Filomat 37:25 (2023), 8489–8503
https://doi.org/10.2298/FIL2325489H

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Classification of second orders symmetric tensors on manifolds through an associated fourth order tensor

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Abstract. For a manifold M admitting a metric g and given a second order symmetric tensor T on M one can construct from g and (the trace-free part of) T a fourth order tensor E on M which is related in a one-to-one way with T and from which T may be readily obtained algebraically. In the case when $\dim M = 4$ this leads to an interesting relationship between the Jordan-Segre algebraic classification of T, viewed as a linear map on the tangent space to M with respect to g, and the Jordan-Segre classification of E, viewed as a linear map on the 6−dimensional vector space of 2−forms to itself (with respect to the usual metric on 2−forms). This paper explores this relationship for each of the three possible signatures for g.

1. Introduction

Let M be a 4−dimensional manifold with metric g of signature either $(+,+,+,+)$ (positive definite), $(-,+,+,+)$ (Lorentz) or $(+,+,-,-)$ (neutral), collectively referred to as (M,g). In an attempt to deal with all signatures simultaneously, if T_mM denotes the tangent space to M at $m \in M$ and $u, v \in T_mM$, $u \cdot v = g(m)(u,v)$ denotes their inner product at m. A non-zero member $u \in T_mM$ is called *spacelike* if $u \cdot u > 0$, *timelike* if $u \cdot u < 0$ and *null* if $u \cdot u = 0$ and the 1−dimensional subspace (*direction*) spanned by u is called, respectively, *spacelike*, *timelike* and *null*. A 2−dimensional subspace (*2-space*) V of T_mM is called *spacelike* if each non-zero member of V is spacelike, or each non-zero member of V is timelike, *timelike* if V contains exactly two, distinct, null directions, *null* if V contains exactly one null direction and *totally null* if each non-zero member of V is null. Thus a totally null 2−space, which can only arise for neutral signature, consists, apart from the zero vector, entirely of null vectors any two of which are orthogonal. The 6−dimensional vector space of 2−forms (usually referred to as *bivectors*) at m is denoted by $\Lambda_m M$. Due to the existence of the metric (and where no confusion could arise) the distinction between the tangent and cotangent spaces will sometimes be ignored as will the index placing on bivectors. Square brackets round indices denote the usual anti-symmetrisation of the indices enclosed. If g has Lorentz signature one may choose a pseudo-orthonormal basis x, y, z, t at $m \in M$ with $x \cdot x = y \cdot y = z \cdot z = -t \cdot t = 1$ and an associated, so-called, *null* basis of vectors l, n, x, y at m given by $\sqrt{2}l = z + t$, $\sqrt{2}n = z - t$ so that l and n are null, $l \cdot n = 1$ and all other inner products are zero. If g has neutral signature one may choose a pseudo-orthonormal basis x, y, s, t at $m \in M$ with $x \cdot x = y \cdot y = -s \cdot s = -t \cdot t = 1$ and an associated *null* basis of (null) vectors l, n, L, N at m given by $\sqrt{2}l = x + t$, $\sqrt{2}n = x - t$, $\sqrt{2}L = y + s$ and $\sqrt{2}N = y - s$ so that $l \cdot n = L \cdot N = 1$ and all other such inner products between basis members are zero.

2020 *Mathematics Subject Classification*. Primary 53C21
Keywords. Tensor classification; Neutral signature.
Received: 18 November 2022; Accepted: 30 November 2022
Communicated by Mića Stanković and Zoran Rakić
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The associated completeness relations are $g_{ab} = z_a z_b - t_a t_b + x_a x_b + y_a y_b = l_a n_b + n_a l_b + x_a x_b + y_a y_b$ (Lorentz signature) and $g_{ab} = x_a x_b + y_a y_b - s_a s_b - t_a t_b = l_a n_b + n_a l_b + L_a N_b + N_a L_b$ (neutral signature). In Lorentz signature the non-zero members of spacelike 2−spaces are always spacelike. If g has positive definite signature an orthonormal basis is usually chosen and only spacelike vectors and spacelike 2-spaces are, in this sense, possible and the situation is somewhat trivial.

For all signatures a bivector $B(m)$ with components $B^{ab}(= -B^{ba})$ necessarily has even matrix rank. If this rank is 2, B is called *simple* and if 4, it is called *non-simple*. If B is simple it may be written as $B^{ab} = u^a v^b - v^a u^b$ for $u, v \in T_m M$ and the 2−space spanned by u and v is uniquely determined by B and called the *blade* of B (and then, unless more precision is required, B or its blade is written $u \wedge v$). A simple bivector is called *spacelike* (respectively, *timelike*, *null* or *totally null*) if its blade is *spacelike* (respectively, *timelike*, *null* or *totally null*). The type of a blade (or a simple bivector) above is sometimes referred to as its *nature*. The set $\Lambda_m M$ admits a (bivector) metric denoted by P so that for $B, B' \in \Lambda_m M$, $P(B, B') \equiv P_{abcd} B^{ab} B'^{cd} = B^{ab} B'_{ab}$ where $P_{abcd} = \frac{1}{2}(g_{ac} g_{bd} - g_{ad} g_{bc})$. One writes $|B|$ for $P(B, B)$ and $B \cdot B'$ for $P(B, B')$. It is easily checked that if g has Lorentz signature, P has signature $(+,+,+,-,-,-)$ whilst if g has neutral signature, the signature of P is $(+,+,-,-,-,-)$. If g is positive definite, P has signature $(+,+,+,+,+,+)$.

For each $B \in \Lambda_m M$, $*$ will denote the Hodge duality operator (linear map) on $\Lambda_m M$ and the dual of B is then $\overset{*}{B}_{ab} = \frac{1}{2}\epsilon_{abcd} B^{cd}$ where $\epsilon_{abcd} \equiv \sqrt{|detg|}\delta_{abcd}$ with δ denoting the usual alternating symbol and $detg$ the determinant of g. For neutral and positive definite signatures $\overset{**}{B} = B$ and so the only eigenvalues of the linear map $*$ are ± 1. Thus $B \cdot B = \overset{*}{B} \cdot \overset{*}{B}$. For Lorentz signature $\overset{**}{B} = -B$ and the corresponding eigenvalues are $\pm i$. In this case $B \cdot B = -\overset{*}{B} \cdot \overset{*}{B}$. In all cases B is simple if and only if $\overset{*}{B}$ is simple and then their blades are mutually orthogonal. Now define the subspaces $\overset{+}{S}_m \equiv \{B \in \Lambda_m M : \overset{*}{B} = B\}$ and $\overset{-}{S}_m \equiv \{B \in \Lambda_m M : \overset{*}{B} = -B\}$ and also the subset $\widetilde{S}_m \equiv \overset{+}{S}_m \cup \overset{-}{S}_m$ of $\Lambda_m M$. For Lorentz signature $\overset{+}{S}_m$ and $\overset{-}{S}_m$ are trivial. For the other two signatures $\overset{+}{S}_m \cap \overset{-}{S}_m = \{0\}$ and $B \in \Lambda_m M \setminus \widetilde{S}_m$ if and only if B and $\overset{*}{B}$ are independent members of $\Lambda_m M$ and any $B \in \Lambda_m M$ may be written in exactly one way as $B = \overset{+}{B} + \overset{-}{B}$ with $\overset{+}{B} \in \overset{+}{S}_m$ and $\overset{-}{B} \in \overset{-}{S}_m$. Thus for these two signatures one has $\Lambda_m M = \overset{+}{S}_m \oplus \overset{-}{S}_m$. For any non-zero $B \in \Lambda_m M$, and g of Lorentz signature, B and $\overset{*}{B}$ are independent. In the positive definite case $\overset{+}{S}_m$ and $\overset{-}{S}_m$ are Lie algebras isomorphic to $o(3)$ whilst for neutral signature $\overset{+}{S}_m$ and $\overset{-}{S}_m$ are Lie algebras isomorphic to $o(1, 2)$.

2. Preliminary Results

It is useful here to collect together some elementary algebraic and geometrical results for later use.

(*i*) For $(+,+,+,+)$ each non-trivial member of $\widetilde{S}_m$ is non-simple.

(*ii*) For $(+,+,-,-)$ each totally null bivector is a simple member of $\widetilde{S}_m$ and all other non-trivial members of $\widetilde{S}_m$ are non-simple. If $B, B' \in \widetilde{S}_m$ are totally null and *each* is in $\overset{+}{S}_m$ or *each* is in $\overset{-}{S}_m$ their blades intersect trivially. Otherwise their blades intersect in a unique null direction. If $B \in \overset{+}{S}_m$ and $B' \in \overset{-}{S}_m$ then if $|B| > 0 < |B'|$ there exist a dual pair of spacelike simple bivectors in the span of B and B', if $|B| < 0 > |B'|$ there exist a dual pair of timelike simple bivectors in the span of B and B' and if $|B| = |B'| = 0$ one may choose a basis, as in the last section, in which $B = l \wedge N$ and $B' = l \wedge L$ [13]. It is also noted from the Lorentz signature $(+,-,-)$ of the 3−dimensional spaces $\overset{+}{S}_m$ and $\overset{-}{S}_m$ that if $B, B' \in \overset{+}{S}_m$ (or $B, B' \in \overset{-}{S}_m$) some linear combination B'' of them satisfies $|B''| < 0$.

(*iii*) For Lorentz and neutral signature, if B is a *null* bivector, $|B| = 0$. For neutral signature and a null bivector B, its unique decomposition $B = \overset{+}{B} + \overset{-}{B}$, with $\overset{+}{B} \in \overset{+}{S}_m$ and $\overset{-}{B} \in \overset{-}{S}_m$, gives rise to unique totally null members $\overset{+}{B}$ and $\overset{-}{B}$ whose unique common (null) direction equals that of B. Conversely the sum and difference of any pair of totally null bivectors $\overset{+}{B} \in \overset{+}{S}_m$ and $\overset{-}{B} \in \overset{-}{S}_m$ give a dual pair of null bivectors whose unique null directions coincide with the intersection of the blades of $\overset{+}{B}$ and $\overset{-}{B}$.

