Filomat 37:26 (2023), 8725–8745
https://doi.org/10.2298/FIL2326725A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Cocycle deformations for weak Hopf algebras

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Abstract. In this paper we introduce a theory of multiplication alteration by two-cocycles for weak Hopf algebras. We show that, just like it happens for Hopf algebras, if H a weak Hopf algebra and H^{σ} its weak Hopf algebra deformation by a 2-cocycle σ, there is a braided monoidal category equivalence between the categories of left-right Yetter-Drinfel'd modules ${}_{H}\mathcal{YD}^{H}$ and ${}_{H^{\sigma}}\mathcal{YD}^{H^{\sigma}}$. As a consequence we get in this context that the category $Rep(D(H))$ of left modules over the Drinfel'd double $D(H)$ can be identified with the category $Rep(D(H^{\sigma}))$ of left modules over the Drinfel'd double $D(H^{\sigma})$.

1. Introduction

Let R be a commutative ring with a unit and denote the tensor product over R by $\otimes$. In [20] we can find one of the first interesting examples of multiplication alteration by a 2-cocycle for R-algebras. In this case Sweedler proved that, if U is an associative unitary R-algebra with a commutative subalgebra A and $\sigma = \sum a_i \otimes b_i \otimes c_i \in A \otimes A \otimes A$ is an Amistur 2-cocycle, then U admits a new associative an unitary product defined by $u \bullet v = \sum a_i\, ub_i\, vc_i$ for all $u, v \in U$. Moreover, if U is central separable, U with the new product is still central separable and is isomorphic to the Rosenberg-Zelinsky central separable algebra obtained from the 2-cocycle σ^{-1} (see [19]). Later, Doi discovered in [8] a new contruction to modify the algebra structure of a bialgebra A over a field $\mathbb{F}$ using an invertible 2-cocycle σ in A. In this case if $\sigma : A \otimes A \to \mathbb{F}$ is the 2-cocycle, the new product on A is defined by

$$a * b = \sum \sigma(a_1 \otimes b_1) a_2 b_2 \sigma^{-1}(a_3 \otimes b_3)$$

for $a, b \in A$. With the new algebra structure and the original coalgebra structure, A is a new bialgebra denoted by A^{σ}, and if A is a Hopf algebra with antipode λ_A, so is A^{σ} with antipode given by

$$\lambda_{A^{\sigma}}(a) = \sum \sigma(a_1 \otimes \lambda_A(a_2)) \lambda_A(a_3) \sigma^{-1}(\lambda_A(a_4) \otimes a_5).$$

2020 *Mathematics Subject Classification*. 18D10, 16W30, 16T05.
Keywords. Weak Hopf algebra, 2-cocycle, Yetter-Drinfel'd module, Drinfel'd double.
Received: 25 September 2019; Accepted: 28 April 2023
Communicated by Dragan S. Djordjević
The present investigation was supported by Ministerio de Ciencia e Innovación of Spain. Agencia Estatal de Investigación. Unión Europea - Fondo Europeo de Desarrollo Regional (FEDER). Grant PID2020-115155GB-I00: Homología, homotopía e invariantes categóricos en grupos y álgebras no asociativas.
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for $a \in A$. This type of deformations appear frequently in relation to the classification problems of semisimple and non semisimple Hopf algebras. For example, as was showed in [4], all Hopf algebras whose coradical is an abelian group algebra are cocycle deformations of their associated graded Hopf algebras.

On the other hand, to study projections of Hopf algebras, Radford [18] establishes conditions that led to the notion of Yetter-Drinfel'd module introduced by Yetter [22] in order to explain the relationship between different theories in mathematics and physics, as low dimensional topology, knots and links, Hopf algebras, quantum integrable systems, and exactly solvable models in statistical mechanics. It is a well-known fact (see [12]) that every Yetter-Drinfel'd module gives rise to a solution of the quantum Yang-Baxter equation, and if H is a finite Hopf algebra in a symmetric monoidal category $\mathcal{C}$, the category ${}_H\mathcal{YD}^H$ of left-right Yetter-Drinfel'd modules is isomorphic to the one of modules over the quantum double (also called the Drinfel'd double), which was originally conceived to find solutions of the Yang-Baxter equation via universal matrices. The connection between Yetter-Drinfel'd modules and cocycle deformations of Hopf algebras was established by Majid and Oeckl [15] by giving a category equivalence between Yetter-Drinfel'd modules for H and those for H^{σ}.

In another vein, weak Hopf algebras (or quantum groupoids in the terminology of Nikshych and Vainerman [17]) were introduced by Böhm, Nill and Szlachányi [5] as a new generalization of Hopf algebras and groupoid algebras. The main difference with other Hopf algebraic constructions is that weak Hopf algebras are coassociative but the coproduct is not required to preserve the unit, equivalently, the counit is not a monoid morphism. There is a close connection between weak Hopf algebras and the theory of algebra extensions, and they have important applications in the study of dynamical twists of Hopf algebras and a deep link with quantum field theories and operator algebras, besides being an useful tool in the study of fusion categories in characteristic zero [10]. In addition there are innumerable examples, such groupoid algebras of finite groupoids and their duals. Also, Hayashi's face algebras (see [11]) are particular instances of weak Hopf algebras, whose counital subalgebras are commutative, and Yamanouchi's generalized Kac algebras [21] are exactly C^*-weak Hopf algebras with involutive antipode.

The main goal of this paper is to get the results related to cocycle deformations and Yetter-Drinfel'd modules cited above in the context of weak Hopf algebras. We have considered a definition of 2-cocycle inspired in the one we had already used in [2] and [3] in order to get the Sweedler cohomology for weak Hopf algebras. Our notion of 2-cocycle is essentially different to the one given by Chen, Zhang and Wang in [6] (see Remark 3.11), and it has the advantage of using a minimum number of conditions similar to those well-known in the Hopf algebra setting.

An outline of the paper is the following: after a section of preliminaries, in Section 3 we introduce a theory of cocycle deformations for weak Hopf algebras. In the main result of this section (Theorem 3.18) we show that, if $\sigma : H \otimes H \to K$ is a normal and convolution invertible 2-cocycle for a weak Hopf algebra H, the cocycle twist H^{σ} is also a weak Hopf algebra. In section 4, using the theory of Yetter-Drinfel'd modules for weak Hopf algebras developed in [1] and [16], we obtain the braided monoidal equivalence between the categories of Yetter-Drinfel'd modules for H and those for H^{σ} which implies a category equivalence between $D(H)$-modules and $D(H^{\sigma})$-modules, being $D(H)$ and $D(H^{\sigma})$ the Drinfel'd doubles associated to the weak Hopf algebras H and H^{σ}, respectively.

2. Preliminaries

From now on $\mathcal{C}$ denotes a strict symmetric category with tensor product denoted by $\otimes$ and unit object K. With c we will denote the natural isomorphism of symmetry and we also assume that $\mathcal{C}$ has equalizers. Then, under these conditions, every idempotent morphism $q : Y \to Y$ splits, i.e., there exist an object Z and morphisms $i : Z \to Y$ and $p : Y \to Z$ such that $q = i \circ p$ and $p \circ i = id_Z$. We denote the class of objects of $\mathcal{C}$ by $|\mathcal{C}|$ and for each object $M \in |\mathcal{C}|$, the identity morphism by $id_M : M \to M$. For simplicity of notation, given objects M, N, P in $\mathcal{C}$ and a morphism $f : M \to N$, we write $P \otimes f$ for $id_P \otimes f$ and $f \otimes P$ for $f \otimes id_P$.

We assume that the reader is familiar with the notion of algebra, coalgebra, module and comodule in a monoidal setting. For an algebra in $\mathcal{C}$, $A = (A, \eta_A, \mu_A)$, $\eta_A : K \to A$ denotes the unit and $\mu_A : A \otimes A \to A$ the product. If A, B are algebras in $\mathcal{C}$, the object $A \otimes B$ is an algebra in $\mathcal{C}$ where $\eta_{A\otimes B} = \eta_A \otimes \eta_B$ and

$\mu_{A\otimes B} = (\mu_A \otimes \mu_B) \circ (A \otimes c_{B,A} \otimes B)$. Similarly, for a coalgebra $D = (D, \varepsilon_D, \delta_D)$, $\varepsilon_D : D \to K$ denotes the counit and $\delta_D : D \to D \otimes D$ the coproduct. When D, E are coalgebras in C, $\delta_{D\otimes E} = (D \otimes c_{D,E} \otimes E) \circ (\delta_D \otimes \delta_E)$ is the coproduct of the coalgebra $D \otimes E$ and $\varepsilon_{D\otimes E} = \varepsilon_D \otimes \varepsilon_E$ its counit.

If A is an algebra, B is a coalgebra and $f : B \to A$, $g : B \to A$ are morphisms, we define the convolution product by $f * g = \mu_A \circ (f \otimes g) \circ \delta_B$.

By weak Hopf algebras we understand the objects introduced in [5], as a generalization of ordinary Hopf algebras. Here we recall the definition in the symmetric monoidal setting.

Definition 2.1. A weak Hopf algebra H is an object in C with an algebra structure (H, η_H, μ_H) and a coalgebra structure $(H, \varepsilon_H, \delta_H)$ such that the following axioms hold:

(a1) $\delta_H \circ \mu_H = (\mu_H \otimes \mu_H) \circ \delta_{H\otimes H}$,

(a2) $\varepsilon_H \circ \mu_H \circ (\mu_H \otimes H) = (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes \delta_H \otimes H)$

$= (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes (c_{H,H} \circ \delta_H) \otimes H)$,

(a3) $(\delta_H \otimes H) \circ \delta_H \circ \eta_H = (H \otimes \mu_H \otimes H) \circ (\delta_H \otimes \delta_H) \circ (\eta_H \otimes \eta_H)$

$= (H \otimes (\mu_H \circ c_{H,H}) \otimes H) \circ (\delta_H \otimes \delta_H) \circ (\eta_H \otimes \eta_H)$.

(a4) There exists a morphism $\lambda_H : H \to H$ in C (called the antipode of H) satisfying:

(a4-1) $id_H * \lambda_H = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H)$,

(a4-2) $\lambda_H * id_H = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H))$,

(a4-3) $\lambda_H * id_H * \lambda_H = \lambda_H$.

To finish this section of preliminaries in the following remark we list the main properties of weak Hopf algebras we will need in this paper.

