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The expressions of the generalized inverses of the block tensor via the C-Product

Hongwei Jin[a], Mengyu He[a], Yuzhen Wang[a,*]

[a]*School of Mathematics and Physics, Guangxi Minzu University, Nanning 530006, China*

Abstract. In this paper, we present the expressions of the generalized inverses of the third-order 2×2 block tensor under the C-Product. Firstly, we give the necessary and sufficient conditions to present some generalized inverses and the Moore-Penrose inverse of the block tensor in Banachiewicz-Schur forms. Next, some results are generalized to the group inverse and the Drazin inverse. Moreover, equivalent conditions for the existence as well as the expressions for the core inverse of the block tensor are obtained. Finally, the results are applied to express the quotient property and the first Sylsvester identity of tensors.

1. Introduction

Operations with tensors, or multiway arrays, have become increasingly prevalent in recent years. A complex tensor $\mathcal{A}$ can be regarded as a multidimensional array of date, which takes the form

$$\mathcal{A} = (a_{i_1 i_2 \dots i_p}) \in \mathbb{C}^{n_1 \times n_2 \times \cdots \times n_p},\ i_j \in \{1, \dots, n_j\},\ j = 1, 2, \dots, p.$$

The order of the above tensor $\mathcal{A}$ is p. In general, vectors and matrices are considered as first-order and second-order tensors, respectively. A third-order tensor can be regard as a "cube" of data. Notice that the orientation of third-order tensors is not unique. So, it is necessary to refer to its slices, i.e., the two-dimensional sections defined by holding two indices constant. We can use horizontal, lateral, and frontal slices defined in [8] to specify the two indices holding constant. In this paper, we mainly focus on the frontal slice, whose Matlab notation is $\mathcal{A}(:, :, i)$ and we denote $\mathcal{A}^{(i)}$ for short. Notice that there are several products of tensors, such as the Einstein product which appears in the theory of relativity [3] and continuum mechanics [10], the T-Product which is introduced in [12] and the C-Product which is defined in [7]. It is also indicated that we can use the C-Product to study the discrete image blurring model and the image restoration model. Moreover, the product can be applied in deep convolutional neural networks [24], image classification [15], matrix networks [22], tensor robust principal component analysis [11] and so on.

Lately, the generalized inverse of tensors with the different tensor product have attracted a lot of study. Sun et al. [23] defined the Moore-Penrose inverse and $\{i\}$-inverses of even order tensors with the Einstein

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* Corresponding author: Yuzhen Wang
Email addresses: jhw_math@126.com (Hongwei Jin), hmy04280611@163.com (Mengyu He), 18778738525@163.com (Yuzhen Wang)

product. Moreover, the explicit formulas of the Moore-Penrose inverse of some block tensors were obtained. Miao et al. [13] presented the definition of generalized tensor function according to the tensor singular value decomposition via the tensor T-Product. Also, the compact singular value decomposition of tensors was introduced, from which the projection operators and Moore-Penrose inverse of tensors were obtained. Miao et al. [14] focused on the tensor decompositions: T-polar, T-LU, T-QR and T-Schur decompositions of tensors. The T-group inverse and T-Drazin inverse which can be viewed as the extension of matrix cases were studied. Krushnachandra et al. [9] gave an expression for the Moore-Penrose inverse of the product of two tensors via the Einstein product. Then a new generalized inverse of a tensor called product Moore-Penrose inverse was introduced. A necessary and sufficient condition for the coincidence of the Moore-Penrose inverse and the product Moore-Penrose inverse was also proposed. Stanimirović PS et al. [21] investigated some basic properties of the range and null space of multidimensional arrays with respect to Einstein tensor product. Computation of tensor outer inverse with prescribed range and kernel of higher order tensors was considered. Results related with the (b, c)-inverses on semigroups were examined in details in a specific semigroup of tensors with a binary associative operation defined as the Einstein tensor product. Ji et al. [4] had a further study on the properties of even-order tensors with Einstein product. The authors defined the index and characterize the invertibility of an even-order square tensor. The notion of the Drazin inverse of a square matrix to an even-order square tensor were extended. An expression for the Drazin inverse through the core-nilpotent decomposition for a tensor of even-order was obtained. Jin et al. [5] established the Moore-Penrose inverse of tensors by using tensor equations with the T-Product. Moreover, the least squares solutions of tensor equations were investigated. Behera et al. [1] studied different generalized inverses of tensors over a commutative ring and a non-commutative ring. The authors also proposed algorithms for computing the inner inverses, the Moore-Penrose inverse, and weighted Moore-Penrose inverse of tensors over a non-commutative ring. Sahoo et al. [19] introduced new representations and characterizations of the outer inverse of tensors through QR decomposition. An effective algorithm for computing outer inverses of tensors was proposed and applied. The power of the proposed method was demonstrated by its application in 3D color image deblurring. Sahoo et al. [20] studied the definitions of the core and core-EP inverses of complex tensors. Some characterizations, representations and properties of the core and core-EP inverses were investigated. The results were verified using specific algebraic approach, based on proposed definitions and previously verified properties.

This paper mainly deals with the generalized inverses of the third-order block tensor with the C-Product.

1.1. The Definition of the C-Product

Let $\mathcal{A} \in \mathbb{C}^{n_1 \times n_2 \times n_3}$ be a third-order tensor. Denote the i^{th} frontal slice of tensor $\mathcal{A}$ as $\mathcal{A}^{(i)}$. The operations **mat** and **ten** are defined as follows [7]. Observe that $\mathbf{mat}(\mathcal{A})$ is the $n_1n_3 \times n_2n_3$ block Toeplitz+Hankel matrix:

$$\mathbf{mat}(\mathcal{A}) = \begin{bmatrix} \mathcal{A}^{(1)} & \mathcal{A}^{(2)} & \cdots & \mathcal{A}^{(n_3-1)} & \mathcal{A}^{(n_3)} \\ \mathcal{A}^{(2)} & \mathcal{A}^{(1)} & \cdots & \mathcal{A}^{(n_3-2)} & \mathcal{A}^{(n_3-1)} \\ \vdots & \vdots & & \vdots & \vdots \\ \mathcal{A}^{(n_3-1)} & \mathcal{A}^{(n_3-2)} & \cdots & \mathcal{A}^{(1)} & \mathcal{A}^{(2)} \\ \mathcal{A}^{(n_3)} & \mathcal{A}^{(n_3-1)} & \cdots & \mathcal{A}^{(2)} & \mathcal{A}^{(1)} \end{bmatrix} + \begin{bmatrix} \mathcal{A}^{(2)} & \mathcal{A}^{(3)} & \cdots & \mathcal{A}^{(n_3)} & O \\ \mathcal{A}^{(3)} & \mathcal{A}^{(4)} & \cdots & O & \mathcal{A}^{(n_3)} \\ \vdots & \vdots & & \vdots & \vdots \\ \mathcal{A}^{(n_3)} & O & \cdots & \mathcal{A}^{(4)} & \mathcal{A}^{(3)} \\ O & \mathcal{A}^{(n_3)} & \cdots & \mathcal{A}^{(3)} & \mathcal{A}^{(2)} \end{bmatrix},$$

where O is the $n_1 \times n_2$ zero matrix. Notice that $\mathbf{ten}(\mathbf{mat}(\mathcal{A})) = \mathcal{A}$, which means that $\mathbf{ten}(\cdot)$ is the inverse operation of the $\mathbf{mat}(\cdot)$. Furthermore, the definition of the cosine transform product was given in [7].

