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On the geometry in the large of Hadamard manifolds

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Abstract. A Hadamard manifold is a simply connected, complete Riemannian manifold with nonpositive sectional curvature. The theory of Hadamard manifolds is a topic that has been more and more intensively studied for more than forty years. In the present paper, we prove Liouville-type theorems for conformal, isometric and harmonic pointwise transformations of metrics of Hadamard manifolds.

1. Introduction

A simply connected complete Riemannian manifold of nonpositive sectional curvature is called a *Hadamard manifold* after the Cartan-Hadamard theorem (see [1, p. 240]). Namely, thanks to this theorem, we know that it is diffeomorphic to a Euclidean space of the same dimension. For example, the hyperbolic space is a Hadamard manifold with negative constant sectional curvature. There is another example of Hadamard manifolds is a Euclidean space of the same dimension which has zero sectional curvature. From the Cartan-Hadamard theorem, there follow several basic properties of Riemannian manifolds of nonpositive curvature. First, from the theorem, we conclude that no compact simply connected manifold admits a metric of nonpositive sectional curvature (see also [2, p. 162]). Second, a Hadamard manifold has an infinite volume, which follows from the Cartan–Hadamard theorem. One can find dozens of papers on the geometry in the large of Hadamard manifolds. But in the present paper, we prove Liouville-type theorems for conformal, isometric and harmonic transformations of metrics of Hadamard manifolds. These theorems complement similar theorems for complete and compact Riemannian manifolds and represent our contribution to the geometry in the large of Hadamard manifolds.
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2. Liouville-type theorems on subharmonic, superharmonic, and convex functions defined on Hadamard manifolds

Let (M, g) be a complete Riemannian manifold. We recall here that a function $f \in C^2(M)$ is *subharmonic* (resp., *superharmonic* and *harmonic*) if it satisfies the differential inequality $\Delta f \geq 0$ (resp., $\Delta f \leq 0$ and $\Delta f = 0$) for the *Beltrami Laplacian* $\Delta f = div\,(grad\, f)$ (see [2, p. 281]). In what follows, we will insist that the function f be in $L^p(M)$ if the $p-$power of the absolute value of f is integrable with respect to the Riemannian measure induced by the given Riemannian metric g. In addition, we recall that if a Riemannian manifold

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(M, g) has infinite volume, then all of the constant functions, except a null function, are not in $L^q(M)$ for any $q \in (0, \infty)$ (see [3, p. 419]). For example, we consider the following famous result of S.-T. Yau (see [4, p. 663]): A nonnegative subharmonic L^q-function for some $q \in (1, \infty)$ on an arbitrary complete Riemannian manifold (M, g), then f is constant. To this, we can add that the constant must be zero if (M, g) has infinite volume. In turn, the following statement holds.

Lemma 2.1. *The Hadamard manifold* (M, g) *does not admit a non-zero non-negative subharmonic* L^q*-function for each* $q \in (0, \infty)$.

Proof. Let f be a non-zero non-negative subharmonic L^q-function for some $q \in (0, \infty)$ defined a Hadamard manifold (M, g). In turn, we known from [4, p. 288] that if a Riemannian manifold (M, g) is complete, simply connected and has non-positive sectional curvature then for each $q \in (0, \infty)$ every nonnegative L^q subharmonic function on (M, g) is constant. Therefore, our function f must be a constant. On the other hand, a Hadamard manifold has an infinite volume, which follows from the Cartan–Hadamard theorem. This forces the constant function f to be zero (see, for example, [3, p. 418]). This completes the proof of our lemma.

Remark 2.2. A similar statement can be found in [3, pp. 419-420]. Namely, the Hadamard manifold (M, g) does not admit a non-constant non-negative subharmonic L^q -function for each $p \in (0, 1]$.

It is obvious that the modulus of a harmonic function is a non-negative subharmonic function, so we can state the obvious corollary of Lemma 2.1.

