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Matrices whose powers are eventually triangular

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Abstract. Square matrices whose powers eventually have some special properties are of both theoretical significance and application value. This paper investigates those complex matrices whose powers are eventually triangular. We completely characterize the eventually triangular complex matrices of order not greater than 4, and extend the results to the nonnegative case. Eventually triangular matrices of order n are also discussed.

1. Introduction

Square matrices whose powers eventually have some special properties are interesting objects of study. It is natural to characterize those complex matrices A for which there exists a positive integer k_0 such that A^k has a certain property for all integers $k \geq k_0$.

Complex matrices whose powers are eventually positive (nonnegative), were introduced by Friedland [6] in 1978. These matrices have applications in control theory [15], and have been studied extensively. See [3, 4, 8, 12, 13, 16, 18] and the references therein. Zaslavsky and Tam [19] introduced the matrices whose powers are eventually irreducible. Hogben [7, 9] introduced the matrices whose powers are eventually reducible, r-cyclic, respectively. In 2019, Ma et al. [11] considered the matrices whose powers eventually have some structural properties. They characterized those complex matrices whose powers are eventually diagonal, Toeplitz, normal, respectively.

In this paper, we focus on another basic kind of matrices: triangular matrices. Triangular matrices have applications in numerical linear algebra [10]. Our aim is to describe the matrices whose powers are eventually triangular. In Sections 2-4, we give a complete characterization of the eventually triangular complex matrices of order not greater than 4, and also describe the eventually triangular nonnegative matrices more clearly. In Section 5, we discuss the eventually triangular matrices of order n in some cases.

A complex square matrix A is called *nilpotent* if there exists a positive integer p such that A^p is a zero matrix. It is clear that triangular matrices and nilpotent matrices are eventually triangular. Note that if a nonsingular matrix A is eventually upper (lower) triangular, that is, there exists a positive integer k_0 such that A^k is upper (lower) triangular for all integers $k \geq k_0$, then $(A^{k_0})^{-1}$ is upper (lower) triangular and thus $A = (A^{k_0})^{-1}A^{k_0+1}$ is upper (lower) triangular.

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2. Eventually triangular matrices of order 2

Eventually triangular complex matrices of order 2 are easy to characterize.

Theorem 2.1. *Let A be a complex matrix of order 2. Then A is eventually upper (lower) triangular if and only if A is either upper (lower) triangular or nilpotent.*

Proof. Consider the two eigenvalues of A. We distinguish three cases.

Case 1. A has two nonzero eigenvalues. Then A is eventually upper (lower) triangular if and only if A is upper (lower) triangular.

Case 2. A has two zero eigenvalues. Then A is eventually triangular if and only if A is nilpotent.

Case 3. A has a zero eigenvalue and a nonzero eigenvalue.

If A is eventually upper triangular, there exists a positive integer k_0 such that A^k is upper triangular for all integers $k \geq k_0$. Then $A^{k_0} = \begin{bmatrix} b_1 & b_2 \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & b_2 \\ 0 & b_1 \end{bmatrix}$ with $b_1 \neq 0$. Suppose $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$. $AA^{k_0} = A^{k_0}A$ implies that $a_3 b_1 = 0$. Since $b_1 \neq 0$, $a_3 = 0$. Thus A is upper triangular.

If A is eventually lower triangular, a similar argument shows that A is lower triangular. This completes the proof. □

A matrix is *nonnegative* if all of its entries are nonnegative real numbers. Nonnegative matrices have many attractive properties, and they are important in a variety of applications [1, 2]. Note that if a nonnegative matrix of order 2 is nilpotent, then it must be strictly upper or lower triangular. As a corollary, we can deduce the eventually triangular nonnegative matrices of order 2.

Corollary 2.2. *Let A be a nonnegative matrix of order 2. Then A is eventually upper (lower) triangular if and only if A is upper (lower) triangular.*

It is clear that a block upper (lower) triangular matrix, with diagonal block being upper (lower) triangular or nilpotent, is eventually triangular. However, we found that eventually triangular matrices may not only have this form, even for the 3×3 case. Consider the matrix $A = \begin{bmatrix} 2 & -1 & -3 \\ -2 & 3 & 5 \\ 2 & -2 & -4 \end{bmatrix}$. A simple computation shows that $A^k = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ for all integers $k \geq 2$, but A is not any block triangular matrix with diagonal block being triangular or nilpotent. Thus it is of interest to study the eventually triangular matrices of order $n \geq 3$.

3. Eventually triangular matrices of order 3

If A is a matrix, $A(i, j)$ denotes its entry in the i-th row and j-th column. The following fact is clear.

Lemma 3.1. *Let A be a complex matrix of order n. If there exists a positive integer k_0 such that A^k is upper triangular for all integers $k \geq k_0$, then given an integer i with $1 \leq i \leq n$, either $A^k(i,i) = 0$ for all integers $k \geq k_0$, or $A^k(i,i) \neq 0$ for all integers $k \geq k_0$.*

Lemma 3.2. *Let A be an eventually upper triangular matrix of order 3, and assume that there exists a positive integer $k_0 \geq 2$ such that A^k is upper triangular for all integers $k \geq k_0$. If $A^{k_0}(2,2) \neq 0$ and $A^{k_0}(1,1) = A^{k_0}(3,3) = 0$, then $A = S\left[\begin{array}{c|cc} a & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{N}} \\ 0 & & \end{array}\right]S^{-1}$, where $a \neq 0$, N is a nilpotent matrix of order 2, and $S = \begin{bmatrix} b & 1 & 0 \\ 1 & 0 & c \\ 0 & 0 & 1 \end{bmatrix}$.*

Proof. Suppose $A^{k_0} = \begin{bmatrix} 0 & b_1 & b_2 \\ 0 & b_3 & b_4 \\ 0 & 0 & 0 \end{bmatrix}$ with $b_3 \neq 0$. Consider the Jordan canonical form of A, denoted as $J(A)$, it follows that $J(A) = \begin{bmatrix} \sqrt[k_0]{b_3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} \sqrt[k_0]{b_3} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Thus $J(A^{k_0}) = \begin{bmatrix} b_3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, i.e., there exists a nonsingular matrix S of order 3 such that $(S^{-1}AS)^{k_0} = S^{-1}A^{k_0}S = \begin{bmatrix} b_3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Since $b_3 \neq 0$, $S^{-1}AS$ has the form $\left[\begin{array}{c|cc} \sqrt[k_0]{b_3} & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{N}} \\ 0 & & \end{array}\right]$, where N is a nilpotent matrix of order 2.

Next we give the matrix S. Since $\operatorname{rank}(A^{k_0}) = \operatorname{rank}(J(A^{k_0})) = 1$, $b_1b_4 = b_2b_3$. A direct computation shows that A^{k_0} has the eigenvectors $[\frac{b_1}{b_3}, 1, 0]^T, [1, 0, 0]^T, [0, -\frac{b_4}{b_3}, 1]^T$ corresponding to the eigenvalues $b_3, 0, 0$. Then $S = \begin{bmatrix} \frac{b_1}{b_3} & 1 & 0 \\ 1 & 0 & -\frac{b_4}{b_3} \\ 0 & 0 & 1 \end{bmatrix}$. Let $a = \sqrt[k_0]{b_3}, b = \frac{b_1}{b_3}, c = -\frac{b_4}{b_3}$. This completes the proof. □

We mentioned in Section 2 that the matrix $A = \begin{bmatrix} 2 & -1 & -3 \\ -2 & 3 & 5 \\ 2 & -2 & -4 \end{bmatrix}$ is eventually upper triangular. It is easy to verify that $A = S\left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{N}} \\ 0 & & \end{array}\right]S^{-1}$, where $N = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$ is nilpotent, and $S = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

The following theorem characterizes the eventually upper triangular complex matrices of order 3.

Theorem 3.3. *Let A be a complex matrix of order 3. Then A is eventually upper triangular if and only if A is one of the following:*

(i) upper triangular;

(ii) nilpotent;

(iii) $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & a_5 & a_6 \\ 0 & a_8 & a_9 \end{bmatrix}$ with $a_1 \neq 0$ and the submatrix $\begin{bmatrix} a_5 & a_6 \\ a_8 & a_9 \end{bmatrix}$ being nilpotent, or $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & a_9 \end{bmatrix}$ with $a_9 \neq 0$ and the submatrix $\begin{bmatrix} a_1 & a_2 \\ a_4 & a_5 \end{bmatrix}$ being nilpotent;

(iv) $A = S\left[\begin{array}{c|cc} a & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{N}} \\ 0 & & \end{array}\right]S^{-1}$, where $a \neq 0$, N is a nilpotent matrix of order 2, and $S = \begin{bmatrix} b & 1 & 0 \\ 1 & 0 & c \\ 0 & 0 & 1 \end{bmatrix}$.

Proof. Consider the three eigenvalues of A. We distinguish four cases.

Case 1. A has three nonzero eigenvalues. Then A is eventually upper triangular if and only if A is upper triangular.

Case 2. A has three zero eigenvalues. Then A is eventually upper triangular if and only if A is nilpotent.

Case 3. A has two nonzero eigenvalues and a zero eigenvalue.

Suppose there exists a positive integer $k_0 \geq 2$ such that A^k is upper triangular for all integers $k \geq k_0$. Then $A^{k_0} = \begin{bmatrix} b_1 & b_2 & b_3 \\ 0 & b_4 & b_5 \\ 0 & 0 & 0 \end{bmatrix}$ with $b_1b_4 \neq 0$, or $\begin{bmatrix} 0 & b_1 & b_2 \\ 0 & b_3 & b_4 \\ 0 & 0 & b_5 \end{bmatrix}$ with $b_3b_5 \neq 0$, or $\begin{bmatrix} b_1 & b_2 & b_3 \\ 0 & 0 & b_4 \\ 0 & 0 & b_5 \end{bmatrix}$ with $b_1b_5 \neq 0$.

If $A^{k_0} = \begin{bmatrix} b_1 & b_2 & b_3 \\ 0 & b_4 & b_5 \\ 0 & 0 & 0 \end{bmatrix}$ with $b_1b_4 \neq 0$, by Lemma 3.1, A^{k_0+1} has the form $\begin{bmatrix} c_1 & c_2 & c_3 \\ 0 & c_4 & c_5 \\ 0 & 0 & 0 \end{bmatrix}$ with $c_1c_4 \neq 0$.

Suppose $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}$. Then by $A \cdot A^{k_0} = A^{k_0} \cdot A = A^{k_0+1}$, we can deduce that $a_4 = a_7 = a_8 = 0$. Thus A is upper triangular.

If $A^{k_0} = \begin{bmatrix} 0 & b_1 & b_2 \\ 0 & b_3 & b_4 \\ 0 & 0 & b_5 \end{bmatrix}$ with $b_3 b_5 \neq 0$, or $\begin{bmatrix} b_1 & b_2 & b_3 \\ 0 & 0 & b_4 \\ 0 & 0 & b_5 \end{bmatrix}$ with $b_1 b_5 \neq 0$, a similar argument shows that A is upper triangular.