Definition 1.1. [7] *Let $\mathcal{A} \in \mathbb{C}^{n_1 \times n_2 \times n_3}$ and $\mathcal{B} \in \mathbb{C}^{n_2 \times n_4 \times n_3}$. The cosine transform product, which is called C-Product for short, is defined as*

$$\mathcal{A} \bigstar \mathcal{B} = \mathbf{ten}(\mathbf{mat}(\mathcal{A})\mathbf{mat}(\mathcal{B})).$$

From the definition, we can see that it is easy to compute $\mathbf{mat}(\mathcal{A})\mathbf{mat}(\mathcal{B})$ by using the technical of the matrices product. In order to compute the C-Product, we must deal with the operation "$\mathbf{ten}(\cdot)$", which can be realized by using the following algorithm.

Algorithm 1.1: Compute **ten**$(\cdot)$ of a matrix

Input: $n_1n_3 \times n_2n_3$ matrix X
Output: $n_1 \times n_2 \times n_3$ tensor $\mathcal{A}$
1. Take the bottom left $n_1 \times n_2$ block of $X \mapsto \mathcal{A}^{(n_3)}$.
2. for $i = n_3 - 1, \ldots, 1$
   [i-th block of first block column of X]$-\mathcal{A}^{(i+1)} \mapsto \mathcal{A}^{(i)}$
   end

There is an alternative method to compute the C-Product by using the face-wise product. The face-wise product of two tensors $\mathcal{A} \in \mathbb{C}^{n_1\times n_2\times n_3}$ and $\mathcal{B} \in \mathbb{C}^{n_2\times n_4\times n_3}$ is defined as

$$(\mathcal{A}\triangle\mathcal{B})^{(i)} = \mathcal{A}^{(i)}\mathcal{B}^{(i)},\ i = 1, 2, ..., n_3.$$

Lemma 1.2. [7] *Let $\mathcal{A} \in \mathbb{C}^{n_1\times n_2\times n_3}$ and $\mathcal{B} \in \mathbb{C}^{n_2\times n_4\times n_3}$. Then,*

$$\mathcal{A}\bigstar\mathcal{B} = L^{-1}(L(\mathcal{A})\triangle L(\mathcal{B})). \tag{1.1}$$

Notice that

$$L(\mathcal{A}) = \mathcal{A} \times_3 M \text{ and } L^{-1}(\mathcal{A}) = \mathcal{A} \times_3 M^{-1},$$

where M is defined in [7, Definition 3.2] and $\mathcal{A} \times_3 M$ means the mode-3 product of a tensor $\mathcal{A} \in \mathbb{C}^{n_1\times n_2\times n_3}$ with the matrix $M \in \mathbb{C}^{n_3\times n_3}$. More precise, we have

$$(\mathcal{A} \times_3 M)_{i_1i_2i_3} = \sum_{i_3=1}^{n_3} a_{i_1i_2i_3}m_{ji_3},\ i_k \in \{1, 2, ..., n_k\},\ k = 1, 2, 3,\ j \in \{1, 2, ..., n_3\}.$$

The reader can consult [8].

Definition 1.3. [7] *Let $L(\mathcal{I}) = \widehat{\mathcal{I}} \in \mathbb{C}^{n_1\times n_1\times n_3}$ be such that $\widehat{\mathcal{I}}^{(i)} = I_{n_1}$, $i = 1, 2, ..., n_3$. Then $\mathcal{I} = L^{-1}(\widehat{\mathcal{I}})$ is the identity tensor.*

Lemma 1.4. [7] *Let $\mathcal{A} \in \mathbb{C}^{n_1\times n_1\times n_3}$ and $\mathcal{I} \in \mathbb{C}^{n_1\times n_1\times n_3}$ be the identity tensor. Then,*

$$\mathcal{I}\bigstar\mathcal{A} = \mathcal{A}\bigstar\mathcal{I} = \mathcal{A}.$$

Definition 1.5. [7] *Let $\mathcal{A} \in \mathbb{C}^{n_1\times n_1\times n_3}$ and $\mathcal{B} \in \mathbb{C}^{n_1\times n_1\times n_3}$. If*

$$\mathcal{A}\bigstar\mathcal{B} = \mathcal{I} \text{ and } \mathcal{B}\bigstar\mathcal{A} = \mathcal{I},$$

then $\mathcal{A}$ is said to be invertible and $\mathcal{B}$ is called the inverse of $\mathcal{A}$, which is denoted by $\mathcal{A}^{-1}$.

Definition 1.6. [7] *Let $\mathcal{A} \in \mathbb{C}^{n_1\times n_2\times n_3}$. Then the conjugate transpose of $\mathcal{A}$, which is denoted by $\mathcal{A}^H$, is such that*

$$L(\mathcal{A}^H)^{(i)} = (L(\mathcal{A}^{(i)}))^H,\ i = 1, 2, \ldots, n_3.$$

1.2. The Genralized Inverse of Tensors with the C-Product

Next, we recall the definition of the Moore-Penrose inverse, the Drazin inverse, the group inverse and the core inverse of a tensor via the C-Product.

Definition 1.7. [6] *Let $\mathcal{A} \in \mathbb{C}^{n_1\times n_2\times n_3}$. The unique tensor $\mathcal{X} \in \mathbb{C}^{n_2\times n_1\times n_3}$ satisfying*

$$(1)\ \mathcal{A}\bigstar\mathcal{X}\bigstar\mathcal{A} = \mathcal{A},\ (2)\ \mathcal{X}\bigstar\mathcal{A}\bigstar\mathcal{X} = \mathcal{X},\ (3)\ (\mathcal{A}\bigstar\mathcal{X})^H = \mathcal{A}\bigstar\mathcal{X},\ (4)\ (\mathcal{X}\bigstar\mathcal{A})^H = \mathcal{X}\bigstar\mathcal{A}, \tag{1.2}$$

is called the Moore-Penrose inverse of the tensor $\mathcal{A}$ and is denoted by $\mathcal{A}^\dagger$.

For any $\mathcal{A} \in \mathbb{C}^{n_1 \times n_2 \times n_3}$, denote $\mathcal{A}\{i, j, \dots, k\}$ the set of all $\mathcal{X} \in \mathbb{C}^{n_2 \times n_1 \times n_3}$ which satisfy equations $(i), (j), \dots, (k)$ of (1.2). In this case, $\mathcal{X}$ is a $\{i, j, \dots, k\}$-inverse of $\mathcal{A}$.

Definition 1.8. [6] *Let $\mathcal{A} \in \mathbb{C}^{n_1 \times n_1 \times n_3}$ and $ind(\mathcal{A}) = k$. The unique tensor $\mathcal{X} \in \mathbb{C}^{n_1 \times n_1 \times n_3}$ satisfying*

$$\mathcal{A}^k \bigstar \mathcal{X} \bigstar \mathcal{A} = \mathcal{A}^k, \ \mathcal{X} \bigstar \mathcal{A} \bigstar \mathcal{X} = \mathcal{X}, \ \mathcal{A} \bigstar \mathcal{X} = \mathcal{X} \bigstar \mathcal{A}, \tag{1.3}$$

is called the Drazin inverse of the tensor $\mathcal{A}$ and is denoted by $\mathcal{A}^D$. In particular, when $k = 1$, the tensor $\mathcal{X}$ is called the group inverse of $\mathcal{A}$ and is denoted by $\mathcal{A}^{\#}$.