Corollary 2.3. *A Hadamard manifold* (M, g) *does not admit a non-zero harmonic* L^q *-function for each* $q \in (0, \infty)$.

A function $f \in C^2(M)$ is called *convex* (see [2, p. 281]) if its Hessian $Hess_g f := \nabla d f$ is positive semi-definite at each point $x \in M$. Then, in particular, we have $\Delta f \geq 0$ for a convex function f and hence f is a subharmonic function. Therefore, the following corollary holds.

Corollary 2.4. *A Hadamard manifold* (M, g) *does not admit non-zero non-negative smooth convex* L^q*-functions for some* $q \in (0, \infty)$.

There are many examples of Hadamard manifolds, and one of them is a Riemannian globally symmetric space (M, g) of non-compact type, which is also simply connected, complete and has nonpositive sectional curvature. That is a prominent example of a Hadamard manifold. Therefore, a new corollary of Lemma 2.1 holds.

Corollary 2.5. *A Riemannian globally symmetric space of noncompact type does not admit a nonzero non-negative subharmonic* (*resp., harmonic and convex*) L^q*-function for* $q \in (0, \infty)$.

Remark 2.6. A simply connected irreducible symmetric space is an Einstein manifold (see [2, p. 386]). In particular, the Einstein constant of a Riemannian globally symmetric space (M, g) of noncompact type is negative and hence $Ric < 0$.

Next, we can prove the following theorem.

Theorem 2.7. *Let* $f \in C^2(M)$ *be a non-negative superharmonic function defined on a complete Riemannian manifold* (M, g) . *If* $f \in L^q(M)$ *for some* $q \in (0, 1)$, *then* f *must be identically constant. Moreover, this constant must be zero if* (M, g) *has infinite volume.*

Proof. S.-T. Yau formulated in [5, p. 607] the following Liouville-type theorem: Let (M, g) be a complete Riemannian manifold and $f \in C^2(M)$ be a nonnegative function such that $(q-1) f \Delta f \geq 0$ where q is a positive number, then for $q \neq 1$, either $\int_M f^q dvol_g = \infty$ or $f =$ constant. This text was a corrected formulation of a theorem that had been proved earlier in his paper [6, p. 664]. In turn, from the theorem above we can conclude that a nonnegative superharmonic L^q -function f defined on a complete Riemannian manifold (M, g) must be a constant function for any $q \in (0, 1)$. If, in addition, (M, g) has infinite volume, then this forces the constant function f to be zero (see, for example, [3, p. 418]).

Next, let (M, g) be a connected and noncompact Riemannian manifold and consider a diffusion process on it, generated by the Laplacian Δ, which is absorbing at infinity. If the probability of the absorption at ∞ in a finite time is zero, then (M, g) is said to be *stochastically complete*. In turn, a classical result by A. Grigor'yan states that on a stochastically complete manifold non-negative superharmonic L^1-functions are necessarily constant (see [7, p. 204]). At the same time, this constant must be zero if (M, g) has infinite volume. Summing up, we can formulate the following theorem.

Theorem 2.7. *Let* $f \in C^2(M)$ *be a non-negative superharmonic function defined on a complete and stochastically* *complete Riemannian manifold* (M, g) *. If* $f \in L^q(M)$ *for some* $q \in (0, 1]$*, then* f *must be identically constant. Moreover, this constant must be zero if* (M, g) *has infinite volume.*

S.-T. Yau proved that an arbitrary complete Riemannian manifold is stochastically complete if its Ricci curvature is bounded from below by a negative constant (see also [7, p. 224]). In this case, we can formulate the following corollary.

Corollary 2.8. *A Hadamard manifold* (M, g) *with the Ricci curvature bounded from below does not admit a non-negative and non-zero superharmonic* L^q*-function for each* $q \in (0, 1)$*.*

Remark 2.9. The last corollary is an analogue of the theorem from [3, pp. 419-420] which we formulated in Remark 2.2.