Remark 2.2. If H is a weak Hopf algebra in C, the antipode λ_H is unique, antimultiplicative, anticomultiplicative and leaves the unit and the counit invariant:

$$\lambda_H \circ \mu_H = \mu_H \circ (\lambda_H \otimes \lambda_H) \circ c_{H,H}, \quad \delta_H \circ \lambda_H = c_{H,H} \circ (\lambda_H \otimes \lambda_H) \circ \delta_H, \tag{1}$$

$$\lambda_H \circ \eta_H = \eta_H, \quad \varepsilon_H \circ \lambda_H = \varepsilon_H. \tag{2}$$

If we define the morphisms Π_H^L (target), Π_H^R (source), $\overline{\Pi}_H^L$ and $\overline{\Pi}_H^R$ by

$$\Pi_H^L = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H),$$

$$\Pi_H^R = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)),$$

$$\overline{\Pi}_H^L = (H \otimes (\varepsilon_H \circ \mu_H)) \circ ((\delta_H \circ \eta_H) \otimes H),$$

$$\overline{\Pi}_H^R = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)),$$

it is straightforward to show (see [5]) that they are idempotent and Π_H^L, Π_H^R satisfy the equalities

$$\Pi_H^L = id_H * \lambda_H, \quad \Pi_H^R = \lambda_H * id_H, \quad \Pi_H^L * id_H = id_H, \quad \Pi_H^R * \lambda_H = \lambda_H. \tag{3}$$

Moreover,

$$\Pi_H^L = \lambda_H \circ \overline{\Pi}_H^L = \overline{\Pi}_H^R \circ \lambda_H, \quad \Pi_H^R = \overline{\Pi}_H^L \circ \lambda_H = \lambda_H \circ \overline{\Pi}_H^R, \tag{4}$$

$$\Pi_H^L \circ \overline{\Pi}_H^L = \Pi_H^L, \quad \Pi_H^L \circ \overline{\Pi}_H^R = \overline{\Pi}_H^R, \quad \Pi_H^R \circ \overline{\Pi}_H^L = \overline{\Pi}_H^L, \quad \Pi_H^R \circ \overline{\Pi}_H^R = \Pi_H^R, \tag{5}$$

$$\overline{\Pi}_H^L \circ \Pi_H^L = \overline{\Pi}_H^L, \quad \overline{\Pi}_H^L \circ \Pi_H^R = \Pi_H^R, \quad \overline{\Pi}_H^R \circ \Pi_H^L = \Pi_H^L, \quad \overline{\Pi}_H^R \circ \Pi_H^R = \overline{\Pi}_H^R. \tag{6}$$

For the morphisms target and source we have the following identities:

$$(H \otimes \Pi_H^L) \circ \delta_H \circ \Pi_H^L = \delta_H \circ \Pi_H^L, \quad (\Pi_H^R \otimes H) \circ \delta_H \circ \Pi_H^R = \delta_H \circ \Pi_H^R, \tag{7}$$

$$(H \otimes \overline{\Pi}_H^R) \circ \delta_H \circ \overline{\Pi}_H^R = \delta_H \circ \overline{\Pi}_H^R, \quad (\overline{\Pi}_H^L \otimes H) \circ \delta_H \circ \overline{\Pi}_H^L = \delta_H \circ \overline{\Pi}_H^L, \tag{8}$$

$$\mu_H \circ (H \otimes \Pi_H^L) = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H), \tag{9}$$

$$(H \otimes \Pi_H^L) \circ \delta_H = (\mu_H \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H), \tag{10}$$

$$\mu_H \circ (\Pi_H^R \otimes H) = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes \delta_H) \tag{11}$$

$$(\Pi_H^R \otimes H) \circ \delta_H = (H \otimes \mu_H) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)) \tag{12}$$

$$\mu_H \circ (\overline{\Pi}_H^R \otimes H) = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes \delta_H), \tag{13}$$

$$\mu_H \circ (H \otimes \overline{\Pi}_H^L) = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (\delta_H \otimes H), \tag{14}$$

$$(\overline{\Pi}_H^L \otimes H) \circ \delta_H = (H \otimes \mu_H) \circ ((\delta_H \circ \eta_H) \otimes H), \tag{15}$$

$$(H \otimes \overline{\Pi}_H^R) \circ \delta_H = (\mu_H \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)), \tag{16}$$

$$\delta_H \circ \eta_H = (\Pi_H^R \otimes H) \circ \delta_H \circ \eta_H = (H \otimes \Pi_H^L) \circ \delta_H \circ \eta_H = (H \otimes \overline{\Pi}_H^R) \circ \delta_H \circ \eta_H = (\overline{\Pi}_H^L \otimes H) \circ \delta_H \circ \eta_H, \tag{17}$$

$$\varepsilon_H \circ \mu_H = \varepsilon_H \circ \mu_H \circ (\Pi_H^R \otimes H) = \varepsilon_H \circ \mu_H \circ (H \otimes \Pi_H^L) = \varepsilon_H \circ \mu_H \circ (\overline{\Pi}_H^R \otimes H) = \varepsilon_H \circ \mu_H \circ (H \otimes \overline{\Pi}_H^L). \tag{18}$$

3. Product alterations by two cocycles for weak Hopf algebras

In this section we prove that, as in the Hopf algebra case (see [9]), 2-cocycles provide a way of altering the product of a weak Hopf algebra to produce another weak Hopf algebra.

Definition 3.1. Let H be a weak Hopf algebra. We will say that a morphism $\sigma : H \otimes H \to K$ is convolution invertible if there exists a morphism $\sigma^{-1} : H \otimes H \to K$ (the convolution inverse of σ) satisfying the following equalities:

(b1) $\sigma * \sigma^{-1} = \sigma^{-1} * \sigma = \varepsilon_H \circ \mu_H$.

(b2) $\sigma * \sigma^{-1} * \sigma = \sigma$.

(b3) $\sigma^{-1} * \sigma * \sigma^{-1} = \sigma^{-1}$.

It is not difficult to see that, if σ is convolution invertible, the inverse σ^{-1} is unique.

Proposition 3.2. *Let H be a weak Hopf algebra and let $\sigma : H \otimes H \to K$ be convolution invertible. Then*

$$\sigma * (\varepsilon_H \circ \mu_H) = \sigma = (\varepsilon_H \circ \mu_H) * \sigma, \tag{19}$$

$$\sigma^{-1} * (\varepsilon_H \circ \mu_H) = \sigma^{-1} = (\varepsilon_H \circ \mu_H) * \sigma^{-1}. \tag{20}$$

As a consequence, the equalities

$$\sigma = \sigma \circ (H \otimes \mu_H) \circ (H \otimes \Pi_H^R \otimes H) \circ (\delta_H \otimes H), \tag{21}$$

$$\sigma = \sigma \circ (\mu_H \otimes H) \circ (H \otimes \overline{\Pi}_H^L \otimes H) \circ (H \otimes (c_{H,H} \circ \delta_H)), \tag{22}$$

$$\sigma = \sigma \circ (\mu_H \otimes H) \circ (H \otimes \Pi_H^L \otimes H) \circ (H \otimes \delta_H), \tag{23}$$

$$\sigma = \sigma \circ (H \otimes \mu_H) \circ (H \otimes \overline{\Pi}_H^R \otimes H) \circ ((c_{H,H} \circ \delta_H) \otimes H), \tag{24}$$

hold, and similar equalities involving σ^{-1} are satisfied.

Proof. We will show (19). Using (b1)-(b2),

$$\sigma * (\varepsilon_H \circ \mu_H) = \sigma * \sigma^{-1} * \sigma = \sigma = \sigma * \sigma^{-1} * \sigma = (\varepsilon_H \circ \mu_H) * \sigma$$

The proof for (20) is similar but using (b3) instead of (b2). Finally, the equalities (21)-(24) are a direct consequence of (9), (11), (13) and (14). □

Proposition 3.3. *Let H be a weak Hopf algebra and let $\sigma : H \otimes H \to K$ be convolution invertible. Assume that the antipode λ_H is an isomorphism. Then*

(i) The following conditions are equivalent:

$$\sigma \circ (\eta_H \otimes H) = \varepsilon_H. \tag{25}$$

$$\sigma \circ (\Pi_H^L \otimes H) \circ \delta_H = \varepsilon_H. \tag{26}$$

$$\sigma \circ c_{H,H} \circ (H \otimes \overline{\Pi}_{H}^{L}) \circ \delta_H = \varepsilon_H. \tag{27}$$

$$\sigma \circ (\Pi_{H}^{R} \otimes \lambda_H) \circ \delta_H = \varepsilon_H. \tag{28}$$

$$\sigma \circ c_{H,H} \circ (\lambda_H^{-1} \otimes \overline{\Pi}_{H}^{R}) \circ \delta_H = \varepsilon_H. \tag{29}$$

(ii) The following conditions are equivalent:

$$\sigma \circ (H \otimes \eta_H) = \varepsilon_H. \tag{30}$$

$$\sigma \circ (H \otimes \Pi_{H}^{R}) \circ \delta_H = \varepsilon_H. \tag{31}$$

$$\sigma \circ c_{H,H} \circ (\overline{\Pi}_{H}^{R} \otimes H) \circ \delta_H = \varepsilon_H. \tag{32}$$

$$\sigma \circ (\lambda_H \otimes \Pi_{H}^{L}) \circ \delta_H = \varepsilon_H. \tag{33}$$

$$\sigma \circ c_{H,H} \circ (\overline{\Pi}_{H}^{L} \otimes \lambda_H^{-1}) \circ \delta_H = \varepsilon_H. \tag{34}$$

And similar equivalences involving σ^{-1} are satisfied.

Proof. We will show part (i). The proof for (ii) follows a similar pattern. By composing with $\eta_H \otimes H$ in (22) and (23), we get that $\sigma \circ (\eta_H \otimes H) = \sigma \circ c_{H,H} \circ (H \otimes \overline{\Pi}_{H}^{L}) \circ \delta_H = \sigma \circ (\Pi_{H}^{L} \otimes H) \circ \delta_H$. Moreover, if we compose in (22) with $\eta_H \otimes \lambda_H$, using (1) and (4) we obtain that $\sigma \circ (\eta_H \otimes \lambda_H) = \sigma \circ (\Pi_{H}^{R} \otimes \lambda_H) \circ \delta_H$. Finally, by composing in (23) with $\eta_H \otimes \lambda_H^{-1}$ we get that $\sigma \circ (\eta_H \otimes \lambda_H^{-1}) = \sigma \circ (\overline{\Pi}_{H}^{R} \otimes \lambda_H^{-1}) \circ c_{H,H} \circ \delta_H$. Then the result follows by (2). □

Definition 3.4. Let H be a weak Hopf algebra with antipode λ_H isomorphism. We will say that a convolution invertible morphism $\sigma : H \otimes H \to K$ is normal if it satisfies any of the equivalent conditions of (i) and (ii) of Proposition 3.3.

