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Curves of stationary acceleration according to alternative frame

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Abstract. This paper investigates curves of stationary acceleration by using alternative frame which includes the principal normal vector, the derivative of principal normal vector and the Darboux vector. Mentioned curves are studied by the way of rigid body motions, that is to say a point in the moving body follows the curve and the alternative frame in the moving body stays aligned with the members of frame. It is determined that in which condition this special motion becomes to stationary acceleration motion. The matrix representations of a constant vector related to velocity vector of the motion which is used to characterize stationary acceleration is obtained by means of alternative frame curvatures. Some examinations are developed with some solutions of differential equations. The main result is attained as: general helix curves with linear curvature and torsion functions are curves of stationary acceleration which are curves in the rigid body motions group SE(3) correlated with robotics. The paths designed as stationary acceleration curves can lead the way to control the end-effectors of robots. Finally, some explanatory examples are imputed.

1. Introduction

A rigid body is a solid body which does not change shape or disfigure. Motion of rigid bodies appears in kinematics and is related to angular velocity and angular acceleration. Rigid body motion has two properties: one of them is "angular velocity and angular acceleration of all lines on arigid body are same" and the other one is "the motion comprises of the translation of an arbitrary point, pursued by a rotation about the point".

Also, robotics deduces from the geometry of rigid body motion because it is supposed that the connections of a robot are rigid bodies. While end-effectors of robot are moving, this movement is described as a curve in group of rigid body motions SE(3). One of the application of robotics is the explanation of geometry of these curves. Finding the best path of end-effector of robot given with a starting point and extreme point can be a remarkable inquiry about this geometry.

In former times, it was said that a screw motion was made by the end-effector of robot. A screw motion is a combination of two motions: a translation along a vector and a rotation about a constant axis(screw axis) which is parallel to that vector. Also, the interpolation problem of this motion was used in computer

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graphics with the help of several algorithms. Despite of applying the rigid body motion in robotics and computer graphics, the geometry of this motion was not adequate.

In computer-aided geometric design, computer graphics and robotics, the interpolation of a set of positions and orientations is found by a smooth motion and this problem was investigated in the work [23] by Zefran and Kumar. They gave methods for computing a smooth motion and these effective methods were for finding trajectories which minimize meaningful functions. These functions were the acceleration of a rigid body as the covariant derivative of its motion and the jerk as the second covariant derivative. Since the covariant derivative is depended on positive definite left-invariant metric on the group and there are many metrics to select on SE(3), there was an elusiveness. Before this work, Noakes et al [13] gave equations for curves in SO(3) with minimum acceleration via the same definition of acceleration.

Then Selig looked over again the ideas of Zefran and Kumar but used bi-variant metric on SE(3) in [15]. Since there are no positive definite bi-variant metric on SE(3), the curves with derived equations are not minimal, they are only stationary. On the other hand, Bottema and Roth [4] gave Frenet-Serret motion which is one of special motions. Selig [15] studied Frenet-Serret and Bishop motion by a unit-speed space curve and gave the condition to be the curve of stationary acceleration. It was indicated that if the curvature and torsion functions of curve are linear functions, then the Frenet-Serret motion determined by this curve has stationary acceleration. Also similar result was given for Bishop motion. Then Selig studied Frenet-Serret and Bishop motions as an application to a simple model of needle steering in [16]. The path of the needle was derived with a kinematic model. It was shown that the kinematic model is resulted in Frenet-Serret motion of needle with constant curvature. Several researches related to Frenet-Serret motions can be found in [1, 8, 10, 11, 18].

Otherwise, while studying in curve theory, frames of curves are so significant to examine their specifications. There are various adapted curve frame examples, most known one is a moving frame called Frenet-Serret frame. An alternative moving frame to Frenet-Serret frame was defined by Uzunoğlu et al in [20]. It is the rotate case of Frenet-Serret frame by taking the principal normal vector N, the derivative of principal normal vector $C = \frac{N'}{\|N'\|}$ and the Darboux vector $W = N \times C$. They defined a new type of slant helix and constant precession curve by means of this frame. Also studies regarding $\{N, C, W\}$ frame are reached in [2, 3, 21, 22].