In conclusion we prove the following theorem (cf. [6, p. 660]).

Theorem 2.10. *Let* $f \in C^2(M)$ *be a superharmonic function on a Hadamard manifold* (M, g) *such that* $\left\| grad\, f \right\| \in L^1(M)$*, then* f *must be a harmonic function.*

Proof. It is well known that if V is a smooth vector field on a complete non-compact and oriented Riemannian manifold (M, g) such that $\| V \| \in L^1(M)$ and $div\, V \leq 0$, then $div\, V = 0$ (see [8, p. 281]). If we suppose $V = grad\, f$ for some $f \in C^2(M)$, then $div\, V = \Delta f$. In this case, the condition $div\, V \leq 0$ can be rewritten as $\Delta f \leq 0$. As a result, we can reformulate the above theorem for the vector field $grad\, f$ and the superharmonic function $f \in C^2(M)$. In order to complete of the proof, we recall that the Hadamard manifold is simply connected and hence orientable.

3. Theorems of Liouville type in the theory of conformal transformations of metrics of Hadamard manifolds

We will consider here the map $id : (M, g) \to (M, \bar{g})$ such that $\bar{g} = e^{2\sigma} g$ which we will call the conformal transformation of the metric of (M, g). Then the scalar curvatures $\bar{s}$ and s of two conformally equivalent metrics $\bar{g} = e^{2\sigma} g$ and g are related by the equality (see [9, p. 271])

$$2\,(n-1)\,\Delta\,\sigma = \left(s - e^{2\sigma}\,\bar{s}\right) - (n-1)\,(n-2)\,\left\| grad\,\sigma \right\|^2 \tag{1}$$

where $\left\| grad\,\sigma \right\|^2 = g\,(d\sigma, d\sigma)$. Therefore, if the inequality $e^{2\sigma}\,\bar{s} \geq s$ holds everywhere on (M, g), then (1) implies the inequality $\Delta\,\sigma \leq 0$. It means that σ is a superharmonic function. In this case, based on Theorem 2.9 and equation (1), we conclude that σ = constant if $\left\| grad\,\sigma \right\| \in L^1(M)$ and (M, g) is a Hadamard manifold. In this case, F is a homothetic transformation. Summarizing the above, we can formulate Theorem 3.1.

Theorem 3.1. *Let* (M, g) *be an n-dimensional* $(n \geq 3)$ *Hadamard manifold and* $id : (M, g) \to (M, \bar{g})$ *be a conformal map with* $\bar{g} = e^{2\sigma} g$*. If the following conditions hold:* $e^{2\sigma}\,\bar{s} \geq s$ *and* $\left\| grad\,\sigma \right\| \in L^1(M)$*, then* $id : (M, g) \to (M, \bar{g})$ *is homothetic.*

Let $\sigma = \ln\, f$ for some positive scalar function $f \in C^2(M)$ then from (2) we obtain the following equation

$$2\,(n-1)\,f\,\Delta\,f = f^2\left(s - f^2\,\bar{s}\right) - (n-1)\,(n-4)\,\left\| grad\,f \right\|^2, \tag{2}$$

where $f^2 = e^{2\sigma}$. Moreover, based on Corollary 2.8 above, we can formulate the following Corollary 3.2.

Corollary 3.2. *An n-dimensional* $(n \geq 4)$ *Hadamard manifold* (M, g) *with the Ricci curvature, bounded from below by some negative constant, does not admit a non-isometric conformal map* $id : (M, g) \to (M, \bar{g})$ *such that* $\bar{g} = e^{2\sigma} g$ *and* $e^{2\sigma}\,\bar{s} \geq s$ *for a* L^q *-function* σ *at least for one* $q \in (0, 1]$*.*

In general, a harmonic function does not transform into a harmonic function. The conditions under which the harmonic functions remain invariant have been studied by Y. Ishii in [10]. He introduced the pointwise conharmonic transformations as a subgroup of the conformal transformations which preserve the harmonicity of a certain class of smooth functions. In particular, Y. Ishii proved that $id : (M, g) \to (M, \bar{g})$ is a conharmonic metric transformation if it is a conformal transformation metric $\bar{g} := F^* g = e^{2\sigma} g$ for a smooth function $\sigma \in C^2(M)$ satisfying the condition $s = e^{2\sigma}\,\bar{s}$. Using Theorem 3.1 we can conclude that the following Liouville-type proposition holds.