(*iv*) For $(+,+,-,-)$ the Lie algebras $\overset{+}{S}_m$ and $\overset{-}{S}_m$ each have signatures $(+,-,-)$ and in terms of the vector bases above one may construct a basis $G = l \wedge N$, $H = n \wedge L$ and $K = l \wedge n - L \wedge N$ for $\overset{+}{S}_m$ and a basis $\bar{G} = l \wedge L$, $\bar{H} = n \wedge N$ and $\bar{K} = l \wedge n + L \wedge N$ for $\overset{-}{S}_m$ with $|K| = |\bar{K}| = -4$, $G \cdot H = \bar{G} \cdot \bar{H} = 2$ and with all other possible inner products between them zero. Further for any bivector $B \in \widetilde{S}_m$ with $|B| < 0$ one may choose a null basis for T_mM so that B may be written as being proportional K or $\bar{K}$ above. Also, given an orientation in T_mM with $J = x \wedge y + s \wedge t \in \overset{+}{S}_m$, and $\bar{J} = x \wedge y - s \wedge t \in \overset{-}{S}_m$, $|J| = |\bar{J}| = 4$. If $\overset{+}{B} \in \overset{+}{S}_m$ and $\overset{-}{B} \in \overset{-}{S}_m$, one has $\overset{+}{B} \cdot \overset{-}{B} = 0$. If B, B' are non-trivial members of $\overset{+}{S}_m$ with $|B| = 0 = B \cdot B'$ then $|B'| \leqslant 0$ and similarly for $\overset{-}{S}_m$. If $B, B' \in \overset{+}{S}_m$ are totally null and $B \cdot B' = 0$ then B and B' are proportional (and similarly for $\overset{-}{S}_m$). Finally if $B \in \widetilde{S}_m$, $B' \in \Lambda_m M$ and $B \cdot B' = 0$ then $B \cdot \overset{*}{B}{}' = 0$.

(*v*) For any signature and a general non-zero member $B \in \Lambda_m M$ the statements that (*a*) B is simple, (*b*) $B_{ab}\overset{*}{B}{}^{bc} = 0$, (*c*) $B \cdot \overset{*}{B} = 0$ and (*d*) $B_{a[b}B_{cd]} = 0$, are equivalent. This leads to the following easily checked results which will be useful later. For neutral signature and $B \in \widetilde{S}_m$, the statements that $|B| = 0$, that B is simple and that B is totally null are equivalent. In addition, a bivector $B \notin \widetilde{S}_m$ (and written as $B = X + Y$ with $X \in \overset{+}{S}_m$ and $Y \in \overset{-}{S}_m$) satisfies $|B| = |X| + |Y|$ and $B \cdot \overset{*}{B} = |X| - |Y|$ and so $|B| = 0$ *and* B is simple $\Leftrightarrow |X| = |Y| = 0 \Leftrightarrow X$ and Y are each totally null $\Leftrightarrow B$ and $\overset{*}{B}$ are null (cf part (*iii*) above). Thus $B \notin \widetilde{S}_m$ is null $\Leftrightarrow$ it is simple with a null vector l in its blade satisfying $B_{ab}l^b = 0$. However, for $B \notin \widetilde{S}_m$ satisfying $|B| = 0$, B need not be null or simple as can be seen by considering $B = J + \bar{K}$ in the notation of (*iv*) above since then $|B| = 0$ and $B \cdot \overset{*}{B} \neq 0$.

Further details regarding these results can be found in [6] (for $(+,+,+,+)$), [7, 12, 13] (for $(+,+,-,-)$ and [8] for $(+,+,+,-)$.

3. Classification of Second Order Symmetric Tensors

For Lorentz signature (and, in particular, for the study of Einstein's general relativity theory) the classification of second order symmetric tensors is well documented [5, 8, 10, 11] and for the case of neutral signature it has been studied in [9, 10]. It is trivial for signature $(+,+,+,+)$ since such tensors are always diagonalisable over $\mathbb{R}$. In general let T be a (non-zero, real) symmetric tensor at $m \in M$. With respect to the metric $g(m)$ let $f' : T_mM \to T_mM$ be the linear map $v^a \to T^a{}_b v^b$ for $v \in T_mM$. Then v is an eigenvector of f' (or T) if $T^a{}_b v^b = \lambda v^a$ (equivalently, $T_{ab}v^b = \lambda g_{ab}v^b = \lambda v_a$) and $\lambda \in \mathbb{C}$ is the associated eigenvalue. (For convenience, an eigenvector or eigenvalue labelled "complex" will be assumed to be *not* real.) A 2−dimensional subspace $W \subset T_mM$ is called an *invariant* 2−*space* for f' (or T) at m if $u \in W \Rightarrow f'(u) \in W$ (and W is invariant if and only if the orthogonal complement of W is invariant). The consequences of this depend on the nature of W (section 1). It is recalled that two eigenvectors with distinct eigenvalues are orthogonal. The algebraic structure of T will be described in terms of Jordan forms and the associated Segre types and Segre notation. In the use of Segre notation a positive integer n inside the Segre symbol {} denotes a *real* elementary divisor of order n whereas the symbol n' denotes a *complex* elementary divisor of order n. Thus, for $n = 1$, the elementary divisor is simple and otherwise it is non-simple. Usually $1'$ and its conjugate are written as the pair "$z\bar{z}$ ". Eigenvalue degeneracies are indicated by enclosing the appropriate digits in the Segre symbol inside round brackets.

For the case of *Lorentz signature* we have (see, for example, [5, 8–10])

Theorem 3.1. *The Jordan form for f' is represented by one of the following Segre types:* $\{1111\}$, $\{z\bar{z}11\}$, $\{211\}$ *or* $\{31\}$ *or one of their possible degeneracies.*

For the case of *neutral signature* we have [9]

Theorem 3.2. *The Jordan form for f' is represented by one of the following Segre types:* $\{1111\}$, $\{z\bar{z}11\}$, $\{z\bar{z}w\bar{w}\}$, $\{211\}$, $\{2z\bar{z}\}$, $\{22\}$, $\{2'2'\}$, $\{31\}$ *or* $\{4\}$ *or one of their possible degeneracies.*

In each of these theorems an eigenvector associated with a non-simple elementary divisor is necessarily null and any real or complex *null* eigenvector is either associated with a non-simple elementary divisor or its eigenvalue is degenerate (that is, its associated eigenspace has dimension ≥ 2). These results are quite general for a symmetric tensor [9, 10]. In the Lorentz case, using the tetrads of section 1 and assuming, for simplicity, that the eigenvalues are non-degenerate, one can list the invariant 2−spaces for f' as follows: $\{1111\}$ $(x \wedge y, x \wedge z, x \wedge t, y \wedge z, y \wedge t, z \wedge t)$, $\{z\bar{z}11\}$ $(x \wedge y, l \wedge n)$, $\{211\}$ $(l \wedge x, l \wedge y, l \wedge n, x \wedge y)$ and $\{31\}$ $(l \wedge x, l \wedge y)$. For neutral signature and again using the tetrads of section 1 and the assumption that the eigenvalues are non-degenerate, one can list the invariant 2−spaces using the canonical forms given in [9] for each type as $\{1111\}$ $(x \wedge y, x \wedge s, x \wedge t, y \wedge s, y \wedge t, s \wedge t)$, $\{z\bar{z}11\}$, $\{z\bar{z}w\bar{w}\}$, $\{2z\bar{z}\}$ $(l \wedge n, L \wedge N)$, $\{211\}$ $(l \wedge y, l \wedge s, l \wedge n, L \wedge N)$, $\{31\}$ $(l \wedge y, l \wedge s)$, $\{2'2'\}$ and $\{4\}$ $(l \wedge L)$ and $\{22\}$ $(l \wedge L, l \wedge n, L \wedge N)$. For either signature, of course, the existence of degeneracies will add extra invariant 2−spaces and they are easily found. It is remarked here that, as described in the next section, interest will focus on the trace-free part $\widetilde{T}$ of T but this has *the same Segre type, including degeneracies, and the same invariant* 2*−space structure as* T.

Two of these cases require further discussion. First, for Lorentz and neutral signature and for a *tracefree* T of (degenerate) Segre type $\{(211)\}$, its single eigenvalue is zero and a canonical form for T is given [8, 9] by $T_{ab} = \lambda l_a l_b$ where l spans the unique null eigendirection of T. In any such canonical form the sign of $\lambda \neq 0$ is fixed and so this type should be thought of as describing two types, $\lambda > 0$ and $\lambda < 0$. Second, for neutral signature and for a tracefree T of (degenerate) type $\{(22)\}$ (and again all eigenvalues are zero), a canonical form in terms of the two orthogonal null eigenvectors l and L is [9]

$$T_{ab} = \mu l_a l_b + \nu L_a L_b + \omega(l_a L_b + L_a l_b) \tag{1}$$

with $\mu, \nu, \omega \in \mathbb{R}$. In this case interests centres on the expression $\omega^2 - \mu\nu$ which must be non-zero in order to preserve the matrix rank of T being 2. The eigenvalue degeneracy in T shows that there is freedom to choose any two independent (necessarily orthogonal, null) eigenvectors in the 2−space $l \wedge L$ represented by $l' \to l + aL$, $L' \to L + bl$ with $a, b \in \mathbb{R}$ and $ab - 1 \neq 0$. The sign of $\omega^2 - \mu\nu$ is invariant under such basis changes and if $\omega^2 > \mu\nu$ one may so choose the basis so that $\mu = \nu = 0$ and $\omega \neq 0$ whilst if $\omega^2 < \mu\nu$ one may choose the basis so that $\omega = 0$ and $\mu\nu > 0$. Regarding (μ, ν, ω) as coordinates in $\mathbb{R}^3$ the sheets of the hyperbola $\omega^2 = \mu\nu$ decompose $\mathbb{R}^3$ into three regions; two outer ones where $\omega^2 < \mu\nu$, distinguished by the conditions $\mu > 0 < \nu$ and $\mu < 0 > \nu$, and an inner one where $\omega^2 > \mu\nu$. Any point in the outer regions may be connected by a base change, as above, to the $w = 0$ plane (preserving the common sign of μ and ν) whilst points in the inner region may be similarly connected to the ω−axis. The hyperbola $\omega^2 = \mu\nu$ itself (away from the origin $\mu = \nu = 0$) is divided into two regions, $\mu > 0 < \nu$ and $\mu < 0 > \nu$, where, in the first region, $T_{ab} = (\sqrt{\mu} l_a + \sqrt{\nu} L_a)(\sqrt{\mu} l_b + \sqrt{\nu} L_b)$ and in the second $T_{ab} = -(\sqrt{|\mu|} l_a - \sqrt{|\nu|} L_a)(\sqrt{|\mu|} l_b - \sqrt{|\nu|} L_b)$ so that these tensors have Segre type $\{(211)\}$ (cf the remarks above).