The definition of acceleration is given as the rate at which velocity changes with respect to time. The acceleration of a rigid body was defined by using covariant derivative by Zefran and Kumar in [23] as $A = \nabla_V V$, where ∇ is the covariant derivative and V is the velocity vector. Selig [15] obtained an expression which concures results given in [23] for the same acceleration. Then, the acceleration along a curve was optimised and the equation $V^{(3)} + \left[\ddot{V}, V\right] = 0$ was obtained for stationary acceleration curves, namely curves with constant acceleration. Also by appropriate calculations, the second derivative of velocity vector was expressed as $\ddot{V} = GXG^{-1}$, where G is the (4×4) matrix representation of rigid body transformations $G(t) \in SE(3)$ and X is a constant vector.

Motivated by these ideas, in this work we focus on a curve given with the alternative $\{N, C, W\}$ frame and its apparatus. We particularly discuss frame motions in three-dimensional Euclidean space and improve $\{N, C, W\}$ frame motions which is the new aspect to extension of kinematic frame motions. We determine stationary acceleration curves via the alternative frame motion, by developing the idea of matrix representation of a constant vector which satisfies above condition $\ddot{V} = GXG^{-1}$. It is established that a special case of general helix curves are curves of stationary acceleration, new curvature and torsion functions belonging to stationary acceleration curves are obtained by solving some differential equations.

2. Differential Geometry Preliminaries

In this section, some definitions, theorems and notions related to differential geometry will be introduced as the basis of the study.

A curve is an object which is similar to line but has not to be straight. It is given with definition topologically that a curve is a continuous function from an interval to topological space. In differential geometry, the differentiable function $\alpha : I \longrightarrow X$ is called a curve for an interval I and differentiable manifold X [19].

The adapted frame of a space curve is the $\{v_1, v_2, v_3\}$ vector collection in three dimensional Euclidean space. Here $v_i, 1 \leq i \leq 3$ constitute orthonormal base of three dimensional Euclidean space such that the tangent of α is v_1 and the vectors v_2, v_3 are selected from plane arbitrarily to be orthogonal to v_1 [5].

For regular curve α, three ortogonal vector fields are called Frenet vectors given by

$$T = \frac{\dot{\alpha}}{\left\|\dot{\alpha}\right\|}, \quad N = B \times T, \quad B = \frac{\dot{\alpha} \times \ddot{\alpha}}{\left\|\dot{\alpha} \times \ddot{\alpha}\right\|} \tag{1}$$

where T, N and B are, the tangent, principal normal and binormal vectors to the curve, respectively.

The curvature and torsion of the curve are calculated in order of

$$\kappa = \frac{\left\|\dot{\alpha} \times \ddot{\alpha}\right\|}{\left\|\dot{\alpha}\right\|^3}, \quad \tau = \frac{\det(\dot{\alpha}, \ddot{\alpha}, \dddot{\alpha})}{\left\|\dot{\alpha} \times \ddot{\alpha}\right\|^2} \tag{2}$$

and Frenet-Serret equations hold as:

$$\begin{aligned} \dot{T} &= \kappa N \\ \dot{N} &= -\kappa T + \tau B \\ \dot{B} &= -\tau N. \end{aligned} \tag{3}$$

One of the new curve frames along a curve is the alternative moving $\{N, C, W\}$ frame given in [20], defined as:

$$C = \frac{\dot{N}}{\left\|\dot{N}\right\|}, \quad W = \frac{\tau T + \kappa B}{\sqrt{\kappa^2 + \tau^2}} \tag{4}$$

such that $W = N \times C$ and N, C, W are, respectively, the unit principal normal vector, the derivative of principal normal vector and the unit Darboux vector. The derivative equations of this alternative frame are:

$$\begin{aligned} \dot{N} &= fC \\ \dot{C} &= -fN + gW \\ \dot{W} &= -gC \end{aligned} \tag{5}$$

where f and g are curvatures of the curve in terms of $\{N, C, W\}$ frame and given by

$$f = \kappa \sqrt{1 + H^2}, \quad g = \sigma f \tag{6}$$

providing that H is the harmonic curvature of the curve and σ is the geodesic curvature of spherical image of principal normal indicatrix curve:

$$\begin{aligned} H &= \frac{\tau}{\kappa} \\ \sigma &= \frac{\kappa^2}{(\kappa^2 + \tau^2)^{3/2}} \dot{\left(\frac{\tau}{\kappa}\right)}. \end{aligned} \tag{7}$$