Corollary 3.3. *Let* $id : (M, g) \to (M, \bar{g})$ *be a conharmonic transformation of an n-dimensional* $(n \geq 3)$ *Hadamard manifold* (M, g) *such that* $\bar{g} = e^{2\sigma} g$ *for some function* $\sigma \in C^2(M)$. *If* $\left\| grad\, \sigma \right\| \in L^1(M)$ *then* $id : (M, g) \to (M, \bar{g})$ *is homothetic.*

Taking into account the equality $s = e^{2\sigma}\bar{s}$, equation (1) can be rewritten in the form $2\,\Delta\sigma = -(n-2)\left\| grad\,\sigma \right\|^2$ where $n \geq 3$. Then we can conclude that σ is a superharmonic function. Then, on the basis of Corollary 2.8, we conclude that the following Liouville-type assertion holds for conharmonic mappings of Hadamard manifolds.

Corollary 3.4. *An n-dimensional* $(n \geq 4)$ *Hadamard manifold* (M, g) *with the Ricci curvature, bounded from below by some negative constant, does not admit a non-isometric conharmonic map* $id : (M, g) \to (M, \bar{g})$ *such that* $\bar{g} = e^{2\sigma} g$ *for a* L^q*-function* σ *at least for one* $q \in (0, 1]$.

A vector field V on (M, g) is called an *infinitesimal conformal transformation* or a *conformal Killing vector field* if a local one-parameter group of infinitesimal transformations generated by the vector field V is a group of conformal transformations of (M, g) (see [9, p. 282]). In this case $L_V g = 2\sigma g$ where L_V is the Lie derivation with respect to V. The function σ is called the *conformal factor* of V and is defined by the equality $n\,\sigma = div\, V$. The vector field V is said to be *infinitesimal homothetic* or *infinitesimal isometric transformation* according as its conformal factor σ is a constant or zero, respectively.

Let V be a conformal Killing vector field, then one can proved that (see [11, p. 25])

$$g(\bar{\Delta} V, X) = Ric\,(V, X) - \frac{n-2}{n}\, X\,(div\, V) \tag{3}$$

where $\bar{\Delta} = -trace_g \nabla^2$ and $X\,(div\,V)$ is the directional derivative of $div\, V$ along an arbitrary smooth vector field X on (M, g). From (3) we obtain the formula

$$\Delta\, e\,(V) = -Ric\,(V, V) - (n-2)\ V\,(\sigma) + \|\nabla V\|^2 \tag{4}$$

for the *energy density function* $\Delta\, e\,(V) = 1/2\,\|V\|^2 := 1/2\, g\,(V, V)$ of the flow generated by the conformal Killing vector field V and $\|\nabla V\|^2 = g\,(\nabla V, \nabla V)$.

If (M, g) is an n-dimensional Hadamard manifold, then, according to the definition of the Ricci tensor, we have $Ric\,(V, V) = \sum_{i=1,\dots,n} \sec(V, e_i) \leq 0$ for any orthonormal frame, $\{e_1, \dots, e_n\}$, of T_xM and for the sectional curvature $\sec\,(Y, Z)$ of the plane spanned by $Y, Z \in T_xM$ at an arbitrary point $x \in M$. If $Ric\,(V, V)$ is not strictly negative and $L_V\,\sigma \leq 0$ everywhere on (M, g), then, based on (4), we conclude that $e\,(V)$ is a subharmonic function. In this case, using Lemma 2.1, we can formulate the following theorem.