Case 4. A has a nonzero eigenvalue and two zero eigenvalues.

Suppose there exists a positive integer $k_0 \geq 2$ such that A^k is upper triangular for all integers $k \geq k_0$. Then $A^{k_0} = \begin{bmatrix} b_1 & b_2 & b_3 \\ 0 & 0 & b_4 \\ 0 & 0 & 0 \end{bmatrix}$ with $b_1 \neq 0$, or $\begin{bmatrix} 0 & b_1 & b_2 \\ 0 & 0 & b_3 \\ 0 & 0 & b_4 \end{bmatrix}$ with $b_4 \neq 0$, or $\begin{bmatrix} 0 & b_1 & b_2 \\ 0 & b_3 & b_4 \\ 0 & 0 & 0 \end{bmatrix}$ with $b_3 \neq 0$.

If $A^{k_0} = \begin{bmatrix} b_1 & b_2 & b_3 \\ 0 & 0 & b_4 \\ 0 & 0 & 0 \end{bmatrix}$ with $b_1 \neq 0$, by Lemma 3.1, A^{k_0+1} has the form $\begin{bmatrix} c_1 & c_2 & c_3 \\ 0 & 0 & c_4 \\ 0 & 0 & 0 \end{bmatrix}$ with $c_1 \neq 0$. Suppose $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}$. By $A \cdot A^{k_0} = A^{k_0} \cdot A = A^{k_0+1}$, we can deduce that $a_4 = a_7 = 0$. Then $\begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & a_5 & a_6 \\ 0 & a_8 & a_9 \end{bmatrix}^{k_0} = \begin{bmatrix} b_1 & b_2 & b_3 \\ 0 & 0 & b_4 \\ 0 & 0 & 0 \end{bmatrix}$ implies $a_1^{k_0} = b_1$ and $\begin{bmatrix} a_5 & a_6 \\ a_8 & a_9 \end{bmatrix}^{k_0} = \begin{bmatrix} 0 & b_4 \\ 0 & 0 \end{bmatrix}$. Thus $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & a_5 & a_6 \\ 0 & a_8 & a_9 \end{bmatrix}$ with $a_1 \neq 0$, and the submatrix $\begin{bmatrix} a_5 & a_6 \\ a_8 & a_9 \end{bmatrix}$ is nilpotent.

If $A^{k_0} = \begin{bmatrix} 0 & b_1 & b_2 \\ 0 & 0 & b_3 \\ 0 & 0 & b_4 \end{bmatrix}$ with $b_4 \neq 0$, a similar argument shows that $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & a_9 \end{bmatrix}$ with $a_9 \neq 0$, and the submatrix $\begin{bmatrix} a_1 & a_2 \\ a_4 & a_5 \end{bmatrix}$ is nilpotent.

If $A^{k_0} = \begin{bmatrix} 0 & b_1 & b_2 \\ 0 & b_3 & b_4 \\ 0 & 0 & 0 \end{bmatrix}$ with $b_3 \neq 0$, by Lemma 3.2, A has the form (iv).

Conversely, when A has any one of the forms (i)–(iv), a direct computation shows that A is eventually upper triangular. This completes the proof. □

Two matrices X and Y are said to be *permutation similar* if there exists a permutation matrix P such that $P^T X P = Y$. Now we can describe the eventually upper triangular nonnegative matrices of order 3 more clearly.

Theorem 3.4. *Let A be a nonnegative matrix of order 3. Then A is eventually upper triangular if and only if A is one of the following:*

(i) upper triangular;

(ii) nilpotent, and thus permutation similar to a strictly upper triangular matrix;

(iii) $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ 0 & a_8 & 0 \end{bmatrix}$ *with* $a_1, a_8 > 0$, *or* $A = \begin{bmatrix} 0 & 0 & a_3 \\ a_4 & 0 & a_6 \\ 0 & 0 & a_9 \end{bmatrix}$ *with* $a_4, a_9 > 0$, *or* $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ b & 0 & 0 \end{bmatrix}$ *with* $a, b > 0$.

Proof. Consider the three eigenvalues of A. We distinguish four cases.

Case 1. A has three nonzero eigenvalues. Then A is eventually upper triangular if and only if A is upper triangular.

Case 2. A has three zero eigenvalues. Then A is eventually upper triangular if and only if A is nilpotent. By Lemma 2.3 in [11], A is permutation similar to a strictly upper triangular matrix.

Case 3. A has two nonzero eigenvalues and a zero eigenvalue. By the Case 3 in the proof of Theorem 3.3, A is upper triangular.

Case 4. A has a nonzero eigenvalue and two zero eigenvalues. Suppose there exists a positive integer $k_0 \geq 2$ such that A^k is upper triangular for all integers $k \geq k_0$.

If $A^{k_0} = \begin{bmatrix} b_1 & b_2 & b_3 \\ 0 & 0 & b_4 \\ 0 & 0 & 0 \end{bmatrix}$ with $b_1 > 0$, by the Case 4 in the proof of Theorem 3.3, $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & a_5 & a_6 \\ 0 & a_8 & a_9 \end{bmatrix}$ with $a_1 > 0$, and the submatrix $\begin{bmatrix} a_5 & a_6 \\ a_8 & a_9 \end{bmatrix}$ is nilpotent. Since A is nonnegative, $a_5 = a_9 = 0$ and $a_6 a_8 = 0$. Thus A is upper triangular or has the first form in (iii).

If $A^{k_0} = \begin{bmatrix} 0 & b_1 & b_2 \\ 0 & 0 & b_3 \\ 0 & 0 & b_4 \end{bmatrix}$ with $b_4 > 0$, a similar argument shows that A is upper triangular or has the second form in (iii).

If $A^{k_0} = \begin{bmatrix} 0 & b_1 & b_2 \\ 0 & b_3 & b_4 \\ 0 & 0 & 0 \end{bmatrix}$ with $b_3 > 0$, either A is eventually diagonal, or there exists a positive integer $k_1 \geq 2$ such that A^{k_1} is upper triangular but not diagonal. In the previous case, by Theorem 2.4 in [11], A is permutation similar to a matrix of the form $\begin{bmatrix} 0 & b & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{bmatrix}$ with $a > 0$. Thus $A = \begin{bmatrix} 0 & 0 & b \\ 0 & a & 0 \\ 0 & 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ b & 0 & 0 \end{bmatrix}$, which implies that A is upper triangular or has the third form in (iii). In the latter case, $A^{k_1} = \begin{bmatrix} 0 & c_1 & c_2 \\ 0 & c_3 & c_4 \\ 0 & 0 & 0 \end{bmatrix}$, where $c_3 > 0$ and at least one of c_1, c_2, c_4 are positive. Suppose $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}$. Then by $A \cdot A^{k_1} = A^{k_1} \cdot A$ and the nonnegativity of A, we can deduce that $a_4 = a_7 = a_8 = 0$. Thus A is upper triangular.

Conversely, when A has any one of the forms (i)–(iii), a direct computation shows that A is eventually upper triangular. This completes the proof. □

Obviously, the eventually lower triangular complex (nonnegative) matrices of order 3 can be obtained from Theorem 3.3 (Theorem 3.4) by taking the transpose.

4. Eventually triangular matrices of order 4

Denote by I_n the identity matrix of order n.

Lemma 4.1. *Let A be an eventually upper triangular matrix of order 4, and assume that there exists a positive integer $k_0 \geq 2$ such that A^k is upper triangular for all integers $k \geq k_0$. If $A^{k_0}(2,2)A^{k_0}(3,3) \neq 0$ and $A^{k_0}(1,1) = A^{k_0}(4,4) = 0$, then* $A = S \left[\begin{array}{cc|cc} a & b & 0 & 0 \\ 0 & c & 0 & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{N}} \\ 0 & 0 & & \end{array}\right] S^{-1}$, *where $ac \neq 0$, N is a nilpotent matrix of order 2, and* $S = \begin{bmatrix} d & e & 1 & 0 \\ 1 & f & 0 & g \\ 0 & 1 & 0 & h \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Proof. Suppose $A^{k_0} = \begin{bmatrix} 0 & b_1 & b_2 & b_3 \\ 0 & b_4 & b_5 & b_6 \\ 0 & 0 & b_7 & b_8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ with $b_4 b_7 \neq 0$. Consider the Jordan canonical form of A, denoted as $J(A)$. We distinguish three cases.

Case 1. $J(A) = \begin{bmatrix} \sqrt[k_0]{b_4} & 0 & 0 & 0 \\ 0 & \sqrt[k_0]{b_7} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} \sqrt[k_0]{b_4} & 0 & 0 & 0 \\ 0 & \sqrt[k_0]{b_7} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ with $\sqrt[k_0]{b_4} \neq \sqrt[k_0]{b_7}$. Then $J(A^{k_0}) = \begin{bmatrix} b_4 & 0 & 0 & 0 \\ 0 & b_7 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, i.e., there exists a nonsingular matrix S of order 4 such that $(S^{-1}AS)^{k_0} = S^{-1}A^{k_0}S = \begin{bmatrix} b_4 & 0 & 0 & 0 \\ 0 & b_7 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Since $b_4b_7 \neq 0$ and $b_4 \neq b_7$, $S^{-1}AS$ has the form $\left[\begin{array}{cc|cc} \sqrt[k_0]{b_4} & 0 & 0 & 0 \\ 0 & \sqrt[k_0]{b_7} & 0 & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{N}} \\ 0 & 0 & & \end{array}\right]$, where N is a nilpotent matrix of order 2. Let $a = \sqrt[k_0]{b_4}, b = 0, c = \sqrt[k_0]{b_7}$.

Next we give the matrix S. Since $\text{rank}(A^{k_0}) = \text{rank}(J(A^{k_0})) = 2$, $b_1(b_5b_8 - b_6b_7) - b_4(b_2b_8 - b_3b_7) = 0$. A direct computation shows that A^{k_0} has the eigenvectors $[\frac{b_1}{b_4}, 1, 0, 0]^T, [\frac{b_2}{b_7} + \frac{b_1b_5}{b_7(b_7-b_4)}, \frac{b_5}{b_7-b_4}, 1, 0]^T, [1, 0, 0, 0]^T, [0, \frac{b_5b_8}{b_4b_7} - \frac{b_6}{b_4}, -\frac{b_8}{b_7}, 1]^T$ corresponding to the eigenvalues $b_4, b_7, 0, 0$. Then $S = \begin{bmatrix} \frac{b_1}{b_4} & \frac{b_2}{b_7} + \frac{b_1b_5}{b_7(b_7-b_4)} & 1 & 0 \\ 1 & \frac{b_5}{b_7-b_4} & 0 & \frac{b_5b_8}{b_4b_7} - \frac{b_6}{b_4} \\ 0 & 1 & 0 & -\frac{b_8}{b_7} \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Let $d = \frac{b_1}{b_4}, e = \frac{b_2}{b_7} + \frac{b_1b_5}{b_7(b_7-b_4)}, f = \frac{b_5}{b_7-b_4}, g = \frac{b_5b_8}{b_4b_7} - \frac{b_6}{b_4}, h = -\frac{b_8}{b_7}$.