Also the relation between H and σ is constructed as:

$$\sigma = \frac{\dot{H}}{\kappa (1 + H^2)^{3/2}}. \tag{8}$$

The Darboux vector (nominately angular velocity vector) in terms of alternative $\{N, C, W\}$ frame is given by

$$D = gN + fW \tag{9}$$

which provides the following relations:

$$D \times N = \dot{N}, \quad D\times = \dot{C}, \quad D \times W = \dot{W}. \tag{10}$$

Besides, a curve is said to be a general helix when $H = \frac{\tau}{\kappa}$ is constant and vice versa.

3. Background of Rigid Body Motions

In this section, some basic notions of screw theory in three-dimensional Euclidean space are introduced for necessity of sequent parts.

Before setting out rigid body motions, the following notions are required to comprehend smoothly.

A Lie group is a topological group which is also a smooth manifold and one of the Lie group representation is spatial transformations, that are used in computer and robotics vision. $SO(3)$ and $SE(3)$ are Lie groups, named respectively as special orthogonal group and special Euclidean group. The elements of $SO(3)$ represent only rotation matrices:

$$SO(3) = \left\{ R \in \mathbb{R}_{3\times 3} \;\mid\; RR^T = R^T R = I, \quad \det(R) = 1 \right\}$$

and $SE(3)$ is the group of rigid transformations consisting of rotation R and translation t:

$$SE(3) = \left\{ G = \begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix} \;\mid\; R \in SO(3), \;\; t \in \mathbb{R}^3 \right\} \subset \mathbb{R}_{4\times 4}.$$

In the sense of transformations, consider a rigid body motion $G(t)$ and an arbitrary point r on the rigid body. The behavior of such a motion on a position $r_0 \in \mathbb{R}_{3\times 1}$ is given by

$$\begin{pmatrix} r(t) \\ 1 \end{pmatrix} = G(t) \begin{pmatrix} r_0 \\ 1 \end{pmatrix} \tag{11}$$

which can also be written as

$$Z(t) = G(t)Y \tag{12}$$

where $Z(t) = \begin{pmatrix} r(t) \\ 1 \end{pmatrix}$ and $Y = \begin{pmatrix} r_0 \\ 1 \end{pmatrix}$. By taking into consideration $Y = G^{-1}(t)Z(t)$, the velocity of the point is given as:

$$\dot{Z}(t) = \dot{G}(t)G^{-1}(t)Z(t). \tag{13}$$

The instantaneous twist of the motion is defined by

$$S_d = \dot{G}(t)G^{-1}(t) \tag{14}$$

which also identify the velocity of the rigid body motion $G(t)$. Thus, by the velocity S_d, the equation (11) holds to

$$\dot{Z}(t) = S_d Z(t). \tag{15}$$

The following equation is derived in terms of 3D vectors, by expanding the equation (13) using partitioned matrix as:

$$\dot{r} = \omega_d \times r + V_d \tag{16}$$

which is the standard form of the velocity vector given by the points on the rigid body motion along an instantaneous screw S_d [15]. Here ω_d is the angular velocity vector of the rigid body. By the way, a screw can be written as a (4×4) matrix:

$$S = \begin{pmatrix} \Omega & u \\ 0 & 0 \end{pmatrix} \tag{17}$$

where u is linear velocity characteristic of the motion and Ω is a (3×3) anti-symmetric matrix, corresponds to angular velocity ω. The matrix of Ω is given in terms of the elements of $\omega = \left(\omega_x, \omega_y, \omega_z\right)$ as:

$$\Omega = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}. \tag{18}$$

Also, the acceleration vector of a rigid body motion is defined in [23] as:

$$A = \nabla_V V \tag{19}$$

where ∇ is the covariant derivative and $V = S_d$ is the velocity vector.

After that, in [15], the acceleration along a curve is optimised and the equation

$$V^{(3)} + \left[\ddot{V}, V\right] = 0 \tag{20}$$

is obtained for stationary acceleration curves which comes to mean curves with constant acceleration.