4. Fourth Order Tensors

From the tensor $T(m)$ and $g(m)$, and for any signature (but still $\dim M = 4$), construct the trace-free symmetric second order tensor $\widetilde{T}$ given by $\widetilde{T}_{ab} = T_{ab} - \frac{1}{4}(T^c{}_c)g_{ab}$ and the following fourth order tensor E at m given by

$$E_{abcd} = \widetilde{T}_{a[c}g_{d]b} + \widetilde{T}_{b[d}g_{c]a} = \frac{1}{2}(\widetilde{T}_{ac}g_{db} - \widetilde{T}_{ad}g_{bc} + \widetilde{T}_{bd}g_{ac} - \widetilde{T}_{bc}g_{ad}). \tag{2}$$

Then E has the properties given by

$$E_{abcd} = -E_{bacd} = -E_{abdc}, \qquad E_{abcd} = E_{cdab}, \qquad E_{a[bcd]} = 0,$$
$$E^c{}_{acb} = \widetilde{T}_{ab} \qquad E^{ab}{}_{ab} = 0. \tag{3}$$

From these equations it is seen that, given $g(m)$, the tensors $E(m)$ and $\widetilde{T}(m)$ are in a one-to-one correspondence with each other (noting the use of $g(m)$ in this double implication).

So far the argument is independent of signature. From now on *the argument will be given for neutral signature*, with the (easier) Lorentz case being dealt with later (the positive definite case being trivial). The tensor E satisfies the following left and right dual signature-independent properties, taken over the first or second pair of skew indices (see, e.g.[4]).

$$^{*}E_{abcd} = -E^{*}_{abcd} \qquad ^{*}E^{*}_{abcd} = -E_{abcd} \tag{4}$$

It is remarked here that a general decomposition of the Riemann curvature tensor *Riem* with components $R^{a}{}_{bcd}$ in terms of the Weyl conformal tensor C with components $C^{a}{}_{bcd}$, the Ricci tensor *Ricc* with components $R_{ab} \equiv R^{c}{}_{acb}$, the Ricci scalar $R \equiv R_{ab}g^{ab}$ and the trace-free Ricci tensor $\widetilde{Ricc}$ with components $\widetilde{R}_{ab} \equiv R_{ab} - \frac{R}{4}g_{ab}$ is

$$R_{abcd} = C_{abcd} + E'_{abcd} + \frac{R}{6}g_{a[c}g_{d]b} \tag{5}$$

and then E' has the form (2) with $\widetilde{T} = \widetilde{Ricc}$. It is also remarked that, given *any* fourth order tensor E satisfying (3) and (4), it may *always* be written in the form (2) with $\widetilde{T}$ as in (3). This can be computed from [4] (page 51). Each of these remarks is independent of signature.] Returning to neutral signature, consider the linear map $f : \Lambda_m M \to \Lambda_m M$ given for $J \in \Lambda_m M$ by $J^{ab} \to E^{ab}{}_{cd}J^{cd}$. Using the shorthand notation EJ for $E^{ab}{}_{cd}J^{cd}$ (or $E_{abcd}J^{cd}$) one can easily show from (4) that $^{*}(EJ) = (^{*}E)J = -(E^{*})J = -E\overset{*}{J}$ and so regarding $\Lambda_m M$ as $\overset{+}{S}_m \oplus \overset{-}{S}_m$ one sees that $f(\overset{+}{S}_m) \equiv \overset{-}{A} \subset \overset{-}{S}_m$ and $f(\overset{-}{S}_m) \equiv \overset{+}{A} \subset \overset{+}{S}_m$ and rank$f$ =dim$\overset{+}{A}$+dim$\overset{-}{A}$. One can thus choose a basis $R, S, Q, \bar{R}, \bar{S}, \bar{Q}$ for $\Lambda_m M$ with $R, S, Q \in \overset{+}{S}_m$ and $\bar{R}, \bar{S}, \bar{Q} \in \overset{-}{S}_m$ and for $C \in \overset{+}{S}_m$ and $D \in \overset{-}{S}_m$ define $(C, D) = C_{ab}D_{cd} + D_{ab}C_{cd}$. Then, using the symmetries of E one can write at m

$$\begin{aligned} E = a(R,\bar{R}) + b(R,\bar{S}) + c(R,\bar{Q}) + d(S,\bar{R}) + e(S,\bar{S}) + f(S,\bar{Q}) \\ +g(Q,\bar{R}) + h(Q,\bar{S}) + k(Q,\bar{Q}) \end{aligned} \tag{6}$$

where $a, b, ...k \in \mathbb{R}$ some of which may be zero (or related) on account of the third relation in (3). This leads to a "symmetry" of the map f with respect to $\overset{+}{S}_m$ and $\overset{-}{S}_m$.

Lemma 4.1. *The range spaces $\overset{+}{A}$ and $\overset{-}{A}$ have the same dimension and hence the map f has even rank.*

Proof. First it is noted from (6) that if dim$\overset{+}{A} = 0$ (or dim$\overset{-}{A} = 0$), $E \equiv 0$. This is clear, for example, by choosing orthonormal bases for $\overset{+}{S}_m$ and $\overset{-}{S}_m$ and then contracting (6) with these basis members. Next rewrite (6) as

$$E = (R, \bar{X}) + (S, \bar{Y}) + (Q, \bar{Z}) \tag{7}$$

where $\bar{X} = a\bar{R} + b\bar{S} + c\bar{Q}$, $\bar{Y} = d\bar{R} + e\bar{S} + f\bar{Q}$ and $\bar{Z} = g\bar{R} + h\bar{S} + k\bar{Q}$. Suppose that rank $E < 6$ so that $EF = 0$ for some bivector F. If $F \notin \widetilde{S}$, $E(F \pm \overset{*}{F}) = 0$ with $0 \neq F + \overset{*}{F} \in \overset{+}{S}$ and $0 \neq F - \overset{*}{F} \in \overset{-}{S}$ and so dim$\overset{+}{A} \leq 2 \geq$dim$\overset{+}{A}$. Now suppose $F \in \widetilde{S}$. If, say, $F \in \overset{+}{S}$ and F is totally null one can choose the above basis R, S, Q for $\overset{+}{S}$ as G, H, K in section 2(*iv*) and with, say, $F = G$. Then a contraction of (7) with H shows that $E = (R, \bar{X}) + (Q, \bar{Z})$ and so dim$\overset{+}{A} \leq 2 \geq$dim$\overset{+}{A}$ (and similarly for $F \in \overset{-}{S}$). If $F \in \overset{+}{S}$ is not totally null then $|F| \neq 0$ and one may extend F to an orthogonal basis for $\Lambda_m M$ and another obvious contraction again shows that dim$\overset{+}{A} \leq 2 \geq$dim$\overset{+}{A}$ and so this latter result always holds if rank$E \neq 6$. Now suppose that, say, dim$\overset{-}{A} = 1$ so that there exist $A, B \in \overset{+}{S}$ with $EA = EB = 0$. If A, B are both totally null choose them as G and H above and contract (7) first with G and then with H to get $E = (Q, Z)$ and so dim$\overset{-}{A} = 1$. If only A is totally null and B not, take $A = G$ and use the basis G, H, K above with $A = G$ so that B may be taken as a linear combination of H and K. Obvious contractions again show that $\dim\overset{-}{A} = 1$. Finally of neither A, nor B is totally null one has $|A| \neq 0 \neq |B|$ and one may extend A, B to an orthogonal basis for $\overset{+}{S}$ and again obtain $\dim\overset{-}{A} = 1$. Thus if $\dim\overset{-}{A} = 1$ one gets $\dim\overset{+}{A} = 1$. This is sufficient to establish the result claimed in the lemma.

5. The relationship between E and T

5.1. *General Theory*

The relationship to be described here seems to have first been noticed in [2, 8] (see also [3]) and used to describe the classification in the case of Lorentz signature. Here one continues with the case of *neutral signature* which is more difficult. First it is noted that, since T is real, f' always admits an invariant 2−space W. To see this note that if a complex eigenvalue for f' exists its real and imaginary parts span such a 2−space, whilst if all eigenvalues are real and correspond to simple elementary divisors any two associated eigenvectors span such a 2−space. Finally if a real eigenvalue with non-simple elementary divisor exists the first two members of its Jordan basis span such a 2−space (see, e.g. [5, 8]). Then the orthogonal "complement" $W^{\perp}$ is another invariant 2−space. (It is remarked that $W^{\perp}$ is not necessarily complementary to W in the sense that $W \cup W^{\perp}$ need not span T_mM: in fact $W^{\perp}$ could equal W). If W is spacelike so also is $W^{\perp}$ and a standard result shows that f' is diagonalisable over $\mathbb{R}$. If W is timelike so also is $W^{\perp}$ and one can write W as $l \wedge n$ for non-orthogonal null vectors l, n (in the notation of section 1) and then either l and n are eigenvectors of f' with equal eigenvalues, or l *or* n spans the only eigendirection in W, or W admits an orthogonal pair of non-null eigenvectors with distinct eigenvalues or a complex conjugate pair of eigenvectors are admitted and which can be chosen as $l \pm in$ following, possibly, a change of basis. The eigenvector structure of $W^{\perp}$ is similar, but independent of that of W. If W is null so also is $W^{\perp}$ and one can write W as $l \wedge y$ and $W^{\perp}$ as $l \wedge s$ (or vice-versa). In this case l spans an eigendirection of f' and W and $W^{\perp}$ may each contain a spacelike (or a timelike) eigendirection or no further eigendirections, with all combinations possible. If W is totally null, $W = W^{\perp} = l \wedge L$, say, and W may contain exactly one or two independent real null eigendirections or there may be two complex conjugate null eigendirections which, for appropriate choice of l and L, are spanned by $l \pm iL$. This last remark follows by noting the eigenvectors are of the form $l + zL$ and $l + \bar{z}L$ for $z \in \mathbb{C}$ and then taking real and imaginary parts followed by a change of basis to achieve the result.