Case 2. $J(A) = \begin{bmatrix} \sqrt[k_0]{b_4} & 0 & 0 & 0 \\ 0 & \sqrt[k_0]{b_7} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} \sqrt[k_0]{b_4} & 0 & 0 & 0 \\ 0 & \sqrt[k_0]{b_7} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ with $\sqrt[k_0]{b_4} = \sqrt[k_0]{b_7}$. Then $b_4 = b_7$ and $J(A^{k_0}) = \begin{bmatrix} b_4 & 0 & 0 & 0 \\ 0 & b_4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, i.e., there exists a nonsingular matrix S of order 4 such that $(S^{-1}AS)^{k_0} = S^{-1}A^{k_0}S = \begin{bmatrix} b_4 & 0 & 0 & 0 \\ 0 & b_4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Since $b_4 \neq 0$, $S^{-1}AS$ has the form $\left[\begin{array}{c|c} B & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \hline \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & N \end{array}\right]$, where $B^{k_0} = b_4I_2$, and N is a nilpotent matrix of order 2.

Next we give the matrix S. Note that $\text{rank}(A^{k_0}) = \text{rank}(J(A^{k_0})) = 2$ and $\text{rank}(b_4I_4 - A^{k_0}) = \text{rank}(b_4I_4 - J(A^{k_0})) = 2$. Then $b_5 = 0$ and thus $b_1b_4b_6 + b_4(b_2b_8 - b_3b_4) = 0$. A direct computation shows that A^{k_0} has the eigenvectors $[\frac{b_1}{b_4}, 1, 0, 0]^T, [\frac{b_2}{b_4}, 0, 1, 0]^T, [1, 0, 0, 0]^T, [0, -\frac{b_6}{b_4}, -\frac{b_8}{b_4}, 1]^T$ corresponding to the eigenvalues $b_4, b_4, 0, 0$. Then $S = \begin{bmatrix} \frac{b_1}{b_4} & \frac{b_2}{b_4} & 1 & 0 \\ 1 & 0 & 0 & -\frac{b_6}{b_4} \\ 0 & 1 & 0 & -\frac{b_8}{b_4} \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Let $d = \frac{b_1}{b_4}, e = \frac{b_2}{b_4}, f = 0, g = -\frac{b_6}{b_4}, h = -\frac{b_8}{b_4}$. Since $A^k = S \left[\begin{array}{c|c} B^k & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \hline \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \end{array}\right] S^{-1}$ is upper triangular for all integers $k \geq k_0$, a direct computation shows that B^k is upper triangular for all integers $k \geq k_0$. By the nonsingularity of B, it follows that B is upper triangular. Let $a = B(1,1), b = B(1,2), c = B(2,2)$.

Case 3. $J(A) = \begin{bmatrix} \sqrt[k_0]{b_4} & 1 & 0 & 0 \\ 0 & \sqrt[k_0]{b_7} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} \sqrt[k_0]{b_4} & 1 & 0 & 0 \\ 0 & \sqrt[k_0]{b_7} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ with $\sqrt[k_0]{b_4} = \sqrt[k_0]{b_7}$. Then $b_4 = b_7$ and

$J(A^{k_0}) = \begin{bmatrix} b_4 & 1 & 0 & 0 \\ 0 & b_4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, i.e., there exists a nonsingular matrix T of order 4 such that $(T^{-1}AT)^{k_0} = T^{-1}A^{k_0}T = \begin{bmatrix} b_4 & 1 & 0 & 0 \\ 0 & b_4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Since $b_4 \neq 0$, $T^{-1}AT$ has the form $\left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{B}} & 0 & 0 \\ & & 0 & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{N}} \\ 0 & 0 & & \end{array}\right]$, where $B^{k_0} = \begin{bmatrix} b_4 & 1 \\ 0 & b_4 \end{bmatrix}$, and N is a nilpotent matrix of order 2.

Now we give the matrix T. Note that $\operatorname{rank}(A^{k_0}) = \operatorname{rank}(J(A^{k_0})) = 2$ and $\operatorname{rank}(b_4I_4 - A^{k_0}) = \operatorname{rank}(b_4I_4 - J(A^{k_0})) = 3$. Then $b_5 \neq 0$ and $b_1(b_5b_8 - b_4b_6) - b_4(b_2b_8 - b_3b_4) = 0$. First A^{k_0} has the eigenvector $\mathbf{s_1} = [\frac{b_1}{b_4}, 1, 0, 0]^T$ corresponding to the eigenvalue b_4. Then by solving $A^{k_0}\mathbf{s_2} = \mathbf{s_1} + b_4\mathbf{s_2}$, we have $\mathbf{s_2} = [\frac{b_2}{b_4b_5} - \frac{b_1}{b_4^2}, 0, \frac{1}{b_5}, 0]^T$. Next A^{k_0} has the eigenvectors $\mathbf{s_3} = [1, 0, 0, 0]^T, \mathbf{s_4} = [0, \frac{b_5b_8}{b_4^2} - \frac{b_6}{b_4}, -\frac{b_8}{b_4}, 1]^T$ corresponding to the eigenvalues $0, 0$. Then $T = [\mathbf{s_1}, \mathbf{s_2}, \mathbf{s_3}, \mathbf{s_4}] = \begin{bmatrix} \frac{b_1}{b_4} & \frac{b_2}{b_4b_5} - \frac{b_1}{b_4^2} & 1 & 0 \\ 1 & 0 & 0 & \frac{b_5b_8}{b_4^2} - \frac{b_6}{b_4} \\ 0 & \frac{1}{b_5} & 0 & -\frac{b_8}{b_4} \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Since $A^k = T\left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{B^k}} & 0 & 0 \\ & & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]T^{-1}$ is upper triangular for all integers $k \geq k_0$, a direct computation shows that B^k is upper triangular for all integers $k \geq k_0$. By the nonsingularity of B, it follows that B is upper triangular.

Let $S = [\mathbf{s_1}, b_5\mathbf{s_2}, \mathbf{s_3}, \mathbf{s_4}] = \begin{bmatrix} \frac{b_1}{b_4} & \frac{b_2}{b_4} - \frac{b_1b_5}{b_4^2} & 1 & 0 \\ 1 & 0 & 0 & \frac{b_5b_8}{b_4^2} - \frac{b_6}{b_4} \\ 0 & 1 & 0 & -\frac{b_8}{b_4} \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Then $S = TD$ with $D = \operatorname{diag}(1, b_5, 1, 1)$. Thus $A = T\left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{B}} & 0 & 0 \\ & & 0 & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{N}} \\ 0 & 0 & & \end{array}\right]T^{-1} = SD^{-1}\left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{B}} & 0 & 0 \\ & & 0 & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{N}} \\ 0 & 0 & & \end{array}\right]DS^{-1} = S\left[\begin{array}{cc|cc} B(1,1) & b_5B(1,2) & 0 & 0 \\ 0 & B(2,2) & 0 & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{N}} \\ 0 & 0 & & \end{array}\right]S^{-1}$.

Let $a = B(1,1), b = b_5B(1,2), c = B(2,2), d = \frac{b_1}{b_4}, e = \frac{b_2}{b_4} - \frac{b_1b_5}{b_4^2}, f = 0, g = \frac{b_5b_8}{b_4^2} - \frac{b_6}{b_4}, h = -\frac{b_8}{b_4}$.

This completes the proof. □

Lemma 4.2. *Let A be an eventually upper triangular matrix of order 4, and assume that there exists a positive integer $k_0 \geq 2$ such that A^k is upper triangular for all integers $k \geq k_0$.*

(i) If $A^{k_0}(2,2) \neq 0$ and $A^{k_0}(1,1) = A^{k_0}(3,3) = A^{k_0}(4,4) = 0$, then $A = S\left[\begin{array}{c|ccc} a & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & N & \\ 0 & & & \end{array}\right]S^{-1}$, where $a \neq 0$, N is a nilpotent matrix of order 3, and $S = \begin{bmatrix} b & 1 & 0 & 0 \\ 1 & 0 & c & d \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$;

(ii) If $A^{k_0}(3,3) \neq 0$ and $A^{k_0}(1,1) = A^{k_0}(2,2) = A^{k_0}(4,4) = 0$, then $A = S\left[\begin{array}{c|ccc} a & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & N & \\ 0 & & & \end{array}\right]S^{-1}$, where $a \neq 0$, N is a nilpotent matrix of order 3, and $S = \begin{bmatrix} b & 1 & 0 & 0 \\ c & 0 & 1 & 0 \\ 1 & 0 & 0 & d \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Proof. (i) If $A^{k_0}(2,2) \neq 0$ and $A^{k_0}(1,1) = A^{k_0}(3,3) = A^{k_0}(4,4) = 0$, by Lemma 3.1, $A^{k_0+1}(2,2) \neq 0$ and $A^{k_0+1}(1,1) = A^{k_0+1}(3,3) = A^{k_0+1}(4,4) = 0$. Suppose $A^{k_0+1} = \begin{bmatrix} 0 & b_1 & b_2 & b_3 \\ 0 & b_4 & b_5 & b_6 \\ 0 & 0 & 0 & b_7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ with $b_4 \neq 0$. Consider the Jordan canonical form of A, denoted as $J(A)$, it follows that $J(A) = \begin{bmatrix} \sqrt[k_0+1]{b_4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} \sqrt[k_0+1]{b_4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} \sqrt[k_0+1]{b_4} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Then $J(A^{k_0+1}) = \begin{bmatrix} b_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ for $k_0 + 1 \geq 3$, i.e., there exists a nonsingular matrix S of order 4 such that $(S^{-1}AS)^{k_0+1} = S^{-1}A^{k_0+1}S = \begin{bmatrix} b_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Since $b_4 \neq 0$, $S^{-1}AS$ has the form $\left[\begin{array}{c|c} \sqrt[k_0+1]{b_4} & 0 \quad 0 \quad 0 \\ \hline 0 & \\ 0 & N \\ 0 & \end{array}\right]$, where N is a nilpotent matrix of order 3. Let $a = \sqrt[k_0+1]{b_4}$.