Then by integrating equation (20) and making proper calculations, it is indicated that, the velocity vector for stationary acceleration curves has the following expression

$$\ddot{V} = GXG^{-1} \tag{21}$$

where G is the (4×4) matrix representation of rigid body transformations $G(t) \in SE(3)$ and X is a constant vector. If the constant vector X is respecified, then:

$$X = G^{-1}\ddot{V}G. \tag{22}$$

For detailed calculations of equations (18), (19) and (20), see the reference [15].

4. Alternative Frame Motion

The paths of rigid body motions can design a curve in terms of kinematics, in the Lie groups. One way specified these paths, is given as Frenet-Serret motion in [4]. Also, Frenet-Serret and Bishop motions are studied in [15] in the sense of mentioned frames. In this section, frame motion will be improved via the alternative moving $\{N, C, W\}$ frame to establish a rigid body motion and its kinematic properties.

Assume that $\alpha(t)$ is a unit speed curve and $\{N, C, W\}$ denotes the alternative frame aligned with the principal normal vector N, the derivative of principal normal vector C and the Darboux vector W. A point in the moving body moves along the curve $\alpha(t)$ and using rigid transformations $SE(3)$, the $\{N, C, W\}$ motion of the curve $\alpha(t)$ is indicated as

$$G(t) = \begin{pmatrix} R(t) & \alpha(t) \\ 0 & 1 \end{pmatrix} \tag{23}$$

where $R(t)$ is the rotation matrix of type (3×3) with column elements N, C, W as

$$R(t) = (N \;\; C \;\; W). \tag{24}$$

Also by using the (3×3) anti-symmetric matrix Ω, corresponding to Darboux vector D given in equation (9), it can be written as:

$$\dot{R} = \Omega R. \tag{25}$$

Theorem 4.1. *Let α be a unit speed cuve and the rigid body motion along the curve α be $\{N, C, W\}$ motion. If the curve α is general helix with linear curvature and torsion functions, then α is curve of stationary acceleration.*

Proof. Our aim is to obtain the vector $X = G^{-1}\ddot{V}G$ given in equation (22), as constant in the $\{N, C, W\}$ motion along the curve α.

Since the Darboux vector, given in equation (9), of the curve α is $D = gN + fW$, we have the following derivatives by using relations in equation (5) as:

$$\begin{aligned} \dot{D} &= \dot{g}N + \dot{f}W \\ \ddot{D} &= \ddot{g}N + (\dot{g}f - \dot{f}g)C + \ddot{f}W. \end{aligned} \tag{26}$$

By using equation (18), the (3×3) anti-symmetric matrix corresponding to $\ddot{D}$ is given by

$$\ddot{\Omega} = \begin{pmatrix} 0 & -\ddot{f} & \dot{g}f - \dot{f}g \\ \ddot{f} & 0 & -\ddot{g} \\ \dot{f}g - \dot{g}f & \ddot{g} & 0 \end{pmatrix}. \tag{27}$$

At the same time, by using equation equation (16) and $\dot{\alpha} = T$, the velocity vector is obtained as:

$$V = T - D \times \alpha. \tag{28}$$

Taking second derivative from equation (28) and using equations (3), (5);

$$\ddot{V} = \dot{\kappa}N + \kappa f C - \ddot{D} \times \alpha - 2\dot{D} \times T - D \times \dot{T} \tag{29}$$

and using equation (26), $B = T \times N$, $N = C \times W$, we have

$$\dot{D} \times T = \frac{\dot{f}\kappa}{f}N - \dot{g}\left(\frac{\tau}{f}C + \frac{\kappa}{f}W\right) \tag{30}$$

$$D \times \dot{T} = \kappa f C. \tag{31}$$

Putting the expressions (30) and (31) into equation (29), we obtain

$$\ddot{V} = \left(\dot{\kappa} - \frac{2\dot{f}\kappa}{f}\right)N + \frac{2\dot{g}\tau}{f}C + \frac{2\dot{g}\kappa}{f}W - \ddot{D} \times \alpha. \tag{32}$$

On the other hand, the inverse matrix and the derivative of G from equation (23) are

$$G^{-1} = \begin{pmatrix} R^T & -R^T\alpha \\ 0 & 1 \end{pmatrix}, \qquad \dot{G} = \begin{pmatrix} \dot{R} & \dot{\alpha} \\ 0 & 0 \end{pmatrix} \tag{33}$$

where $R^T = R^{-1}$, since R is the rotation matrix.