Proposition 3.5. *Let H be a weak Hopf algebra and let $\sigma : H \otimes H \to K$ be a convolution invertible morphism. Assume that the antipode λ_H is an isomorphism. Then σ is normal if and only if so is σ^{-1}.*

Proof. Assume that σ is normal. Then

$$\sigma^{-1} \circ (\eta_H \otimes H)$$

$$= (\sigma^{-1} * (\varepsilon_H \circ \mu_H)) \circ (\eta_H \otimes H) \text{ (by (20))}$$

$$= ((\sigma^{-1} \circ c_{H,H} \circ (H \otimes \overline{\Pi}_{H}^{L}) \circ \delta_H) \otimes \varepsilon_H) \circ \delta_H \text{ (H coalgebra and by the definition of $\overline{\Pi}_{H}^{L}$)}$$

$$= ((\sigma^{-1} \circ c_{H,H} \circ (H \otimes \overline{\Pi}_{H}^{L}) \circ \delta_H) \otimes (\sigma \circ (\Pi_{H}^{L} \otimes H) \circ \delta_H)) \circ \delta_H \text{ (by (26))}$$

$= ((\sigma^{-1} \circ c_{H,H}) \otimes (((\varepsilon_H \circ \mu_H) \otimes \sigma) \circ (H \otimes c_{H,H} \otimes H))) \circ (H \otimes ((H \otimes \mu_H \otimes H) \circ ((\delta_H \circ \eta_H) \otimes (\delta_H \circ \eta_H))) \otimes \delta_H)$

$\circ \delta_H$ (by (10) and by the definition of $\overline{\Pi}_H^L$)

$= ((\sigma^{-1} \circ c_{H,H}) \otimes ((\varepsilon_H \circ \mu_H) * \sigma)) \circ (H \otimes (\delta_H \circ \eta_H) \otimes H) \circ \delta_H$ (by (a3))

$= (\sigma^{-1} * \sigma) \circ (\eta_H \otimes H)$ (by (19))

$= \varepsilon_H$ (by (b1)).

The equality $\sigma^{-1} \circ (H \otimes \eta_H) = \varepsilon_H$ follows a similar pattern but using (31) for σ^{-1} and (32). Finally, by exchanging the roles of σ and σ^{-1} we get the only if part. □

Proposition 3.6. *Let H be a weak Hopf algebra and let $\sigma : H \otimes H \to K$ be a normal and convolution invertible morphism. Assume that the antipode λ_H is an isomorphism. Then we have that*

$$(\mu_H \otimes \sigma) \circ \delta_{H \otimes H} \circ (\eta_H \otimes H) = id_H, \tag{35}$$

$$(\sigma \otimes \mu_H) \circ \delta_{H \otimes H} \circ (\eta_H \otimes H) = id_H, \tag{36}$$

$$(\mu \otimes \sigma) \circ \delta_{H \otimes H} \circ (H \otimes \eta_H) = id_H, \tag{37}$$

$$(\sigma \otimes \mu_H) \circ \delta_{H \otimes H} \circ (H \otimes \eta_H) = id_H, \tag{38}$$

and similar equalities involving σ^{-1} are satisfied.

Proof. The proof follows easily by Proposition 3.3 and by (10), (15), (16) and (12). □

Proposition 3.7. *Let H be a weak Hopf algebra and let $\sigma : H \otimes H \to K$ be a normal and convolution invertible morphism. Assume that the antipode λ_H is an isomorphism. Then, for $\Pi \in \{\Pi_H^R, \Pi_H^L, \overline{\Pi}_H^R, \overline{\Pi}_H^L\}$,*

$$(\sigma \otimes \mu_H) \circ \delta_{H \otimes H} \circ (\Pi \otimes H) = \mu_H \circ (\Pi \otimes H), \tag{39}$$

$$(\mu_H \otimes \sigma) \circ \delta_{H \otimes H} \circ (\Pi \otimes H) = \mu_H \circ (\Pi \otimes H), \tag{40}$$

$$(\sigma \otimes \mu_H) \circ \delta_{H \otimes H} \circ (H \otimes \Pi) = \mu_H \circ (H \otimes \Pi), \tag{41}$$

$$(\mu_H \otimes \sigma) \circ \delta_{H \otimes H} \circ (H \otimes \Pi) = \mu_H \circ (H \otimes \Pi), \tag{42}$$

and similar equalities involving σ^{-1} are satisfied.

Proof. We begin by showing (39) for Π_H^R. Indeed,

$(\sigma \otimes \mu_H) \circ \delta_{H \otimes H} \circ (\Pi_H^R \otimes H)$

$= ((\sigma \circ (\Pi_H^R \otimes H)) \otimes \mu_H) \circ \delta_{H \otimes H} \circ (\Pi_H^R \otimes H)$ (by (7))

$$= \mu_H \circ (\Pi_H^R \otimes ((\sigma \otimes \mu_H) \circ \delta_{H\otimes H} \circ (\eta_H \otimes H))) \text{ (by (12))}$$

$$= \mu_H \circ (\Pi_H^R \otimes H) \text{ (by (36))},$$

and we get the equality (39) for Π_H^R. As a consequence, using (19) and (18), we have that $\sigma \circ (\Pi_H^R \otimes H) = \varepsilon_H \circ \mu_H$ and then

$$(\mu_H \otimes \sigma) \circ \delta_{H\otimes H} \circ (\Pi_H^R \otimes H)$$

$$= ((\mu_H \circ (\Pi_H^R \otimes H)) \otimes \sigma) \circ \delta_{H\otimes H} \circ (\Pi_H^R \otimes H)$$

$$= (H \otimes (((\varepsilon_H \circ \mu_H) \otimes \sigma) \circ \delta_{H\otimes H})) \circ (c_{H,H} \otimes H) \circ (\Pi_H^R \otimes \delta_H)$$

$$= (H \otimes \sigma) \circ (c_{H,H} \otimes H) \circ (\Pi_H^R \otimes \delta_H)$$

$$= (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes \delta_H)$$

$$= \mu_H \circ (\Pi_H^R \otimes H).$$

By composing with $\overline{\Pi}_H^L \otimes H$ and using (5) we obtain the equalities for $\overline{\Pi}_H^L$. In a similar way we get (39) and (40) for $\overline{\Pi}_H^R$ and the equalities for Π_H^L follow by composing with $\Pi_H^L \otimes H$.

On the other hand,

$$(\mu_H \otimes \sigma) \circ \delta_{H\otimes H} \circ (H \otimes \Pi_H^L)$$

$$= (\mu_H \otimes (\sigma \circ (H \otimes \Pi_H^L))) \circ \delta_{H\otimes H} \circ (H \otimes \Pi_H^L) \text{ (by (7))}$$

$$= \mu_H \circ (((\mu_H \otimes \sigma) \circ \delta_{H\otimes H} \circ (H \otimes \eta_H)) \otimes \Pi_H^L) \text{ (by (10))}$$

$$= \mu_H \circ (H \otimes \Pi_H^L) \text{ (by (37))},$$

and we get the equality (42) for Π_H^L. As a consequence, using (19) and (18) we have that $\sigma \circ (H \otimes \Pi_H^R) = \varepsilon_H \circ \mu_H$ and then

$$(\sigma \otimes \mu_H) \circ \delta_{H\otimes H} \circ (H \otimes \Pi_H^L)$$

$$= (\sigma \otimes (\mu_H \circ (H \otimes \Pi_H^L))) \circ \delta_{H\otimes H} \circ (H \otimes \Pi_H^L)$$

$$= (((\sigma \otimes (\varepsilon_H \circ \mu_H)) \circ \delta_{H\otimes H}) \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes \Pi_H^L)$$

$$= (\sigma \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes \Pi_H^L)$$

$$= ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes \Pi_H^L)$$

$$= \mu_H \circ (H \otimes \Pi_H^L).$$

By composing with $H \otimes \overline{\Pi}_H^R$ and using (5) we obtain the equalities for $\overline{\Pi}_H^R$. In a similar way we get (42) and (41) for $\overline{\Pi}_H^L$ and the equalities for Π_H^R follow by composing with $H \otimes \Pi_H^R$. □

Corollary 3.8. *Let H be a weak Hopf algebra and let $\sigma : H \otimes H \to K$ be a normal and convolution invertible morphism. Assume that the antipode λ_H is an isomorphism. Then*

$$\varepsilon_H \circ \mu_H = \sigma \circ (\Pi_H^R \otimes H) = \sigma \circ (\overline{\Pi}_H^R \otimes H) = \sigma \circ (H \otimes \Pi_H^L) = \sigma \circ (H \otimes \overline{\Pi}_H^L), \tag{43}$$

$$\varepsilon_H \circ \mu_H \circ (\lambda_H \otimes H) = \sigma \circ (\Pi_H^L \otimes H) = \sigma \circ c_{H,H} \circ (H \otimes \overline{\Pi}_H^L), \tag{44}$$

$$\varepsilon_H \circ \mu_H \circ (H \otimes \lambda_H) = \sigma \circ (H \otimes \Pi_H^R) = \sigma \circ c_{H,H} \circ (\overline{\Pi}_H^R \otimes H), \tag{45}$$

and similar equalities involving σ^{-1} are satisfied.

As a consequence, we have the following expressions for the Π morphisms:

$$\Pi_H^L = (\sigma \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H) = (\sigma^{-1} \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H), \tag{46}$$

$$\Pi_H^R = (H \otimes \sigma) \circ (c_{H,H} \otimes H)) \circ (H \otimes (\delta_H \circ \eta_H)) = (H \otimes \sigma^{-1}) \circ (c_{H,H} \otimes H)) \circ (H \otimes (\delta_H \circ \eta_H)), \tag{47}$$

$$\overline{\Pi}_H^L = (H \otimes \sigma) \circ ((\delta_H \circ \eta_H) \otimes H) = (H \otimes \sigma^{-1}) \circ ((\delta_H \circ \eta_H) \otimes H), \tag{48}$$

$$\overline{\Pi}_H^R = (\sigma \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)) = (\sigma^{-1} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)). \tag{49}$$

Proof. The equalities of (43) follow by composing in (39) and (42) with ε_H and using (18). Moreover, taking into account (4) and by composing with $\lambda_H \otimes H$ we get the first equality of (44), and the second one follows because

$$\begin{aligned} & \varepsilon_H \circ \mu_H \circ (\lambda_H \otimes H) \\ = & \ \varepsilon_H \circ \mu_H \circ (\lambda_H \otimes \Pi_H^L) \\ = & \ \varepsilon_H \circ \mu_H \circ (\lambda_H \otimes (\lambda_H \circ \overline{\Pi}_H^L)) \\ = & \ \varepsilon_H \circ \mu_H \circ c_{H,H} \circ (H \otimes \overline{\Pi}_H^L) \\ = & \ \sigma \circ c_{H,H} \circ (H \otimes \overline{\Pi}_H^L). \end{aligned}$$

The proof for the equalities of (45) is similar. Finally, (46)-(49) can be easily obtained using (43) and (17) and the proof is complete. □