Next we give the matrix S. Note that $\mathrm{rank}(A^{k_0+1}) = \mathrm{rank}(J(A^{k_0+1})) = 1$. Then $b_7 = 0$, $b_1b_5 - b_2b_4 = 0$ and $b_1b_6 - b_3b_4 = 0$. A direct computation shows that A^{k_0+1} has the eigenvectors $[\frac{b_1}{b_4}, 1, 0, 0]^T$, $[1, 0, 0, 0]^T$, $[0, -\frac{b_5}{b_4}, 1, 0]^T$, $[0, -\frac{b_6}{b_4}, 0, 1]^T$ corresponding to the eigenvalues $b_4, 0, 0, 0$. Then $S = \begin{bmatrix} \frac{b_1}{b_4} & 1 & 0 & 0 \\ 1 & 0 & -\frac{b_5}{b_4} & -\frac{b_6}{b_4} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Let $b = \frac{b_1}{b_4}, c = -\frac{b_5}{b_4}, d = -\frac{b_6}{b_4}$.

(ii) If $A^{k_0}(3,3) \neq 0$ and $A^{k_0}(1,1) = A^{k_0}(2,2) = A^{k_0}(4,4) = 0$, a similar argument shows that $A = S\left[\begin{array}{c|c} \sqrt[k_0+1]{A^{k_0+1}(3,3)} & 0 \quad 0 \quad 0 \\ \hline 0 & \\ 0 & N \\ 0 & \end{array}\right]S^{-1}$, where $A^{k_0+1}(3,3) \neq 0$, N is a nilpotent matrix of order 3 and $S = \begin{bmatrix} \frac{A^{k_0+1}(1,3)}{A^{k_0+1}(3,3)} & 1 & 0 & 0 \\ \frac{A^{k_0+1}(2,3)}{A^{k_0+1}(3,3)} & 0 & 1 & 0 \\ 1 & 0 & 0 & -\frac{A^{k_0+1}(3,4)}{A^{k_0+1}(3,3)} \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Let $a = \sqrt[k_0+1]{A^{k_0+1}(3,3)}, b = \frac{A^{k_0+1}(1,3)}{A^{k_0+1}(3,3)}, c = \frac{A^{k_0+1}(2,3)}{A^{k_0+1}(3,3)}, d = -\frac{A^{k_0+1}(3,4)}{A^{k_0+1}(3,3)}$. □

The following theorem characterizes the eventually upper triangular complex matrices of order 4.

Theorem 4.3. *Let A be a complex matrix of order 4. Then A is eventually upper triangular if and only if A is one of the following:*

(i) upper triangular;

(ii) nilpotent;

(iii) $A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & a_6 & a_7 & a_8 \\ 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & a_{15} & a_{16} \end{bmatrix}$ *with* $a_1a_6 \neq 0$ *and the submatrix* $\begin{bmatrix} a_{11} & a_{12} \\ a_{15} & a_{16} \end{bmatrix}$ *being nilpotent, or* $A =$

$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & 0 & a_{16} \end{bmatrix}$ *with* $a_{11}a_{16} \neq 0$ *and the submatrix* $\begin{bmatrix} a_1 & a_2 \\ a_5 & a_6 \end{bmatrix}$ *being nilpotent, or* $A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & a_6 & a_7 & a_8 \\ 0 & a_{10} & a_{11} & a_{12} \\ 0 & 0 & 0 & a_{16} \end{bmatrix}$ *with* $a_1a_{16} \neq 0$ *and the submatrix* $\begin{bmatrix} a_6 & a_7 \\ a_{10} & a_{11} \end{bmatrix}$ *being nilpotent;*

(iv) $A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & a_6 & a_7 & a_8 \\ 0 & a_{10} & a_{11} & a_{12} \\ 0 & a_{14} & a_{15} & a_{16} \end{bmatrix}$ *with* $a_1 \neq 0$ *and the submatrix* $\begin{bmatrix} a_6 & a_7 & a_8 \\ a_{10} & a_{11} & a_{12} \\ a_{14} & a_{15} & a_{16} \end{bmatrix} = S \left[\begin{array}{c|cc} a & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{N}} \\ 0 & & \end{array}\right] S^{-1}$, *or* $A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ 0 & 0 & 0 & a_{16} \end{bmatrix}$ *with* $a_{16} \neq 0$ *and the submatrix* $\begin{bmatrix} a_1 & a_2 & a_3 \\ a_5 & a_6 & a_7 \\ a_9 & a_{10} & a_{11} \end{bmatrix} = S \left[\begin{array}{c|cc} a & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{N}} \\ 0 & & \end{array}\right] S^{-1}$, *where* $a \neq 0$, *N is a nilpotent matrix of order 2, and* $S = \begin{bmatrix} b & 1 & 0 \\ 1 & 0 & c \\ 0 & 0 & 1 \end{bmatrix}$;

(v) $A = S \left[\begin{array}{cc|cc} a & b & 0 & 0 \\ 0 & c & 0 & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{N}} \\ 0 & 0 & & \end{array}\right] S^{-1}$, *where* $ac \neq 0$, *N is a nilpotent matrix of order 2, and* $S = \begin{bmatrix} d & e & 1 & 0 \\ 1 & f & 0 & g \\ 0 & 1 & 0 & h \\ 0 & 0 & 0 & 1 \end{bmatrix}$;

(vi) $A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & a_6 & a_7 & a_8 \\ 0 & a_{10} & a_{11} & a_{12} \\ 0 & a_{14} & a_{15} & a_{16} \end{bmatrix}$ *with* $a_1 \neq 0$ *and the submatrix* $\begin{bmatrix} a_6 & a_7 & a_8 \\ a_{10} & a_{11} & a_{12} \\ a_{14} & a_{15} & a_{16} \end{bmatrix}$ *being nilpotent, or* $A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ 0 & 0 & 0 & a_{16} \end{bmatrix}$ *with* $a_{16} \neq 0$ *and the submatrix* $\begin{bmatrix} a_1 & a_2 & a_3 \\ a_5 & a_6 & a_7 \\ a_9 & a_{10} & a_{11} \end{bmatrix}$ *being nilpotent;*

(vii) $A = S \left[\begin{array}{c|ccc} a & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & N & \\ 0 & & & \end{array}\right] S^{-1}$, *where* $a \neq 0$, *N is a nilpotent matrix of order 3,* $S = \begin{bmatrix} b & 1 & 0 & 0 \\ 1 & 0 & c & d \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ *or* $\begin{bmatrix} b & 1 & 0 & 0 \\ c & 0 & 1 & 0 \\ 1 & 0 & 0 & d \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Proof. Consider the four eigenvalues of A. We distinguish five cases.

Case 1. A has four nonzero eigenvalues. Then A is eventually upper triangular if and only if A is upper triangular.

Case 2. A has four zero eigenvalues. Then A is eventually upper triangular if and only if A is nilpotent.

Case 3. A has three nonzero eigenvalues and a zero eigenvalue.

Suppose there exists a positive integer $k_0 \geq 2$ such that A^k is upper triangular for all integers $k \geq k_0$. Then $A^{k_0} = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ 0 & b_5 & b_6 & b_7 \\ 0 & 0 & b_8 & b_9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ with $b_1b_5b_8 \neq 0$, or $\begin{bmatrix} 0 & b_1 & b_2 & b_3 \\ 0 & b_4 & b_5 & b_6 \\ 0 & 0 & b_7 & b_8 \\ 0 & 0 & 0 & b_9 \end{bmatrix}$ with $b_4b_7b_9 \neq 0$, or $\begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ 0 & 0 & b_5 & b_6 \\ 0 & 0 & b_7 & b_8 \\ 0 & 0 & 0 & b_9 \end{bmatrix}$

with $b_1b_7b_9 \neq 0$, or $\begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ 0 & b_5 & b_6 & b_7 \\ 0 & 0 & 0 & b_8 \\ 0 & 0 & 0 & b_9 \end{bmatrix}$ with $b_1b_5b_9 \neq 0$.

If $A^{k_0} = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ 0 & b_5 & b_6 & b_7 \\ 0 & 0 & b_8 & b_9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ with $b_1b_5b_8 \neq 0$, by Lemma 3.1, A^{k_0+1} has the form $\begin{bmatrix} c_1 & c_2 & c_3 & c_4 \\ 0 & c_5 & c_6 & c_7 \\ 0 & 0 & c_8 & c_9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ with $c_1c_5c_8 \neq 0$. Suppose $A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix}$. Then by $A \cdot A^{k_0} = A^{k_0} \cdot A = A^{k_0+1}$, we can deduce that $a_5 = a_9 = a_{10} = a_{13} = a_{14} = a_{15} = 0$. Thus A is upper triangular.

If $A^{k_0} = \begin{bmatrix} 0 & b_1 & b_2 & b_3 \\ 0 & b_4 & b_5 & b_6 \\ 0 & 0 & b_7 & b_8 \\ 0 & 0 & 0 & b_9 \end{bmatrix}$ with $b_4b_7b_9 \neq 0$, or $\begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ 0 & 0 & b_5 & b_6 \\ 0 & 0 & b_7 & b_8 \\ 0 & 0 & 0 & b_9 \end{bmatrix}$ with $b_1b_7b_9 \neq 0$, or $\begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ 0 & b_5 & b_6 & b_7 \\ 0 & 0 & 0 & b_8 \\ 0 & 0 & 0 & b_9 \end{bmatrix}$ with $b_1b_5b_9 \neq 0$, a similar argument shows that A is upper triangular.

Case 4. A has two nonzero eigenvalues and two zero eigenvalues.

Suppose there exists a positive integer $k_0 \geq 2$ such that A^k is upper triangular for all integers $k \geq k_0$. Then $A^{k_0} = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ 0 & b_5 & b_6 & b_7 \\ 0 & 0 & 0 & b_8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ with $b_1b_5 \neq 0$, or $\begin{bmatrix} 0 & b_1 & b_2 & b_3 \\ 0 & 0 & b_4 & b_5 \\ 0 & 0 & b_6 & b_7 \\ 0 & 0 & 0 & b_8 \end{bmatrix}$ with $b_6b_8 \neq 0$, or $\begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ 0 & 0 & b_5 & b_6 \\ 0 & 0 & 0 & b_7 \\ 0 & 0 & 0 & b_8 \end{bmatrix}$ with $b_1b_8 \neq 0$, or $\begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ 0 & 0 & b_5 & b_6 \\ 0 & 0 & b_7 & b_8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ with $b_1b_7 \neq 0$, or $\begin{bmatrix} 0 & b_1 & b_2 & b_3 \\ 0 & b_4 & b_5 & b_6 \\ 0 & 0 & 0 & b_7 \\ 0 & 0 & 0 & b_8 \end{bmatrix}$ with $b_4b_8 \neq 0$, or $\begin{bmatrix} 0 & b_1 & b_2 & b_3 \\ 0 & b_4 & b_5 & b_6 \\ 0 & 0 & b_7 & b_8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ with $b_4b_7 \neq 0$.

