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Global existence and exponential decay for Thermoelastic System with nonlinear distributed delay

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Abstract. In this work, we consider the thermoelastic system with a nonlinear distributed delay. Under appropriate hypothesis on the weight function of the delay we prove well-posedness by using the Feado-Galerkin method. Also we establish an exponential decay result by introducing a suitable Lyapunov functional.

1. Introduction

Consider the following thermoelastic system with a nonlinear distributed delay:

$$\begin{cases} u_{tt}(x,t) - au_{xx}(x,t) + bv_x(x,t) + \mu_1|u_t(x,t)|^{m-2}u_t(x,t) & \\ + \int_{\tau_1}^{\tau_2} \mu(s)|u_t(x,t-s)|^{m-2}u_t(x,t-s)ds = 0, & (x,s,t) \in (0,L)\times(\tau_1,\tau_2)\times\mathbb{R}_+, \\ v_t(x,t) - dv_{xx}(x,t) + bu_{xt}(x,t) = 0, & (x,t) \in (0,L)\times\mathbb{R}_+, \\ u(0,t) = u(L,t) = 0 \qquad v_x(0,t) = v_x(L,t) = 0 & t \in \mathbb{R}_+, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ v(x,0) = v_0(x), & x \in (0,L), \\ u_t(x,-t) = u_2(x,t), & (x,t) \in (0,L)\times(0,\tau_2). \end{cases} \tag{1}$$

Where x and t represents the space and the time variable, respectively, $u(x,t)$ and $v(x,t)$ are the displacement and the temperature, subscripts mean partial derivatives. $a,\ b,\ d,\ \mu_1,\ \tau_1,\ \tau_2$, L are some positive constants and $m \geq 2$, the function $u_0,\ u_1,\ v_0$ and u_2 are the initial data and $\mu(s)$ is a bounded function.

Several authors have addressed the problem of stability of classical thermoelastic systems, and many results have been established in this regard. We mention, for example, Recently, Moulay et all [12] studied the problem (1) by adding the delay and replacing the memory damping with the Kelvin Voigt damping of the form $cu_{xxt}(x,t)$ for some real positive number c. Then the system writes as follows:

$$\begin{cases} u_{tt}(x,t) - au_{xx}(x,t-\tau) + bv_x(x,t) - cu_{xxt}(x,t) = 0, & (0,L)\times(0,\infty), \\ v_t(x,t) - dv_{xx}(x,t) + bu_{xt}(x,t) = 0, & (0,L)\times(0,\infty). \end{cases} \tag{2}$$

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They proved that the problem is well-posedness using semigroup theory and exponential stability using Lyapunov's method. in the absence of Kelvin Voigt damping ($c = 0$) in the system (2), Racke proved in [18] that, the internal time delay leads to ill-posedness and unstable even if τ is relatively small. See also [1, 6, 19].

Recently, Hao and Wang [7] dealing with an abstract thermoelastic system with infinite memory and Kelvin Voigt damping of the form $Bu_t(x,t)$:

$$\begin{cases} u_{tt} + Au + Bu_t + \int_0^\infty h(s)Au(s)ds - A^\alpha v = 0, & t \geq 0, \\ v_t + kA^\beta v + A^\alpha u_t = 0, & t \geq 0, \end{cases} \tag{3}$$

where A and B are self-adjoint linear positive definite operators with domain $D(A) \subset D(B)$. They have given the well-posedness and the general decay rate system by semigroup theory and perturbed energy.

In [5], Ferhat and Hakem considered the weak viscoelastic wave equation in bounded domain with dynamic boundary conditions and nonlinear delay term:

$$\begin{cases} u_{tt} - \Delta_x u(x,t) - \delta\Delta u_t(x,t) - \sigma(t)\displaystyle\int_0^t g(t-s)\Delta u(x,s)\,ds = 0 \text{ in } \Omega \times (0,+\infty), \\ u = 0 \text{ on } \Gamma_0 \times (0,+\infty), \\ u_{tt} + a\left[\dfrac{\partial u}{\partial \nu}(x,t) + \delta\dfrac{\partial u_t}{\partial \nu}(x,t) - \sigma(t)\displaystyle\int_0^t g(t-s)\Delta u(x,s)\dfrac{\partial u}{\partial \nu}(x,s)\,ds \right. \\ \left. +\mu_1|u_t(x,t)|^{m-1}u_t(x,t) + \mu_2|u_t(x,t-\tau)|^{m-1}u_t(x,t-\tau)\right] = 0 \text{ on } \Gamma_1 \times (0,+\infty) \\ u(x,0) = u_0(x) \quad u'(x,0) = u_1(x) \text{ on } \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau) \text{ on } \Gamma_1 \times (0,+\infty). \end{cases} \tag{4}$$

Under suitable conditions on the initial data and the relaxation function, they proved global existence and general decay of energy.