From equations (14), (25) and (33), we get

$$V = \dot{G}G^{-1} = \begin{pmatrix} \Omega & T - D \times \alpha \\ 0 & 1 \end{pmatrix} \tag{34}$$

where $\Omega\alpha = D \times \alpha$.

Taking second derivative of equation (34) and using equations (30), (31), we obtain

$$\ddot{V} = \begin{pmatrix} \ddot{\Omega} & a_1N + a_2C + a_3W - \ddot{D} \times \alpha \\ 0 & 0 \end{pmatrix} \tag{35}$$

where $a_1 = \dot{\kappa} - \frac{2\dot{f}\kappa}{f}$, $a_2 = \frac{2\dot{g}\tau}{f}$ and $a_3 = \frac{2\dot{g}\kappa}{f}$. If it is looked closely, it can be noticed that equation (32) and (35) are overlaped.

Considering the equations (33), (35) and $\ddot{\Omega}\alpha = \ddot{D} \times \alpha$, we finally have

$$X = G^{-1}\ddot{V}G = \begin{pmatrix} R^T\ddot{\Omega}R & a_1R^TN + a_2R^TC + a_3R^TW \\ 0 & 0 \end{pmatrix}. \tag{36}$$

In equation (36), if we use equation (24) and the properties of vector and dot products of N, C, W, the expression can be extended to

$$X = G^{-1}\ddot{V}G = \begin{pmatrix} 0 & -\ddot{f} & \dot{g}f - \dot{f}g & \dot{\kappa} - \frac{2\dot{f}\kappa}{f} \\ \ddot{f} & 0 & -\ddot{g} & \frac{2\dot{g}\tau}{f} \\ \dot{f}g - \dot{g}f & \ddot{g} & 0 & \frac{2\dot{g}\kappa}{f} \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{37}$$

Now if the curve α is general helix, then $\frac{\tau}{\kappa}$ =constant (taking $\kappa = at$ and $\tau = \mu t$ where a, μ are constants and t is the parameter of α). From the equation (6) and (7), the curvatures of the curve α in terms of the alternative frame are:

$$f = \lambda at \text{ and } g = 0 \tag{38}$$

where λ is constant. The following expressions are derived from equation (38) as:

$$\begin{aligned} \dot{f} &= \lambda a \\ \ddot{f} &= 0 \\ \dot{g} &= 0 \\ \ddot{g} &= 0. \end{aligned} \tag{39}$$

Thus, using equation (38) and (39), in equation (37), we have

$$X = \begin{pmatrix} 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which is constant. This completes the proof. □

Theorem 4.2. *Let $\alpha(t)$ be a unit speed cuve and the rigid body motion along the curve α be $\{N, C, W\}$ motion. If the curvatures in terms of alternative frame of the curve α are $f = at + b$ and $g = \lambda$, then the curve α is curve of stationary acceleration with curvature and torsion*

$$\begin{aligned} \kappa &= E(at+b) + D(at+b)^2 \\ \tau &= (E(at+b) + D(at+b)^2)\sqrt{\frac{Y(t)^2}{1 - Y(t)^2}} \end{aligned}$$

where $a, b, c, D, E = -\frac{c}{a}, \lambda, N, M$ are constants and $Y(t) = E\lambda t + D\lambda\left(\frac{at^2}{2} + bt\right) + N - M$.

Proof. By using similar method in the previous proof, we consider the vector X in equation (37) to be constant.

If the curvatures are $f = at + b$ and $g = \lambda$, then we have

$$\begin{aligned} \dot{f} &= a \\ \ddot{f} &= 0 \\ \dot{g} &= 0 \\ \ddot{g} &= 0. \end{aligned} \tag{40}$$

To get a constant from the first row and fourth column element of the matrix in equation (37), we solve the following differential equation

$$\dot{\kappa} - \frac{2\dot{f}\kappa}{f} = c \tag{41}$$

where c is a constant. By using $f = at + b$, the equation (41) becomes a linear differential equation

$$\dot{\kappa} + p(t)\kappa = c \tag{42}$$

where $p(t) = -\frac{2a}{at+b}$. Solving the equation (42), we obtain the curvature as:

$$\kappa = E(at + b) + D(at + b)^2 \tag{43}$$

with b, $E = -\frac{c}{a}$ and D constants.