A bivector $B \in \Lambda_mM$ (or in the complexification of Λ_mM) is an *eigenbivector* of E with eigenvalue $\alpha \in \mathbb{C}$ if $EB = \alpha B$. The convention will be adopted that the terms *eigenbivector* and *eigenvalue* will be applied to *real* such objects whilst a complex eigenbivector (taken as *not* being proportional to a real bivector) and a complex (that is, *not* real) eigenvalue will be declared as such. [It is remarked that the results so far hold, where appropriate, for all signatures.]

The following remarks, which apply in neutral signature, are important in resolving the classification of E. The other signatures will be briefly dealt with later.

(*a*) If $B \in \Lambda_mM$ is (real and) simple the blade of B is an invariant 2−space for T (or f') if and only if B is a (simple) eigenbivector of E (or f). To see this suppose $B = p \wedge q$ for $p, q \in T_mM$ so that one has $f'(p)$ and $f'(q)$ each linear combinations of p and q. A substitution into (2) and some elementary algebra then shows that EB is a multiple of B, and conversely. Note here that one can choose $p.q = 0$ for each case of B.

(*b*) Since T must admit an invariant 2−space, E (or f) always admits a simple (real) eigenbivector. If $B \in \Lambda_mM$ is a (real) eigenbivector of E with eigenvalue $0 \neq \alpha \in \mathbb{R}$ then B is necessarily simple. And $\overset{*}{B}$ is also (simple) and is an independent eigenbivector of E with eigenvalue $-\alpha$ and the blades of B and $\overset{*}{B}$ are invariant for T (cf the above remark about invariant 2−spaces for T occurring in orthogonal pairs). This follows by taking the left dual of the equation $EB = \alpha B$ to get $({}^{*}E)B = \alpha\overset{*}{B}$ and so using (4) $E\overset{*}{B} = -\alpha\overset{*}{B}$. Use of the second relation in (3) gives, in an obvious notation, $BE\overset{*}{B} = \overset{*}{B}EB$ and so $\alpha(B \cdot \overset{*}{B}) = 0$ and the result finally follows from section 2(*v*). If $B \notin \widetilde{S}_m$ is an eigenbivector of E then so also is $\overset{*}{B}$ and B and $\overset{*}{B}$ are independent eigenbivectors with eigenvalues $\pm\beta$ ($\beta \in \mathbb{R}$). If an eigenbivector of E lies in $\widetilde{S}_m$ its eigenvalue is zero since then $EB = \alpha B$ gives, after taking duals, $EB = -\alpha B$ (or use the above results regarding $\overset{+}{A}$ and $\overset{-}{A}$). Thus the (real) eigenbivectors of E either occur in (independent) dual pairs (possibly with zero eigenvalue) or constitute $2n(n \in \mathbb{N})$ members of $\widetilde{S}_m$, n in $\overset{+}{S}_n$ and n in $\overset{-}{S}_n$ (necessarily with zero eigenvalue) from lemma 4.1.

(*c*) If T is diagonalisable over $\mathbb{R}$ it admits four, independent, non-null, mutually orthogonal eigenvectors which, pairwise, give rise to six distinct spacelike or timelike invariant 2−spaces for T and hence to six independent simple non-null eigenbivectors for E which is then diagonalisable over $\mathbb{R}$. From an earlier remark if E admits one (and hence two orthogonal) simple spacelike eigenbivectors their blades are invariant for T, and hence T and E are diagonalisable (over $\mathbb{R}$).

(*d*) If B is a real or complex eigenbivector of E and $|B| = 0$, either B arises from a non-simple elementary divisor or its eigenvalue is degenerate [9, 10]. If a real or complex eigenbivector B arises from a non-simple elementary divisor of order $n \geq 2$, $|B| = 0$. If either B is real and not in $\widetilde{S}_m$ or B is complex but not a (complex) linear combination of members of $\overset{+}{S}_n$ or of $\overset{-}{S}_n$, B and $\overset{*}{B}$ are independent complex eigenbivectors arising from elementary divisors of the same order n and so $n \leq 3$. To see this note that, from the Jordan canonical form, one has $EB = \alpha B$ and the existence of a complex bivector A such that $EA = \alpha A + B$. Since, in an obvious notation, $AEB = BEA$, one immediately obtains $|B| = 0$. If B is associated with an elementary divisor of order n the spanning members of a Jordan block for B, when acted upon by the duality operator, yield spanning members of a Jordan block for $\overset{*}{B}$. This follows since if $B, A, ..$ span the Jordan block for B consider the spans $U = \mathrm{Sp}(B, A, ...)$ and $V = \mathrm{Sp}(\overset{*}{B}, \overset{*}{A}, ...)$ each of which is invariant for the map f. Then if $W \equiv U \cap V$, which is invariant for f, is non-empty it must contain B since B is a member of any invariant subspace of U. But since B and $\overset{*}{B}$ are independent and $\overset{*}{B}$ is the only eigenbivector of E in V, $B \notin V$ and so $U \cap V = \{0\}$ and $B, A, ...\overset{*}{B}, \overset{*}{A}, ...$ are independent and the orders of the elementary divisors are equal (and ≤ 3).

(*e*) Let P and $\overset{*}{P}$ be a pair of real null eigenbivectors of E with associated elementary divisors simple and with real eigenvalues α and $-\alpha$. If the pair $(P, \overset{*}{P})$ is the only such pair the eigenvalue is degenerate ($\alpha = 0$), and one may replace $(P, \overset{*}{P})$ by the pair of totally null eigenbivectors $P \pm \overset{*}{P}$ of E and this applies to any such pair with zero eigenvalue. If a pair exists with non-zero eigenvalues and simple elementary divisors then, by degeneracy, at least one other null pair must exist (not necessarily with simple elementary divisors) so that $(P, \overset{*}{P})$ and $(Q, \overset{*}{Q})$ are two independent pairs of null eigenbivectors with eigenvalue pairs $(\alpha, -\alpha)$ and $(\beta, -\beta)$, $(\alpha \neq 0)$. If each has an associated simple elementary divisor then, by the necessary eigenvalue degeneracy, $\alpha = \pm\beta$ and (say $\alpha = \beta$). Then $P, Q, \overset{*}{P}$ and $\overset{*}{Q}$ are independent and, since $\alpha \neq -\alpha$, $P \cdot \overset{*}{Q} = \overset{*}{P} \cdot Q = 0$. Suppose, in addition, $P \cdot Q = 0$. Then $\overset{*}{P} \cdot \overset{*}{Q} = 0$ and $P + \overset{*}{P}$ and $Q + \overset{*}{Q}$ are then orthogonal and totally null members of $\overset{+}{S}_n$. Thus they are proportional and so one obtains the contradiction that $P, \overset{*}{P}, Q$ and $\overset{*}{Q}$ are dependent. If $P \cdot Q \neq 0$, $P \pm Q$ are independent eigenbivectors of E with non-zero eigenvalues and are hence simple with $|P \pm Q|$ each non-zero and with opposite signs since $|P| = |Q| = 0$. It follows that $(P \pm Q)$ give a spacelike and a timelike pair of eigenbivectors. Thus if two duals pair of null eigenbivectors exist with non-zero eigenvalues and simple elementary divisors, a pair of spacelike and a pair of timelike (simple) eigenbivectors exist.

(*f*) Suppose f admits a complex eigenbivector B so that $EB = zB$ with $z = a + ib \in \mathbb{C}$ with $b \neq 0$. Then $E\bar{B} = \bar{z}\bar{B}$, $E\overset{*}{B} = -z\overset{*}{B}$ and $E\overset{*}{\bar{B}} = -\bar{z}\overset{*}{\bar{B}}$ so that $B, \bar{B}, \overset{*}{B}$ and $\overset{*}{\bar{B}}$ are also eigenbivectors of E. If $a \neq 0$ these complex eigenvalues are distinct and the associated four eigenbivectors are independent. If the eigenvalues are $\pm ib$ and $\pm id$ $(0 \neq b \neq \pm d \neq 0)$ again four independent eigenbivectors occur. In each of these cases, recalling that E must admit at least one real eigenvalue, the corresponding elementary divisors are simple and the Segre type for f is $\{11z\bar{z}w\bar{w}\}$ or $\{2z\bar{z}w\bar{w}\}$. The latter is impossible because the non-simple elementary divisor has eigenvalue zero and the rank of E would be 5, contradicting lemma 4.1. If the complex eigenbivector is $B = C + iD$ for real bivectors C, D, $(C + iD) \cdot (C - iD) = 0$ and so $|C| + |D| = 0$. These results together with (*d*) complete the discussion when complex eigenvalues occur.

The following Segre types for E, whatever the eigenvalue degeneracies, are now impossible: $\{z\bar{z}w\bar{w}\rho\bar{\rho}\}$, {21111}, $\{211z\bar{z}\}$, $\{2z\bar{z}w\bar{w}\}$, {2′1111}, $\{2'11z\bar{z}\}$, $\{2'z\bar{z}w\bar{w}\}$, $\{2'2'z\bar{z}\}$, {2′211}, $\{2'2z\bar{z}\}$, {222}, {2′2′2}, {2′22}, {2′2′2′}, {321}, {32′1}, {3′111}, $\{3'1z\bar{z}\}$, {3′21}, {3′2′1}, {3′3′}, {411}, {4′11}, $\{4z\bar{z}\}$, $\{4'z\bar{z}\}$, {42′}, {4′2}, {4′2′}, {5′1}, {6} and {6′}. These follow from (b), (f) and by noting the zero eigenvalues which arise and using lemma 4.1. Thus the remaining possibilities for E are {111111}, $\{1111z\bar{z}\}$, $\{11z\bar{z}w\bar{w}\}$, {2211}, {2′2′11}, {3111}, $\{31z\bar{z}\}$, $\{22z\bar{z}\}$, {42}, {51} and {33}. Of these, {42} and $\{22z\bar{z}\}$ will be eliminated later.

5.2. Timelike Eigenbivectors

A number of the remaining possibilities permitted by part (g) above admit an orthogonal, dual pair of simple, timelike eigenbivectors and once this is established they can be conveniently handled together.