Recently [2], the authors examined a viscoelastic Kirchhoff equation with distributed delay and Balakrishnan Taylor damping:

$$\begin{cases} |u_t|^p u_{tt} - (\zeta_0 + \zeta_1\|\nabla u\|_2^2 u_{xx}(x,t) + \sigma(\nabla u, \nabla u_t)_{L^2(\Omega)} - \Delta u_{tt} \\ + \int_0^t h(t-\varrho)\Delta u(\varrho)d\varrho + \beta_1|u_t(t)|^{m-2}u_t(t) \\ + \int_{\tau_1}^{\tau_2} |\beta_2(s)||u_t(t-s)|^{m-2}u_t(t-s)ds = 0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & \text{in } \Omega, \\ u_t(x,-t) = f_0(x,t), & \text{in } \Omega \times (0,\tau_2), \\ u(x,t) = 0, & \text{in } \partial\Omega \times (0,\infty). \end{cases} \tag{5}$$

Under suitable hypothesis they proved general decay of energy.

The paper is organized as follows.In Section 2, we give some materials needed for our work and state our main results. The well-posedness of the problem is analyzed in Section 3, by using Faedo-Galerkin method. In Section 4, we prove the exponential decay of the energy when time goes to infinity.

2. Preliminaries

An integrating the second equation (1) over $(0, L)$ while respecting the boundary condition, gives

$$\int_0^L v(x,t)dx = r, \qquad \forall t \geq 0, \tag{6}$$

where $r = \int_0^L v_0(x)dx$. by posing

$$\tilde{v}(x,t) = v(x,t) - \frac{r}{L}, \qquad \forall (x,t) \in (0,L) \times \mathbb{R}_+, \tag{7}$$

we find

$$\int_0^L \tilde{v}(x,t)dx = 0, \qquad \forall t \geq 0. \tag{8}$$

Moreover, (u, v) and $(u, \tilde{v})$ satisfy the same problem (1). So in what follows we deal with $\tilde{v}$ while keeping the notation v for simplicity.

The weight function of the delay $\mu : [\tau_1, \tau_2] \to \mathbb{R}_+$ satisfying

$$\int_{\tau_1}^{\tau_2} \mu(s)ds < \mu_1. \tag{9}$$

We define the energy of problem (1) by

$$E(t) \;=\; \frac{1}{2}\int_0^L \left\{ u_t^2 + au_x^2 + v^2 + 2\frac{m-1}{m}\int_0^1\int_{\tau_1}^{\tau_2} s\mu(s)|y(x,p,s,t)|^m dsdp \right\} dx, \tag{10}$$

Lemma 2.1. *The energy functional $E(t)$ satisfies*

$$E'(t) = -d\int_0^L v_x^2 dx - \eta_1 \int_0^L |u_t|^m dx. \tag{11}$$

where $\eta_1 = \mu_1 - \frac{1}{m}\left(\int_{\tau_1}^{\tau_2} \mu(s)ds\right)$.

Proof. Multiplying the first two equations in (18) by u_t and v_t, respectively, then integrating over $(0, L)$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^L \left\{ u_t^2 + au_x^2 + v^2 \right\} dx \qquad (12)\\ =\;& -d\int_0^L v_x^2 dx - \mu_1\int_0^L |u_t|^m dx - \int_0^L u_t \int_{\tau_1}^{\tau_2} \mu(s)|y(1,s)|^{m-2}y(1,s)dsdx \end{aligned}$$

and by applying Young's inequality to the last term of the above relation, we have

$$\begin{aligned} &\int_0^L u_t \int_{\tau_1}^{\tau_2} \mu(s)|y(1,s)|^{m-2}y(1,s)dsdx \qquad (13)\\ \leq\;& \frac{1}{m}\left(\int_{\tau_1}^{\tau_2} \mu(s)ds\right)\int_0^L |u_t|^m dx + \frac{m-1}{m}\int_0^L\int_{\tau_1}^{\tau_2} \mu(s)|y(x,1,s,t)|^m dsdx. \end{aligned}$$

Now, multiplying the third equation in (18) by $\mu(s)|y(x,p,s,t)|^{m-2}y(x,p,s,t)$ and integrating over $(0, L) \times (0, 1) \times (\tau_1, \tau_2)$, we get

$$\begin{aligned} &\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s\mu(s)|y|^{m-2}yy_t(x,p,s,t)dsdpdx\\ =\;& \frac{1}{m}\frac{d}{dt}\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s\mu(s)|y(x,p,s,t)|^m dsdpdx \qquad (14)\\ =\;& -\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} \mu(s)|y|^{m-2}yy_p(x,p,s,t)dsdpdx\\ =\;& -\frac{1}{m}\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} \mu(s)\frac{\partial}{\partial p}|y(x,p,s,t)|^m dsdpdx\\ =\;& -\frac{1}{m}\int_0^L\int_{\tau_1}^{\tau_2} \mu(s)\left[|y(x,1,s,t)|^m - |u_t|^m\right] dsdx\\ =\;& \frac{1}{m}\left(\int_{\tau_1}^{\tau_2} \mu(s)ds\right)\int_0^L |u_t|^m dx - \frac{1}{m}\int_0^L\int_{\tau_1}^{\tau_2} \mu(s)|y(x,1,s,t)|^m dsdx \end{aligned}$$