For the definition of the core inverse of a tensor, we generalize Theorem 2.14 in [18] and obtain the following definition. We also indicate that the core inverse we defined has similar forms as the core inverse under the Einstein product studied by Sahoo et al. [20].

Definition 1.9. *Let $\mathcal{A} \in \mathbb{C}^{n_1 \times n_1 \times n_3}$. The unique tensor $\mathcal{X} \in \mathbb{C}^{n_1 \times n_1 \times n_3}$ is the core inverse of $\mathcal{A}$ if and only if $\mathcal{X}$ satisfies that*

$$(\mathcal{A} \bigstar \mathcal{X})^H = \mathcal{A} \bigstar \mathcal{X}, \ \mathcal{X} \bigstar \mathcal{A}^2 = \mathcal{A}, \ \mathcal{A} \bigstar \mathcal{X}^2 = \mathcal{X}. \tag{1.4}$$

The core inverse of $\mathcal{A}$ is denoted by $\mathcal{A}^{\textcircled{\#}}$.

1.3. The Generalized Schur Complement of Tensors

Let $\mathcal{M} \in \mathbb{C}^{(m_1+m_2) \times (n_1+n_2) \times p}$. Then, $\mathcal{M}$ can be partitioned as the following form

$$\mathcal{M} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \in \mathbb{C}^{(m_1+m_2) \times (n_1+n_2) \times p}, \tag{1.5}$$

where $\mathcal{A} \in \mathbb{C}^{m_1 \times n_1 \times p}$, $\mathcal{B} \in \mathbb{C}^{m_1 \times n_2 \times p}$, $\mathcal{C} \in \mathbb{C}^{m_2 \times n_1 \times p}$, $\mathcal{D} \in \mathbb{C}^{m_2 \times n_2 \times p}$.

The operations of partitioned tensors are given as follows.

Lemma 1.10. *The following statements are true.*

(1) $\mathcal{X} \bigstar \begin{bmatrix} \mathcal{A} & \mathcal{B} \end{bmatrix} = \begin{bmatrix} \mathcal{X} \bigstar \mathcal{A} & \mathcal{X} \bigstar \mathcal{B} \end{bmatrix}$, *where* $\mathcal{A} \in \mathbb{C}^{m_1 \times n_1 \times p}$, $\mathcal{B} \in \mathbb{C}^{m_1 \times n_2 \times p}$, $\mathcal{X} \in \mathbb{C}^{a \times m_1 \times p}$.

(2) $\begin{bmatrix} \mathcal{C} \\ \mathcal{D} \end{bmatrix} \bigstar \mathcal{Y} = \begin{bmatrix} \mathcal{C} \bigstar \mathcal{Y} \\ \mathcal{D} \bigstar \mathcal{Y} \end{bmatrix}$, *where* $\mathcal{C} \in \mathbb{C}^{m_1 \times n_1 \times p}$, $\mathcal{D} \in \mathbb{C}^{m_2 \times n_1 \times p}$, $\mathcal{Y} \in \mathbb{C}^{n_1 \times a \times p}$.

(3) $\begin{bmatrix} \mathcal{A} & \mathcal{B} \end{bmatrix} \bigstar \begin{bmatrix} \mathcal{C} \\ \mathcal{D} \end{bmatrix} = \mathcal{A} \bigstar \mathcal{C} + \mathcal{B} \bigstar \mathcal{D}$, *where* $\mathcal{A} \in \mathbb{C}^{m_1 \times n_1 \times p}$, $\mathcal{B} \in \mathbb{C}^{m_1 \times n_2 \times p}$, $\mathcal{C} \in \mathbb{C}^{n_1 \times m_2 \times p}$, $\mathcal{D} \in \mathbb{C}^{n_2 \times m_2 \times p}$.

(4) $\begin{bmatrix} \mathcal{A} \\ \mathcal{B} \end{bmatrix} \bigstar \begin{bmatrix} \mathcal{C} & \mathcal{D} \end{bmatrix} = \begin{bmatrix} \mathcal{A} \bigstar \mathcal{C} & \mathcal{A} \bigstar \mathcal{D} \\ \mathcal{B} \bigstar \mathcal{C} & \mathcal{B} \bigstar \mathcal{D} \end{bmatrix}$, *where* $\mathcal{A} \in \mathbb{C}^{m_1 \times n_1 \times p}$, $\mathcal{B} \in \mathbb{C}^{m_2 \times n_1 \times p}$, $\mathcal{C} \in \mathbb{C}^{n_1 \times m_1 \times p}$, $\mathcal{D} \in \mathbb{C}^{n_1 \times m_2 \times p}$.

(5) $\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \bigstar \begin{bmatrix} \mathcal{E} \\ \mathcal{F} \end{bmatrix} = \begin{bmatrix} \mathcal{A} \bigstar \mathcal{E} + \mathcal{B} \bigstar \mathcal{F} \\ \mathcal{C} \bigstar \mathcal{E} + \mathcal{D} \bigstar \mathcal{F} \end{bmatrix}$, *where* $\mathcal{A} \in \mathbb{C}^{m_1 \times n_1 \times p}$, $\mathcal{B} \in \mathbb{C}^{m_1 \times n_2 \times p}$, $\mathcal{C} \in \mathbb{C}^{m_2 \times n_1 \times p}$, $\mathcal{D} \in \mathbb{C}^{m_2 \times n_2 \times p}$, $\mathcal{E} \in \mathbb{C}^{n_1 \times m_1 \times p}$, $\mathcal{F} \in \mathbb{C}^{n_2 \times m_1 \times p}$.

(6) $\begin{bmatrix} \mathcal{G} & \mathcal{H} \end{bmatrix} \bigstar \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} = [\mathcal{G} \bigstar \mathcal{A} + \mathcal{H} \bigstar \mathcal{C} \ \ \mathcal{G} \bigstar \mathcal{B} + \mathcal{H} \bigstar \mathcal{D}]$, *where* $\mathcal{A} \in \mathbb{C}^{m_1 \times n_1 \times p}$, $\mathcal{B} \in \mathbb{C}^{m_1 \times n_2 \times p}$, $\mathcal{C} \in \mathbb{C}^{m_2 \times n_1 \times p}$, $\mathcal{D} \in \mathbb{C}^{m_2 \times n_2 \times p}$, $\mathcal{G} \in \mathbb{C}^{n_1 \times m_1 \times p}$, $\mathcal{H} \in \mathbb{C}^{n_1 \times m_2 \times p}$.

(7) $\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \bigstar \begin{bmatrix} \mathcal{E} & \mathcal{F} \\ \mathcal{G} & \mathcal{H} \end{bmatrix} = \begin{bmatrix} \mathcal{A} \bigstar \mathcal{E} + \mathcal{B} \bigstar \mathcal{G} & \mathcal{A} \bigstar \mathcal{F} + \mathcal{B} \bigstar \mathcal{H} \\ \mathcal{C} \bigstar \mathcal{E} + \mathcal{D} \bigstar \mathcal{G} & \mathcal{C} \bigstar \mathcal{F} + \mathcal{D} \bigstar \mathcal{H} \end{bmatrix}$, *where* $\mathcal{A} \in \mathbb{C}^{m_1 \times n_1 \times p}$, $\mathcal{B} \in \mathbb{C}^{m_1 \times n_2 \times p}$, $\mathcal{C} \in \mathbb{C}^{m_2 \times n_1 \times p}$, $\mathcal{D} \in \mathbb{C}^{m_2 \times n_2 \times p}$, $\mathcal{E} \in \mathbb{C}^{n_1 \times m_1 \times p}$, $\mathcal{F} \in \mathbb{C}^{n_1 \times m_2 \times p}$, $\mathcal{G} \in \mathbb{C}^{n_2 \times m_1 \times p}$, $\mathcal{H} \in \mathbb{C}^{n_2 \times m_2 \times p}$.