Proposition 3.9. *Let H be a weak Hopf algebra, and let $\sigma : H \otimes H \to K$ be a normal and convolution invertible morphism. Define $\partial^1(\sigma) = \varepsilon_H \otimes \sigma$, $\partial^2(\sigma) = \sigma \circ (\mu_H \otimes H)$, $\partial^3(\sigma) = \sigma \circ (H \otimes \mu_H)$, $\partial^4(\sigma) = \sigma \otimes \varepsilon_H$ and $e = \varepsilon_H \circ \mu_H \circ (H \otimes \mu_H)$. Then the following equalities hold:*

$$\partial^1(\sigma) * \partial^1(\sigma^{-1}) * e = e = e * \partial^1(\sigma) * \partial^1(\sigma^{-1}), \tag{50}$$

$$\partial^1(\sigma^{-1}) * \partial^1(\sigma) * e = e = e * \partial^1(\sigma^{-1}) * \partial^1(\sigma), \tag{51}$$

$$\partial^2(\sigma) * \partial^2(\sigma^{-1}) = e = \partial^2(\sigma^{-1}) * \partial^2(\sigma) \tag{52}$$

$$\partial^2(\sigma) * e = \partial^2(\sigma) = e * \partial^2(\sigma), \quad \partial^2(\sigma^{-1}) * e = \partial^2(\sigma^{-1}) = e * \partial^2(\sigma^{-1}) \tag{53}$$

$$\partial^3(\sigma) * \partial^3(\sigma^{-1}) = e = \partial^3(\sigma^{-1}) * \partial^3(\sigma) \tag{54}$$

$$\partial^3(\sigma) * e = \partial^3(\sigma) = e * \partial^3(\sigma), \quad \partial^3(\sigma^{-1}) * e = \partial^3(\sigma^{-1}) = e * \partial^3(\sigma^{-1}) \tag{55}$$

$$\partial^4(\sigma) * \partial^4(\sigma^{-1}) * e = e = e * \partial^4(\sigma) * \partial^4(\sigma^{-1}), \tag{56}$$

$$\partial^4(\sigma^{-1}) * \partial^4(\sigma) * e = e = e * \partial^4(\sigma^{-1}) * \partial^4(\sigma), \tag{57}$$

Proof. First of all, note that

$$\partial^1(\sigma) * e = \sigma \circ (\mu_H \otimes H) \circ (\Pi_H^R \otimes H \otimes H). \tag{58}$$

$$e * \partial^1(\sigma) = \sigma \circ (\mu_H \otimes H) \circ (\overline{\Pi}_H^R \otimes H \otimes H), \tag{59}$$

and similar equalities involving σ^{-1} are satisfied. We will show (58), the other is similar. Indeed,

$$\begin{aligned} &\partial^1(\sigma) * e \\ &= (\varepsilon_H \otimes \sigma \otimes ((\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes (c_{H,H} \circ \delta_H) \otimes H))) \circ \delta_{H\otimes H\otimes H} \text{ (by (a2))} \\ &= (\sigma * (\varepsilon_H \circ \mu_H)) \circ (((H \otimes (\varepsilon \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes \delta_H)) \otimes H) \text{ (H is a coalgebra and naturality)} \\ &= \sigma \circ (\mu_H \otimes H) \circ (\Pi_H^R \otimes H \otimes H) \text{ (by (11) and (20)).} \end{aligned}$$

Then

$$\begin{aligned} &\partial^1(\sigma) * \partial^1(\sigma^{-1}) * e \\ &= (\varepsilon_H \otimes \sigma \otimes (\sigma^{-1} \circ (\mu_H \otimes H) \circ (\Pi_H^R \otimes H \otimes H))) \circ \delta_{H\otimes H\otimes H} \\ &= (\sigma * \sigma^{-1}) \circ (((H \otimes (\varepsilon \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes \delta_H)) \otimes H) \\ &= \varepsilon_H \circ \mu_H \circ (\Pi_H^R \otimes \mu_H) \\ &= e. \end{aligned}$$

The other equality of (50) and the ones of (51) follow a similar pattern.

On the other hand, using (a1) and by naturality,

$$\partial^2(\sigma) * \partial^2(\sigma^{-1}) = (\sigma * \sigma^{-1}) \circ (\mu_H \otimes H) = e = (\sigma^{-1} * \sigma) \circ (\mu_H \otimes H) = \partial^2(\sigma^{-1}) * \partial^2(\sigma).$$

Moreover,

$$\partial^2(\sigma) * e = (\sigma * (\varepsilon_H \circ \mu_H)) \circ (\mu_H \otimes H) = \partial^2(\sigma) = ((\varepsilon_H \circ \mu_H) * \sigma) \circ (\mu_H \otimes H) = e * \partial^2(\sigma),$$

and with similar computations we can obtain (54). Finally, taking into account that the equalities

$$\partial^4(\sigma) * e = \sigma \circ (H \otimes \mu_H) \circ (H \otimes H \otimes \overline{\Pi}_H^L) \tag{60}$$

$$e * \partial^4(\sigma) = \sigma \circ (H \otimes \mu_H) \circ (H \otimes H \otimes \Pi_H^L), \tag{61}$$

and the ones corresponding for σ^{-1} hold, it is not difficult to see (56) and (57). □

Now we recall the notion of (normal) 2-cocycle (see [2] for details).

Definition 3.10. Let H be a weak Hopf algebra, and let $\sigma : H\otimes H \to K$ be a convolution invertible morphism. We say that σ is a 2-cocycle if the equality

$$\partial^1(\sigma) * \partial^3(\sigma) = \partial^4(\sigma) * \partial^2(\sigma) \tag{62}$$

holds.

Equivalently, a convolution invertible morphism $\sigma : H \otimes H \to K$ is a 2-cocycle if

$$\sigma \circ (H \otimes ((\sigma \otimes \mu_H) \circ \delta_{H\otimes H})) = \sigma \circ (((\sigma \otimes \mu_H) \circ \delta_{H\otimes H}) \otimes H). \tag{63}$$

The 2-cocycle σ is called normal if it satisfies the normal condition, i.e.; the conditions (25) and (30) hold.

Remark 3.11. We want to point out that there is an alternative definition of 2-cocycle given by Chen, Zhang and Wang (see [6]) in which other conditions are requested. Although some of their conditions can be obtained from those of our definition and viceversa, there is a very important difference involving the convolution inverse morphism of the 2-cocycle, since in our definition $\sigma^{-1} * \sigma = \varepsilon_H \circ \mu_H = \sigma * \sigma^{-1}$, while in the one of [6] $\sigma^{-1} * \sigma = \varepsilon_H \circ \mu_H$ but $\sigma * \sigma^{-1} = \varepsilon_H \circ \mu_H \circ c_{H,H}$. This is a fundamental point to establish that both definitions are essentially different, since in the case that one could be obtained from the other, we would have that $\varepsilon_H \circ \mu_H = \varepsilon_H \circ \mu_H \circ c_{H,H}$, and this equality is not true in general in the weak Hopf algebra setting (consider for example a groupoid algebra). Therefore, both alternatives determine different ways and procedures. We consider that our choice has the advantage of using a minimum number of conditions similar to those well-known in Hopf algebras, and in addition the conditions about the convolution inverse of the 2-cocycle in our definition are very close to the ones given by the third author of the quoted paper in an article with Liu and Shen (see [13]).

Proposition 3.12. *Let H be a weak Hopf algebra, and let $\sigma : H \otimes H \to K$ be a normal and convolution invertible 2-cocycle. Then the equalities*

$$\partial^1(\sigma) * e = e * \partial^1(\sigma), \tag{64}$$

$$\partial^4(\sigma) * e = e * \partial^4(\sigma), \tag{65}$$

hold.

Proof. Indeed,

$$\begin{aligned}
&\partial^1(\sigma) * e \\
&= \sigma \circ (\mu_H \otimes H) \circ (\Pi^R_H \otimes H \otimes H) \text{ (by (58))} \\
&= \sigma \circ (((\sigma \otimes \mu_H) \circ \delta_{H\otimes H}) \otimes H) \circ (\Pi^R_H \otimes H \otimes H) \text{ (by (39))} \\
&= \sigma \circ (H \otimes ((\sigma \otimes \mu_H) \circ \delta_{H\otimes H})) \circ (\Pi^R_H \otimes H \otimes H) \text{ (by (63))} \\
&= \sigma \circ (H \otimes ((\sigma \otimes \mu_H) \circ \delta_{H\otimes H})) \circ (\overline{\Pi}^R_H \otimes H \otimes H) \text{ (by (43))} \\
&= \sigma \circ (((\sigma \otimes \mu_H) \circ \delta_{H\otimes H}) \otimes H) \circ (\overline{\Pi}^R_H \otimes H \otimes H) \text{ (by (63))} \\
&= \sigma \circ (\mu_H \otimes H) \circ (\overline{\Pi}^R_H \otimes H \otimes H) \text{ (by (39))} \\
&= e * \partial^1(\sigma) \text{ (by (58))},
\end{aligned}$$

and we obtain (64). The proof for the equality (65) follows a similar pattern and we get the details to the reader. □

Proposition 3.13. *Let H be a weak Hopf algebra, and let $\sigma : H\otimes H \to K$ be a normal and convolution invertible 2-cocycle. Then the following conditions are equivalent:*

(i)

$$\partial^3(\sigma) * \partial^2(\sigma^{-1}) = \partial^1(\sigma^{-1}) * \partial^4(\sigma), \tag{66}$$

$$\partial^2(\sigma) * \partial^3(\sigma^{-1}) = \partial^4(\sigma^{-1}) * \partial^1(\sigma). \tag{67}$$

(ii)

$$\partial^1(\sigma) * e = e * \partial^1(\sigma), \tag{68}$$

$$\partial^4(\sigma) * e = e * \partial^4(\sigma). \tag{69}$$

(iii)

$$\partial^1(\sigma^{-1}) * e = e * \partial^1(\sigma^{-1}), \tag{70}$$

$$\partial^4(\sigma^{-1}) * e = e * \partial^4(\sigma^{-1}). \tag{71}$$

Proof. $(i) \Rightarrow (ii)$ Using (54), condition of 2-cocycle, (67) and (59) for σ^{-1},

$$\begin{aligned}
&\partial^1(\sigma) * e\\
&= \partial^1(\sigma) * \partial^3(\sigma) * \partial^3(\sigma^{-1})\\
&= \partial^4(\sigma) * \partial^2(\sigma) * \partial^3(\sigma^{-1})\\
&= \partial^4(\sigma) * \partial^4(\sigma^{-1}) * \partial^1(\sigma)\\
&= ((\sigma * \sigma^{-1}) \otimes \varepsilon_H) * \partial^1(\sigma)\\
&= ((\varepsilon_H \circ \mu_H) \otimes \sigma) \circ (H \otimes \delta_H \otimes H)\\
&= \sigma \circ (\mu_H \otimes H) \circ (\overline{\Pi}_H^R \otimes H \otimes H)\\
&= e * \partial^1(\sigma),
\end{aligned}$$

and in a similar way but using (66) and (53) instead of (67) and (55), respectively, we get (65).