Lemma 5.1. *If E has general (that is, ignoring degeneracies) Segre type* {111111}, $\{1111z\bar{z}\}$, $\{11z\bar{z}w\bar{w}\}$, {2211}, {2′2′11} *or* {3111} *at $m \in M$ it admits an orthogonal dual pair of simple, timelike eigenbivectors.*

Throughout this proof it is noted that if E admits (an orthogonal pair of) simple spacelike eigenbivectors, they are invariant for T and so T is diagonalisable and thus admits an orthogonal pair of timelike invariant 2−spaces. It then follows that E admits a pair of simple timelike eigenbivectors. *So it will be supposed throughout this proof that, for the above types, E does not admit any (simple) spacelike or timelike eigenbivectors and a contradiction will be achieved for each of the above types.* With this assumption the (real) eigenbivectors of E consist of either simple, null, dual pairs together with eigenbivectors in $\widetilde{S}_m$ each with zero eigenvalue and with equal numbers of independent ones in each of $\overset{+}{S}_m$ and $\overset{-}{S}_m$ by lemma 4.1. (It is remarked that one cannot rule out a non-simple eigenbivector B with $B \notin \widetilde{S}_m$ with necessarily zero eigenvalue, $EB = 0$. But then $B \pm \overset{*}{B}$ would constitute an independent pair of eigenbivectors of E in $\widetilde{S}_m$ with zero eigenvalue.) The results listed above will be freely used in what is to follow.

If E has type {111111} then, by assumption, its eigenbivectors are either in $\widetilde{S}_m$ with zero eigenvalue or occur in null pairs. If an example of the latter occurs, and it is the only one, its necessarily degenerate eigenvalue is zero, and then this pair can be replaced with two totally null eigenbivectors in $\widetilde{S}_m$. Similar comments apply if other null pairs exist with zero eigenvalues. Otherwise there exist two independent such null pairs $P, \overset{*}{P}$ and $Q, \overset{*}{Q}$ with (equal) eigenvalues $\pm\alpha \neq 0$ and a contradiction follows since now pairs of spacelike and timelike eigenbivectors exist for E (see 5.1(e) above). So all eigenbivectors may be assumed to be in $\widetilde{S}_m$ each with zero eigenvalue. Since E is diagonalisable over $\mathbb{R}$ the contradiction $E = 0$ follows and so this type for E admits a timelike pair of simple eigenbivectors.

If E has type {2211} let the simple elementary divisors correspond to a null pair of eigenbivectors R and $\overset{*}{R}$ with associated eigenvalues $\pm\alpha$ or a totally null pair of eigenbivectors, one in each of $\overset{\pm}{S}_m$ with eigenvalues (regarded with an abuse of notation as $\pm\alpha = 0$) and similarly let the non-simple elementary divisors correspond to a null pair of eigenbivectors P and $\overset{*}{P}$ with eigenvalues $\pm\beta$ or a pair of totally null eigenbivectors, one in each of $\overset{\pm}{S}_m$ with eigenvalue zero. By degeneracy, one has either $\alpha = 0 \neq \beta$, $\alpha = \beta = 0$ or $\alpha = \pm\beta \neq 0$. In the first case P and $\overset{*}{P}$ are each orthogonal to R and $\overset{*}{R}$ and so, as before, $P + \overset{*}{P}$ is proportional to $R + \overset{*}{R}$ and independence is contradicted. In the second case one has two independent eigenbivectors in $\overset{+}{S}_m$ and two in $\overset{-}{S}_m$ all with zero eigenvalue and, as a consequence one has eigenbivectors B and B' with $|B| < 0 > |B'|$ (see section 2(ii)) and the contradiction that a pair of simple, timelike eigenbivectors for E exists. The third case also leads to a similar contradiction from the results above.

If E has type $\{1111z\bar{z}\}$ one uses the argument in the type {111111} case and the results above to show E admits two independent eigenbivectors in each of $\overset{+}{S}_m$ and $\overset{-}{S}_m$ each with zero eigenvalue and the result follows as in the last proof.

If E has type $\{3111\}$ the non-simple elementary divisor arises from a totally null eigenbivector say $G \in \overset{+}{S}_m$ with zero eigenvalue whilst the simple ones may be chosen as $P \in \overset{+}{S}_m$ and $Q, R \in \overset{-}{S}_m$, each with zero eigenvalue. Again the result follows by considering $\mathrm{Sp}(G, P)$ or $\mathrm{Sp}(Q, R)$ as in the $\{2211\}$ case.

If E is of type $\{11z\bar{z}w\bar{w}\}$ one has a complex eigenbivector pair $A \pm iB$ (with eigenvalues $a \pm ib$, ($a, b \in \mathbb{R}$ and $b \neq 0$) and another complex eigenbivector pair $C \pm iD$. The real (simple) elementary divisors arise either directly from a pair of eigenbivectors $X, Y \in \widetilde{S}_m$ or from a pair of null eigenbivectors with (degenerate) eigenvalue zero and which can also be regarded as giving rise to a pair of eigenbivectors $X \in \overset{+}{S}_m$ and $Y \in \overset{-}{S}_m$ but in this case they are necessarily totally null. However, E must admit a simple real eigenbivector and so one may assume that X is a totally null eigenbivector in $\overset{+}{S}_m$ with zero eigenvalue. The orthogonality of the complex eigenbivectors $A \pm iB$ gives $|A| + |B| = 0$ and their orthogonality with X gives $X \cdot A = X \cdot B = 0$ and hence $X \cdot \overset{*}{A} = X \cdot \overset{*}{B} = 0$. Thus $A \pm \overset{*}{A}$ and $B \pm \overset{*}{B}$ are orthogonal to X and, since X is totally null, it follows that $|A \pm \overset{*}{A}| \leq 0$ and $|B \pm \overset{*}{B}| \leq 0$. Since $A \cdot A = \overset{*}{A} \cdot \overset{*}{A}$ these give $|A| \leqslant 0$, $|B| \leqslant 0$ and so, since $|A| + |B| = 0$, $|A| = |B| = 0$. Then since $|A \pm \overset{*}{A}| \leq 0$ and $|B \pm \overset{*}{B}| \leq 0$, $A \cdot \overset{*}{A} = B \cdot \overset{*}{B} = 0$ and so A and B are simple. So $|A + \overset{*}{A}| = |B + \overset{*}{B}| = 0$ and hence $A + \overset{*}{A}$ and $B + \overset{*}{B}$ are real multiples of X. Now $E(A + iB) = z(A + iB)$ and $E(\overset{*}{A} + i\overset{*}{B}) = -z(\overset{*}{A} + i\overset{*}{B})$ with $z = a + ib \neq 0$. So adding these and using the fact that $EX = 0$ gives $z(A - \overset{*}{A} + iB - i\overset{*}{B}) = 0$ and so $A = \overset{*}{A}$ and $B = \overset{*}{B}$. A back substitution into the eigen equations for E then gives the contradiction that $z = 0$ ($EA = EB = 0$). So, again, E admits a pair of simple timelike eigenbivectors.

If E is of type $\{2'2'11\}$ there are exactly two distinct complex eigenvalues $\pm ib$ (lemma 2(d)) and they have non-simple elementary divisors corresponding to the complex eigenbivectors $A \pm iB$, the latter being orthogonal and $|A \pm iB| = 0$. Thus $|A| = |B| = A \cdot B = 0$. Since $E(A \pm iB) = \pm ib(A \pm iB)$ it follows that $E(\overset{*}{A} \pm i\overset{*}{B}) = \mp ib(\overset{*}{A} \pm i\overset{*}{B})$ and so $\overset{*}{A} + i\overset{*}{B} = \mu(A - iB)$ for $\mu \in \mathbb{C}$. Thus $A \cdot (\overset{*}{A} + i\overset{*}{B}) = B \cdot (\overset{*}{A} + i\overset{*}{B}) = 0$ and hence $A \cdot \overset{*}{A} = B \cdot \overset{*}{B} = A \cdot \overset{*}{B} = B \cdot \overset{*}{A} = 0$ and so $|A \pm \overset{*}{A}| = |B \pm \overset{*}{B}| = (A \pm \overset{*}{A}) \cdot (B \pm \overset{*}{B}) = 0$ (since $A \cdot B = 0 \Rightarrow \overset{*}{A} \cdot \overset{*}{B} = 0$) showing that $(A \pm \overset{*}{A})$ and $(B \pm \overset{*}{B})$ are totally null with $(A + \overset{*}{A})$ (respectively $(A - \overset{*}{A})$) proportional to $(B + \overset{*}{B})$ (respectively, $(B - \overset{*}{B})$). Now consider the eigenbivectors arising from the simple elementary divisors. If they are a null dual pair their eigenvalues are degenerate and hence zero and so they may be taken as a totally null pair, $R \in \overset{+}{S}_m$ and $S \in \overset{-}{S}_m$. Otherwise they are general members $R \in \overset{+}{S}_m$ and $S \in \overset{-}{S}_m$ also with zero eigenvalues. (It is impossible that they both lie in $\overset{+}{S}_m$ or both in $\overset{-}{S}_m$ since then the rank of the map f associated with E would be 2 (lemma 4.1) but, in fact, it is 4.) They satisfy $A \cdot R = A \cdot S = B \cdot R = B \cdot S = 0$ and hence $\overset{*}{A} \cdot R = \overset{*}{A} \cdot S = \overset{*}{B} \cdot R = \overset{*}{B} \cdot S = 0$ (section 2(iv)) and so $R \cdot (A + \overset{*}{A}) = S \cdot (A - \overset{*}{A}) = R \cdot (B + \overset{*}{B}) = S \cdot (B - \overset{*}{B}) = 0$. So, since $A \pm \overset{*}{A}$ and $B \pm \overset{*}{B}$ are totally null, $|R| \leqslant 0$ and $|S| \leqslant 0$. The equality option here is forbidden since then R is a multiple of $A + \overset{*}{A}$ and also of $B + \overset{*}{B}$ and then the complex eigen equation gives $EA = -bB$, $EB = bA$, $E\overset{*}{A} = b\overset{*}{B}$ and $E\overset{*}{B} = -b\overset{*}{A}$. Adding and remembering that $A + \overset{*}{A}$ and $B + \overset{*}{B}$ are multiples of R and that $ER = 0$ shows that $A = \overset{*}{A}$ and $B = \overset{*}{B}$ so that $A, B \in \overset{+}{S}_m$. but then one calculates that $E(A + iB) = 0$ giving the contradiction that $b = 0$. A similar contradiction arises from the equality option and the bivector S. Thus $|R| < 0$ and $|S| < 0$ and so R and S lead to a dual pair of simple timelike eigenbivectors (section 2(ii)) which is a contradiction to the initial assumption. □