Proof. (1) By Lemma 1.2,

$$\begin{aligned} \mathcal{X}\bigstar\begin{bmatrix} \mathcal{A} & \mathcal{B} \end{bmatrix} &= L^{-1}(L(\mathcal{X})\triangle L(\begin{bmatrix} \mathcal{A} & \mathcal{B} \end{bmatrix})) \\ &= \begin{bmatrix} L^{-1}(L(\mathcal{X})\triangle L(\mathcal{A})) & L^{-1}(L(\mathcal{X})\triangle L(\mathcal{B})) \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{X}\bigstar\mathcal{A} & \mathcal{X}\bigstar\mathcal{B} \end{bmatrix}. \end{aligned}$$

(2)-(7) follow similarly. □

The Schur complements and generalized Schur complements, were studied by a lot of researchers, and have applications in matrix theory, statistics, electrical network theory, discrete-time regulator problem, sophisticated techniques and some other fields. So, it is natural to have a deep study on the Schur complements of tensors.

Let

$$\mathcal{M} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \in \mathbb{C}^{(m_1+m_2)\times(n_1+n_2)\times p}, \tag{1.6}$$

where $\mathcal{A} \in \mathbb{C}^{m_1\times n_1\times p}$, $\mathcal{B} \in \mathbb{C}^{m_1\times n_2\times p}$, $\mathcal{C} \in \mathbb{C}^{m_2\times n_1\times p}$, $\mathcal{D} \in \mathbb{C}^{m_2\times n_2\times p}$.

When $m_1 = n_1$ and $\mathcal{A}$ is invertible, the Schur complement of $\mathcal{A}$ in $\mathcal{M}$ is defined by $\mathcal{S} = \mathcal{D} - \mathcal{C}\bigstar\mathcal{A}^{-1}\bigstar\mathcal{B}$. Similarly, if $m_2 = n_2$ and $\mathcal{D}$ is invertible, then the Schur complement of $\mathcal{D}$ in $\mathcal{M}$ is defined by $\mathcal{T} = \mathcal{A} - \mathcal{B}\bigstar\mathcal{D}^{-1}\bigstar\mathcal{C}$.

Notice that if $\mathcal{A} \in \mathbb{C}^{m_1\times m_1\times p}$ is invertible, then $\mathcal{M}$ is invertible if and only if $\mathcal{S}$ is invertible. In which case, the inverse of $\mathcal{M}$ is given by

$$\mathcal{M}^{-1} = \begin{bmatrix} \mathcal{A}^{-1} + \mathcal{A}^{-1}\bigstar\mathcal{B}\bigstar\mathcal{S}^{-1}\bigstar\mathcal{C}\bigstar\mathcal{A}^{-1} & -\mathcal{A}^{-1}\bigstar\mathcal{B}\bigstar\mathcal{S}^{-1} \\ -\mathcal{S}^{-1}\bigstar\mathcal{C}\bigstar\mathcal{A}^{-1} & \mathcal{S}^{-1} \end{bmatrix} \in \mathbb{C}^{(m_1+m_2)\times(m_1+m_2)\times p},$$

which is called the Banachiewicz-Schur forms of the inverse of $\mathcal{M}$.

Indeed, if $\mathcal{A}$ is invertible, then the Schur complement of $\mathcal{A}$ in $\mathcal{M}$ is $\mathcal{S} = \mathcal{D} - \mathcal{C}\bigstar\mathcal{A}^{-1}\bigstar\mathcal{B}$. Since

$$\begin{bmatrix} \mathcal{I}_1 & \mathcal{O}_1 \\ -\mathcal{C}\bigstar\mathcal{A}^{-1} & \mathcal{I}_2 \end{bmatrix}\bigstar\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}\bigstar\begin{bmatrix} \mathcal{I}_1 & -\mathcal{A}^{-1}\bigstar\mathcal{B} \\ \mathcal{O}_2 & \mathcal{I}_2 \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{O}_1 \\ \mathcal{O}_2 & \mathcal{S} \end{bmatrix},$$

where $\mathcal{I}_1, \mathcal{I}_2$ are the identity tensors with proper size and $\mathcal{O}_1, \mathcal{O}_2$ are the zero tensors with proper size, we have $\mathcal{M}$ is invertible if and only if $\mathcal{S}$ is invertible. Therefore, the inverse of $\mathcal{M}$ is

$$\begin{aligned} \mathcal{M}^{-1} &= \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}^{-1} = \begin{bmatrix} \mathcal{I}_1 & -\mathcal{A}^{-1}\bigstar\mathcal{B} \\ \mathcal{O}_2 & \mathcal{I}_2 \end{bmatrix}\bigstar\begin{bmatrix} \mathcal{A}^{-1} & \mathcal{O}_1 \\ \mathcal{O}_2 & \mathcal{S}^{-1} \end{bmatrix}\bigstar\begin{bmatrix} \mathcal{I}_1 & \mathcal{O}_1 \\ -\mathcal{C}\bigstar\mathcal{A}^{-1} & \mathcal{I}_2 \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{A}^{-1} + \mathcal{A}^{-1}\bigstar\mathcal{B}\bigstar\mathcal{S}^{-1}\bigstar\mathcal{C}\bigstar\mathcal{A}^{-1} & -\mathcal{A}^{-1}\bigstar\mathcal{B}\bigstar\mathcal{S}^{-1} \\ -\mathcal{S}^{-1}\bigstar\mathcal{C}\bigstar\mathcal{A}^{-1} & \mathcal{S}^{-1} \end{bmatrix}. \end{aligned}$$

Dually, if $\mathcal{D} \in \mathbb{C}^{m_2\times m_2\times p}$ is invertible, then $\mathcal{M}$ is invertible if and only if $\mathcal{T}$ is invertible. In which case, $\mathcal{M}^{-1}$ can be expressed as

$$\mathcal{M}^{-1} = \begin{bmatrix} \mathcal{T}^{-1} & -\mathcal{T}^{-1}\bigstar\mathcal{B}\bigstar\mathcal{D}^{-1} \\ -\mathcal{D}^{-1}\bigstar\mathcal{C}\bigstar\mathcal{T}^{-1} & \mathcal{D}^{-1} + \mathcal{D}^{-1}\bigstar\mathcal{C}\bigstar\mathcal{T}^{-1}\bigstar\mathcal{B}\bigstar\mathcal{D}^{-1} \end{bmatrix}.$$

More general, when $\mathcal{A}$ is not invertible, for the tensor given by (1.6) and some fixed generalized inverse $\mathcal{A}^{-} \in \mathcal{A}\{1\}$, the generalized Schur complement of $\mathcal{A}$ in $\mathcal{M}$ is defined by

$$\mathcal{S} = \mathcal{D} - \mathcal{C}\bigstar\mathcal{A}^{-}\bigstar\mathcal{B}.$$

Similarly, for some fixed $\mathcal{D}^- \in \mathcal{D}\{1\}$, the generalized Schur complement of $\mathcal{D}$ in $\mathcal{M}$ is defined by

$$\mathcal{T} = \mathcal{A} - \mathcal{B}\bigstar\mathcal{D}^-\bigstar\mathcal{C}.$$

The work is organized as follows. In section 2, we give necessary and sufficient conditions to present $\{1\}, \{2\}, \{1,2\}, \{1,2,3\}, \{1,2,4\}$-inverses and the Moore-Penrose inverse of the 2×2 block tensor $\mathcal{M}$ in the Banachiewicz-Schur forms. In section 3, we present some results when the group and Drazin inverse of the block tensor can be represented in the generalized Schur form. In section 4, we establish several expressions of the core inverse of the block tensor. In section 5, we apply the results of section 4 to express the quotient property and the first Syslvester identity based on the core inverse of tensors.