$(ii) \Rightarrow (iii)$ Indeed, (70) follows because by (50),

$$\partial^1(\sigma^{-1}) * e = \partial^1(\sigma^{-1}) * e * \partial^1(\sigma) * \partial^1(\sigma^{-1}) = \partial^1(\sigma^{-1}) * \partial^1(\sigma) * e * \partial^1(\sigma^{-1}) = e * \partial^1(\sigma^{-1}),$$

and in a similar way we get (71).

Moreover, if we assume (iii), it is easy to obtain (ii). Indeed,

$$\partial^1(\sigma) * e = \partial^1(\sigma) * e * \partial^1(\sigma^{-1}) * \partial^1(\sigma) = \partial^1(\sigma) * \partial^1(\sigma^{-1}) * e * \partial^1(\sigma) = e * \partial^1(\sigma),$$

and by similar computations $\partial^4(\sigma) * e = e * \partial^4(\sigma)$.

To complete the proof we get $(iii) \Rightarrow (ii)$. First of all, note that the equalities

$$\partial^1(\sigma^{-1}) * e * e * \partial^4(\sigma) = \partial^1(\sigma^{-1}) * \partial^4(\sigma), \tag{72}$$

$$\partial^4(\sigma^{-1}) * e * e * \partial^1(\sigma) = \partial^4(\sigma^{-1}) * \partial^1(\sigma), \tag{73}$$

hold. We will show (72), the proof for (73) is similar. Indeed,

$$\begin{aligned}
&\partial^1(\sigma^{-1}) * e * e * \partial^4(\sigma)\\
&= (\sigma^{-1} \circ (\mu_H \otimes H) \circ (\Pi_H^R \otimes H \otimes H)) * (\sigma \circ (H \otimes \mu_H) \circ (H \otimes H \otimes \Pi_H^L)) \text{ (by (58) and (61))}\\
&= (\sigma^{-1} \otimes \sigma) \circ (((H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes \delta_H)) \otimes H \otimes H \otimes (((\varepsilon_H \circ \mu_H) \otimes H)\\
&\quad \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H))) \circ \delta_{H\otimes H\otimes H} \text{ (by (9) and (11))}\\
&= (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes \delta_H \otimes H) \circ (H \otimes H \otimes \sigma^{-1} \otimes H) \circ (H \otimes (c_{H,H} \circ \delta_H) \otimes \sigma \otimes \delta_H) \circ (\delta_{H\otimes H} \otimes H)\\
&\quad \text{(by naturality)}\\
&= (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes (c_{H,H} \circ \delta_H) \otimes H) \circ (H \otimes H \otimes \sigma^{-1} \otimes H)\\
&\quad \circ (H \otimes (c_{H,H} \circ \delta_H) \otimes \sigma \otimes \delta_H) \circ (\delta_{H\otimes H} \otimes H) \text{ (by (a2))}\\
&= (((\varepsilon_H \circ \mu_H) * \sigma) \otimes (\sigma^{-1} * (\varepsilon_H \circ \mu_H))) \circ (H \otimes (c_{H,H} \circ \delta_H) \otimes H) \text{ (by naturality)}\\
&= \partial^1(\sigma^{-1}) * \partial^4(\sigma) \text{ (by (19) and (20)).}
\end{aligned}$$

As a consequence, using (ii), (iii), (52), (55), the condition of 2-cocycle and (51) we have that

$$\begin{aligned}
&\partial^1(\sigma^{-1}) * \partial^4(\sigma)\\
&= \partial^1(\sigma^{-1}) * e * e * \partial^4(\sigma)\\
&= e * \partial^1(\sigma^{-1}) * \partial^4(\sigma) * e\\
&= e * \partial^1(\sigma^{-1}) * \partial^4(\sigma) * \partial^2(\sigma) * \partial^2(\sigma^{-1})\\
&= e * \partial^1(\sigma^{-1}) * \partial^1(\sigma) * \partial^3(\sigma) * \partial^2(\sigma^{-1})\\
&= e * \partial^3(\sigma) * \partial^2(\sigma^{-1})\\
&= \partial^3(\sigma) * \partial^2(\sigma^{-1}).
\end{aligned}$$

Finally,

$$\begin{aligned}
&\partial^4(\sigma^{-1}) * \partial^1(\sigma)\\
&= \partial^4(\sigma^{-1}) * e * e * \partial^1(\sigma)\\
&= e * \partial^4(\sigma^{-1}) * \partial^1(\sigma) * e\\
&= e * \partial^4(\sigma^{-1}) * \partial^1(\sigma) * \partial^3(\sigma) * \partial^3(\sigma^{-1})\\
&= e * \partial^4(\sigma^{-1}) * \partial^4(\sigma) * \partial^2(\sigma) * \partial^3(\sigma^{-1})\\
&= e * \partial^2(\sigma) * \partial^3(\sigma^{-1})\\
&= \partial^2(\sigma) * \partial^3(\sigma^{-1}),
\end{aligned}$$

and the proof is complete. □

Proposition 3.14. *Let H be a weak Hopf algebra, and let $\sigma : H\otimes H \to K$ be a normal and convolution invertible 2-cocycle. Then the equality*

$$\partial^3(\sigma^{-1}) * \partial^1(\sigma^{-1}) = \partial^2(\sigma^{-1}) * \partial^4(\sigma^{-1}) \tag{74}$$

holds.

Proof. Indeed,

$$\begin{aligned}
&\partial^3(\sigma^{-1}) * \partial^1(\sigma^{-1})\\
&= e * \partial^3(\sigma^{-1}) * \partial^1(\sigma^{-1}) \text{ (by (55))}\\
&= \partial^2(\sigma^{-1}) * \partial^2(\sigma) * \partial^3(\sigma^{-1}) * \partial^1(\sigma^{-1}) \text{ (by (52))}\\
&= \partial^2(\sigma^{-1}) * \partial^4(\sigma^{-1}) * \partial^1(\sigma) * \partial^1(\sigma^{-1}) \text{ (by (67))}\\
&= \partial^2(\sigma^{-1}) * e * \partial^4(\sigma^{-1}) * \partial^1(\sigma) * \partial^1(\sigma^{-1}) \text{ (by (53))}\\
&= \partial^2(\sigma^{-1}) * \partial^4(\sigma^{-1}) * e * \partial^1(\sigma) * \partial^1(\sigma^{-1}) \text{ (by (65))}\\
&= \partial^2(\sigma^{-1}) * \partial^4(\sigma^{-1}) * e \text{ (by (50))}\\
&= \partial^2(\sigma^{-1}) * e * \partial^4(\sigma^{-1}) \text{ (by (65))}\\
&= \partial^2(\sigma^{-1}) * \partial^4(\sigma^{-1}) \text{ (by (53)).}
\end{aligned}$$

□

Proposition 3.15. *Let H be a weak Hopf algebra and let σ be a normal and convolution invertible 2-cocycle. Define the product μ_{H^σ} as*

$$\mu_{H^\sigma} = (\sigma\otimes\mu_H\otimes\sigma^{-1})\circ(H\otimes H\otimes\delta_{H\otimes H})\circ\delta_{H\otimes H}.$$

Then $H^\sigma = (H, \eta_{H^\sigma} = \eta_H, \mu_{H^\sigma}, \varepsilon_{H^\sigma} = \varepsilon_H, \delta_{H^\sigma} = \delta_H)$ is an algebra coalgebra and the equality

$$\delta_{H^\sigma}\circ\mu_{H^\sigma} = (\mu_{H^\sigma}\otimes\mu_{H^\sigma})\circ\delta_{H^\sigma\otimes H^\sigma} \tag{75}$$

holds.

Proof. We begin by showing that H^σ is an algebra. Indeed, using (46),

$$\begin{aligned}
&\mu_{H^\sigma}\circ(\eta_{H^\sigma}\otimes H)\\
&= (\mu_H\otimes\sigma^{-1})\circ\delta_{H\otimes H}\circ(\Pi_H^L\otimes H)\circ\delta_H\\
&= \Pi_H^L * id_H\\
&= id_H,
\end{aligned}$$

and in a similar way we get that $\mu_{H^\sigma}\circ(H\otimes\eta_{H^\sigma}) = id_H$. As far as the associativity of the product, the proof is identical that the well-known one for Hopf algebras ([8], Theorem 1.6).

On the other hand, it is obvious that H^σ is a coalgebra. Finally,

$$\begin{aligned}
&(\mu_{H^\sigma}\otimes\mu_{H^\sigma})\circ\delta_{H^\sigma\otimes H^\sigma}\\
&= (\sigma\otimes\mu_H\otimes(\sigma^{-1}*\sigma)\otimes\mu_H\otimes\sigma^{-1})\circ(\delta_{H\otimes H}\otimes H\otimes H\otimes\delta_{H\otimes H})\circ(H\otimes H\otimes\delta_{H\otimes H})\circ\delta_{H\otimes H}\\
&= (\sigma\otimes\mu_H\otimes(\varepsilon_H\circ\mu_H)\otimes\mu_H\otimes\sigma^{-1})\circ(\delta_{H\otimes H}\otimes H\otimes H\otimes\delta_{H\otimes H})\circ(H\otimes H\otimes\delta_{H\otimes H})\circ\delta_{H\otimes H}
\end{aligned}$$

$= (\sigma \otimes \mu_H \otimes \mu_H \otimes \sigma^{-1}) \circ (\delta_{H\otimes H} \otimes \delta_{H\otimes H}) \circ \delta_{H\otimes H}$

$= \delta_{H^\sigma} \circ \mu_{H^\sigma},$

and the proof is complete. □

Remark 3.16. In the same conditions of Proposition 3.15, by (19),

$$\varepsilon_H \circ \mu_{H^\sigma} = \varepsilon_H \circ \mu_H. \tag{76}$$

As a consequence the Π morphisms of H and H^σ coincide, that is to say, for $i \in \{L, R\}$, the equalities

$$\Pi^i_{H^\sigma} = \Pi^i_H \tag{77}$$

and

$$\overline{\Pi}^i_{H^\sigma} = \overline{\Pi}^i_H \tag{78}$$

hold.

Lemma 3.17. *Let H be a weak Hopf algebra and let $\sigma : H\otimes H \to K$ be a normal and convolution invertible morphism. Assume that the antipode λ_H is an isomorphism. Then, for $\Pi \in \{\Pi^R_H, \Pi^L_H, \overline{\Pi}^R_H, \overline{\Pi}^L_H\}$, the following equalities hold:*

$$\mu_{H^\sigma} \circ (H \otimes \Pi) = \mu_H \circ (H \otimes \Pi) \tag{79}$$

$$\mu_{H^\sigma} \circ (\Pi \otimes H) = \mu_H \circ (\Pi \otimes H) \tag{80}$$

Proof. The result follows easily using Proposition 3.7 and by equalities (7) and (8). □

The following theorem is the main result of this section. We will show that, under suitable conditions, H^σ is also a weak Hopf algebra we will call the cocycle twist of H.