Combining (10) and (12)-(14), we obtain (11). □

Remark 2.2. *From the assumption* (9) *we can conclude that the energy* $E(t)$ *of the system* (1) *is nonincreasing for all* $t \geq 0$,

$$E(t) \leq E(0), \qquad \forall t \geq 0. \tag{15}$$

3. Well-posedness

In this section, we prove the existence and the uniqueness solution of system (1) by using Faedo-Galerkin method.

Based on the approach of [4], we introduce a new variable

$$\begin{cases} y(x,p,s,t) = u_t(x,t-sp), & \forall (x,p,t,s) \in (0,L)\times(0,1)\times(\tau_1,\tau_2)\times\mathbb{R}_+, \\ y(x,p,s,0) = y_0(sp) = u_2(x,sp), & \forall (x,p,s) \in (0,L)\times(0,1)\times(\tau_1,\tau_2). \end{cases} \tag{16}$$

It is clear that

$$\begin{cases} y_p(x,p,s,t) + s y_t(x,p,s,t) = 0, & \forall (x,p,t,s) \in (0,L)\times(0,1)\times(\tau_1,\tau_2)\times\mathbb{R}_+, \\ y(x,0,s,t) = u_t(x,t), & \forall (x,s,t) \in (0,L)\times(\tau_1,\tau_2)\times\mathbb{R}_+. \end{cases} \tag{17}$$

Thus, system (1) becomes

$$\begin{cases} u_{tt}(x,t) - a u_{xx}(x,t) + b v_x(x,t) + \mu_1 |u_t(x,t)|^{m-2} u_t(x,t) & \\ + \int_{\tau_1}^{\tau_2} \mu(s) |y(x,1,s,t)|^{m-2} y(x,1,s,t) ds = 0, & (x,s,t) \in (0,L)\times(\tau_1,\tau_2)\times\mathbb{R}_+, \\ v_t(x,t) - d v_{xx}(x,t) + b u_{xt}(x,t) = 0, & (x,t) \in (0,L)\times\mathbb{R}_+, \\ y_p(x,p,s,t) + s y_t(x,p,s,t) = 0, & (x,p,s,t) \in (0,L)\times(0,1)\times(\tau_1,\tau_2)\times\mathbb{R}_+, \\ u(0,t) = u(L,t) = 0 \qquad v_x(0,t) = v_x(L,t) = 0 & t \in \mathbb{R}_+ \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ v(x,0) = v_0(x), & x \in (0,L), \\ y(x,p,s,0) = u_2(x,ps), & (x,p,s) \in (0,L)\times(0,1)\times(0,\tau_2), \end{cases} \tag{18}$$

with condition (8).

The existence and uniqueness result is stated as follows:

Theorem 3.1. *Assume that* (9) *hold.*

Then given $(u_0, v_0, u_2) \in (H^2(0,L)\cap H_0^1(0,L))^2 \times L^2((0,L), H_0^1((0,1),(\tau_1,\tau_2))$ *,*$(u_1,u_2) \in L^2(0,L)\times L^2((0,L),(0,1),(\tau_1,\tau_2))$*, there exists a unique weak solution* u, v, y *of problem* (18) *such that*

$$(u,v) \in C([0,+\infty[, H^2(0,L)\cap H_0^1(0,L)) \cap C^1([0,+\infty[, L^2(0,L)),$$

$$y \in C([0,+\infty[; L^2((0,L), H_0^1((0,1),(\tau_1,\tau_2)).$$

Proof. We divide the proof of Theorem3.1 into two steps:the Faedo-Galerkin approximation and the energy estimates.

Step 1 :Faedo-Galerkin approximation.

We construct approximations of the solution (u, v, y) by the Faedo-Galerkin method as follows. For $n \geq 1$, let $W_n = \text{span}\,\{w_1, \ldots\ldots, w_i\}$ be a Hilbertian basis of the space H_0^1. Now, we we define for $1 \leq i \leq n$ the sequence $\varphi_i(x,\rho)$ as follows:

$$\varphi_i(x,0) = w_i(x)$$

Then we may extend $\varphi_i(x,\rho)$ over $L^2((0,1),\Omega)$ and denote $V_n = \text{span}\,\{\varphi_1, ...,\varphi_i\}$. We choose sequences $(u_0^n), (u_1^n), (v_0^n), (v_1^n)$ in W_n and (y_0^n) in V_n such that
$(u_0^n, v_0^n, u_1^n, v_1^n) \to (u_0, v_0, u_1, v_1)$ strongly in $H^2(\Omega) \cap H_0^1(\Omega)$ and $z_0^n \to u_2$ strongly in $L^2((0,1), H_0^1(\Omega))$ as $n \to \infty$.