2. The Moore-Penrose Inverse of the 2 × 2 Block Tensor

In this section, we will establish the necessary and sufficient conditions of the Banachiewicz-Schur forms of the generalized inverse of $\mathcal{M}$. In the following, we omit $\bigstar$ in the C-Product of two tensors.

Let $\mathcal{M} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \in \mathbb{C}^{(m_1+m_2)\times(n_1+n_2)\times p}$. Then,

$$\mathcal{X} = \begin{bmatrix} \mathcal{A}^- + \mathcal{A}^-\mathcal{B}\mathcal{S}^-\mathcal{C}\mathcal{A}^- & -\mathcal{A}^-\mathcal{B}\mathcal{S}^- \\ -\mathcal{S}^-\mathcal{C}\mathcal{A}^- & \mathcal{S}^- \end{bmatrix} \in \mathbb{C}^{(n_1+n_2)\times(m_1+m_2)\times p},$$

where $\mathcal{A}^- \in \mathcal{A}\{1\}$, $\mathcal{S}^- \in \mathcal{S}\{1\}$, $\mathcal{A}^- + \mathcal{A}^-\mathcal{B}\mathcal{S}^-\mathcal{C}\mathcal{A}^- \in \mathbb{C}^{n_1\times m_1\times p}$, $-\mathcal{A}^-\mathcal{B}\mathcal{S}^- \in \mathbb{C}^{n_1\times m_2\times p}$, $-\mathcal{S}^-\mathcal{C}\mathcal{A}^- \in \mathbb{C}^{n_2\times m_1\times p}$ and $\mathcal{S}^- \in \mathbb{C}^{n_2\times m_2\times p}$.

Now, we suppose that the sets $\mathcal{N}_1\{i,j,k\}$ and $\mathcal{N}_2\{i,j,k\}$ are expressed as the following.

$$\mathcal{N}_1\{i,j,k\} = \{\mathcal{X} = \begin{bmatrix} \mathcal{A}^- + \mathcal{A}^-\mathcal{B}\mathcal{S}^-\mathcal{C}\mathcal{A}^- & -\mathcal{A}^-\mathcal{B}\mathcal{S}^- \\ -\mathcal{S}^-\mathcal{C}\mathcal{A}^- & \mathcal{S}^- \end{bmatrix} : \ \mathcal{A}^- \in \mathcal{A}\{i,j,k\},\ \mathcal{S} = \mathcal{D} - \mathcal{C}\mathcal{A}^-\mathcal{B},\ \mathcal{S}^- \in \mathcal{S}\{i,j,k\}\}.$$

$$\mathcal{N}_2\{i,j,k\} = \{\mathcal{Y} = \begin{bmatrix} \mathcal{T}^- & -\mathcal{T}^-\mathcal{B}\mathcal{D}^- \\ -\mathcal{D}^-\mathcal{C}\mathcal{T}^- & \mathcal{D}^- + \mathcal{D}^-\mathcal{C}\mathcal{T}^-\mathcal{B}\mathcal{D}^- \end{bmatrix} : \ \mathcal{D}^- \in \mathcal{D}\{i,j,k\},\ \mathcal{T} = \mathcal{A} - \mathcal{B}\mathcal{D}^-\mathcal{C},\ \mathcal{T}^- \in \mathcal{T}\{i,j,k\}\}.$$

Denote $\Gamma_\alpha = \mathcal{I} - \alpha^-\alpha$ and $\Upsilon_\alpha = \mathcal{I} - \alpha\alpha^-$, where $\alpha = \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{S}, \mathcal{T}$, $\alpha^- \in \alpha\{i,j,k\}$.

Theorem 2.1. *Let* $\mathcal{M} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \in \mathbb{C}^{(m_1+m_2)\times(n_1+n_2)\times p}$. *Then,*

$$\mathcal{N}_1\{1\} \subseteq \mathcal{M}\{1\}$$

if and only if

$$\Upsilon_{\mathcal{A}}\mathcal{B}\Gamma_{\mathcal{S}} = 0, \quad \Upsilon_{\mathcal{S}}\mathcal{C}\Gamma_{\mathcal{A}} = 0, \quad \Upsilon_{\mathcal{A}}\mathcal{B}\mathcal{S}^-\mathcal{C}\Gamma_{\mathcal{A}} = 0,$$

for some $\mathcal{A}^- \in \mathcal{A}\{1\}$, $\mathcal{S}^- \in \mathcal{S}\{1\}$.

Proof. Let

$$\mathcal{X} = \begin{bmatrix} \mathcal{A}^- + \mathcal{A}^-\mathcal{B}\mathcal{S}^-\mathcal{C}\mathcal{A}^- & -\mathcal{A}^-\mathcal{B}\mathcal{S}^- \\ -\mathcal{S}^-\mathcal{C}\mathcal{A}^- & \mathcal{S}^- \end{bmatrix} \in \mathcal{N}_1\{1\},$$

where $\mathcal{A}^- \in \mathcal{A}\{1\}$, $\mathcal{S} = \mathcal{D} - \mathcal{C}\mathcal{A}^-\mathcal{B}$, $\mathcal{S}^- \in \mathcal{S}\{1\}$. Assume

$$\mathcal{M}\mathcal{X}\mathcal{M} = \mathcal{Z} = \begin{bmatrix} \mathcal{Z}_1 & \mathcal{Z}_2 \\ \mathcal{Z}_3 & \mathcal{Z}_4 \end{bmatrix}.$$

By some computations, we have

$$\mathcal{Z}_1 = \mathcal{A} + \Upsilon_{\mathcal{A}}\mathcal{B}\mathcal{S}^-\mathcal{C}\Gamma_{\mathcal{A}},\ \mathcal{Z}_2 = \mathcal{A}\mathcal{A}^-\mathcal{B} + \Upsilon_{\mathcal{A}}\mathcal{B}\mathcal{S}^-\mathcal{S},$$

$$\mathcal{Z}_3 = \mathcal{C}\mathcal{A}^-\mathcal{A} + \mathcal{S}\mathcal{S}^-\mathcal{C}\Gamma_{\mathcal{A}},\ \mathcal{Z}_4 = \mathcal{C}\mathcal{A}^-\mathcal{B} + \mathcal{S}.$$

Then,

$$\begin{aligned} \mathcal{X} \in \mathcal{M}\{1\},\ \text{that is } \mathcal{N}_1\{1\} \subseteq \mathcal{M}\{1\} \quad &\Leftrightarrow \quad \mathcal{Z}_1 = \mathcal{A},\ \mathcal{Z}_2 = \mathcal{B},\ \mathcal{Z}_3 = \mathcal{C},\ \mathcal{Z}_4 = \mathcal{D}. \\ &\Leftrightarrow \quad \Upsilon_{\mathcal{A}}\mathcal{B}\mathcal{S}^-\mathcal{C}\Gamma_{\mathcal{A}} = 0,\ \Upsilon_{\mathcal{A}}\mathcal{B}\Gamma_{\mathcal{S}} = 0,\ \Upsilon_{\mathcal{S}}\mathcal{C}\Gamma_{\mathcal{A}} = 0. \end{aligned}$$