Proposition 3.18. *Let H be a weak Hopf algebra and let σ be a normal and convolution invertible 2-cocycle. Then H^σ is a weak Hopf algebra with antipode*

$$\lambda_{H^\sigma} = ((\sigma \circ (H \otimes \lambda_H) \circ \delta_H) \otimes \lambda_H \otimes (\sigma^{-1} \circ (H \otimes \lambda_H) \circ \delta_H)) \circ (\delta_H \otimes H) \circ \delta_H.$$

Moreover, σ^{-1} is a convolution invertible 2-cocycle for H^σ, and $(H^\sigma)^{\sigma^{-1}} = H$.

Proof. By Proposition 3.15, to get that H^σ is a weak Hopf algebra we only need to see (a2)-(a4). Then

$\varepsilon_{H^\sigma} \circ \mu_{H^\sigma} \circ (\mu_{H^\sigma} \otimes H)$

$= \varepsilon_H \circ \mu_H \circ (\mu_{H^\sigma} \otimes H)$ (by (76))

$= \varepsilon_H \circ \mu_H \circ (\mu_{H^\sigma} \otimes \Pi^L_H)$ (by (18))

$= \varepsilon_{H^\sigma} \circ \mu_{H^\sigma} \circ (\mu_{H^\sigma} \otimes \Pi^L_H)$ (by (76))

$= \varepsilon_{H^\sigma} \circ \mu_{H^\sigma} \circ (H \otimes \mu_{H^\sigma}) \circ (H \otimes H \otimes \Pi^L_H)$ (by associativity)

$= \varepsilon_H \circ \mu_H \circ (H \otimes \mu_H) \circ (H \otimes H \otimes \Pi^L_H)$ (by (76) and (79))

$= \varepsilon_H \circ \mu_H \circ (H \otimes \mu_H)$ (by (18)), and we obtain (a2). Moreover, using (17) and (80) it is not difficult to get (a3) because

$$(H \otimes \mu_{H^\sigma} \otimes H) \circ ((\delta_H \circ \eta_H) \otimes (\delta_H \circ \eta_H)) = (H \otimes \mu_H \otimes H) \circ ((\delta_H \circ \eta_H) \otimes (\delta_H \circ \eta_H)). \quad (81)$$

Now we will show that λ_{H^σ} is the antipode of H^σ. Indeed,

$$
\begin{aligned}
& id_{H^\sigma} * \lambda_{H^\sigma} \\
&= (\mu_H \otimes \sigma^{-1}) \circ (\sigma \otimes \delta_{H\otimes H}) \circ \delta_{H\otimes H} \circ (H \otimes \sigma \otimes H \otimes \sigma^{-1}) \\
&\quad \circ (H \otimes H \otimes (c_{H,H} \circ \delta_H \circ \lambda_H) \otimes \lambda_H \otimes H) \circ (H \otimes \delta_H \otimes \delta_H) \circ (H \otimes \delta_H) \circ \delta_H \\
&\quad \text{(anticomultiplicativity of } \lambda_H) \\
&= (((\sigma \otimes \mu_H) \circ \delta_{H\otimes H}) \otimes (\sigma^{-1} * \sigma)) \circ (\delta_{H\otimes H} \otimes \sigma^{-1}) \circ (H \otimes \lambda_H \otimes \lambda_H \otimes H) \circ (\delta_H \otimes \delta_H) \circ \delta_H \\
&\quad \text{(by naturality)} \\
&= (((\sigma \otimes \mu_H) \circ \delta_{H\otimes H}) \otimes (\varepsilon_H \circ \mu_H)) \circ (\delta_{H\otimes H} \otimes \sigma^{-1}) \circ (H \otimes \lambda_H \otimes \lambda_H \otimes H) \circ (\delta_H \otimes \delta_H) \circ \delta_H \\
&\quad (\sigma \text{ is convolution invertible}) \\
&= (\sigma \otimes \mu_H) \circ (\delta_{H\otimes H} \otimes \sigma^{-1}) \circ (H \otimes \lambda_H \otimes \lambda_H \otimes H) \circ (\delta_H \otimes \delta_H) \circ \delta_H \ \text{(} H \text{ weak Hopf algebra)} \\
&= (\Pi_H^L \otimes \sigma \otimes \sigma^{-1}) \circ ((c_{H,H} \circ \delta_H) \otimes \lambda_H \otimes \lambda_H \otimes H) \circ (\delta_H \otimes \delta_H) \circ \delta_H \ \text{(anticomultiplicativity of } \lambda_H) \\
&= (\Pi_H^L \otimes (\partial^1(\sigma^{-1}) * \partial^4(\sigma))) \circ ((c_{H,H} \circ \delta_H) \otimes \lambda_H \otimes H) \circ (H \otimes \delta_H) \circ \delta_H \ \text{(anticomultiplicativity of } \lambda_H) \\
&= (\Pi_H^L \otimes (\partial^3(\sigma) * \partial^2(\sigma^{-1}))) \circ ((c_{H,H} \circ \delta_H) \otimes \lambda_H \otimes H) \circ (H \otimes \delta_H) \circ \delta_H \ \text{(by (66))} \\
&= (\Pi_H^L \otimes \sigma \otimes \sigma^{-1}) \circ (H \otimes H \otimes H \otimes \mu_H \otimes H) \circ (H \otimes H \otimes c_{H,H} \otimes H \otimes H) \\
&\quad \circ (H \otimes \delta_H \otimes (c_{H,H} \circ (\lambda_H \otimes \Pi_H^R) \circ \delta_H) \otimes H) \circ ((c_{H,H} \circ \delta_H) \otimes \delta_H) \circ \delta_H \ \text{(anticomultiplicativity of } \lambda_H) \\
&= (\Pi_H^L \otimes (\varepsilon_H \circ \mu_H) \otimes \sigma^{-1}) \circ (H \otimes H \otimes H \otimes \mu_H \otimes H) \circ (H \otimes H \otimes c_{H,H} \otimes H \otimes H) \\
&\quad \circ (H \otimes \delta_H \otimes (c_{H,H} \circ (\lambda_H \otimes \lambda_H) \circ \delta_H) \otimes H) \circ ((c_{H,H} \circ \delta_H) \otimes \delta_H) \circ \delta_H \ \text{(by (45))} \\
&= (\Pi_H^L \otimes (\varepsilon_H \circ \mu_H) \otimes (\sigma^{-1} \circ (\mu_H \otimes H))) \circ (H \otimes \delta_{H\otimes H} \otimes H) \circ ((c_{H,H} \circ \delta_H) \otimes ((\lambda_H \otimes H) \circ \delta_H)) \circ \delta_H \\
&\quad \text{(anticomultiplicativity of } \lambda_H) \\
&= (\Pi_H^L \otimes \sigma^{-1}) \circ (H \otimes (\mu_H \circ (H \otimes \lambda_H)) \otimes H) \circ ((c_{H,H} \circ \delta_H) \otimes \delta_H) \circ \delta_H \ \text{(} H \text{ weak Hopf algebra)} \\
&= (H \otimes \sigma^{-1}) \circ (\mu_{H\otimes H} \otimes H) \circ ((c_{H,H} \circ \delta_H) \otimes ((\lambda_H \otimes \lambda_H) \circ \delta_H) \otimes H) \circ (\delta_H \otimes H) \circ \delta_H \ \text{(definition of } \Pi_H^L) \\
&= (H \otimes \sigma^{-1}) \circ (\mu_{H\otimes H} \otimes H) \circ ((c_{H,H} \circ \delta_H) \otimes (c_{H,H} \circ \delta_H \circ \lambda_H) \otimes H) \circ (\delta_H \otimes H) \circ \delta_H \\
&\quad \text{(} H \text{ weak Hopf algebra and anticomultiplicativity of } \lambda_H) \\
&= (H \otimes \sigma^{-1}) \circ ((c_{H,H} \circ \delta_H \circ \Pi_H^L) \otimes H) \circ \delta_H \ \text{(} H \text{ weak Hopf algebra)} \\
&= ((((\varepsilon_H \circ \mu_H) \otimes \sigma^{-1}) \circ \delta_{H\otimes H}) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H) \ \text{(definition of } \Pi_H^L \text{ and coassociativity)} \\
&= (\sigma^{-1} \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H) \ \text{(by (20)} \\
&= \Pi_H^L \ \text{(by (46))} \\
&= \Pi_{H^\sigma}^L \ \text{(by (77)),}
\end{aligned}
$$

and with similar computations, we get that $\lambda_{H^\sigma} * id_{H^\sigma} = \Pi_{H^\sigma}^R$. Finally, using (79),

$$
\begin{aligned}
&\lambda_{H^\sigma} * id_{H^\sigma} * \lambda_{H^\sigma} \\
&= \lambda_{H^\sigma} * \Pi^L_{H^\sigma} \\
&= \lambda_{H^\sigma} * \Pi^L_{H} \\
&= ((\sigma \circ (H \otimes \lambda_H)) \otimes \sigma^{-1} \otimes (\mu_H \circ (H \otimes \Pi^L_H))) \circ (\delta_H \otimes (\delta_{H\otimes H} \circ (\lambda_H \otimes H))) \circ (\delta_H \otimes H) \circ \delta_H \\
&= ((\sigma \circ (H \otimes \lambda_H)) \otimes ((\sigma^{-1} \otimes (\varepsilon_H \circ \mu_H)) \circ \delta_{H\otimes H}) \otimes H) \circ (\delta_H \otimes H \otimes c_{H,H}) \circ (H \otimes (\delta_H \circ \lambda_H) \otimes H) \\
&\quad \circ (\delta_H \otimes H) \circ \delta_H \\
&= (\sigma \otimes \sigma^{-1} \otimes H) \circ (H \otimes \lambda_H \otimes H \otimes c_{H,H}) \circ (\delta_H \otimes (\delta_H \circ \lambda_H) \otimes H) \circ (\delta_H \otimes H) \circ \delta_H \\
&= (\sigma \otimes \sigma^{-1} \otimes H) \circ (H \otimes \lambda_H \otimes H \otimes c_{H,H}) \circ (\delta_H \otimes ((\lambda_H \otimes \lambda_H) \circ c_{H,H} \circ \delta_H) \otimes H) \circ (\delta_H \otimes H) \circ \delta_H \\
&= \lambda_{H^\sigma}.
\end{aligned}
$$