We search the approximate solutions

$$u^n(x,t) = \sum_{i=1}^{n} f_i^n(t) w_i(x), \quad v^n(x,t) = \sum_{i=1}^{n} h_i^n(t) w_i(x) \quad and \quad y^n(x,\rho,t) = \sum_{i=1}^{n} k_i^n(t)\varphi_i(x,\rho)$$

to the finite dimensional Cauchy problem:

$$\begin{cases} \displaystyle\int_0^L u_{tt}^n w_i dx + a\int_0^L (u^n)_x (w_i)_x dx + b\int_0^L (v^n)_x w_i dx + \mu_1 \int_0^L |(u^n)_t|^{m-2} u_t^n w_i dx \\ \displaystyle + \int_0^L \int_{\tau_1}^{\tau_2} \mu(s) |y^n(x,1,t)|^{m-2} y^n(x,1,t) w_i ds dx = 0, \\ \displaystyle\int_0^L v_t^n w_i dx + d\int_0^L (v^n)_x (w_i)_x dx + b\int_0^L (u_t^n)_x w_i dx = 0, \\ (u^n(0), v^n(0)) = (u_0^n, v_0^n) \quad (u_t^n(0), v_t^n(0)) = (u_1^n, v_1^n), \end{cases} \tag{19}$$

and

$$\begin{cases} \displaystyle\int_0^L (y_p^n(x,p,s,t) + s y_t^n(x,p,s,t))\varphi_i dx = 0, \\ \qquad\qquad y^n(x,p,s,0) = y_0^n. \end{cases} \tag{20}$$

According to the standard theory of ordinary differential equations, the finite dimensional problem (19)-(20) has solution $f_i^n(t), h_i^n(t), k_i^n(t)$ defined on $[0,t)$. The a priori estimates that follow imply that in fact $t_n = T$.

Step 2: Energy estimates. Multiplying the first and the second equation of (19) by $\left(f_i^n(t)\right)'$ and $h_i^n(t)$ respectively, we obtain:

$$\begin{aligned} &\int_0^L u_{tt}^n u_t^n dx + a\int_0^L (u^n)_x (u_t^n)_x dx + b\int_0^L (v^n)_x u_t^n dx + \mu_1 \int_0^L |(u^n)_t|^m dx \\ &+ \int_0^L \int_{\tau_1}^{\tau_2} \mu(s)|y^n(x,1,t)|^{m-2} y^n(x,1,t) u_t^n ds dx = 0. \end{aligned} \tag{21}$$

and

$$\int_0^L v_t^n v^n dx + d\int_0^L (v_x^n)^2 dx + b\int_0^L (u_t^n)_x v^n dx = 0. \tag{22}$$

Multiplying the first equation of (20) by $(m-1)\mu(s)|y^n(x,p,s,t)|^{m-2} k_i^n(t)$ and integrating over $(0,t) \times (0,1) \times (\tau_1, \tau_2)$, we get

$$\begin{aligned} &(m-1)\int_0^t \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} \mu(s)|y^n|^{m-2} y^n y_p^n(x,p,s,t) ds dp dx d\sigma = \\ &\qquad \frac{m-1}{m}\int_0^t \int_0^L \int_{\tau_1}^{\tau_2} \mu(s)\Big(|y^n(x,1,s,t)|^m - |y^n(x,0,s,t)|^m\Big) ds dx d\sigma, \end{aligned} \tag{23}$$

and

$$(m-1)\int_0^t\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s\mu(s)|y^n(x,p,s,t)|^{m-2}y^n y_t^n(x,p,s,t)dsdpdxdt = \frac{m-1}{m}\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s\mu(s)\Big(|y^n(x,p,s,t)|^m-|y^n(x,p,s,0)|^m\Big)dsdpdx. \tag{24}$$

Integrating (21) and (22) over $(0,t)$, taking into account (23),(24), we obtain

$$\begin{aligned}&\mathcal{E}_n(t)+\mu_1\int_0^t\int_0^L|(u^n)_t|^m dxd\sigma+\int_0^L\int_{\tau_1}^{\tau_2}\mu(s)|y^n(x,1,t)|^{m-2}y^n(x,1,t)u_t^n dsdxd\sigma+d\int_0^t\int_0^L(v_x^n)^2dxd\sigma\\ &\frac{m-1}{m}\int_0^t\int_0^L\int_{\tau_1}^{\tau_2}\mu(s)|y^n(x,1,s,t)|^m dsdxd\sigma-\frac{m-1}{m}\Big(\int_{\tau_1}^{\tau_2}\mu(s)ds\Big)\int_0^t\int_0^L|(u^n)_t)|^m dsdxd\sigma\\ &=\mathcal{E}_n(0),\end{aligned} \tag{25}$$