□

Theorem 2.2. *Let* $\mathcal{M} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \in \mathbb{C}^{(m_1+m_2)\times(n_1+n_2)\times p}$. *Then*

$$\mathcal{N}_1\{2\} \subseteq \mathcal{M}\{2\}.$$

Proof. Suppose

$$\mathcal{X} = \begin{bmatrix} \mathcal{A}^- + \mathcal{A}^-\mathcal{B}\mathcal{S}^-\mathcal{C}\mathcal{A}^- & -\mathcal{A}^-\mathcal{B}\mathcal{S}^- \\ -\mathcal{S}^-\mathcal{C}\mathcal{A}^- & \mathcal{S}^- \end{bmatrix} \in \mathcal{N}_1\{2\},$$

where $\mathcal{A}^- \in \mathcal{A}\{2\}$, $\mathcal{S} = \mathcal{D} - \mathcal{C}\mathcal{A}^-\mathcal{B}$, $\mathcal{S}^- \in \mathcal{S}\{2\}$. Let

$$\mathcal{X}\mathcal{M}\mathcal{X} = \mathcal{Z} = \begin{bmatrix} \mathcal{Z}_1 & \mathcal{Z}_2 \\ \mathcal{Z}_3 & \mathcal{Z}_4 \end{bmatrix}.$$

Some computations show that

$$\begin{aligned} \mathcal{X}\mathcal{M}\mathcal{X} &= \begin{bmatrix} \mathcal{A}^- + \mathcal{A}^-\mathcal{B}\mathcal{S}^-\mathcal{C}\mathcal{A}^- & -\mathcal{A}^-\mathcal{B}\mathcal{S}^- \\ -\mathcal{S}^-\mathcal{C}\mathcal{A}^- & \mathcal{S}^- \end{bmatrix} \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \begin{bmatrix} \mathcal{A}^- + \mathcal{A}^-\mathcal{B}\mathcal{S}^-\mathcal{C}\mathcal{A}^- & -\mathcal{A}^-\mathcal{B}\mathcal{S}^- \\ -\mathcal{S}^-\mathcal{C}\mathcal{A}^- & \mathcal{S}^- \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{A}^-\mathcal{A} - \mathcal{A}^-\mathcal{B}\mathcal{S}^-\mathcal{C}\Gamma_{\mathcal{A}} & \mathcal{A}^-\mathcal{B} - \mathcal{A}^-\mathcal{B}\mathcal{S}^-\mathcal{S} \\ \mathcal{S}^-\mathcal{C}\Gamma_{\mathcal{A}} & \mathcal{S}^-\mathcal{S} \end{bmatrix} \begin{bmatrix} \mathcal{A}^- + \mathcal{A}^-\mathcal{B}\mathcal{S}^-\mathcal{C}\mathcal{A}^- & -\mathcal{A}^-\mathcal{B}\mathcal{S}^- \\ -\mathcal{S}^-\mathcal{C}\mathcal{A}^- & \mathcal{S}^- \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{A}^- + \mathcal{A}^-\mathcal{B}\mathcal{S}^-\mathcal{C}\mathcal{A}^- & -\mathcal{A}^-\mathcal{B}\mathcal{S}^- \\ -\mathcal{S}^-\mathcal{C}\mathcal{A}^- & \mathcal{S}^- \end{bmatrix} = \mathcal{X}, \end{aligned}$$

that is $\mathcal{X} \in \mathcal{M}\{2\}$. Therefore, $\mathcal{N}_1\{2\} \subseteq \mathcal{M}\{2\}$. □

An immediate consequence of Theorems 2.1 and Theorems 2.2 is the following result.

Theorem 2.3. *Let* $\mathcal{M} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \in \mathbb{C}^{(m_1+m_2)\times(n_1+n_2)\times p}$. *Then*

$$\mathcal{N}_1\{1,2\} \subseteq \mathcal{M}\{1,2\}$$

if and only if

$$\Upsilon_{\mathcal{A}}\mathcal{B}\Gamma_{\mathcal{S}} = 0, \quad \Upsilon_{\mathcal{S}}\mathcal{C}\Gamma_{\mathcal{A}} = 0, \quad \Upsilon_{\mathcal{A}}\mathcal{B}\mathcal{S}^-\mathcal{C}\Gamma_{\mathcal{A}} = 0,$$

for some $\mathcal{A}^- \in \mathcal{A}\{1,2\}$, $\mathcal{S}^- \in \mathcal{S}\{1,2\}$.

Proof. It follows by Theorem 2.1 and Theorem 2.2. □

Theorem 2.4. *Let* $\mathcal{M} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \in \mathbb{C}^{(m_1+m_2)\times(n_1+n_2)\times p}$. *Then*

$$\mathcal{N}_1\{1,2,3\} \subseteq \mathcal{M}\{1,2,3\}$$

if and only if

$$\Upsilon_{\mathcal{A}}\mathcal{B} = 0, \quad \Upsilon_{\mathcal{S}}\mathcal{C} = 0,$$

for some $\mathcal{A}^- \in \mathcal{A}\{1,2,3\}$, $\mathcal{S}^- \in \mathcal{S}\{1,2,3\}$.

Proof. Let

$$\mathcal{X} = \begin{bmatrix} \mathcal{A}^- + \mathcal{A}^-\mathcal{B}\mathcal{S}^-\mathcal{C}\mathcal{A}^- & -\mathcal{A}^-\mathcal{B}\mathcal{S}^- \\ -\mathcal{S}^-\mathcal{C}\mathcal{A}^- & \mathcal{S}^- \end{bmatrix} \in \mathcal{N}_1\{1,2,3\},$$

where $\mathcal{A}^- \in \mathcal{A}\{1,2,3\}$, $\mathcal{S} = \mathcal{D} - \mathcal{C}\mathcal{A}^-\mathcal{B}$, $\mathcal{S}^- \in \mathcal{S}\{1,2,3\}$. By Theorem 2.3, $\mathcal{X} \in \mathcal{M}\{1,2\}$ if and only if

$$\Upsilon_{\mathcal{A}}\mathcal{B}\Gamma_{\mathcal{S}} = 0, \quad \Upsilon_{\mathcal{S}}\mathcal{C}\Gamma_{\mathcal{A}} = 0, \quad \Upsilon_{\mathcal{A}}\mathcal{B}\mathcal{S}^-\mathcal{C}\Gamma_{\mathcal{A}} = 0. \tag{2.1}$$

Therefore, $\mathcal{X} \in \mathcal{M}\{1,2,3\}$ if and only if $(\mathcal{M}\mathcal{X})^H = \mathcal{M}\mathcal{X}$ and (2.1) holds. Now, we will compute $\mathcal{M}\mathcal{X}$. By some operations, it shows that

$$\begin{aligned} \mathcal{M}\mathcal{X} &= \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \begin{bmatrix} \mathcal{A}^- + \mathcal{A}^-\mathcal{B}\mathcal{S}^-\mathcal{C}\mathcal{A}^- & -\mathcal{A}^-\mathcal{B}\mathcal{S}^- \\ -\mathcal{S}^-\mathcal{C}\mathcal{A}^- & \mathcal{S}^- \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{A}\mathcal{A}^- - \Upsilon_{\mathcal{A}}\mathcal{B}\mathcal{S}^-\mathcal{C}\mathcal{A}^- & \Upsilon_{\mathcal{A}}\mathcal{B}\mathcal{S}^- \\ \Upsilon_{\mathcal{S}}\mathcal{C}\mathcal{A}^- & \mathcal{S}\mathcal{S}^- \end{bmatrix}. \end{aligned}$$

By $(\mathcal{M}\mathcal{X})^H = \mathcal{M}\mathcal{X}$, we have $\Upsilon_{\mathcal{S}}\mathcal{C}\mathcal{A}^- = (\Upsilon_{\mathcal{A}}\mathcal{B}\mathcal{S}^-)^H$. Postmultiplying $\mathcal{A}\mathcal{A}^-$ from the both sides of this equation leads to $\Upsilon_{\mathcal{S}}\mathcal{C}\mathcal{A}^- = 0$.