To finish the proof, using (b1) it is easy to see that $(H^\sigma)^{\sigma^{-1}} = H$. Moreover, by (76) and Proposition 3.5, $\sigma^{-1} : H \otimes H \to K$ is a normal and convolution invertible morphism with inverse σ. Finally, condition (63) holds because

$$
\begin{aligned}
&\sigma^{-1} \circ (H \otimes ((\sigma^{-1} \otimes \mu_{H^\sigma}) \circ \delta_{H\otimes H})) \\
&= \sigma^{-1} \circ (H \otimes (\sigma^{-1} * \sigma) \otimes \mu_H \otimes \sigma^{-1}) \circ (H \otimes H \otimes H \otimes \delta_{H\otimes H}) \circ (H \otimes \delta_{H\otimes H}) \text{ (by the definition of } \mu_{H^\sigma}) \\
&= \sigma^{-1} \circ (H \otimes (\varepsilon_H \circ \mu_H) \otimes \mu_H \otimes \sigma^{-1}) \circ (H \otimes H \otimes H \otimes \delta_{H\otimes H}) \circ (H \otimes \delta_{H\otimes H}) \text{ (by (b1))} \\
&= \sigma^{-1} \circ (H \otimes ((\mu_H \otimes \sigma^{-1}) \circ \delta_{H\otimes H})) \text{ (by (a1))} \\
&= \sigma^{-1} \circ (((\mu_H \otimes \sigma^{-1}) \circ \delta_{H\otimes H}) \otimes H) \text{ (by (66))} \\
&= \sigma^{-1} \circ ((\sigma^{-1} * \sigma) \otimes \mu_H \otimes \sigma^{-1} \otimes H) \circ (H \otimes H \otimes \delta_{H\otimes H} \otimes H) \circ (\delta_{H\otimes H} \otimes H) \text{ (by (a1) and (b1))} \\
&= \sigma^{-1} \circ (((\sigma^{-1} \otimes \mu_{H^\sigma}) \circ \delta_{H\otimes H}) \otimes H) \text{ (by the definition of } \mu_{H^\sigma}),
\end{aligned}
$$

and the proof is complete. □

4. The Yetter-Drinfel'd module category of a cocycle deformation

In this section we consider the categories of left-right Yetter-Drinfel'd modules over the weak Hopf algebras H and H^σ and show that they are equivalent. As a consequence we get an equivalence between modules for their Drinfel'd doubles.

First of all we recall the notion of left-right Yetter-Drinfel'd module in the weak Hopf algebra setting. We want to point that the properties of Yetter-Drinfel'd modules remain valid with slight changes regardless of the side where we work (left-left, left-right, right-left or right-right), so that we will cite our convenience different papers without taking into account the side on which their results are obtained.

Definition 4.1. A left-right Yetter-Drinfel'd module over a weak Hopf algebra H is a triple $M = (M, \varphi_M, \rho_M)$ such that

(b1) (M, φ_M) is a left H-module.

(b2) (M, ρ_M) is a right H-comodule.

(b3) $(\varphi_M \otimes \mu_H) \circ (H \otimes c_{H,M} \otimes H) \circ (\delta_H \otimes \rho_M) = (M \otimes \mu_H) \circ (\rho_M \otimes H) \circ c_{H,M} \circ (H \otimes \varphi_M) \circ (\delta_H \otimes M)$

(b4) $\rho_M = (\varphi_M \otimes \mu_H) \circ (H \otimes c_{H,M} \otimes H) \circ (\delta_H \otimes \rho_M) \circ (\eta_H \otimes M)$

If M and N are left-right Yetter-Drinfel'd modules over H, we say that $f : M \to N$ is a morphism of left-right Yetter-Drinfel'd modules if f is a morphism of left H-modules and right H-comodules. In what follows we will denote by ${}_H\mathcal{YD}^H$ the category of left-right Yetter-Drinfel'd modules over H.

Remark 4.2. The condition (b4) (see [1]) can be restated as

$$\varphi_M = (M \otimes \varepsilon_H) \circ (\varphi_M \otimes \mu_H) \circ (H \otimes c_{H,M} \otimes H) \circ (\delta_H \otimes \rho_M) \tag{82}$$

Moreover, conditions (b3) and (b4) (see [16]) are equivalent to

$$\rho_M \circ \varphi_M = (M \otimes (\mu_H \circ c_{H,H})) \circ (c_{H,M} \otimes H) \circ (\lambda_H^{-1} \otimes \varphi_M \otimes \mu_H) \circ (\delta_H \otimes c_{H,M} \otimes H) \circ (\delta_H \otimes \rho_M) \tag{83}$$

Let H be a weak Hopf algebra and let (M, φ_M) and (N, φ_N) be left H-modules. Following [1], the morphism

$$\nabla_{M\otimes N} = \varphi_{M\otimes N} \circ (\eta_H \otimes M \otimes N) : M \otimes N \to M \otimes N$$

is an idempotent, where $\varphi_{M\otimes N} = (\varphi_M \otimes \varphi_N) \circ (H \otimes c_{H,M} \otimes N) \circ (\delta_H \otimes M \otimes N)$. Moreover, if we denote by $M \times N$ the image of $\nabla_{M\otimes N}$ and by $p_{M\otimes N} : M \otimes N \to M \times N$, $i_{M\otimes N} : M \times N \to M \otimes N$ the morphisms such that $i_{M\otimes N} \circ p_{M\otimes N} = \nabla_{M\otimes N}$ and $p_{M\otimes N} \circ i_{M\otimes N} = id_{M\times N}$, the object $M \times N$ is a left H-module with action

$$\varphi_{M\times N} = p_{M\otimes N} \circ \varphi_{M\otimes N} \circ (H \otimes i_{M\otimes N}) : H \otimes (M \times N) \to M \times N.$$

Moreover, the following equalities hold:

$$\varphi_{M\otimes N} = \nabla_{M\otimes N} \circ \varphi_{M\otimes N} = \varphi_{M\otimes N} \circ (H \otimes \nabla_{M\otimes N}) \tag{84}$$

In a similar way, if (M, ρ_M) and (N, ρ_N) are right H-comodules, the morphism

$$\Delta_{M\otimes N} = (M \otimes N \otimes \varepsilon_H) \circ \rho_{M\otimes N} : M \otimes N \to M \otimes N$$

is an idempotent, where $\rho_{M\otimes N} = (M \otimes N \otimes (\mu_H \circ c_{H,H})) \circ (M \otimes c_{H,N} \otimes H) \circ (\rho_M \otimes \rho_N)$. Moreover, if we denote by $M \odot N$ the image of $\Delta_{M\otimes N}$ and by $p'_{M\otimes N} : M \otimes N \to M \odot N$, $i'_{M\otimes N} : M \odot N \to M \otimes N$ the morphisms such that $i'_{M\otimes N} \circ p'_{M\otimes N} = \Delta_{M\otimes N}$ and $p'_{M\otimes N} \circ i'_{M\otimes N} = id_{M\odot N}$, the object $M \odot N$ is a right H-comodule with coaction

$$\rho_{M\odot N} = (p'_{M\otimes N} \otimes H) \circ \rho_{M\otimes N} \circ i'_{M\otimes N} : M \odot N \to (M \odot N) \otimes H,$$

and the equalities

$$\rho_{M\otimes N} = \rho_{M\otimes N} \circ \Delta_{M\otimes N} = (\Delta_{M\otimes N} \otimes H) \circ \rho_{M\otimes N} \tag{85}$$

hold.

On the other hand, by [16], if σ is a normal and convolution invertible 2-cocycle and M, N are in ${}_H\mathcal{YD}^H$, then $\nabla_{M\otimes N} = \Delta_{M\otimes N}$ and $M \times N = M \odot N$ is in ${}_H\mathcal{YD}^H$ with the action and coaction defined above. Moreover ${}_H\mathcal{YD}^H$ is a braided monoidal category with braiding $t_{M,N} : M \times N \to N \times M$ defined by

$$t_{M,N} = p_{N\otimes M} \circ (N \otimes \varphi_M) \circ (\rho_N \otimes M) \circ c_{M,N} \circ i_{M\otimes N},$$

with inverse

$$t^{-1}_{M,N} = p_{M\otimes N} \circ c_{N,M} \circ (N \otimes \varphi_M) \circ (N \otimes \lambda_H \otimes M) \circ (\rho_N \otimes M) \circ i_{N\otimes M}.$$

Proposition 4.3. *Let H be a weak Hopf algebra and let σ be a normal and convolution invertible 2-cocycle. If $M = (M, \varphi_M, \rho_M)$ is a left-right Yetter-Drinfel'd module over H then $M^\sigma = (M, \varphi^\sigma_M, \rho_M)$ is a left-right Yetter-Drinfel'd module over H^σ, being*

$$\varphi^\sigma_M = (M \otimes \sigma) \circ (\rho_M \otimes H) \circ c_{H,M} \circ (H \otimes \varphi_M \otimes \sigma^{-1}) \circ (\delta_H \otimes c_{H,M} \otimes H) \circ (\delta_H \otimes \rho_M)$$

Proof. It is obvious that (M, ρ_M) is a right H-comodule. To get that (M, φ_M^{σ}) is a left H-module, the equality $\varphi_M^{\sigma} \circ (H \otimes \varphi_M^{\sigma}) = \varphi_M^{\sigma} \circ (\mu_{H^{\sigma}} \otimes M)$ follows with similar computations to the ones developed for Hopf algebras in [7]. A far as the unity,

$$\varphi_M^{\sigma} \circ (\eta_H \otimes M)$$

$$= (M \otimes \sigma) \circ (\rho_M \otimes H) \circ c_{H,M} \circ (\overline{\Pi}_H^L \otimes \varphi_M \otimes \sigma^{-1}) \circ (\delta_H \otimes c_{H,M} \otimes H) \circ (H \otimes \overline{\Pi}_H^R \otimes \rho_M) \circ ((\delta_H \circ \eta_H) \otimes M)$$

(by (17))

$$= (M \otimes \varepsilon_H) \circ (((M \otimes \mu_H) \circ (\rho_M \otimes H) \circ c_{H,M} \circ (H \otimes \varphi_M) \circ (\delta_H \otimes M)) \otimes (\varepsilon_H \circ \mu_H))$$
$$\circ (H \otimes c_{H,M} \otimes H) \circ ((\delta_H \circ \eta_H) \otimes \rho_M) \text{ (by (43))}$$

$$= (M \otimes \varepsilon_H) \circ (((\varphi_M \otimes \mu_H) \circ (H \otimes c_{H,M} \otimes H) \circ (\delta_H \otimes \rho_M)) \otimes (\varepsilon_H \circ \mu_H)) \circ (H \otimes c_{H,M} \otimes H) \circ ((\delta_H \circ \eta_H) \otimes \rho_M)$$

(by (b3))

$$= (M \otimes \varepsilon_H) \circ (\varphi_M \otimes \mu_H) \circ (H \otimes c_{H,M} \otimes H) \circ (\delta_H \otimes \rho_M) \circ (\eta_H \otimes M)$$