where

$$\begin{aligned}\mathcal{E}_n(t)=&\frac{1}{2}\int_0^L(u_t^n)^2(x,t)dx+\frac{a}{2}\int_0^L(u_x^n)^2(x,t)dx+\frac{1}{2}\int_0^L(v^n)^2dx\\ &+\frac{m-1}{m}\int_0^L\int_0^1\int_{\tau_1}^{\tau_2}s\mu(s)|y^n(x,p,s,t)|^m dsdpdx.\end{aligned} \tag{26}$$

Young's inequality gives us that

$$\mathcal{E}_n(t)+(\mu_1-\Big(\int_{\tau_1}^{\tau_2}\mu(s)ds\Big))\int_0^t\int_0^L|(u^n)_t)|^m dxd\sigma+d\int_0^t\int_0^L(v_x^n)^2dxd\sigma\le\mathcal{E}_n(0). \tag{27}$$

Consequently, using that $\int_{\tau_1}^{\tau_2}\mu(s)ds<\mu_1$, we have the following estimate:

$$\mathcal{E}_n(t)\le\mathcal{E}_n(0). \tag{28}$$

Now, since the sequences $\Big(u_0^n\Big)_{n\in\mathbb{N}}$, $\Big(u_1^n\Big)_{n\in\mathbb{N}}$, $\Big(v_0^n\Big)_{n\in\mathbb{N}}$, $\Big(y_0^n\Big)_{n\in\mathbb{N}}$ converge and we can find a positive constant c independent of n such that

$$\mathcal{E}_n(t)\le c. \tag{29}$$

Therefore, the estimate (29) together with (28) give us, for all $n\in\mathbb{N}$, $t_n=T$, we deduce

$$\begin{array}{lll}\Big(u^n\Big)_{n\in\mathbb{N}} & \text{is bounded in} & L^\infty(0,T;H_0^1(0,L)),\\ \Big(v^n\Big)_{n\in\mathbb{N}} & \text{is bounded in} & L^\infty(0,T;H_0^1(0,L)),\\ \Big(u_t^n\Big)_{n\in\mathbb{N}} & \text{is bounded in} & L^\infty(0,T;H_0^1(0,L)),\\ \Big(y^n\Big)_{n\in\mathbb{N}} & \text{is bounded in} & L^\infty(0,T;L^m((0,L),(0,1),(\tau_1,\tau_2)).\end{array} \tag{30}$$

Consequently, we conclude that

$$\begin{array}{lll}u^n\rightharpoonup u & \text{weakly star in} & L^\infty(0,T;H_0^1(0,L)),\\ v^n\rightharpoonup v & \text{weakly star in} & L^\infty(0,T;H_0^1(0,L)),\\ u_t^n\rightharpoonup u_t & \text{weakly star in} & L^\infty(0,T;H_0^1(0,L)),\\ y^n\rightharpoonup y & \text{weakly star in} & L^\infty(0,T;L^2((0,L),(0,1),(\tau_1,\tau_2)).\end{array} \tag{31}$$

From (30), we have $\left(u^n\right)_{n\in\mathbb{N}}$, $\left(v^n\right)_{n\in\mathbb{N}}$ are bounded in $L^\infty(0,T;H_0^1(0,L))$ and $\left(y^n\right)_{n\in\mathbb{N}}$ is bounded in $L^\infty(0,T;L^2((0,L),H_0^1((0,1),(\tau_1,\tau_2)))$. Then $\left(u^n\right)_{n\in\mathbb{N}}$, $\left(v^n\right)_{n\in\mathbb{N}}$ are bounded in $L^2(0,T;H_0^1(0,L))$, and $\left(y^n\right)_{n\in\mathbb{N}}$ is bounded in $L^2(0,T;L^2((0,L),(0,1),(\tau_1,\tau_2))$. Consequently,$\left(u^n\right)_{n\in\mathbb{N}}$, $\left(v^n\right)_{n\in\mathbb{N}}$ are bounded in $H^1(0,T;H^1(0,L))$ and $\left(y^n\right)_{n\in\mathbb{N}}$ is bounded in $L^2(0,T;L^2((0,L),(0,1),(\tau_1,\tau_2))$. Since the embedding

$$L^\infty(0,T;L^2(0,L)) \hookrightarrow L^2(0,T;L^2(0,L))$$

$$H^1(0,T;H^1(0,L)) \hookrightarrow L^2(0,T;L^2(0,L))$$

is compact, using Aubin-Lion's theorem [15], we can extract subsequences $\left(u^k\right)_{k\in\mathbb{N}}$ of $\left(u^n\right)_{n\in\mathbb{N}}$, $\left(v^k\right)_{k\in\mathbb{N}}$ of $\left(v^n\right)_{n\in\mathbb{N}}$ and $\left(y^k\right)_{k\in\mathbb{N}}$ of $\left(y^n\right)_{n\in\mathbb{N}}$ such that

$$u^k \to u \quad \text{strongly in} \quad L^2(0,T;L^2(0,L)),$$

$$v^k \to v \quad \text{strongly in} \quad L^2(0,T;L^2(0,L))$$

and

$$y^k \to y \quad \text{strongly in} \quad L^2(0,T;L^2((0,L),(0,1),(\tau_1,\tau_2)))$$