Theorem 3.5. *Let V be a conformal Killing vector field on n-dimensional* $(n \geq 3)$ *Hadamard manifold* (M, g). *If the following conditions hold:*

(i) the energy density function $e\,(V) \in L^q(M)$ *at least for one* $q \in (0, \infty)$;

(ii) $V\,(\sigma) \leq 0$ *for the conformal factor* σ *of* V;

(iii) $Ric\,(V, V) \leq 0$,

then V is identically zero.

Remark 3.6. The condition $L_V\,\sigma \leq 0$, which is equivalent to the inequality $L_V\,(div\,V) \leq 0$. In turn, the last inequality means that $d\,vol_g$ is a nonincreasing function along trajectories of the flow generated by the vector field V.

An interesting particular case of a conformal Killing vector field is when its dual 1-form is closed (see [8, p. 280]). In this case, it is said to be a *closed conformal Killing vector field*, or, in other words, a *concircular vector field* (see [9, p. 168]). In turn, concircular vector fields appeared in the study of *concircular transformations*, i.e., conformal transformations preserving geodesic circles. The following theorem holds for these vector fields.

Theorem 3.7. *Let Vbe a closed conformal Killing vector field on n-dimensional* $(n \geq 3)$ *Hadamard manifold* (M, g). *If the following conditions hold:*

(i) the energy density function $e\,(V) \in L^q(M)$ *at least for one* $q \in (0, \infty)$;

(ii) $Ric\,(V, V) \leq 0$,

then V is identically zero.

Proof. Let V be a closed conformal Killing vector field on a Riemannian manifold (M, g), then from [8, p. 282] we have the formula

$$div\ (\sigma V) = -\frac{1}{n-1} Ric\ (V, V) + n\,\sigma^2, \tag{5}$$

where $div\ (\sigma V) = \Delta\, e\ (V)$. If $Ric\ (V, V) \le 0$, then from (5) we conclude that $\Delta\, e\ (V)$ is a subharmonic function. Let now (M, g) be an n-dimensional ($n \ge 3$) Hadamard manifold, then, using Lemma 2.1, we can formulate Theorem 3.7.

Remark 3.8. First, Theorem 3.7 completes of the following Proposition 2.3 from [8, p. 282]: Let (M, g) be an n-dimensional complete, simply connected Riemannian manifold with $Ric \le 0$, and let V be a closed conformal vector field on (M, g), with conformal factor σ. If $\|\sigma V\| \in L^1(M)$, then V is parallel and $Ric\ (V, V) = 0$. Second, thanks to *Poincare lemma*, every closed form of degree 1 is exact on any manifold diffeomorphic to a Euclidean space of the same dimension. Therefore, the dual form for a closed conformal Killing vector field is exact on a Hadamard manifold. Therefore, the words "closed conformal Killing vector" in Theorem 3.7 can be replaced by "gradient conformal Killing vector".

4. Theorems of Liouville type in the theory of isometric transformations of metrics of Hadamard manifolds

Let (M, g) be a complete Riemannian manifold of dimension $n \ge 2$ and $d\ (x, y)$ be the *distance function* defined by g for any $x, y \in M$ (see details in Section 3.2.2 of monograph [2]). If $F : (M, g) \to (M, g)$ is a *isometric transformation* of (M, g) then it preserves the distance function $d\ (x, y)$, i.e., $d\ (x, y) = d\ (F(x), F(y))$ for any $x, y \in M$ (see [2, p. 202]). Let (M, g) be a Hadamard manifold, then the square d_F^2 of the *displacement function* $d_F(x) = d\,(x, F(x))$ is smooth and convex (see [1, p. 246]). Therefore, thanks to Corollary 2.4, we can formulate the following theorem