Each of the types in Lemma 5.1 were shown to admit a dual pair of simple timelike eigenbivectors which may be taken as $l \wedge n$ and $L \wedge N$ in the usual basis and with respective eigenvalues, say, α and $-\alpha$. So for each of these types choose a basis $G, H, K, \bar{G}, \bar{H}, \bar{K}$ for $\Lambda_m M$ as in section 2 with $K = l \wedge n - L \wedge N$ and $\bar{K} = l \wedge n + L \wedge N$ and expand E as

$$
\begin{aligned}
E = a(G, \bar{G}) + b(G, \bar{H}) + c(G, \bar{K}) + d(H, \bar{G}) + e(H, \bar{H}) + f(H, \bar{K}) \\
+ g(K, \bar{G}) + h(K, \bar{H}) + k(K, \bar{K}) \qquad (8)
\end{aligned}
$$

with $a, ..., k \in \mathbb{R}$. One has $f(K) = \alpha\bar{K}$ and $f(\bar{K}) = \alpha K$ and so $g = h = c = f = 0$ and $k = -\frac{\alpha}{4}$. Now f maps the (bivector) subspace spanned by $G, H, \bar{G}$ and $\bar{H}$ into itself (and also the subspace spanned by $K, \bar{K}$ into itself) and one can consider its (restricted) characteristic equation for the former subspace. One finds

$$f(G) = 2d\bar{G} + 2e\bar{H}, \qquad f(\bar{G}) = 2bG + 2eH,$$
$$f(H) = 2a\bar{G} + 2b\bar{H}, \qquad f(\bar{H}) = 2aG + 2dH, \tag{9}$$

and so the characteristic equation for the eigenvalues λ is

$$\lambda^4 - 2(ae + bd)\lambda^2 + (bd - ae)^2 = 0 \tag{10}$$

with solution $\lambda^2 = (ae + bd) \pm 2\sqrt{(ae)(bd)}$ (in addition to the two real ones from the timelike eigenbivectors and which could each be zero). The solutions may be divided into six simple cases (up to exchanges of the products ae and bd); (i) $ae > 0 < bd$, (ii) $ae > 0 > bd$, (iii) $ae < 0 > bd$, (iv) $ae = 0 < bd$, (v) $ae = 0 > bd$ and (vi) $ae = bd = 0$. For (i) $\lambda^2 = (\sqrt{ae} \pm \sqrt{bd})^2$ and either four distinct real solutions for λ arise ($ae \neq bd$) or real eigenvalues of the form $\pm 2\sqrt{ae}, 0, 0$ ($ae = bd$). In this case E is, up to degeneracies, of type $\{111111\}$. In (ii) four distinct complex solutions for λ occur and E is of type $\{11z\bar{z}w\bar{w}\}$ or $\{(11)z\bar{z}w\bar{w}\}$. In (iii) $\lambda^2 = -(\sqrt{|ae|} \pm \sqrt{|bd|})^2$ and so either four distinct complex solutions for λ arise of the form $\pm i\mu, \pm i\nu$, ($\mu, \nu \in \mathbb{R}$, $ae \neq bd$) or the solutions for λ are of the form $\pm i\mu, 0, 0$ ($ae = bd$). Thus E is of type $\{11z\bar{z}w\bar{w}\}$, $\{11(11)z\bar{z}\}$ or $\{(1111)z\bar{z}\}$. For (iv) $\lambda^2 = bd > 0$ and there are two (repeated) real solutions for λ and E is, up to degeneracies, of type $\{(11)(11)11\}$ ($a = e = 0$) and of type $\{2211\}$ if $e = 0 \neq a$ with a Jordan basis containing $(G \pm \mu\bar{G})$ and $(H \pm \mu^{-1}\bar{H})$ ($\mu = \sqrt{\frac{d}{b}}$) from (9) in the latter case and with $(G \pm \mu\bar{G})$ a pair of null eigenbivectors. For (v) $\lambda^2 = bd < 0$ and there are two (repeated) complex solutions for λ of the form $\pm ir$ ($0 \neq r \in \mathbb{R}$) and E is, up to degeneracies, of type $\{11(zz)(\bar{z}\bar{z})\}$ ($a = e = 0$) or $\{2'2'11\}$ ($a \neq 0 = e$) with $(G + i\nu\bar{G})$ and $(H + i\nu^{-1}\bar{H})$ ($\nu = \sqrt{|\frac{d}{b}|}$), together with their conjugates, part of a Jordan basis for E in the latter case and $(G + i\nu\bar{G})$ complex eigenbivectors for E.

For (vi), if $a = e = b = d = 0$, E has type $\{(1111)11\}$ or its degeneracy whilst if $e = b = d = 0 \neq a$, E has type $\{(22)11\}$ or $\{(2211)\}$ with Jordan basis containing $G, \bar{H}, \bar{G}, H$. If $a \neq 0 \neq d, e = b = 0$, E has type $\{(31)11\}$ or $\{(3111)\}$ with Jordan basis containing $\bar{G}$, $aG + dH$, $\bar{H}$ and $aG - dH$. It is noted that the general type $\{3111\}$ must contain (at least) the degeneracy $\{(31)11\}$.

Now from (8) with $g = h = c = f = 0, k = \frac{-\alpha}{4}$, as above, and the definitions of $G, H, K, \bar{G}, \bar{H}, \bar{K}$ from section 2, and recalling that $\widetilde{T}_{ab} = E^c{}_{acb}$, one finds

$$\widetilde{T}_{ab} = 2al_al_b + 2bN_aN_b + 2dL_aL_b + 2en_an_b - \frac{\alpha}{2}(-l_an_b - n_al_b + L_aN_b + N_aL_b) \tag{11}$$

Thus the linear map f' from $\widetilde{T}$ satisfies in the l, n, L, N basis

$$f'(l) = 2e.n + \frac{\alpha}{2}l, \qquad f'(n) = 2a.l + \frac{\alpha}{2}n$$
$$f'(L) = 2b.N - \frac{\alpha}{2}L, \qquad f'(N) = 2d.L - \frac{\alpha}{2}N \tag{12}$$

Thus if $ae > 0$ (respectively, $ae < 0$), it is easily checked that $\widetilde{T}$ admits two distinct real (respectively complex) eigenvalues for the plane $l \wedge n$ and similarly for bd and the plane $L \wedge N$. If, however, $a = e = 0$, $\widetilde{T}$ admits $l \wedge n$ as an $\frac{\alpha}{4}$-eigenspace (and similarly for the plane $L \wedge N$). If exactly one of a and e is zero then $l \wedge n$ contains only one real eigendirection for $\widetilde{T}$ and yields a real non-simple elementary divisor of order two in that plane. If exactly one of a and e *and* exactly one of b and d is zero $\widetilde{T}$ has Segre type $\{22\}$.

It follows from this argument (and those to follow) that if E is diagonalisable over $\mathbb{R}$ so also is $\widetilde{T}$, and conversely from section 5.1. Similarly, if E is diagonalisable over $\mathbb{C}$ but not over $\mathbb{R}$ the same is true for $\widetilde{T}$ and conversely. This permits a complete correspondence between the type of E and that of $\widetilde{T}$. It will be included in the (more complete) list below.

5.3. Further Cases

Lemma 5.2. *The type* $\{22z\bar{z}\}$ *for* E *is impossible.*

Proof. In this type let the complex eigenbivectors be $A \pm iB$ with eigenvalues $\pm ib$ ($b \in \mathbb{R}$). The two non-simple elementary divisors either correspond to a dual pair of null eigenbivectors P and $\overset{*}{P}$ or to two totally null eigenbivectors $R, S \in \widetilde{S}_m$ with zero eigenvalue. In the latter case one must take $R \in \overset{+}{S}_m$ and $S \in \overset{-}{S}_m$. If the complex eigenbivectors are $A \pm iB$ one sees, as in an earlier case, that $|A| + |B| = 0$ and that, taking both cases together, $A + \overset{*}{A}$ and $B + \overset{*}{B}$ are each totally null and proportional to R (or to $P + \overset{*}{P}$) and $A - \overset{*}{A}$ and $B - \overset{*}{B}$ are proportional to S (or to $P - \overset{*}{P}$). From this follows the contradiction that $A + iB$, R and S (or $A + iB$, P and $\overset{*}{P}$) are dependent. □

Now consider the remaining cases for E. They are types $\{31z\bar{z}\}$, $\{51\}$ and $\{33\}$ (together with type $\{42\}$ which will be shown to be impossible) and they do not admit a simple pair of timelike eigenbivectors. If E has type $\{31z\bar{z}\}$ the non-simple elementary divisor corresponds to a totally null eigenbivector in $R \in \overset{+}{S}_m$) with zero eigenvalue, the (real) simple elementary divisor must correspond to an eigenbivector $S \in \overset{-}{S}_m$ also with zero eigenvalue and there is also a complex pair of $A \pm iB$ satisfying $|A| + |B| = 0$. Thus the Segre type is necessarily the degenerate case $\{(31)z\bar{z}\}$. As before, $R \cdot (A + \overset{*}{A}) = R \cdot (B + \overset{*}{B}) = 0$ and $S \cdot (A - \overset{*}{A}) = S \cdot (B - \overset{*}{B}) = 0$. If $|S| = 0$, so that S is also totally null, one gets $|A \pm \overset{*}{A}| \leqslant 0$ and $|B \pm \overset{*}{B}| \leqslant 0$. Then, as before, one is lead to a contradiction. So suppose $|S| > 0$. This gives $|A - \overset{*}{A}| < 0$ and $|A + \overset{*}{A}| \leq 0$ and similarly for B which, when added, contradict $|A| + |B| = 0$. So $|S| < 0$ and one can choose a basis including $R \equiv G$ and $S \equiv \bar{K}$ as in (8) and use the results $EG = 0$, $E\bar{K} = 0$ to get $c = d = e = f = k = 0$ and so

$$\begin{aligned} &EG = 0, \qquad EH = 2a\bar{G} + 2b\bar{H}, \qquad EK = -4g\bar{G} - 4h\bar{H}, \\ &E\bar{K} = 0, \qquad E\bar{G} = 2bG + 2hK, \qquad E\bar{H} = 2aG + 2gK. \end{aligned} \tag{13}$$

and the characteristic equation for E is easily calculated as $\lambda^4(\lambda^2 + 16gh) = 0$. Clearly one here needs $gh > 0$ for complex eigenvalues for E. The tensor $\widetilde{T}$ can be recovered from $\widetilde{T}_{ab} = E^c{}_{acb}$ and its only eigenvectors are two independent complex null vectors in the 2−space spanned by l and N and thus has type $\{2'2'\}$. (If $gh < 0$ one reproduces one of the previous cases when E has type $\{(31)11\}$ and if $gh = 0$ the next case applies.)