Therefore,

$$u^k \to u \quad \text{strongly and a.e} \quad (0,T)\times(0,L),$$

$$v^k \to v \quad \text{strongly and a.e} \quad (0,T)\times(0,L),$$

and

$$y^k \to y \quad \text{strongly and a.e} \quad (0,T)\times(0,L)\times(0,1)\times(\tau_1,\tau_2).$$

The proof now can be completed arguing as in Theorem 3.1 of [15]

Uniqueness.

Let (u_1,v_1,y_1) and (u_2,v_2,y_2) be two solutions of problem (1) Then $(u,v)=(u_1-u_2,v_1-v_2,y_1-y_2)$ satisfies

$$\begin{cases} u_{tt}(x,t)-au_{xx}(x,t)+bv_x(x,t)+\mu_1|u_t(x,t)|^{m-2}u_t(x,t) & \\ +\int_{\tau_1}^{\tau_2}\mu(s)|u_t(x,t-s)|^{m-2}u_t(x,t-s)ds=0, & \\ v_t(x,t)-dv_{xx}(x,t)+bu_{xt}(x,t)=0, & (x,t)\in(0,L)\times\mathbb{R}_+, \\ u(0,t)=u(L,t)=0 \qquad v_x(0,t)=v_x(L,t)=0 & t\in\mathbb{R}_+ \\ u(x,0)=0,\ u_t(x,0)=0,\ v(x,0)=0, & x\in(0,L), \\ y_0=0, & (x,t)\in(0,L)\times(0,\tau_2). \end{cases} \tag{32}$$

Following Lemma 2.1, the energy function associated to the problem (32) satisfies $E'(t)\le 0$. Then $E(t)=E(0)=0$, we deduce that $u=v=0$. The proof is complete. □

4. Decay of solutions

In this section we study the asymptotic behavior of solutions. For this we use the method of Lyapunov. In order to prove the decay of energy, we define the Lyapunov candidate function by

$$L(t) = E(t) + k_1 V_1(t) + k_2 V_2(t) + k_3 V_3(t), \tag{33}$$

where k_1, k_2 and k_3 are positive constants, $E(t)$ is the energy given by (10) and

$$V_1(t) = \int_0^L u u_t dx, \tag{34}$$

$$V_2(t) = \int_0^L u_t \int_0^x v(z,t) dz dx, \tag{35}$$

$$V_3(t) = \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s\mu(s) e^{-ps} |y(x,p,s,t)|^m ds dp dx, \tag{36}$$

Proposition 4.1. *There exist two positive constants a and b, such that*

$$aE(t) \le L(t) \le bE(t), \qquad t \ge 0. \tag{37}$$

Proof. Using Young's inequality, Cauchy-Schwarz's inequality and Poincare's inequality, we obtain (37). □

Lemma 4.2. *Let $V_1(t)$, the functional given by* (34), *then, for $t \ge 0$, its time derivative, yield*

$$\begin{aligned} V_1'(t) &= \int_0^L u_t^2 dx - \left(a - \frac{\delta_1 b}{4} - \eta_2\right) \int_0^L u_x^2 dx \\ &+ \frac{b}{\delta_1} \int_0^L v^2 dx + \frac{m-1}{m\delta_2} \mu_1 \int_0^L |u_t|^m dx \\ &+ \frac{m-1}{m\delta_2} \int_0^L \int_{\tau_1}^{\tau_2} \mu(s) |y(x,1,s,t)|^m ds dx. \end{aligned} \tag{38}$$

Where $\eta_2 = \frac{\delta_2 c_*}{4m}\left(\mu_1 + \int_{\tau_1}^{\tau_2} \mu(s) ds\right)$, δ_1 and δ_1 are small positive constants.