If $A^{k_0} = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ 0 & b_5 & b_6 & b_7 \\ 0 & 0 & 0 & b_8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ with $b_1b_5 \neq 0$, by Lemma 3.1, A^{k_0+1} has the form $\begin{bmatrix} c_1 & c_2 & c_3 & c_4 \\ 0 & c_5 & c_6 & c_7 \\ 0 & 0 & 0 & c_8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ with $c_1c_5 \neq 0$. Suppose $A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix}$. By $A \cdot A^{k_0} = A^{k_0} \cdot A = A^{k_0+1}$, we can deduce that $a_5 = a_9 = a_{10} = a_{13} = a_{14} = 0$. Then $\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & a_6 & a_7 & a_8 \\ 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & a_{15} & a_{16} \end{bmatrix}^{k_0} = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ 0 & b_5 & b_6 & b_7 \\ 0 & 0 & 0 & b_8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ implies $a_1^{k_0} = b_1, a_6^{k_0} = b_5$ and $\begin{bmatrix} a_{11} & a_{12} \\ a_{15} & a_{16} \end{bmatrix}^{k_0} = \begin{bmatrix} 0 & b_8 \\ 0 & 0 \end{bmatrix}$. Thus $A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & a_6 & a_7 & a_8 \\ 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & a_{15} & a_{16} \end{bmatrix}$ with $a_1a_6 \neq 0$, and the submatrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{15} & a_{16} \end{bmatrix}$ is nilpotent.

If $A^{k_0} = \begin{bmatrix} 0 & b_1 & b_2 & b_3 \\ 0 & 0 & b_4 & b_5 \\ 0 & 0 & b_6 & b_7 \\ 0 & 0 & 0 & b_8 \end{bmatrix}$ with $b_6b_8 \neq 0$, a similar argument shows that $A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & 0 & a_{16} \end{bmatrix}$ with

$a_{11}a_{16} \neq 0$, and the submatrix $\begin{bmatrix} a_1 & a_2 \\ a_5 & a_6 \end{bmatrix}$ is nilpotent.

If $A^{k_0} = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ 0 & 0 & b_5 & b_6 \\ 0 & 0 & 0 & b_7 \\ 0 & 0 & 0 & b_8 \end{bmatrix}$ with $b_1 b_8 \neq 0$, a similar argument shows that $A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & a_6 & a_7 & a_8 \\ 0 & a_{10} & a_{11} & a_{12} \\ 0 & 0 & 0 & a_{16} \end{bmatrix}$ with $a_1 a_{16} \neq 0$, and the submatrix $\begin{bmatrix} a_6 & a_7 \\ a_{10} & a_{11} \end{bmatrix}$ is nilpotent.

If $A^{k_0} = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ 0 & 0 & b_5 & b_6 \\ 0 & 0 & b_7 & b_8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ with $b_1 b_7 \neq 0$, by Lemma 3.1, A^{k_0+1} has the form $\begin{bmatrix} c_1 & c_2 & c_3 & c_4 \\ 0 & 0 & c_5 & c_6 \\ 0 & 0 & c_7 & c_8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ with $c_1 c_7 \neq 0$. Suppose $A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix}$. By $A \cdot A^{k_0} = A^{k_0} \cdot A = A^{k_0+1}$, we can deduce that $a_5 = a_9 = a_{13} = 0$. Then $A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & a_6 & a_7 & a_8 \\ 0 & a_{10} & a_{11} & a_{12} \\ 0 & a_{14} & a_{15} & a_{16} \end{bmatrix}$ with $a_1 = \sqrt[k_0]{b_1} \neq 0$, and the submatrix $\begin{bmatrix} a_6 & a_7 & a_8 \\ a_{10} & a_{11} & a_{12} \\ a_{14} & a_{15} & a_{16} \end{bmatrix}$ satisfies $\begin{bmatrix} a_6 & a_7 & a_8 \\ a_{10} & a_{11} & a_{12} \\ a_{14} & a_{15} & a_{16} \end{bmatrix}^{k_0} = \begin{bmatrix} 0 & b_5 & b_6 \\ 0 & b_7 & b_8 \\ 0 & 0 & 0 \end{bmatrix}$ with $b_7 \neq 0$. By Lemma 3.2, $\begin{bmatrix} a_6 & a_7 & a_8 \\ a_{10} & a_{11} & a_{12} \\ a_{14} & a_{15} & a_{16} \end{bmatrix} = S \left[\begin{array}{c|cc} a & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{N}} \\ 0 & & \end{array}\right] S^{-1}$, where $a \neq 0$, N is a nilpotent matrix of order 2, and $S = \begin{bmatrix} b & 1 & 0 \\ 1 & 0 & c \\ 0 & 0 & 1 \end{bmatrix}$.

If $A^{k_0} = \begin{bmatrix} 0 & b_1 & b_2 & b_3 \\ 0 & b_4 & b_5 & b_6 \\ 0 & 0 & 0 & b_7 \\ 0 & 0 & 0 & b_8 \end{bmatrix}$ with $b_4 b_8 \neq 0$, a similar argument shows that $A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ 0 & 0 & 0 & a_{16} \end{bmatrix}$ with $a_{16} \neq 0$, and the submatrix $\begin{bmatrix} a_1 & a_2 & a_3 \\ a_5 & a_6 & a_7 \\ a_9 & a_{10} & a_{11} \end{bmatrix} = S \left[\begin{array}{c|cc} a & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{N}} \\ 0 & & \end{array}\right] S^{-1}$, where $a \neq 0$, N is a nilpotent matrix of order 2, and $S = \begin{bmatrix} b & 1 & 0 \\ 1 & 0 & c \\ 0 & 0 & 1 \end{bmatrix}$.

If $A^{k_0} = \begin{bmatrix} 0 & b_1 & b_2 & b_3 \\ 0 & b_4 & b_5 & b_6 \\ 0 & 0 & b_7 & b_8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ with $b_4 b_7 \neq 0$, by Lemma 4.1, A has the form (v).

Case 5. A has a nonzero eigenvalue and three zero eigenvalues.

Suppose there exists a positive integer $k_0 \geq 2$ such that A^k is upper triangular for all integers $k \geq k_0$. Then $A^{k_0} = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ 0 & 0 & b_5 & b_6 \\ 0 & 0 & 0 & b_7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ with $b_1 \neq 0$, or $\begin{bmatrix} 0 & b_1 & b_2 & b_3 \\ 0 & 0 & b_4 & b_5 \\ 0 & 0 & 0 & b_6 \\ 0 & 0 & 0 & b_7 \end{bmatrix}$ with $b_7 \neq 0$, or $\begin{bmatrix} 0 & b_1 & b_2 & b_3 \\ 0 & b_4 & b_5 & b_6 \\ 0 & 0 & 0 & b_7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ with $b_4 \neq 0$, or $\begin{bmatrix} 0 & b_1 & b_2 & b_3 \\ 0 & 0 & b_4 & b_5 \\ 0 & 0 & b_6 & b_7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ with $b_6 \neq 0$.

If $A^{k_0} = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ 0 & 0 & b_5 & b_6 \\ 0 & 0 & 0 & b_7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ with $b_1 \neq 0$, by Lemma 3.1, A^{k_0+1} has the form $\begin{bmatrix} c_1 & c_2 & c_3 & c_4 \\ 0 & 0 & c_5 & c_6 \\ 0 & 0 & 0 & c_7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ with $c_1 \neq 0$.
Suppose $A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix}$. By $A \cdot A^{k_0} = A^{k_0} \cdot A = A^{k_0+1}$, we can deduce that $a_5 = a_9 = a_{13} = 0$. Then $\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & a_6 & a_7 & a_8 \\ 0 & a_{10} & a_{11} & a_{12} \\ 0 & a_{14} & a_{15} & a_{16} \end{bmatrix}^{k_0} = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ 0 & 0 & b_5 & b_6 \\ 0 & 0 & 0 & b_7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ implies $a_1^{k_0} = b_1$ and $\begin{bmatrix} a_6 & a_7 & a_8 \\ a_{10} & a_{11} & a_{12} \\ a_{14} & a_{15} & a_{16} \end{bmatrix}^{k_0} = \begin{bmatrix} 0 & b_5 & b_6 \\ 0 & 0 & b_7 \\ 0 & 0 & 0 \end{bmatrix}$. Thus $A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & a_6 & a_7 & a_8 \\ 0 & a_{10} & a_{11} & a_{12} \\ 0 & a_{14} & a_{15} & a_{16} \end{bmatrix}$ with $a_1 \neq 0$, and the submatrix $\begin{bmatrix} a_6 & a_7 & a_8 \\ a_{10} & a_{11} & a_{12} \\ a_{14} & a_{15} & a_{16} \end{bmatrix}$ is nilpotent.

If $A^{k_0} = \begin{bmatrix} 0 & b_1 & b_2 & b_3 \\ 0 & 0 & b_4 & b_5 \\ 0 & 0 & 0 & b_6 \\ 0 & 0 & 0 & b_7 \end{bmatrix}$ with $b_7 \neq 0$, a similar argument shows that $A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ 0 & 0 & 0 & a_{16} \end{bmatrix}$ with $a_{16} \neq 0$, and the submatrix $\begin{bmatrix} a_1 & a_2 & a_3 \\ a_5 & a_6 & a_7 \\ a_9 & a_{10} & a_{11} \end{bmatrix}$ is nilpotent.

If $A^{k_0} = \begin{bmatrix} 0 & b_1 & b_2 & b_3 \\ 0 & b_4 & b_5 & b_6 \\ 0 & 0 & 0 & b_7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ with $b_4 \neq 0$, or $\begin{bmatrix} 0 & b_1 & b_2 & b_3 \\ 0 & 0 & b_4 & b_5 \\ 0 & 0 & b_6 & b_7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ with $b_6 \neq 0$, by Lemma 4.2, A has the form (vii).

Conversely, when A has any one of the forms (i)–(vii), a direct computation shows that A is eventually upper triangular. This completes the proof. □

Lemma 4.4. *Let A be an eventually upper triangular nonnegative matrix of order 4, and assume that there exists a positive integer $k_0 \geq 2$ such that A^k is upper triangular for all integers $k \geq k_0$. If $A^{k_0}(2,2), A^{k_0}(3,3) > 0$ and $A^{k_0}(1,1) = A^{k_0}(4,4) = 0$, then A is either upper triangular or has the form* $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & a_6 & a_7 & 0 \\ 0 & 0 & a_{11} & 0 \\ a_{13} & 0 & 0 & 0 \end{bmatrix}$ *with $a_6, a_{11}, a_{13} > 0$.*

Proof. Suppose $A^{k_0} = \begin{bmatrix} 0 & b_1 & b_2 & b_3 \\ 0 & b_4 & b_5 & b_6 \\ 0 & 0 & b_7 & b_8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ with $b_4, b_7 > 0$. By Lemma 3.1, A^{k_0+1} has the form $\begin{bmatrix} 0 & c_1 & c_2 & c_3 \\ 0 & c_4 & c_5 & c_6 \\ 0 & 0 & c_7 & c_8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ with $c_4, c_7 > 0$. Suppose $A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix}$. Then by $A \cdot A^{k_0} = A^{k_0} \cdot A = A^{k_0+1}$ and the nonnegativity of A, we can deduce that $a_5 = a_9 = a_{10} = a_{14} = a_{15} = 0$.