If E has type $\{51\}$ the non-simple elementary divisor must correspond to a totally null eigenbivector taken as $G \in \overset{+}{S}_m$ with eigenvalue zero and the simple elementary divisor to an eigenbivector Q necessarily in $\overset{-}{S}_m$ (otherwise E would have rank 2 and a contradiction would follow) and with zero eigenvalue. This type is then necessarily $\{(51)\}$ since $EG = EQ = 0$. For the non-simple elementary divisor of order 5 one has, from its Jordan basis, bivectors A, B, C, D satisfying $EG = 0$, $EA = G$, $EB = A$, $EC = B$ and $ED = C$. From these one has the relations $BEG = GEB$ ($\Rightarrow G \cdot A = 0$), $BEQ = QEB$ ($\Rightarrow Q \cdot A = 0$), $CEG = GEC$ ($\Rightarrow G \cdot B = 0$) and $BEA = AEB$ ($\Rightarrow G \cdot B = |A|$) and hence $|A| = |\overset{*}{A}| = 0$). (It is noted that only the relations for a non-simple elementary divisor of order 4 are used here.) It follows that $G \cdot A = G \cdot \overset{*}{A} = Q \cdot A = Q \cdot \overset{*}{A} = 0$ and so $G \cdot (A + \overset{*}{A}) = Q \cdot (A - \overset{*}{A}) = 0$ and so $|A + \overset{*}{A}| \leq 0$. Suppose Q is a totally null member of $\overset{+}{S}_m$. Then $|A - \overset{*}{A}| \leq 0$ and it follows that $A \cdot \overset{*}{A} = 0$ and hence that $|A \pm \overset{*}{A}| = 0$. This means that A is a linear combination of G and Q and one obtains the contradiction $EA = 0$. (It is noted here that in the case when E has Segre type $\{42\}$ Q is necessarily totally null and so *this type for E is impossible* and this completes the claims made in lemma 2.) Continuing in the case $\{51\}$ and assuming $|Q| > 0$ one finds $|A - \overset{*}{A}| < 0$ (from the Lorentz signature in $\overset{-}{S}_m$) but since it is already known that $|A + \overset{*}{A}| \leq 0$ and that $|A| = |\overset{*}{A}| = 0$, one obtains a contradiction on the value of $A \cdot \overset{*}{A}$. Thus $|Q| < 0$. With $G = l \wedge N$ one can choose $Q = \bar{K}$ in the notation of section 2(iv) and, in (8) obtain $c = d = e = f = k = 0$ and hence (13), as in the previous case. Now the product gh cannot be positive or negative since these choices lead to types for E different from {51} as shown in the last paragraph. So $gh = 0$. Thus either $g = 0$ or $h = 0$, but not both, since $g = h = 0$ gives the contradiction $EK = 0$ in (13). A consideration of the tensor $\widetilde{T}$, using (3), shows that it admits a single (necessarily real, null) eigenvector and hence has Segre type {4}.

The final case is when E has Segre type {33}. Here the two independent eigenbivectors can be taken as a null, dual pair Q and $\overset{*}{Q}$ with non-zero eigenvalues $\pm\alpha$ (lemma 2(a, c)) or, if $\alpha = 0$, as $G \in \overset{+}{S}_m$ and $\bar{G} \in \overset{-}{S}_m$ (the {(33)} case). Thus $EQ = \alpha Q$ and $E\overset{*}{Q} = -\alpha\overset{*}{Q}$, that is, $E(Q \pm \overset{*}{Q}) = \alpha(Q \mp \overset{*}{Q})$. Then with $G \equiv (Q + \overset{*}{Q})$ and $\bar{G} \equiv (Q - \overset{*}{Q})$ one has $EG = \alpha\bar{G}$ and $E\bar{G} = \alpha G$ and, in (8), $e = f = h = 0$, $2d = \alpha$ and $2b = \alpha$. Suppose $\alpha \neq 0$. Then a change of bivector basis of the form $G' = G$, $H' = H + \mu^2 G - \mu K$ and $K' = K - 2\mu G$ for $\mu \in \mathbb{R}$ (and similarly for $\bar{G}$, $\bar{H}$ and $\bar{K}$) shows that one may choose μ so that in the new basis $c = g$. If $c = g = 0$ one gets the contradiction that $K \pm \bar{K}$ are eigenbivectors and so one may assume $c = g \neq 0$. A consideration of the characteristic equation for E then shows that in order to have two triply repeated eigenvalues $\pm\alpha$ one must take $4k = \pm\alpha$. However, the choice $4k = \alpha$ reveals that $(K + \bar{K}) + \frac{2c}{\alpha}(G + \bar{G})$ is also an eigenbivector of E and this contradiction shows that $4k = -\alpha$. (Alternatively, one could consider the minimal polynomial of E.) A Jordan basis for E is then $G + \bar{G}$, $K + \bar{K}$, $H + \bar{H}$, $G - \bar{G}$, $K - \bar{K}$ and $H - \bar{H}$. In the case that $\alpha = 0$ the minimal polynomial condition that $f \circ f \circ f$ is the zero map (but $f \circ f$ and f are not) can be checked to be equivalent to $ck = 0$ and hence to $k = 0$ since $c \neq 0$. The resulting expression for T in the corresponding tangent space basis l, n, L, N, after modification to l, n, y, s where $y = \frac{1}{2}(L + N)$ and $s = \frac{1}{2}(L - N)$, gives $f'(l) = \frac{\alpha}{4}l$, $f'(n) = al - 2cy + \frac{\alpha}{4}n$, $f'(y) = \frac{\alpha}{4}y - cl$, $f'(s) = -\frac{3\alpha}{4}s$ and so T has Segre type {31} with Jordan basis l, y, n, s. If $\alpha = 0$ E has Segre type {(33)} and T has Segre type {(31)}.

5.4. The Main Result

The relationship between the Segre types of E and T can now be given in terms of the cases (i)–(vi) of section 5.2, where a simple pair of timelike eigenbivectors for E exist, and the remaining cases of section 5.3. The Segre types will be given up to degeneracies, these being easily calculated.

Theorem 5.3. *For case (i) E has (Segre) type {111111} and T has type {1111}.*
For case (ii) E has type $\{11z\bar{z}w\bar{w}\}$ *and T has type* $\{z\bar{z}11\}$.
For case (iii) E has type $\{11z\bar{z}w\bar{w}\}$ *or* $\{1111z\bar{z}\}$ *and* $\widetilde{T}$ *has type* $\{z\bar{z}w\bar{w}\}$
For case (iv) E has type {2211} and T has type {(211)} ($a = 0 \neq e$)*, or, E has type {(11)(11)11} and T has type {(11)11}* ($a = e = 0$).
For case (v) E has type {2′2′11} and T has type $\{2z\bar{z}\}$ ($a = 0 \neq e$)*, or E has type* $\{11(zz)(\bar{z}\bar{z})\}$ *and T has type* $\{z\bar{z}(11)\}$ ($a = e = 0$).
For case (vi) E has type {(31)11} and $\widetilde{T}$ *has type {22} or E has type {(3111)} and* $\widetilde{T}$ *has type {(22)},* ($e = 0 \neq a, b = 0 \neq d$)*, or, E has type {(22)11} and T has type {2(11)}* ($= b = d = 0 \neq e$)*, or, E has type {(1111)11} and T has type {(11)(11)}* ($a = b = d = e = 0$).
If E has type $\{31z\bar{z}\}$*, (necessarily* $\{(31)z\bar{z}\}$*), T has type {2′2′}*
If E has type {51}, T has type {4}
If E has type {33}, T has type {31}.
These constitute all the possibilities for E and T.

It is easily checked that the eigenbivector structure of E is consistent with the remarks in section 5.1 and the description of the invariant 2−spaces of T. For each of the types {(51)} and $\{31z\bar{z}\}$ a single, simple (necessarily totally null and with associated elementary divisor non-simple) eigenbivector is admitted corresponding to the fact that when T has either type {4} or {2′2′} a single (necessarily totally null) invariant 2−space exists. (In each of these cases for E the only other (real) eigenbivector is a non-simple member of $\overset{-}{S}_m$ with associated simple elementary divisor.) These two cases may be distinguished by the fact that this single invariant 2−space for T admits either a single (real) independent null eigenvector for T (type {4}) or no such eigenvectors (type {2′2′}). The type $\{11z\bar{z}w\bar{w}\}$ for E (corresponding to types $\{z\bar{z}11\}$ and $\{z\bar{z}w\bar{w}\}$ for T—cases (ii) and (iii)) and the type {2′2′11} for E (corresponding to type $\{2z\bar{z}\}$ for T—case (v)) each have a single pair of (real, timelike) eigenbivectors for E (a single pair of timelike invariant 2−spaces, U and V, for T). These can be distinguished by noticing that the invariant 2−spaces for T yield either exactly two real eigendirections for T in U and exactly two complex ones in V ($\{z\bar{z}11\}$), no real eigendirections for T in either U or V ($\{z\bar{z}w\bar{w}\}$) or a single real eigendirection for T in U and none in V ($\{2z\bar{z}\}$). The situations where U and V admit different eigenstructures from these lead to extra eigenbivectors (extra invariant 2−spaces) for E (T).