Proof. Deriving $V_1(t)$, taking the system (1) and integrating by parts, we obtain

$$\begin{aligned} V_1'(t) &= \int_0^L u_t^2 dx - a \int_0^L u_x^2 dx + b \int_0^L v u_x dx \\ &- \mu_1 \int_0^L |u_t(x,t)|^{m-2} u_t u(x,t) dx \\ &- \int_0^L u \int_{\tau_1}^{\tau_2} \mu(s) |y(x,1,s,t)|^{m-2} y(x,1,s,t) ds dx. \end{aligned} \tag{39}$$

By exploiting Young's and Poincare's inequality, the last three terms in the right hand side of the above inequality gives

$$b \int_0^L v u_x dx \le \frac{b}{\delta_1} \int_0^L v^2 dx + \frac{\delta_1 b}{4} \int_0^L u_x^2 dx \tag{40}$$

$$\begin{aligned} &\int_0^L |u_t(x,t)|^{m-2} u_t u(x,t) ds dx \\ \le\ & \frac{\delta_2 c_*}{4m} \int_0^L |u_x|^2 dx + \frac{m-1}{m\delta_2} \int_0^L |u_t|^m dx. \end{aligned} \tag{41}$$

$$\begin{aligned}
&\int_0^L u \int_{\tau_1}^{\tau_2} \mu(s)|y(1,s)|^{m-2} y(1,s) ds dx \qquad (42)\\
\leq\ & \frac{\delta_2 c_*}{4m}\left(\int_{\tau_1}^{\tau_2}\mu(s)ds\right)\int_0^L |u_x|^2 dx + \frac{m-1}{m\delta_2}\int_0^L\int_{\tau_1}^{\tau_2}\mu(s)|y(x,1,s,t)|^m ds dx.
\end{aligned}$$

for δ_1 and δ_2 any small positive constants.

Substituting (40)-(42) into (39), we obtain (38). □

Lemma 4.3. *Let $V_2(t)$ the functional given by (35), then for $t \geq 0$, its derivative satisfies*

$$\begin{aligned}
V_2'(t) &= -\left(b - \frac{\delta_3 d}{4}\right)\int_0^L u_t^2 dx + \left(b + \frac{a}{\delta_3} + \eta_3\right)\int_0^L v^2 dx\\
&+\frac{\delta_3 a}{4}\int_0^L u_x^2 dx + \frac{d}{\delta_3}\int_0^L v_x^2 dx + \frac{m-1}{m}\delta_4\mu_1\int_0^L |u_t|^m dx\\
&+\frac{m-1}{m\delta_2}\int_0^L\int_{\tau_1}^{\tau_2}\mu(s)|y(x,1,s,t)|^m ds dx \qquad (43)
\end{aligned}$$

where $\eta_3 = \frac{Lc_*}{4m}\left(\frac{\mu_1}{\delta_4} + \delta_2\int_{\tau_1}^{\tau_2}\mu(s)ds\right)$, δ_3 and δ_4 are small positive constants.

Proof. By differentiating $V_2(t)$ and using system (1) we obtain

$$\begin{aligned}
V_2'(t) &= d\int_0^L u_t \int_0^x v_{xx} dz dx - b\int_0^L u_t \int_0^x u_{xt} dz dx\\
&+\int_0^L \left(a u_{xx} - b v_x - \mu_1 |u_t|^{m-2} u_t\right)\int_0^x v(z,t) dz dx\\
&-\int_0^L\int_{\tau_1}^{\tau_2}\mu(s)|y(x,1,s,t)|^{m-2} y(x,1,s,t) ds \int_0^x v(z,t) dz dx \qquad (44)
\end{aligned}$$

Taking into account the boundary conditions and (8), integrating by parts the terms of (44), we have

$$\begin{aligned}
V_2'(t) &= d\int_0^L v_x u_t dx - b\int_0^L u_t^2 dx + b\int_0^L v^2 dx - a\int_0^L u_x v dx\\
&-\mu_1\int_0^L |u_t|^{m-2} u_t \int_0^x v(z,t) dz dx\\
&-\int_0^L\int_{\tau_1}^{\tau_2}\mu(s)|y(x,1,s,t)|^{m-2} y(x,1,s,t) ds \int_0^x v(z,t) dz dx \qquad (45)
\end{aligned}$$

Similar to (40)-(42), we estimate the terms of (45), as

$$d\int_0^L v_x u_t dx \leq \frac{d}{\delta_3}\int_0^L v_x^2 dx + \frac{\delta_3 d}{4}\int_0^L u_t^2 dx \qquad (46)$$

$$a\int_0^L u_x v dx \leq \frac{a}{\delta_3}\int_0^L v^2 dx + \frac{\delta_3 a}{4}\int_0^L u_x^2 dx \qquad (47)$$

$$\begin{aligned}
&\int_0^L |u_t|^{m-2} u_t \int_0^x v(z,t) dz dx \qquad (48)\\
\leq\ & \frac{Lc_*}{4m\delta_4}\int_0^L v^2 dx + \frac{m-1}{m}\delta_4\int_0^L |u_t|^m dx
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^L \int_{\tau_1}^{\tau_2} \mu(s)|y(x,1,s,t)|^{m-2} y(x,1,s,t)ds \int_0^x v(z,t)dzdx \\
\leq \quad & \frac{\delta_2 L c_*}{4m}\left(\int_{\tau_1}^{\tau_2} \mu(s)ds\right)\int_0^L v^2 dx + \frac{m-1}{m\delta_2}\int_0^L \int_{\tau_1}^{\tau_2} \mu(s)|y(x,1,s,t)|^m dsdx.
\end{aligned} \tag{49}$$

for δ_3 and δ_4 any small positive constants.
Substituting (46)–(49) into (44), we obtain (43). □