If $a_{13} = 0$, then A is upper triangular.

If $a_{13} > 0$, then by $A \cdot A^{k_0} = A^{k_0} \cdot A$ and the nonnegativity of A, we have $b_1 = b_2 = b_3 = b_6 = b_8 = 0$, and thus $a_2 = a_3 = a_8 = a_{12} = 0$. $A^{k_0}(1,1) = \sum_{1 \leq i_1, \ldots, i_{k_0-1} \leq 4} A(1,i_1)A(i_1,i_2) \cdots A(i_{k_0-1},1) = A(1,1)^{k_0} + \cdots = 0$ implies

$A(1,1) = a_1 = 0$. $A^{k_0}(4,4) = \sum\limits_{1\le i_1,\ldots,i_{k_0-1}\le 4} A(4,i_1)A(i_1,i_2)\cdots A(i_{k_0-1},4) = A(4,4)^{k_0} + \cdots = 0$ implies $A(4,4) = a_{16} = 0$. $A^{2k_0}(1,1) = \sum\limits_{1\le i_1,\ldots,i_{2k_0-1}\le 4} A(1,i_1)A(i_1,i_2)\cdots A(i_{2k_0-1},1) = (A(1,4)A(4,1))^{k_0} + \cdots = 0$ implies $A(1,4) = a_4 = 0$. Thus $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & a_6 & a_7 & 0 \\ 0 & 0 & a_{11} & 0 \\ a_{13} & 0 & 0 & 0 \end{bmatrix}$. Since $b_4 = a_6^{k_0}$ and $b_7 = a_{11}^{k_0}$, we have $a_6, a_{11} > 0$. □

Lemma 4.5. *Let A be an eventually upper triangular nonnegative matrix of order 4, and assume that there exists a positive integer $k_0 \ge 2$ such that A^k is upper triangular for all integers $k \ge k_0$. If $A^{k_0}(2,2) > 0$ and $A^{k_0}(1,1) = A^{k_0}(3,3) = A^{k_0}(4,4) = 0$, then A is either upper triangular or has one of the following forms:* $\begin{bmatrix} 0 & a_2 & a_3 & a_4 \\ 0 & a_6 & a_7 & a_8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_{15} & 0 \end{bmatrix}$ *with $a_6, a_{15} > 0$;* $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & a_6 & 0 & 0 \\ a_9 & 0 & 0 & a_{12} \\ a_{13} & 0 & a_{15} & 0 \end{bmatrix}$ *with $a_6, a_9, a_{13} > 0$ and $a_{12}a_{15} = 0$;* $\begin{bmatrix} 0 & 0 & a_3 & 0 \\ 0 & a_6 & a_7 & 0 \\ 0 & 0 & 0 & 0 \\ a_{13} & 0 & a_{15} & 0 \end{bmatrix}$ *with $a_6, a_{13} > 0$;* $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & a_6 & 0 & 0 \\ 0 & 0 & 0 & a_{12} \\ a_{13} & 0 & 0 & 0 \end{bmatrix}$ *with $a_6, a_{12}, a_{13} > 0$;* $\begin{bmatrix} 0 & 0 & 0 & a_4 \\ 0 & a_6 & 0 & a_8 \\ a_9 & 0 & 0 & a_{12} \\ 0 & 0 & 0 & 0 \end{bmatrix}$ *with $a_6, a_9 > 0$;* $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & a_6 & 0 & 0 \\ a_9 & 0 & 0 & 0 \\ 0 & 0 & a_{15} & 0 \end{bmatrix}$ *with $a_6, a_9, a_{15} > 0$.*

Proof. Suppose $A^{k_0} = \begin{bmatrix} 0 & b_1 & b_2 & b_3 \\ 0 & b_4 & b_5 & b_6 \\ 0 & 0 & 0 & b_7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ with $b_4 > 0$. By Lemma 3.1, A^{k_0+1} has the form $\begin{bmatrix} 0 & c_1 & c_2 & c_3 \\ 0 & c_4 & c_5 & c_6 \\ 0 & 0 & 0 & c_7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ with $c_4 > 0$. Suppose $A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix}$. Then by $A \cdot A^{k_0} = A^{k_0} \cdot A = A^{k_0+1}$ and the nonnegativity of A, we can deduce that $a_5 = a_{10} = a_{14} = 0$. $A^{k_0}(1,1) = \sum\limits_{1\le i_1,\ldots,i_{k_0-1}\le 4} A(1,i_1)A(i_1,i_2)\cdots A(i_{k_0-1},1) = A(1,1)^{k_0} + \cdots = 0$ implies $A(1,1) = a_1 = 0$. $A^{k_0}(3,3) = \sum\limits_{1\le i_1,\ldots,i_{k_0-1}\le 4} A(3,i_1)A(i_1,i_2)\cdots A(i_{k_0-1},3) = A(3,3)^{k_0} + \cdots = 0$ implies $A(3,3) = a_{11} = 0$. $A^{k_0}(4,4) = \sum\limits_{1\le i_1,\ldots,i_{k_0-1}\le 4} A(4,i_1)A(i_1,i_2)\cdots A(i_{k_0-1},4) = A(4,4)^{k_0} + \cdots = 0$ implies $A(4,4) = a_{16} = 0$. $A^{2k_0}(3,3) = \sum\limits_{1\le i_1,\ldots,i_{2k_0-1}\le 4} A(3,i_1)A(i_1,i_2)\cdots A(i_{2k_0-1},3) = (A(3,4)A(4,3))^{k_0} + \cdots = 0$ implies $A(3,4)A(4,3) = a_{12}a_{15} = 0$. Next we distinguish four cases.

Case 1. $a_9 = a_{13} = 0$. Then $b_4 = A^{k_0}(2,2) = \sum\limits_{1\le i_1,\ldots,i_{k_0-1}\le 4} A(2,i_1)A(i_1,i_2)\cdots A(i_{k_0-1},2) = A(2,2)^{k_0} > 0$ implies $a_6 > 0$. Thus A is either upper triangular or has the form $\begin{bmatrix} 0 & a_2 & a_3 & a_4 \\ 0 & a_6 & a_7 & a_8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_{15} & 0 \end{bmatrix}$ with $a_6, a_{15} > 0$.

Case 2. $a_9, a_{13} > 0$. Then $A^{2k_0}(3,3) = \sum\limits_{1\le i_1,\ldots,i_{2k_0-1}\le 4} A(3,i_1)A(i_1,i_2)\cdots A(i_{2k_0-1},3) = (A(3,1)A(1,3))^{k_0} + \cdots = 0$ implies $A(1,3) = a_3 = 0$. $A^{2k_0}(4,4) = \sum\limits_{1\le i_1,\ldots,i_{2k_0-1}\le 4} A(4,i_1)A(i_1,i_2)\cdots A(i_{2k_0-1},4) = (A(4,1)A(1,4))^{k_0} + \cdots = 0$ implies $A(1,4) = a_4 = 0$.

We assert that $a_2 = 0$. To the contrary, assume $a_2 > 0$. Then $A^{k_0+1}(3,2) = \sum\limits_{1\le i_1,\ldots,i_{k_0}\le 4} A(3,i_1)A(i_1,i_2)\cdots A(i_{k_0},2) = A(3,1)A(1,2)A(2,2)^{k_0-1} + \cdots = 0$ implies $A(2,2) = a_6 = 0$, $A^{3k_0}(3,3) = \sum\limits_{1\le i_1,\ldots,i_{3k_0-1}\le 4} A(3,i_1)A(i_1,i_2)\cdots A(i_{3k_0-1},3) =$

$(A(3,1)A(1,2)A(2,3))^{k_0}+\cdots=0$ implies $A(2,3)=a_7=0$, $A^{3k_0}(4,4)=\sum_{1\le i_1,\dots,i_{3k_0-1}\le 4} A(4,i_1)A(i_1,i_2)\cdots A(i_{3k_0-1},4)=(A(4,1)A(1,2)A(2,4))^{k_0}+\cdots=0$ implies $A(2,4)=a_8=0$. Thus $a_5=a_6=a_7=a_8=0$ implies $b_4=A^{k_0}(2,2)=\sum_{1\le i_1,\dots,i_{k_0-1}\le 4} A(2,i_1)A(i_1,i_2)\cdots A(i_{k_0-1},2)=0$, which is a contradiction.

Since $a_2=a_{10}=a_{14}=0$, $A^{k_0}(2,2)=\sum_{1\le i_1,\dots,i_{k_0-1}\le 4} A(2,i_1)A(i_1,i_2)\cdots A(i_{k_0-1},2)>0$ implies $A(2,2)=a_6>0$. Then $A^{k_0+1}(2,1)=\sum_{1\le i_1,\dots,i_{k_0}\le 4} A(2,i_1)A(i_1,i_2)\cdots A(i_{k_0},1)=A(2,2)^{k_0-1}A(2,3)A(3,1)+A(2,2)^{k_0-1}A(2,4)A(4,1)+\cdots=0$ implies $A(2,3)=a_7=0$ and $A(2,4)=a_8=0$. Thus $A=\begin{bmatrix}0&0&0&0\\0&a_6&0&0\\a_9&0&0&a_{12}\\a_{13}&0&a_{15}&0\end{bmatrix}$ with $a_6,a_9,a_{13}>0$ and $a_{12}a_{15}=0$.

Case 3. $a_9=0, a_{13}>0$. Then $A^{2k_0}(4,4)=\sum_{1\le i_1,\dots,i_{2k_0-1}\le 4} A(4,i_1)A(i_1,i_2)\cdots A(i_{2k_0-1},4)=(A(4,1)A(1,4))^{k_0}+\cdots=0$ implies $A(1,4)=a_4=0$. $A^{3k_0}(4,4)=\sum_{1\le i_1,\dots,i_{3k_0-1}\le 4} A(4,i_1)A(i_1,i_2)\cdots A(i_{3k_0-1},4)=(A(4,1)A(1,2)A(2,4))^{k_0}+(A(4,1)A(1,3)A(3,4))^{k_0}+\cdots=0$ implies $A(1,2)A(2,4)=a_2a_8=0$ and $A(1,3)A(3,4)=a_3a_{12}=0$. $A^{4k_0}(4,4)=\sum_{1\le i_1,\dots,i_{4k_0-1}\le 4} A(4,i_1)A(i_1,i_2)\cdots A(i_{4k_0-1},4)=(A(4,1)A(1,2)A(2,3)A(3,4))^{k_0}+\cdots=0$ implies $A(1,2)A(2,3)A(3,4)=a_2a_7a_{12}=0$.