6. Principal Null Directions

In this section it is shown how the algebraic relationship between E and $\widetilde{T}$ may be revealed (albeit it in a coarser, but sometimes more useful, form) by the use of principal null directions. The approach is similar to that used in the classification of the Weyl tensor for this signature [7] and was inspired by the work of L. Bel [1]. Suppose throughout this section that $E(m) \neq 0$ and consider the following equations for $0 \neq k \in T_mM$, some non-zero 1−form p at m and $\alpha \in \mathbb{R}$

$$E_{abcd}k^a k^c = \alpha k_b k_d \tag{14}$$

$$k_{[e}E_{a]bc[d}k_{f]}k^b k^c = 0 \qquad \Leftrightarrow \qquad E_{abcd}k^a k^c = k_b p_d + p_b k_d \tag{15}$$

In (14), if $\alpha \neq 0$, a contraction with k^b shows that k is necessarily null. However, if $\alpha = 0$ k will be *assumed* to be null. [This assumption is necessary since in this latter case k may not be null. To see this choose $x, y \in T_pM$ with $x \cdot x = y \cdot y = 1$ and $x \cdot y = 0$ and take $\widetilde{T}_{ab} = x_a y_b + y_a x_b$. Then $\widetilde{T}$ is symmetric and trace-free and (14) holds with $\alpha = 0$ and $k = y$ which is non-null. It is noted that this example applies to both Lorentz and neutral signatures.]

If the situation (14) holds (with k assumed null if $\alpha = 0$) the null direction spanned by k is called a *repeated principal null direction* (*repeated pnd*) for $E(m)$. The two equations in (15) are equivalent but here will be looked at in a slightly different way. If the second equation in (15) is augmented with the condition that p_a is *not* proportional to k_a and the first in (15) is augmented with the condition that (14) does *not* hold for k then these (augmented) restrictions are also equivalent. A contraction of the second (augmented) equation in (15) with k^b then shows that k is necessarily null and $p_a k^a = 0$. A solution k of either equation in (15) (whether augmented or not) is said to span a *general principal null direction* (*general pnd*) for $E(m)$. Thus a repeated pnd is a general pnd but a general pnd is not necessarily a repeated pnd. The following results link these special null directions of $E(m) \neq 0$ with the algebraic structure of $\widetilde{T}(m) \neq 0$ at m.

Theorem 6.1. (*i*) *If $E_{abcd}k^d = 0$ for $0 \neq k \in T_mM$ then k is null and this equation holds $\Leftrightarrow$ $\widetilde{T}$ has Segre type* $\{(211)\}$ *with unique null eigendirection spanned by k and eigenvalue zero.*

(*ii*) *If $E_{abcd}k^d = B_{ab}k_c$ for some non-zero bivector B and $0 \neq k \in T_mM$ then k is null, B is null or totally null with k in its blade and $\widetilde{T}$ has Segre type* $\{(31)\}$ *(for B null) or* $\{(22)\}$ *(for B totally null) where in each case k is a null eigenvector of T with eigenvalue zero. If B is null k is unique up to a scaling. Conversely if $\widetilde{T}$ has type* $\{(31)\}$ *(with necessarily zero eigenvalue since it is tracefree) with k a null eigenvector of $\widetilde{T}$ then the above equation holds with B null and k is unique up to a scaling. [The situation when B is totally null is a little more complicated.]*

(*iii*) *k is a repeated pnd of E if and only if k is a null eigenvector of $\widetilde{T}$,*

(*iv*) *If k is a general pnd of E, k satisfies $\widetilde{T}_{ab}k^a k^b = 0$. If k is null and satisfies $\widetilde{T}_{ab}k^a k^b = 0$ then k is a repeated or a general pnd of E.*

Proof. Part (*i*) follows from (2) by computing $E_{abcd}k^d$ and setting the result equal to zero. (The fact that k is necessarily null follows from the fact that ${}^*E_{abcd}k^d = 0$ is now also true (see lemma 1 in [7]). By performing contractions one first achieves $\widetilde{T}_{ab}k^a k^b = 0$, then $\widetilde{T}_{ab}k^b = 0$ and then, by a back substitution, $\widetilde{T}_{ac}k_b = \widetilde{T}_{bc}k_a$. The Segre type $\{(211)\}$ for $\widetilde{T}$ and the fact that the eigenvalue is zero then follows. The converse is trivial from (2).

For (*ii*) since $B \neq 0$ a contraction of the given condition with k^c shows that k is null and the condition $E_{[abc]d} = 0$ gives $B_{[ab}k_{c]} = 0$ and this is equivalent to B being simple with k in its blade. But one also has

${}^*E_{abcd}k^d = \overset{*}{B}_{ab}k_c$ and so $\overset{*}{B}_{[ab}k_{c]} = 0$ showing that k lies in the blade of $\overset{*}{B}$ also. Since k is null, it follows that B is null or totally null and so $B_{ab}k^b = 0$. If B is null choose a basis l, n, y, s in T_mM so that $k = l$ and $B = l \wedge y$ (the case $B = l \wedge s$ is similar). The fourth equation in (3) shows that $\widetilde{T}_{ab}l^b = 0$. Thus $(\widetilde{T}_{ac}l_b - \widetilde{T}_{bc}l_a) = 2B_{ab}l_c$ and a contraction with n^b gives $\widetilde{T}_{ac} = l_a n'_c - 2y_a l_c$ where $n'_a = \widetilde{T}_{ab}n^b$. Obvious contractions then reveal that $\widetilde{T}_{ab}y^b = -2l_a$ and $\widetilde{T}_{ab}s^b = 0$. It then follows that $l, y, n + \frac{a}{2}y, s$, where $a = \widetilde{T}_{ab}n^a n^b$, is a Jordan basis for $\widetilde{T}$ with the latter having Segre type {(31)} with eigenvalue zero. Since l is the unique null eigendirection for $\widetilde{T}$ the choice $k = l$ is unique. The converse is established by a back substitution. If B is totally null, say $B = l \wedge N$ in the usual basis, one finds $\widetilde{T}_{ab}l^b = \widetilde{T}_{ab}N^b = 0$, $\widetilde{T}_{ab}L^b = -2l_a$ and that $l, L, N, n + \frac{a}{2}L$ is a Jordan basis for $\widetilde{T}$ with the latter having Segre type {(22)} with eigenvalue zero.

For (*iii*) (14) holds with k null and a substitution into (2) gives $T_{ab}k^a k^b = 0$ and finally that k is an eigenvector of T. The converse is trivial. For (*iv*) the proof is almost the same. □

□

7. The Lorentz Case

It is interesting to compare the above results with those in the case of Lorentz signature and for which a (very brief) discussion was given in [2, 3, 8]). For this signature the proof is much easier since B and $\overset{*}{B}$ are independent for any real bivector B (that is, the algebras $\overset{+}{S}_m$ and $\overset{-}{S}_m$ are trivial). Also $\overset{**}{B} = -B$, ${}^*E = -E^*$ and ${}^*E^* = E$ for this signature. It is however, remarked that in the tangent space geometry, the orthogonal complement of a spacelike 2−space is timelike, and vice versa, and that there are no totally null 2−spaces. The equivalence between invariant 2−spaces of $\widetilde{T}$ and simple (real) eigenbivectors of E still holds and so E always admits a simple real eigenbivector B, $EB = \alpha B$ ($\alpha \in \mathbb{R}$). But for *any* real eigenbivector B of E with $EB = \alpha B$, $\overset{*}{B}$ is *always an independent* real eigenbivector of E, $E\overset{*}{B} = -\alpha\overset{*}{B}$. Thus E cannot have rank 5. It is easily checked from this that E has even rank. In fact if its rank is 3 one has independent bivectors $A, \overset{*}{A}, B$ with $EA = E\overset{*}{A} = EB = 0$. But then $E\overset{*}{B} = 0$ and $A, B, \overset{*}{A}, \overset{*}{B}$ are independent otherwise there exists $a, b, c, d \in \mathbb{R}$, not all zero, (but with $b \neq 0 \neq d$) such that $aA + bB + c\overset{*}{A} + d\overset{*}{B} = 0$. On taking the dual of this equation and substituting back one finds the contradiction $b = d = 0$. A similar extension of this argument shows that E cannot have rank 1 and the claimed result follows.

If E has a complex eigenvalue $z \equiv \alpha + i\beta$ ($\alpha, \beta \in \mathbb{R}, \beta \neq 0$) and $EB = zB$ then $\bar{B}, \overset{*}{B}$ and $\overset{*}{\bar{B}}$ are eigenbivectors with eigenvalues $\bar{z}, -z, -\bar{z}$. If $\alpha \neq 0$, $B, \bar{B}, \overset{*}{B}$ and $\overset{*}{\bar{B}}$ are clearly independent whilst if $\alpha = 0$, these eigenbivectors are again independent unless $\overset{*}{B} = \lambda\bar{B}$ for $\lambda \in \mathbb{C}$ (and hence $\overset{*}{\bar{B}} = \bar{\lambda}B$) from which one easily finds, using the (Lorentz) relation $\overset{**}{B} = -B$, the contradiction $|\lambda| < 0$. It follows that any complex eigenvalue is associated with a simple elementary divisor and, should one exist, the Segre type is $\{11z\bar{z}w\bar{w}\}$ or its degeneracy. Further if E admits a (necessarily real) eigenbivector B corresponding to a non-simple elementary divisor of order n for some positive integer n then $\overset{*}{B}$ is also an eigenbivector corresponding to a non-simple elementary divisor of order n (and $n \leq 3$). Also any eigenbivector corresponding to a non-simple elementary divisor is null and any null eigenbivector either corresponds to a non-simple elementary divisor or its eigenvalue is degenerate. It follows that the only possibilities for the Segre type of E are {111111}, $\{11z\bar{z}w\bar{w}\}$, {2211} and {33}, together with their possible degeneracies. Using (3) these types can be checked to correspond, respectively, to the Segre types {1111}, $\{11z\bar{z}\}$, {211} and {31}, or their possible degeneracies, for $\widetilde{T}$ (cf theorem 3.1). A discussion of the principal null directions for this case can be found in [8] and is essentially the same as for neutral signature (with the case of Segre type {(22)} excluded in the latter case).

It is finally remarked that in the positive definite case E and T are both diagonalisable over $\mathbb{R}$ and so this case is trivial, merely requiring degeneracies to be calculated.

8. Acknowledgements

The author gratefully acknowledges the financial support and hospitality of the organisers of the geometrical seminar XXI in Belgrade where this paper was given and a travel grant from the journal MDPI (Mathematics Section). He also thanks Bahar Kırık for her help in preparing the presentation.

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