alpha_b] A,$$

where $\vartheta = \theta + \mu_\alpha \omega^\alpha - \lambda_a \omega^a$.

This means that the projection $N_{m-1} + dN_{m-1} \xrightarrow{N_{n-m}} N_{m-1}$ is performed. □

Theorem 7.2. *The simple subobject* $\overset{B01}{\Gamma}_2 = \{\overset{01}{\Gamma}{}^\alpha_{\beta\gamma}, \overset{01}{\Gamma}{}^\alpha_{\beta a}, \overset{01}{\Gamma}{}^{\alpha a}_{\beta\gamma}\}$ *of connection object* $\overset{B01}{\Gamma}$ *is characterized by the central projection of the plane* $C_{n-m-1} + dC_{n-m-1}$ *adjacent to the equipping plane* C_{n-m-1} *onto the initial plane* C_{n-m-1} *from the generating plane* P^*_m *(the center of projection), i.e.,*

$$\overset{B01}{\Gamma}_2 : \qquad C_{n-m-1} + dC_{n-m-1} \xrightarrow{P^*_m} C_{n-m-1} \tag{16}$$

Proof. The plane C_{n-m-1} is determined by the points $B_\alpha = A_\alpha + \lambda^a_\alpha A_a + \lambda_\alpha A$, whose displacements are determined by the expression

$$dB_\alpha = \vartheta B_\alpha + \overset{01}{\tilde{\omega}}{}^\beta_\alpha B_\beta + (\overset{01}{\nabla} \lambda^a_\alpha + \lambda^a_\beta \lambda^a_\alpha \lambda_b \omega^\beta - \mu_\alpha \omega^a + \lambda^a_\beta \lambda^b_\alpha \omega^\beta_b) A_a + (\overset{01}{\nabla} \lambda_\alpha - \lambda_\alpha \mu_\beta \omega^\beta + \lambda_\alpha \lambda_a \omega^a) A.$$

This means that the projection $C_{n-m-1} + dC_{n-m-1} \xrightarrow{P^*_m} C_{n-m-1}$ is performed. □

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