Lemma 4.4. *Let $V_3(t)$ the functional given by* (36)*, then for $t \geq 0$, its derivative satisfies*

$$\begin{aligned}
V_3'(t) \quad \leq \quad & m\int_{\tau_1}^{\tau_2} \mu(s)ds \int_0^L |u_t|^m dx - me^{-\tau_2}\int_0^L \int_{\tau_1}^{\tau_2} \mu(s)|y(x,1,s,t)|^m dsdx \\
& -m\int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s\mu(s)e^{-ps}|y(x,p,s,t)|^m dsdpdx.
\end{aligned} \tag{50}$$

where δ_3 and δ_3 are small positive constants.

Proof. Deriving $V_3(t)$, using the identity (17) and integrating by parts, we have

$$\begin{aligned}
V_3'(t) \quad = \quad & m\int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s\mu(s)e^{-ps}|y(x,p,s,t)|^{m-2} yy_t(x,p,s,t)dsdpdx \\
= \quad & -m\int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} \mu(s)e^{-ps}\frac{\partial}{\partial p}|y(x,p,s,t)|^m dsdpdx \\
= \quad & m\int_0^L \int_{\tau_1}^{\tau_2} \mu(s)e^{-ps}\left[|y(x,0,s,t)|^m - |y(x,1,s,t)|^m\right] dsdx \\
& -m\int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s\mu(s)e^{-ps}|y(x,p,s,t)|^m dsdpdx.
\end{aligned}$$

Using the fact that $e^{-ps} \leq 1$, for all $ps \in [0, \tau_2]$, we obtain (50). □

Theorem 4.5. *Assume that **(H1)** and **(H2)** hold. Then, there exist three positive constants α and C such that*

$$E(t) \leq Ce^{-\alpha t} \qquad \forall t \geq 0. \tag{51}$$

Proof. From (11), (33), (38), (43), (50) and the fact that

$$-\int_0^L v_x^2 dx \leq -\frac{1}{L^2}\int_0^L v^2 dx.$$

The derivative of $L(t)$ given

$$\begin{aligned} L'(t) \leq & -\left[\left(d-\frac{dk_2}{\delta_3}\right)\frac{1}{L^2}-\frac{bk_1}{\delta_1}-\left(b+\frac{a}{\delta_3}+\eta_3\right)k_2\right]\int_0^L v^2dx \\ & -\left[\eta_1-\frac{m-1}{m\delta_2}\mu_1k_1-\frac{m-1}{m}\delta_4\mu_1k_2-m\left(\int_{\tau_1}^{\tau_2}\mu(s)ds\right)k_3\right]\int_0^L |u_t|^m dx \\ & -\left[\left(b-\frac{\delta_3 d}{4}\right)k_2-k_1\right]\int_0^L u_t^2dx \\ & -\left[\left(a-\frac{\delta_1 b}{4}-\eta_2\right)k_1-\frac{\delta_3 a}{4}k_2\right]\int_0^L u_x^2dx \\ & -\left[mk_3e^{-\tau_2}-\frac{m-1}{m\delta_2}k_1-\frac{m-1}{m\delta_2}k_2\right]\int_0^L\int_{\tau_1}^{\tau_2}\mu(s)|y(x,1,s,t)|^m dsdx \\ & -mk_3\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s\mu(s)e^{-ps}|y(x,p,s,t)|^m dsdpdx \end{aligned} \tag{52}$$

Now, we pick $\delta_i, i = 1,2,3,4, k_1, k_2$ and k_3 small enough such that coefficients on the right hand side of (52) are all strictly negative. Then, there exist a positive constant λ_1 such that

$$L'(t) \leq -\lambda_1 E(t) \qquad \forall t \geq 0. \tag{53}$$

Using equivalence relation (37) , we have

$$L'(t) \leq -\lambda L(t) \qquad \forall t \geq 0. \tag{54}$$

where $\lambda = \dfrac{\lambda_1}{b}$. Multiplying inequality (54) by $e^{\lambda t}$ and integrating over $(0,t)$, we have

$$L(t) \leq L(0)e^{-\lambda t} \qquad \forall t \geq 0. \tag{55}$$

Using (56) and the left inequality of the relationship (37), we get

$$E(t) \leq Ce^{-\lambda t} \qquad \forall t \geq 0, \tag{56}$$

where $C = L(0)/a$. The proof is complete. □

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