Again using $(\Upsilon_{\mathcal{S}}\mathcal{C}\mathcal{A}^-)^H = \Upsilon_{\mathcal{A}}\mathcal{B}\mathcal{S}^-$, postmultiplying $\mathcal{S}\mathcal{S}^-$ from the both sides of the equation gives $\Upsilon_{\mathcal{A}}\mathcal{B}\mathcal{S}^- = 0$. This means $(\mathcal{M}\mathcal{X})^H = \mathcal{M}\mathcal{X}$ if and only if

$$\Upsilon_{\mathcal{S}}\mathcal{C}\mathcal{A}^- = 0 \text{ and } \Upsilon_{\mathcal{A}}\mathcal{B}\mathcal{S}^- = 0. \tag{2.2}$$

Therefore, $\mathcal{X} \in \mathcal{M}\{1,2,3\}$ if and only if (2.1) and (2.2) hold. By (2.1) and (2.2), since $\Upsilon_{\mathcal{A}}\mathcal{B}\Gamma_{\mathcal{S}} = \Upsilon_{\mathcal{A}}\mathcal{B}\mathcal{S}^- = 0$, one has

$$\Upsilon_{\mathcal{A}}\mathcal{B} = \Upsilon_{\mathcal{A}}\mathcal{B}\Gamma_{\mathcal{S}} + \Upsilon_{\mathcal{A}}\mathcal{B}\mathcal{S}^-\mathcal{S} = 0 + 0 = 0. \tag{2.3}$$

Similarly, $\Upsilon_{\mathcal{S}}\mathcal{C} = 0$. Conversely, if $\Upsilon_{\mathcal{A}}\mathcal{B} = 0$ and $\Upsilon_{\mathcal{S}}\mathcal{C} = 0$, then (2.1) and (2.2) are true. Hence,

$$\mathcal{N}_1\{1,2,3\} \subseteq \mathcal{M}\{1,2,3\}$$

if and only if

$$\Upsilon_{\mathcal{A}}\mathcal{B} = 0, \quad \Upsilon_{\mathcal{S}}\mathcal{C} = 0,$$

for some $\mathcal{A}^- \in \mathcal{A}\{1,2,3\}$, $\mathcal{S}^- \in \mathcal{S}\{1,2,3\}$. □

In a similar way, we have the following theorem concerning the $\{1,2,4\}$-inverse of $\mathcal{M}$.

Theorem 2.5. *Let* $\mathcal{M} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \in \mathbb{C}^{(m_1+m_2)\times(n_1+n_2)\times p}$. *Then*

$$\mathcal{N}_1\{1,2,4\} \subseteq \mathcal{M}\{1,2,4\}$$

if and only if

$$\mathcal{B}\Gamma_{\mathcal{S}} = 0, \quad \mathcal{C}\Gamma_{\mathcal{A}} = 0,$$

for some $\mathcal{A}^- \in \mathcal{A}\{1,2,4\}$, $\mathcal{S}^- \in \mathcal{S}\{1,2,4\}$.

By using Theorem 2.4 and Theorem 2.5, we get the following desired result.

Theorem 2.6. *Let* $\mathcal{M} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \in \mathbb{C}^{(m_1+m_2)\times(n_1+n_2)\times p}$. *Then*

$$\mathcal{M}^\dagger = \begin{bmatrix} \mathcal{A}^\dagger + \mathcal{A}^\dagger\mathcal{B}\mathcal{S}^\dagger\mathcal{C}\mathcal{A}^\dagger & -\mathcal{A}^\dagger\mathcal{B}\mathcal{S}^\dagger \\ -\mathcal{S}^\dagger\mathcal{C}\mathcal{A}^\dagger & \mathcal{S}^\dagger \end{bmatrix} \tag{2.4}$$

if and only if

$$\Upsilon_{\mathcal{A}}\mathcal{B} = 0, \quad \mathcal{C}\Gamma_{\mathcal{A}} = 0, \quad \mathcal{B}\Gamma_{\mathcal{S}} = 0, \quad \Upsilon_{\mathcal{S}}\mathcal{C} = 0,$$

where $\mathcal{A}^- = \mathcal{A}^\dagger$, $\mathcal{S}^- = \mathcal{S}^\dagger$, $\mathcal{S} = \mathcal{D} - \mathcal{C}\mathcal{A}^\dagger\mathcal{B}$.

Dually, we have the following theorem.

Theorem 2.7. *Let* $\mathcal{M} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \in \mathbb{C}^{(m_1+m_2)\times(n_1+n_2)\times p}$. *Then,*

$$\mathcal{M}^{\dagger} = \begin{bmatrix} \mathcal{T}^{\dagger} & -\mathcal{T}^{\dagger}\mathcal{B}\mathcal{D}^{\dagger} \\ -\mathcal{D}^{\dagger}\mathcal{C}\mathcal{T}^{\dagger} & \mathcal{D}^{\dagger} + \mathcal{D}^{\dagger}\mathcal{C}\mathcal{T}^{\dagger}\mathcal{B}\mathcal{D}^{\dagger} \end{bmatrix} \tag{2.5}$$

if and only if

$$\Upsilon_{\mathcal{D}}\mathcal{C} = 0, \quad \mathcal{B}\Gamma_{\mathcal{D}} = 0, \quad \mathcal{C}\Gamma_{\mathcal{T}} = 0, \quad \Upsilon_{\mathcal{T}}\mathcal{B} = 0,$$

where $\mathcal{D}^{-} = \mathcal{D}^{\dagger}$, $\mathcal{T}^{-} = \mathcal{T}^{\dagger}$, $\mathcal{T} = \mathcal{A} - \mathcal{B}\mathcal{D}^{\dagger}\mathcal{C}$.

3. The Drazin Inverse of the 2 × 2 Block Tensor

In this section, we will present some results which characterizes when the group and Drazin inverse of a partitioned tensor can be expressed in the Banachiewicz-Schur forms.

Theorem 3.1. *Let* $\mathcal{M} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \in \mathbb{C}^{(m_1+m_2)\times(m_1+m_2)\times p}$. *Then,*

$$\mathcal{M}^{\#} = \begin{bmatrix} \mathcal{A}^{\#} + \mathcal{A}^{\#}\mathcal{B}\mathcal{S}^{\#}\mathcal{C}\mathcal{A}^{\#} & -\mathcal{A}^{\#}\mathcal{B}\mathcal{S}^{\#} \\ -\mathcal{S}^{\#}\mathcal{C}\mathcal{A}^{\#} & \mathcal{S}^{\#} \end{bmatrix} \tag{3.1}$$

if and only if

$$\Upsilon_{\mathcal{A}}\mathcal{B} = 0, \quad \mathcal{C}\Gamma_{\mathcal{A}} = 0, \quad \mathcal{B}\Gamma_{\mathcal{S}} = 0, \quad \Upsilon_{\mathcal{S}}\mathcal{C} = 0,$$

where $\mathcal{A}^{-} = \mathcal{A}^{\#}$, $\mathcal{S}^{-} = \mathcal{S}^{\#}$, $\mathcal{S} = \mathcal{D} - \mathcal{C}\mathcal{A}^{\#}\mathcal{B}$.