(because M is a right H-comodule and by (a1))

$$= (M \otimes \varepsilon_H) \circ \rho_M \text{ (by (b4))}$$

$$= id_M,$$

and we have (b2). To get (b3), the proof given in [7] for Hopf algebras works well in our setting by using (a1) instead of the invertibility of the cocycle σ. Finally,

$$(\varphi_M^{\sigma} \otimes \mu_{H^{\sigma}}) \circ (H \otimes c_{H,M} \otimes H) \circ (\delta_H \otimes \rho_M) \circ (\eta_H \otimes M)$$

$$= (M \otimes (\varepsilon_H \circ \mu_H)) \circ (\rho_M \otimes H) \circ c_{H,M} \circ (\overline{\Pi}_H^L \otimes \varphi_M)) \otimes \sigma^{-1} \otimes (\mu_{H^{\sigma}} \circ (\Pi_H^L \otimes H))) \circ (((\delta_H \otimes c_{H,M} \otimes H)$$
$$\circ (\delta_H \otimes \rho_M)) \otimes H \otimes H) \circ (H \otimes c_{H,M} \otimes H) \circ ((\delta_H \circ \eta_H) \otimes \rho_M) \text{ (by (17))}$$

$$= (((M \otimes \varepsilon_H) \circ (M \otimes \mu_H) \circ (\rho_M \otimes H) \circ c_{H,M} \circ (H \otimes \varphi_M) \circ (\delta_H \otimes M)) \otimes \sigma^{-1} \otimes (\mu_H \circ (\Pi_H^L \otimes H)))$$
$$\circ (((H \otimes c_{H,M} \otimes H) \circ (\delta_H \otimes \rho_M)) \otimes H \otimes H) \circ (H \otimes c_{H,M} \otimes H) \circ ((\delta_H \circ \eta_H) \otimes \rho_M) \text{ (by (43) and (80))}$$

$$= (((M \otimes \varepsilon_H) \circ (\varphi_M \otimes \mu_H) \circ (H \otimes c_{H,M} \otimes H) \circ (\delta_H \otimes \rho_M)) \otimes \sigma^{-1} \otimes (\mu_H \circ (\Pi_H^L \otimes H)))$$
$$\circ (((H \otimes c_{H,M} \otimes H) \circ (\delta_H \otimes \rho_M)) \otimes H \otimes H) \circ (H \otimes c_{H,M} \otimes H) \circ ((\delta_H \circ \eta_H) \otimes \rho_M) \text{ (by (b3) and (17))}$$

$$= (\varphi_M \otimes \sigma^{-1} \otimes \mu_H) \circ (((H \otimes c_{H,M} \otimes H) \circ (\delta_H \otimes \rho_M)) \otimes H \otimes H) \circ ((H \otimes c_{H,M} \otimes H) \circ ((\delta_H \circ \eta_H) \otimes \rho_M))$$

(by (82))

$$= (\varphi_M \otimes ((\sigma^{-1} \otimes \mu_H) \circ \delta_{H \otimes H} \circ (\Pi_H^L \otimes H))) \circ ((H \otimes c_{H,M} \otimes H) \circ ((\delta_H \circ \eta_H) \otimes \rho_M))$$

(by coassociativity, (17) and condition of right H-comodule)

$$= (\varphi_M \otimes \mu_H) \circ (H \otimes c_{H,M} \otimes H) \circ (\delta_H \otimes \rho_M) \circ (\eta_H \otimes M) \text{ (by (39))}$$

$$= \rho_M \text{ (by (b4))},$$

and the proof is complete. □

Now we will show that ${}_H\mathcal{YD}^H$ and ${}_{H^{\sigma}}\mathcal{YD}^{H^{\sigma}}$ are equivalent as braided monoidal categories.

Theorem 4.4. *Let H be a weak Hopf algebra and let σ be a normal and convolution invertible* 2*-cocycle. Then there is a braided monoidal category isomorphism between ${}_H\mathcal{YD}^H$ and ${}_{H^{\sigma}}\mathcal{YD}^{H^{\sigma}}$.*

Proof. We define the covariant functor

$$F :_H \mathcal{YD}^H \to_{H^\sigma} \mathcal{YD}^{H^\sigma}$$

by $F(M) = M^\sigma$ on the objects and by the identity on the morphisms. By Proposition 4.3, M^σ is an object in ${}_{H^\sigma}\mathcal{YD}^{H^\sigma}$. To get that F is monoidal, note that if M and N are two objects in ${}_H\mathcal{YD}^H$, by (76) the idempotent morphisms $\Delta_{M^\sigma\otimes N^\sigma}$ and $\Delta_{(M\otimes N)^\sigma}$ coincide and then we can define $\Phi_{M,N} : M^\sigma \times N^\sigma \to (M\times N)^\sigma$ as

$$\Phi_{M,N} = (p_{(M\otimes N)^\sigma} \otimes (\sigma^{-1} \circ c_{H,H})) \circ (M \otimes c_{H,N} \otimes H) \circ (\rho_M \otimes \rho_N) \circ i_{M^\sigma\otimes N^\sigma}.$$

Using (20) it is easy to see that

$$\Phi_{M,N} \circ p_{M^\sigma\otimes N^\sigma} = (p_{(M\otimes N)^\sigma} \otimes (\sigma^{-1} \circ c_{H,H})) \circ (M \otimes c_{H,N} \otimes H) \circ (\rho_M \otimes \rho_N) \tag{86}$$

and

$$i_{(M\otimes N)^\sigma} \circ \Phi_{M,N} = (M \otimes N)^\sigma \otimes (\sigma^{-1} \circ c_{H,H})) \circ (M \otimes c_{H,N} \otimes H) \circ (\rho_M \otimes \rho_N) \circ i_{M^\sigma\otimes N^\sigma}, \tag{87}$$

and using this fact and (b1) we can see in a similar way that in the case of Hopf algebras that $\Phi_{M,N}$ is a morphism in ${}_{H^\sigma}\mathcal{YD}^{H^\sigma}$ and commutes with the braidings.

Moreover $\Phi_{M,N}$ is an isomorphism with inverse

$$\Phi^{-1}_{M,N} = (p_{M^\sigma\otimes N^\sigma} \otimes (\sigma \circ c_{H,H})) \circ (M \otimes c_{H,N} \otimes H) \circ (\rho_M \otimes \rho_N) \circ i_{(M\otimes N)^\sigma}.$$

Indeed, using (87) and (b1),

$$\begin{aligned}
&\Phi^{-1}_{M,N} \circ \Phi_{M,N}\\
&= (((p_{M^\sigma\otimes N^\sigma} \otimes (\sigma \circ c_{H,H})) \circ (M \otimes c_{H,N} \otimes H) \circ (\rho_M \otimes \rho_N) \circ (\Delta_{(M\otimes N)^\sigma} \otimes (\sigma^{-1} \circ c_{H,H}))\\
&\quad \circ(M \otimes c_{H,N} \otimes H) \circ (\rho_M \otimes \rho_N) \circ i_{M^\sigma\otimes N^\sigma}\\
&= (p_{M^\sigma\otimes N^\sigma} \otimes ((\sigma * \sigma^{-1}) \circ c_{H,H})) \circ (M \otimes c_{H,N} \otimes H) \circ (\rho_M \otimes \rho_N) \circ i_{M^\sigma\otimes N^\sigma}\\
&= p_{M^\sigma\otimes N^\sigma} \circ \Delta_{M^\sigma\otimes N^\sigma} \circ i_{M^\sigma\otimes N^\sigma}\\
&= id_{M^\sigma\times N^\sigma},
\end{aligned}$$

and in a similar way $\Phi_{M,N} \circ \Phi^{-1}_{M,N} = id_{(M\times N)^\sigma}$.

To get the inverse functor of F note that, by Proposition 3.18, σ^{-1} is a 2-cocycle on H^σ and $(H^\sigma)^{\sigma^{-1}} = H$ and then ${}_{(H^\sigma)^{\sigma^{-1}}}\mathcal{YD}^{(H^\sigma)^{\sigma^{-1}}} =_H \mathcal{YD}^H$. As a consequence we can define the functor

$$G :_{H^\sigma} \mathcal{YD}^{H^\sigma} \to_H \mathcal{YD}^H$$

by $G(M) = M^{\sigma^{-1}}$ on the objects, and by the identity on the morphisms. Then, if M is in ${}_H\mathcal{YD}^H$,

$$\begin{aligned}
&(\varphi^\sigma_M)^{\sigma^{-1}}\\
&= (M \otimes \sigma^{-1}) \circ (\rho_M \otimes H) \circ c_{H,M} \circ (H \otimes \varphi^\sigma_M \otimes \sigma) \circ (\delta_H \otimes c_{H,M} \otimes H) \circ (\delta_H \otimes \rho_M)\\
&= (M \otimes (\sigma^{-1} * \sigma)) \circ (\rho_M \otimes H) \circ c_{H,M} \circ (H \otimes \varphi_M \otimes (\sigma^{-1} * \sigma)) \circ (\delta_H \otimes c_{H,M} \otimes H) \circ (\delta_H \otimes \rho_M)\\
&\quad \text{(by naturality)}\\
&= (M \otimes (\varepsilon_H \circ \mu_H)) \circ (\rho_M \otimes H) \circ c_{H,M} \circ (H \otimes \varphi_M \otimes (\varepsilon_H \circ \mu_H)) \circ (\delta_H \otimes c_{H,M} \otimes H) \circ (\delta_H \otimes \rho_M)\\
&\quad \text{(by (b1))}
\end{aligned}$$

$= (M \otimes (\varepsilon_H \circ \mu_H)) \circ (\rho_M \otimes H) \circ c_{H,M} \circ (H \otimes \varphi_M) \circ (\delta_H \otimes M)$ (by (82))

$= \varphi_M$ (by (82)).

As a consequence $G \circ F$ is equal to the identity functor on the braided monoidal category ${}_H\mathcal{YD}^H$, and in a similar way $F \circ G$ is equal to the identity functor on the braided monoidal category ${}_{H^\sigma}\mathcal{YD}^{H^\sigma}$. Therefore ${}_H\mathcal{YD}^H$ and ${}_{H^\sigma}\mathcal{YD}^{H^\sigma}$ are isomorphic braided monoidal categories and this finishes the proof. □

It is a well-known fact the close connection between Yetter-Drinfel'd modules and the Drinfel'd double in the sense that ${}_H\mathcal{YD}^H$ can be identified with the category $Rep(D(H))$ of left H-modules over the Drinfel'd double $D(H)$ (see [14] for Hopf algebras or [16] for weak Hopf algebras). As a consequence we get the following corollary.

Corollary 4.5. *Let H be a weak Hopf algebra and let σ be a normal and convolution invertible 2-cocycle. Then there is a category equivalence between $Rep(D(H))$ and $Rep(D(H^\sigma))$*

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