From the equation (6), (7), $f = at + b$ and $g = \lambda$, we have

$$\frac{\lambda}{at + b} = \frac{\kappa^2}{(\kappa^2 + \tau^2)^{3/2}} \dot{\left(\frac{\tau}{\kappa}\right)}. \tag{44}$$

By letting $h(t) = \frac{\lambda}{at+b}$ and $Z(t) = \frac{\tau}{\kappa}$ and making appropriate collocations in equation (44), we get the differential equation

$$\kappa(t)h(t) = \frac{1}{(1 + Z(t)^2)^{3/2}} \frac{dZ}{dt}. \tag{45}$$

Integrating equation (45) by substituting $Z = \tan s$ and using equation (43), we find

$$\frac{Z}{\sqrt{1 + Z^2}} = E\lambda t + D\lambda\left(\frac{at^2}{2} + bt\right) + N - M \tag{46}$$

where $a, b, c, D, E = -\frac{c}{a}, \lambda, N, M$ are constants.

By using $Z(t) = \frac{\tau}{\kappa}$, equation (43) and letting $Y(t) = E\lambda t + D\lambda\left(\frac{at^2}{2} + bt\right) + N - M$ in equation (46), we obtain the results clearly as:

$$\tau = (E(at + b) + D(at + b)^2)\sqrt{\frac{Y(t)^2}{1 - Y(t)^2}}. \tag{47}$$

Thus equations (43) and (46) bring to completion the proof. □

Example 4.3. *Consider the unit speed general helix curve $\alpha(t)$ with curvature and torsion*

$$\kappa(t) = 3t \quad \textit{and} \quad \tau(t) = 4t.$$

The figure is diagrammatized for $-5 \le t \le 5$. The path given in Figure 1 is stationary acceleration curve.

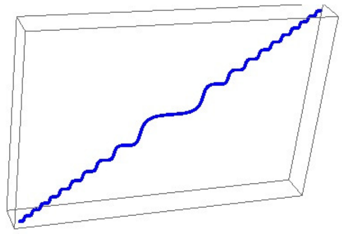

Figure 1: The curve $\alpha(t)$ with κ and τ

Example 4.4. *Consider the unit speed curve $\alpha(t)$ with curvature and torsion*

$$\kappa = t^2 + t$$

$$\tau = (t^2 + t)\sqrt{\frac{2t^2}{4 - t^4}}$$

where the curvature and torsion are written by the constants $a = 1$, $b = 1$, $c = 1$, $D = 1$, $N = 1$, $M = 1$, $\lambda = 1$, in Theorem 2. The figure is diagrammatized for $-1 \leq t \leq 1$. The path given in Figure 2 is stationary acceleration curve.

Figure 2: The curve $\alpha(t)$ with κ and τ

5. Conclusion and Remarks

The alternative $\{N, C, W\}$ frame motions have been studied in three dimensional Euclidean space. These type of motions are designated by a unit speed curve. It has been shown that if the curve determined by these motions is a general helix with linear functions of curvature and torsion, then the curve is stationary acceleration curve. Also a stationary acceleration curve is given with determined curvature and torsion functions which are obtained by solving some differential equations, are achieved as the elements of the matrix related to stationary acceleration curve condition. By the notion of the robot motion which describes a rigid body motion, these stationary acceleration curve paths can lead the way to robots as a control of end-effectors. Namely, the path can be taken as a curve of general helix with linear curvature and torsion function or a curve with curvature and torsion given in Theorem 2, which provides the stationary acceleration curve.

In [15], Selig gave the stationary acceleration curve condition as only linear functions of curvature and torsion, by using Frenet-Serret motion. In this paper, we indicated that condition as a general helix with linear curvature and torsion functions by using $\{N, C, W\}$ frame motion, thus we concluded the stationary acceleration curves with general helix case. Also we gave another case which involves determined curvature and torsion functions.