Assume that $a_6=0$. Since $a_{12}a_{15}=0$, $b_4^2=A^{2k_0}(2,2)=\sum_{1\le i_1,\dots,i_{2k_0-1}\le 4} A(2,i_1)A(i_1,i_2)\cdots A(i_{2k_0-1},2)=A(2,3)A(3,4)A(4,1)A(1,2)\cdots+A(2,3)A(3,4)A(4,1)A(1,3)\cdots+A(2,3)A(3,4)A(4,3)\cdots+A(2,4)A(4,1)A(1,2)\cdots+A(2,4)A(4,1)A(1,3)A(3,4)\cdots+A(2,4)A(4,3)A(3,4)\cdots=a_2a_7a_{12}a_{13}\cdots+a_3a_{12}a_7a_{13}\cdots+a_{12}a_{15}a_7\cdots+a_2a_8a_{13}\cdots+a_3a_{12}a_8a_{13}\cdots+a_{12}a_{15}a_8\cdots=0$, which is a contradiction. Thus $a_6>0$.

Then $A^{k_0}(4,2)=\sum_{1\le i_1,\dots,i_{k_0-1}\le 4} A(4,i_1)A(i_1,i_2)\cdots A(i_{k_0-1},2)=A(4,1)A(1,2)A(2,2)^{k_0-2}+\cdots=0$ implies $A(1,2)=a_2=0$, $A^{k_0}(2,1)=\sum_{1\le i_1,\dots,i_{k_0-1}\le 4} A(2,i_1)A(i_1,i_2)\cdots A(i_{k_0-1},1)=A(2,2)^{k_0-2}A(2,4)A(4,1)+\cdots=0$ implies $A(2,4)=a_8=0$. When $a_{12}=0$, $A=\begin{bmatrix}0&0&a_3&0\\0&a_6&a_7&0\\0&0&0&0\\a_{13}&0&a_{15}&0\end{bmatrix}$ with $a_6,a_{13}>0$. When $a_{12}>0$, $a_3a_{12}=a_{12}a_{15}=0$ implies $a_3=a_{15}=0$. $A^{k_0+1}(2,1)=\sum_{1\le i_1,\dots,i_{k_0}\le 4} A(2,i_1)A(i_1,i_2)\cdots A(i_{k_0},1)=A(2,2)^{k_0-2}A(2,3)A(3,4)A(4,1)+\cdots=0$ implies $A(2,3)=a_7=0$. Thus $A=\begin{bmatrix}0&0&0&0\\0&a_6&0&0\\0&0&0&a_{12}\\a_{13}&0&0&0\end{bmatrix}$ with $a_6,a_{12},a_{13}>0$.

Case 4. $a_9>0, a_{13}=0$. A similar argument as in Case 3 shows that $A=\begin{bmatrix}0&0&0&a_4\\0&a_6&0&a_8\\a_9&0&0&a_{12}\\0&0&0&0\end{bmatrix}$ with $a_6,a_9>0$, or $A=\begin{bmatrix}0&0&0&0\\0&a_6&0&0\\a_9&0&0&0\\0&0&a_{15}&0\end{bmatrix}$ with $a_6,a_9,a_{15}>0$. $\square$

Now we can describe the eventually upper triangular nonnegative matrices of order 4 more clearly.

Theorem 4.6. *Let A be a nonnegative matrix of order 4. Then A is eventually upper triangular if and only if A is one of the following:*

(i) upper triangular;

(ii) nilpotent, and thus permutation similar to a strictly upper triangular matrix;

(iii) $A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & a_6 & a_7 & a_8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_{15} & 0 \end{bmatrix}$ *with* $a_1, a_6, a_{15} > 0$, *or* $A = \begin{bmatrix} 0 & 0 & a_3 & a_4 \\ a_5 & 0 & a_7 & a_8 \\ 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & 0 & a_{16} \end{bmatrix}$ *with* $a_5, a_{11}, a_{16} > 0$, *or* $A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & a_8 \\ 0 & a_{10} & 0 & a_{12} \\ 0 & 0 & 0 & a_{16} \end{bmatrix}$ *with* $a_1, a_{10}, a_{16} > 0$;

(iv) $A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_{11} & 0 \\ 0 & a_{14} & 0 & 0 \end{bmatrix}$ *with* $a_1, a_{11}, a_{14} > 0$, *or* $A = \begin{bmatrix} 0 & 0 & 0 & a_4 \\ 0 & a_6 & 0 & a_8 \\ a_9 & 0 & 0 & a_{12} \\ 0 & 0 & 0 & a_{16} \end{bmatrix}$ *with* $a_6, a_9, a_{16} > 0$, *or* $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & a_6 & a_7 & 0 \\ 0 & 0 & a_{11} & 0 \\ a_{13} & 0 & 0 & 0 \end{bmatrix}$ *with* $a_6, a_{11}, a_{13} > 0$;

(v) $A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & a_6 & a_7 & a_8 \\ 0 & a_{10} & a_{11} & a_{12} \\ 0 & a_{14} & a_{15} & a_{16} \end{bmatrix}$ *with* $a_1 > 0$ *and the submatrix* $\begin{bmatrix} a_6 & a_7 & a_8 \\ a_{10} & a_{11} & a_{12} \\ a_{14} & a_{15} & a_{16} \end{bmatrix}$ *being nilpotent, or* $A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ 0 & 0 & 0 & a_{16} \end{bmatrix}$ *with* $a_{16} > 0$ *and the submatrix* $\begin{bmatrix} a_1 & a_2 & a_3 \\ a_5 & a_6 & a_7 \\ a_9 & a_{10} & a_{11} \end{bmatrix}$ *being nilpotent;*

(vi) $A = \begin{bmatrix} 0 & a_2 & a_3 & a_4 \\ 0 & a_6 & a_7 & a_8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_{15} & 0 \end{bmatrix}$ *with* $a_6, a_{15} > 0$, *or* $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & a_6 & 0 & 0 \\ a_9 & 0 & 0 & a_{12} \\ a_{13} & 0 & a_{15} & 0 \end{bmatrix}$ *with* $a_6, a_9, a_{13} > 0$ *and* $a_{12}a_{15} = 0$, *or* $A = \begin{bmatrix} 0 & 0 & a_3 & 0 \\ 0 & a_6 & a_7 & 0 \\ 0 & 0 & 0 & 0 \\ a_{13} & 0 & a_{15} & 0 \end{bmatrix}$ *with* $a_6, a_{13} > 0$, *or* $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & a_6 & 0 & 0 \\ 0 & 0 & 0 & a_{12} \\ a_{13} & 0 & 0 & 0 \end{bmatrix}$ *with* $a_6, a_{12}, a_{13} > 0$, *or* $A = \begin{bmatrix} 0 & 0 & 0 & a_4 \\ 0 & a_6 & 0 & a_8 \\ a_9 & 0 & 0 & a_{12} \\ 0 & 0 & 0 & 0 \end{bmatrix}$ *with* $a_6, a_9 > 0$, *or* $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & a_6 & 0 & 0 \\ a_9 & 0 & 0 & 0 \\ 0 & 0 & a_{15} & 0 \end{bmatrix}$ *with* $a_6, a_9, a_{15} > 0$;

(vii) $A = \begin{bmatrix} 0 & 0 & a_3 & a_4 \\ a_5 & 0 & a_7 & a_8 \\ 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & 0 & 0 \end{bmatrix}$ *with* $a_5, a_{11} > 0$, *or* $A = \begin{bmatrix} 0 & a_2 & 0 & 0 \\ a_5 & 0 & 0 & 0 \\ 0 & 0 & a_{11} & 0 \\ a_{13} & a_{14} & 0 & 0 \end{bmatrix}$ *with* $a_{11}, a_{13}, a_{14} > 0$ *and* $a_2a_5 = 0$, *or* $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ a_5 & 0 & a_7 & a_8 \\ 0 & 0 & a_{11} & 0 \\ a_{13} & 0 & 0 & 0 \end{bmatrix}$ *with* $a_{11}, a_{13} > 0$, *or* $A = \begin{bmatrix} 0 & a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_{11} & 0 \\ a_{13} & 0 & 0 & 0 \end{bmatrix}$ *with* $a_2, a_{11}, a_{13} > 0$, *or* $A = \begin{bmatrix} 0 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_{11} & 0 \\ 0 & a_{14} & 0 & 0 \end{bmatrix}$ *with* $a_{11}, a_{14} > 0$, *or* $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ a_5 & 0 & 0 & 0 \\ 0 & 0 & a_{11} & 0 \\ 0 & a_{14} & 0 & 0 \end{bmatrix}$ *with* $a_5, a_{11}, a_{14} > 0$.

Proof. Consider the four eigenvalues of A. We distinguish five cases.

Case 1. A has four nonzero eigenvalues. Then A is eventually upper triangular if and only if A is upper triangular.

Case 2. A has four zero eigenvalues. Then A is eventually upper triangular if and only if A is nilpotent. By Lemma 2.3 in [11], A is permutation similar to a strictly upper triangular matrix.

Case 3. A has three nonzero eigenvalues and a zero eigenvalue. By the Case 3 in the proof of Theorem 4.3, A is upper triangular.

Case 4. A has two nonzero eigenvalues and two zero eigenvalues. Suppose there exists a positive integer $k_0 \geq 2$ such that A^k is upper triangular for all integers $k \geq k_0$.

If $A^{k_0} = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ 0 & b_5 & b_6 & b_7 \\ 0 & 0 & 0 & b_8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ with $b_1, b_5 > 0$, by the Case 4 in the proof of Theorem 4.3, $A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & a_6 & a_7 & a_8 \\ 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & a_{15} & a_{16} \end{bmatrix}$ with $a_1, a_6 > 0$, and the submatrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{15} & a_{16} \end{bmatrix}$ is nilpotent. Since A is nonnegative, $a_{11} = a_{16} = 0$ and $a_{12}a_{15} = 0$. Thus A is upper triangular or has the first form in (iii).

If $A^{k_0} = \begin{bmatrix} 0 & b_1 & b_2 & b_3 \\ 0 & 0 & b_4 & b_5 \\ 0 & 0 & b_6 & b_7 \\ 0 & 0 & 0 & b_8 \end{bmatrix}$ with $b_6, b_8 > 0$, a similar argument shows that A is upper triangular or has the second form in (iii).

If $A^{k_0} = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ 0 & 0 & b_5 & b_6 \\ 0 & 0 & 0 & b_7 \\ 0 & 0 & 0 & b_8 \end{bmatrix}$ with $b_1, b_8 > 0$, a similar argument shows that A is upper triangular or has the third form in (iii).