Theorem 4.1. *Let d_F be the displacement function of an isometric self-diffeomorphisms $F : (M, g) \to (M, g)$ of a Hadamard manifold (M, g). If its square is a L^q-function at least for one $q \in (0, \infty)$, then F is the constant map.*

A vector field V on (M, g) is called an *infinitesimal isometric* or *Killing vector field* (see [2, p. 313]) if a local one-parameter group of infinitesimal transformations generated by the vector field V is a group of pointwise isometric transformations of (M, g). In this case, we have $L_V\, g = 0$ where L_V is the Lie derivation with respect to V. One can proved that (see [2, p. 318])

$$\left(Hess_g e\,(V)\right)(X, X) = \|\nabla_X V\|^2 - g\,(R\,(V, X)\ X,\ V) \tag{6}$$

for the energy density function $\Delta\, e\ (V) = 1/2\ \|V\|^2$ of the flow generated by the Killing vector field V and an arbitrary smooth $X \in TM$. At the same time, by the definition of a Hadamard manifold (M, g) we have $g\,(R\,(V, X)\ X,\ V) \le 0$. In this case, from (6) we obtain the inequality $\left(Hess_g e\,(V)\right) \ge 0$. Therefore, $e\ (V)$ is a non-negative convex function. As a result, we get Corollary 4.2.

Corollary 4.2. *The Hadamard manifold (M, g) does not admit a non-zero Killing vector field such that its energy density function is a L^q-function at least for one $q \in (0, \infty)$.*

On the other hand, from (4) we obtain (see also [2, p. 318])

$$\Delta\, e\ (V) = -Ric\ (V, V) + \|\nabla V\|^2\,. \tag{7}$$

If we suppose that $Ric\ (V, V) \le 0$ on (M, g), then (7) implies the inequality $\Delta\, e\ (V) \ge 0$. In this case, based on Lemma 2.1, we can formulate the following Corollary 4.3.

Corollary 4.3. *Let V be a Killing vector field on n-dimensional ($n \ge 3$) Hadamard manifold (M, g). If the following conditions hold:*

(i) the energy density function $e\ (V) \in L^q(M)$ at least for one $q \in (0, \infty)$;

(ii) $Ric\ (V, V) \le 0$,

then V is identically zero.

Remark 4.4. Corollary 4.3 generalizes the classical result on the Killing vector field on a compact Riemannian manifold (see [2, p. 319]).

5. Theorems of Liouville type in the theory of harmonic transformations of metrics of Hadamard manifolds

Consider an n-dimensional ($n \geq 3$) smooth manifold M with two Riemannian metrics g and $\bar{g}$. We denote a globally definite tensor field $T = \bar{\nabla} - \nabla$ which is called the deformation tensor of the Levi-Civita connection ∇ into the Levi-Civita connection $\overline{\nabla}$. A map $id : (M, g) \to (M, \bar{g})$ is said to be *harmonic transformation* if $trace_g T = 0$ (see [12, p. 295]).

Let us now prove the following Liouville-type theorem, supplementing the above result of S.-T. Yau and R. Schoen (see also [13, p. 337]).

Theorem 5.1. *Let $(M, \bar{g})$ be an n-dimensional ($n \geq 2$) Hadamard manifold and g be another complete Riemannian metric on M such that its Ricci tensor is non-negative. Then the harmonic map $id : (M, g) \to (M, \bar{g})$ is a constant map if its energy density $e\ (id)$ is a L^q-function for at least one $q \in (0, \infty)$.*