For further investigation, the rigid body motions can be studied with other curve frames and also in different spaces. Some new conditions can be obtained for stationary acceleration curves.

References

[1] N. Abazarı, M. Bohner, I. Sağer and Y. Yaylı, *Stationary acceleration of Frenet curves*, J. of Inequa. and App. **92** (2017), DOI 10.1186/s13660-017-1354-7.
[2] Ş. Alıç and B. Yılmaz, *Smarandache curves according to alternative frame in E^3*, Journal of Universal Mathematics **4(2)** (2021), 140–156.
[3] M. Bekar and Y. Yaylı, *Slant helix curves and acceleration centers*, Gazi University Journal of Science **32(1)** (2019), 256–271.
[4] O. Bottema and B. Roth, *Theoretical Kinematics*, Dover Publications, New York, 1979.
[5] M. P. Do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice Hall, Englewood Cliffs, NJ, 1976.
[6] R. Featherstone, *The acceleration vector of a rigid body*, Int. J. Robotics Res. **20** (2001), 841–846.
[7] D. Marsh, *Applied Geometry for Computer Graphics and CAD*, (2nd edition), Springer, 2005.
[8] C. Ekici, Y. Ünlütürk, M. Dede and B. S. Ryuh, *On motion of robot end-effector using the curvature theory of timelike ruled surfaces with timelike rulings*, Mathematical Problems in Engineering **2008** (2008), DOI:10.1155/2008/362783.
[9] F. Freudenstein, *Kinematics: past, present and future*, Mechanism and Machine Theory **8(2)** (1973), 151–160.
[10] D. Kahveci and Y. Yaylı, *Persistent rigid body motions on slant helices*, Int. J. Geom. Methods Mod. Phys. **16(12)** (2019), 1950193.
[11] D. Kahveci and Y. Yaylı, *Geometric kinematics of persistent rigid motions in three-dimensional Minkowski space*, Mechanism and Machine Theory **167** (2022), 104535.
[12] A. Karger and J. Novak, *Space Kinematics and Lie Groups*, Gordon and Breach Science Publishers, New York, 1978.
[13] L. Noakes, G. Heinzinger and B. Paden, *Cubic splines on curved space*, IMA J. of Math. Control Inform. **6** (1989), 465–473.
[14] B. F. Schutz, *Geometrical Methods of Mathematical Physics*, Cambridge University Press, Cambridge, 1980.
[15] J. M. Selig, *Curves os stationary acceleration in SE(3)*, IMA J. of Math. Control Inform. **24** (2007), 95–113.
[16] J. M. Selig, *Characterisation of Frenet-Serret and Bishop motions with applications to needle steering*, Robotica **31(6)** (2013), 981–992.
[17] J. M. Selig, *Geometric Fundamental of Robotics*, (2nd edition), Springer, New York, 2005.
[18] J. M. Selig and Y. Wu, *Interpolated rigid-body motions and robotics*, Proc. of the 2006 IEEE/RSJ International Conference on Intelligent Robots and Systems (2006), 1086–1091.
[19] D. J. Struik, *Lectures on Classical Differential Geometry*, Dover, New York, 1988.
[20] B. Uzunoğlu, İ. Gök and Y. Yaylı, *A new approach on curves of constant precession*, Applied Mathematics and Computation **275** (2016), 317–323.
[21] B. Yılmaz, Ç. Ramis and Y. Yaylı, *On developable ruled surface of the principal-direction curve*, Konuralp Journal of Mathematics **5(2)** (2017), 172–180.
[22] B. Yılmaz and A. Has, *Alternative partner curves in the Euclidean 3-space*, Comm. Fac. of Sci. Uni. of Ankara Series A1 Math. and Sta. **69(1)** (2020), 900–909.
[23] M. Zefran and V. Kumar, *Two methods for interpolating rigid body motions*, Proceedings of the IEEE International Conference on Robotics and Automation, Leuven, Belgium **4** (1998), 2922–2927.
[24] M. Zefran, V. Kumar and C. Croke, *On the generation of smooth three-dimensional rigid body motions*, IEEE Trans. Robotics Autom. **12** (1998), 576–589.