If $A^{k_0} = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ 0 & 0 & b_5 & b_6 \\ 0 & 0 & b_7 & b_8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ with $b_1, b_7 > 0$, by the Case 4 in the proof of Theorem 4.3, $A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & a_6 & a_7 & a_8 \\ 0 & a_{10} & a_{11} & a_{12} \\ 0 & a_{14} & a_{15} & a_{16} \end{bmatrix}$ with $a_1 > 0$, and the submatrix $\begin{bmatrix} a_6 & a_7 & a_8 \\ a_{10} & a_{11} & a_{12} \\ a_{14} & a_{15} & a_{16} \end{bmatrix}$ satisfies $\begin{bmatrix} a_6 & a_7 & a_8 \\ a_{10} & a_{11} & a_{12} \\ a_{14} & a_{15} & a_{16} \end{bmatrix}^{k_0} = \begin{bmatrix} 0 & b_5 & b_6 \\ 0 & b_7 & b_8 \\ 0 & 0 & 0 \end{bmatrix}$. By the Case 4 in the proof of Theorem 3.4, $\begin{bmatrix} a_6 & a_7 & a_8 \\ a_{10} & a_{11} & a_{12} \\ a_{14} & a_{15} & a_{16} \end{bmatrix}$ is upper triangular or has the form $\begin{bmatrix} 0 & 0 & 0 \\ 0 & a_{11} & 0 \\ a_{14} & 0 & 0 \end{bmatrix}$ with $a_{11}, a_{14} > 0$. Thus A is upper triangular or has the first form in (iv).

If $A^{k_0} = \begin{bmatrix} 0 & b_1 & b_2 & b_3 \\ 0 & b_4 & b_5 & b_6 \\ 0 & 0 & 0 & b_7 \\ 0 & 0 & 0 & b_8 \end{bmatrix}$ with $b_4, b_8 > 0$, a similar argument shows that A is upper triangular or has the second form in (iv).

If $A^{k_0} = \begin{bmatrix} 0 & b_1 & b_2 & b_3 \\ 0 & b_4 & b_5 & b_6 \\ 0 & 0 & b_7 & b_8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ with $b_4, b_7 > 0$, by Lemma 4.4, A is upper triangular or has the third form in (iv).

Case 5. A has a nonzero eigenvalue and three zero eigenvalues. Suppose there exists a positive integer $k_0 \geq 2$ such that A^k is upper triangular for all integers $k \geq k_0$.

If $A^{k_0} = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ 0 & 0 & b_5 & b_6 \\ 0 & 0 & 0 & b_7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ with $b_1 > 0$, or $A^{k_0} = \begin{bmatrix} 0 & b_1 & b_2 & b_3 \\ 0 & 0 & b_4 & b_5 \\ 0 & 0 & 0 & b_6 \\ 0 & 0 & 0 & b_7 \end{bmatrix}$ with $b_7 > 0$, by the Case 5 in the proof of Theorem 4.3, A is upper triangular or has the form (v).

If $A^{k_0} = \begin{bmatrix} 0 & b_1 & b_2 & b_3 \\ 0 & b_4 & b_5 & b_6 \\ 0 & 0 & 0 & b_7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ with $b_4 > 0$, by Lemma 4.5, A is upper triangular or has the form (vi).

If $A^{k_0} = \begin{bmatrix} 0 & b_1 & b_2 & b_3 \\ 0 & 0 & b_4 & b_5 \\ 0 & 0 & b_6 & b_7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ with $b_6 > 0$, using a similar argument as in the proof of Lemma 4.5, we can show that A is upper triangular or has the form (vii).

Conversely, when A has any one of the forms (i)–(vii), a direct computation shows that A is eventually upper triangular. This completes the proof. □

Obviously, the eventually lower triangular complex (nonnegative) matrices of order 4 can be obtained from Theorem 4.3 (Theorem 4.6) by taking the transpose.

5. Eventually triangular matrices of order n and future work

The eventually triangular matrices of orders $n = 2, 3$ and 4 have been completely characterized. We can see that many cases have been discussed when $n = 4$, and the computations would be more complicated as n increases. In some cases, for example, when the matrix of order n has exactly n, $n-1$, 1 and 0 zero eigenvalue(s), respectively, the results concerning the eventually triangular matrices of order $n \leq 4$ can be expanded to that of the general order n.

Theorem 5.1. *Let A be a complex matrix of order n. Then the following statements hold.*

(i) When A has n zero eigenvalues, A is eventually upper triangular if and only if A is nilpotent;

(ii) When A has n nonzero eigenvalues, A is eventually upper triangular if and only if A is upper triangular;

(iii) When A has $n-1$ nonzero eigenvalues and a zero eigenvalue, A is eventually upper triangular if and only if A is upper triangular;

(iv) When A has $n-1$ zero eigenvalues and a nonzero eigenvalue, A is eventually upper triangular if and only if A is one of the following:

(1) $A = \left[\begin{array}{c|ccc} a & b_1 & \cdots & b_{n-1} \\ \hline 0 & & & \\ \vdots & & N & \\ 0 & & & \end{array}\right]$ *or* $\left[\begin{array}{ccc|c} & & & b_1 \\ & N & & \vdots \\ & & & b_{n-1} \\ \hline 0 & \cdots & 0 & a \end{array}\right]$*, where $a \neq 0$ and N is a nilpotent matrix of order $n-1$;*

(2) $A = S_i \left[\begin{array}{c|ccc} a & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & N & \\ 0 & & & \end{array}\right] S_i^{-1}$*, where $a \neq 0$, N is a nilpotent matrix of order $n-1$, and $S_i =$*

$$\begin{bmatrix} b_1 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ b_2 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{i-1} & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 & b_i & b_{i+1} & \cdots & b_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, i = 2, 3, \ldots, n-1.$$

Proof. (i) and (ii) are obvious.

(iii) This can be shown in a similar way that the Case 3 in the proof of Theorem 4.3 is being proved.

(iv) Suppose there exists a positive integer $k_0 \geq 2$ such that A^k is upper triangular for all integers $k \geq k_0$. Then $A^{k_0}(i,i) \neq 0$ for some i with $1 \leq i \leq n$, and $A^{k_0}(j,j) = 0$ for all $j \neq i$ with $1 \leq j \leq n$. We distinguish three cases.

Case 1. $A^{k_0}(1,1) \neq 0$. Using a similar argument as the Case 5 in the proof of Theorem 4.3, we can deduce that A has the first form in (1).

Case 2. $A^{k_0}(n,n) \neq 0$. Using a similar argument as the Case 5 in the proof of Theorem 4.3, we can deduce that A has the second form in (1).

Case 3. $A^{k_0}(i,i) \neq 0$ for some i with $2 \leq i \leq n-1$. Let a be the only nonzero eigenvalue of A. Since $k_0 + n - 3 \geq n - 1$, the Jordan canonical form of A^{k_0+n-3}, denoted as $J(A^{k_0+n-3})$, has the form $\begin{bmatrix} a^{k_0+n-3} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$. Thus there exists a nonsingular matrix S_i of order n such that $(S_i^{-1}AS_i)^{k_0+n-3} = S_i^{-1}A^{k_0+n-3}S_i = \begin{bmatrix} a^{k_0+n-3} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$. Since $a \neq 0$, $S_i^{-1}AS_i$ has the form $\left[\begin{array}{c|ccc} a & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & N & \\ 0 & & & \end{array}\right]$, where N is a nilpotent matrix of order $n-1$.

Next we give the matrix S_i. By Lemma 3.1, $A^{k_0+n-3}(i,i) \neq 0$ and $A^{k_0+n-3}(j,j) = 0$ for all $j \neq i$. It is clear that $A^{k_0+n-3}(i,i) = a^{k_0+n-3}$ is the only nonzero eigenvalue of A^{k_0+n-3}. Since $\mathrm{rank}(A^{k_0+n-3}) = 1$, each 2×2 minor of A^{k_0+n-3} is zero. For $r = 1, 2, \ldots, n$, let $\mathbf{e_r}$ be the column vector of dimension n whose only nonzero component is the r-th component equal to 1. A direct computation shows that A^{k_0+n-3} has the eigenvectors $\mathbf{s_1} = \frac{A^{k_0+n-3}(1,i)}{a^{k_0+n-3}}\mathbf{e_1} + \frac{A^{k_0+n-3}(2,i)}{a^{k_0+n-3}}\mathbf{e_2} + \cdots + \frac{A^{k_0+n-3}(i-1,i)}{a^{k_0+n-3}}\mathbf{e_{i-1}} + \mathbf{e_i}$, $\mathbf{s_2} = \mathbf{e_1}$, $\mathbf{s_3} = \mathbf{e_2}$, $\ldots$, $\mathbf{s_i} = \mathbf{e_{i-1}}$, $\mathbf{s_{i+1}} = -\frac{A^{k_0+n-3}(i,i+1)}{a^{k_0+n-3}}\mathbf{e_i} + \mathbf{e_{i+1}}$, $\mathbf{s_{i+2}} = -\frac{A^{k_0+n-3}(i,i+2)}{a^{k_0+n-3}}\mathbf{e_i} + \mathbf{e_{i+2}}$, $\ldots$, $\mathbf{s_n} = -\frac{A^{k_0+n-3}(i,n)}{a^{k_0+n-3}}\mathbf{e_i} + \mathbf{e_n}$ corresponding to the eigenvalues $a^{k_0+n-3}, 0, \ldots, 0$. Then $S_i = [\mathbf{s_1}, \mathbf{s_2}, \ldots, \mathbf{s_n}]$. Let $b_1 = \frac{A^{k_0+n-3}(1,i)}{a^{k_0+n-3}}$, $b_2 = \frac{A^{k_0+n-3}(2,i)}{a^{k_0+n-3}}$, $\ldots$, $b_{i-1} = \frac{A^{k_0+n-3}(i-1,i)}{a^{k_0+n-3}}$, $b_i = -\frac{A^{k_0+n-3}(i,i+1)}{a^{k_0+n-3}}$, $b_{i+1} = -\frac{A^{k_0+n-3}(i,i+2)}{a^{k_0+n-3}}$, $\ldots$, $b_{n-1} = -\frac{A^{k_0+n-3}(i,n)}{a^{k_0+n-3}}$.

Conversely, when A has the form (1) or (2), a direct computation shows that A is eventually upper triangular. This completes the proof. □

When an eventually triangular matrix A of order n has exactly m zero eigenvalues with $2 \leq m \leq n-2$, it seems difficult to give a complete characterization of A. Although the results for $n = 4$ might shed some light on the solutions for large n, it is necessary to develop more useful techniques to study this problem.

For an eventually triangular matrix A of order n, it is also interesting to determine the positive integer k_0 such that A^k is triangular for all integers $k \geq k_0$. When A has exactly m zero eigenvalues, by the characterization of the eventually triangular matrices of orders 2, 3 and 4 in Theorems 2.1, 3.3 and 4.3, we can verify that A^k must be triangular for all integers $k \geq m$. We suspect that this statement holds for the eventually triangular matrices of order n.

Problem 5.2. *Let A be an eventually triangular matrix of order n, and suppose A has exactly m zero eigenvalues. Is it true that A^k must be triangular for all integers $k \geq m$?*

As a generalization of matrix theory, topics on tensors have drawn much people's attention in recent years [5, 14, 17]. It is natural to consider those tensors whose powers eventually have certain properties. We leave them for further study.

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Conflicts of Interest

The authors declare no conflict of interest.

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