Proof. Using the general theory of harmonic mappings (see, for example, [7]), we proved in [14, p. 110-111] that the map $id : (M, g) \to (M, \bar{g})$ is harmonic if and only if the following equation holds:

$$\Delta e(id) = Q(id) + \|T\|^2, \tag{8}$$

where $\Delta e(id) = \Delta(trace_g \bar{g})$ is the Laplacian of the energy density of the harmonic map $id : (M, g) \to (M, \bar{g})$ and $Q\ (id) = g\ (Ric, \bar{g}) - trace_g \left(trace_g \overline{Riem} \right)$ for the Riemannian curvature tensor $\overline{Riem}$ of the metric $\bar{g}$. From (8) we conclude that $Q\ (id) \geq 0$ holds if the Ricci curvature of g is nonnegative and the sectional curvature of $\bar{g}$ is nonpositive (see also [14, p. 110-111]). In this case, from (8), we obtain that $\Delta\, e\ (id) \geq 0$ under the above curvature assumptions and hence $e\ (id)$ is a subharmonic function on (M, g). At the same time, recall that every non-negative subharmonic L^q-function for any $q \in (0, \infty)$ is constant on a complete Riemannian manifold (M, g) with non-negative Ricci curvature (see [4, p. 288]). In turn, this constant must be equal to zero everywhere on a complete manifold (M, g) with infinite volume (see [5, p. 667]). To conclude the proof, we note that a simply connected manifold M with a complete Riemannian metric $\bar{g}$ of nonpositive sectional curvature is a Hadamard manifold.

The vector field V is an infinitesimal harmonic transformation in (M, g) if the local one-parameter group of infinitesimal pointwise transformations generated by the vector field V is a group of harmonic transformations (see [12, p. 295]). Analytic characteristic of such vector field V has the form (see also [12, p. 295])

$$\Delta\, e\ (V) = -Ric\ (V, V) + \|\nabla V\|^2 . \tag{9}$$

In this case, using our Lemma 2.1 we can prove the following Theorem 5.2.

Theorem 5.2. *Let V be an infinitesimal harmonic transformation on n-dimensional ($n \geq 3$) Hadamard manifold (M, g). If the following conditions hold:*
(i) the energy density function $e\ (V) \in L^q\ (M)$ at least for one $q \in (0, \infty)$;
(ii) $Ric\ (V, V) \leq 0$,
then V is identically zero.

Suppose that (M, g) is a complete Riemannian manifold such that the equation

$$-2Ric = 2\lambda\, g + L_V g \tag{10}$$

holds for some constant λ and some complete vector field V on M. In this case, we say g is a *Ricci soliton* (see [15, pp. 37-38]). The Ricci soliton g is said to be steady if $\lambda = 0$, shrinking if $\lambda < 0$, and expanding if $\lambda > 0$. If the vector field V is zero or is a Killing vector field, then the Ricci soliton g becomes Einstein. In this case, if (M, g) is a Hadamard manifold, from (10) we obtain the inequality $Ric = -\lambda\, g \leq 0$. This means that the Ricci soliton g is steady or expanding. Moreover, if g is steady then $Ric \equiv 0$. In the last case, from the conditions $Ric \equiv 0$ and $\sec \leq 0$ we obtain $\sec \equiv 0$, i.e., the sectional curvature of (M, g) vanishes identically. Therefore, (M, g) is a flat Riemannian manifold. Moreover, (M, g) is simply connected, and hence isometric to a Euclidean space of the same dimension.

In turn, we proved that the following theorem holds: The vector field V of a Ricci soliton g is an infinitesimal harmonic transformation (see [16, p. 474]). Therefore, the validity of the following statement is obvious as a corollary of Theorem 5.2.

Corollary 5.3. *Let (M, g) be a Hadamard manifold (M, g) and the metric g be a Ricci soliton with a smooth vector field V such that its energy density function $e\ (V)$ is a L^q-function at least for one $q \in (0, \infty)$ and $Ric\,(V, V) \leq 0$. Then*
(i) g cannot be a shrinking soliton;
(ii) if g is a steady soliton, then (M, g) is isometric to a Euclidean space of the same dimension;
(iii) if g is a expanding soliton, then (M, g) is an Einstein manifold with negative Einstein constant and hence $Ric < 0$.

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