Proof. Let

$$\mathcal{X} = \begin{bmatrix} \mathcal{A}^{\#} + \mathcal{A}^{\#}\mathcal{B}\mathcal{S}^{\#}\mathcal{C}\mathcal{A}^{\#} & -\mathcal{A}^{\#}\mathcal{B}\mathcal{S}^{\#} \\ -\mathcal{S}^{\#}\mathcal{C}\mathcal{A}^{\#} & \mathcal{S}^{\#} \end{bmatrix}.$$

By Theorem 2.3, $\mathcal{X} \in \mathcal{M}\{1,2\}$ if and only if

$$\Upsilon_{\mathcal{A}}\mathcal{B}\Gamma_{\mathcal{S}} = 0, \quad \Upsilon_{\mathcal{S}}\mathcal{C}\Gamma_{\mathcal{A}} = 0, \quad \Upsilon_{\mathcal{A}}\mathcal{B}\mathcal{S}^{-}\mathcal{C}\Gamma_{\mathcal{A}} = 0. \tag{3.2}$$

where $\mathcal{A}^{-} = \mathcal{A}^{\#}$, $\mathcal{S}^{-} = \mathcal{S}^{\#}$, $\mathcal{S} = \mathcal{D} - \mathcal{C}\mathcal{A}^{\#}\mathcal{B}$. Furthermore, $\mathcal{X} = \mathcal{M}^{\#}$ if and only if $\mathcal{M}\mathcal{X} = \mathcal{X}\mathcal{M}$ and (3.2) holds. Simple computations show that

$$\mathcal{M}\mathcal{X} = \begin{bmatrix} \mathcal{A}\mathcal{A}^{\#} - \Upsilon_{\mathcal{A}}\mathcal{B}\mathcal{S}^{\#}\mathcal{C}\mathcal{A}^{\#} & \Upsilon_{\mathcal{A}}\mathcal{B}\mathcal{S}^{\#} \\ \Upsilon_{\mathcal{S}}\mathcal{C}\mathcal{A}^{\#} & \mathcal{S}\mathcal{S}^{\#} \end{bmatrix}$$

and

$$\mathcal{X}\mathcal{M} = \begin{bmatrix} \mathcal{A}^{\#}\mathcal{A} - \mathcal{A}^{\#}\mathcal{B}\mathcal{S}^{\#}\mathcal{C}\Gamma_{\mathcal{A}} & \mathcal{A}^{\#}\mathcal{B}\Gamma_{\mathcal{S}} \\ \mathcal{S}^{\#}\mathcal{C}\Gamma_{\mathcal{A}} & \mathcal{S}^{\#}\mathcal{S} \end{bmatrix}.$$

If $\mathcal{M}\mathcal{X} = \mathcal{X}\mathcal{M}$, one has $\Upsilon_{\mathcal{A}}\mathcal{B}\mathcal{S}^{\#} = \mathcal{A}^{\#}\mathcal{B}\Gamma_{\mathcal{S}}$. Postmultiplication by $\mathcal{S}$ from both sides of the equation gives $\Upsilon_{\mathcal{A}}\mathcal{B}\mathcal{S}^{\#}\mathcal{S} = \mathcal{A}^{\#}\mathcal{B}\Gamma_{\mathcal{S}}\mathcal{S} = 0$. Premultiplication by $\mathcal{A}$ from both sides of the equation gives $\mathcal{A}\Upsilon_{\mathcal{A}}\mathcal{B}\mathcal{S}^{\#} = \mathcal{A}\mathcal{A}^{\#}\mathcal{B}\Gamma_{\mathcal{S}} = 0$. Notice that

$$\Upsilon_{\mathcal{A}}\mathcal{B}\Gamma_{\mathcal{S}} = \Upsilon_{\mathcal{A}}\mathcal{B} - \Upsilon_{\mathcal{A}}\mathcal{B}\mathcal{S}^{\#}\mathcal{S} = \mathcal{B}\Gamma_{\mathcal{S}} - \mathcal{A}\mathcal{A}^{\#}\mathcal{B}\Gamma_{\mathcal{S}} = 0.$$

Thus, $\Upsilon_{\mathcal{A}}\mathcal{B} = 0$ and $\mathcal{B}\Gamma_{\mathcal{S}} = 0$. Similarly, by $\Upsilon_{\mathcal{S}}\mathcal{C}\mathcal{A}^{\#} = \mathcal{S}^{\#}\mathcal{C}\Gamma_{\mathcal{A}}$ and $\Upsilon_{\mathcal{S}}\mathcal{C}\Gamma_{\mathcal{A}} = 0$, one has

$$\mathcal{C}\Gamma_{\mathcal{A}} = 0, \text{ and } \Upsilon_{\mathcal{S}}\mathcal{C} = 0.$$

Conversely, if

$$\Upsilon_{\mathcal{A}}\mathcal{B} = 0, \quad \mathcal{C}\Gamma_{\mathcal{A}} = 0, \quad \mathcal{B}\Gamma_{\mathcal{S}} = 0, \quad \Upsilon_{\mathcal{S}}\mathcal{C} = 0,$$

where $\mathcal{A}^{-} = \mathcal{A}^{\#}$, $\mathcal{S}^{-} = \mathcal{S}^{\#}$, $\mathcal{S} = \mathcal{D} - \mathcal{C}\mathcal{A}^{\#}\mathcal{B}$, it is clear that $\mathcal{M}\mathcal{X} = \mathcal{X}\mathcal{M}$ and (3.2) holds. Thus, the proof is finished. □

If $\mathcal{S} = \mathcal{D} - \mathcal{C}\mathcal{A}^{\#}\mathcal{B}$ is invertible, we can get the following result.

Corollary 3.2. *Let* $\mathcal{M} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \in \mathbb{C}^{(m_1+m_2)\times(m_1+m_2)\times p}$. *If* $\Upsilon_{\mathcal{A}}\mathcal{B} = 0$, $\mathcal{C}\Gamma_{\mathcal{A}} = 0$, *where* $\mathcal{A}^- = \mathcal{A}^{\#}$, *and* $\mathcal{S} = \mathcal{D} - \mathcal{C}\mathcal{A}^{\#}\mathcal{B}$ *is invertible, then*

$$\mathcal{M}^{\#} = \begin{bmatrix} \mathcal{A}^{\#} + \mathcal{A}^{\#}\mathcal{B}\mathcal{S}^{-1}\mathcal{C}\mathcal{A}^{\#} & -\mathcal{A}^{\#}\mathcal{B}\mathcal{S}^{-1} \\ -\mathcal{S}^{-1}\mathcal{C}\mathcal{A}^{\#} & \mathcal{S}^{-1} \end{bmatrix}.$$

Dually, we have the following theorems.

Theorem 3.3. *Let* $\mathcal{M} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \in \mathbb{C}^{(m_1+m_2)\times(m_1+m_2)\times p}$. *Then,*

$$\mathcal{M}^{\#} = \begin{bmatrix} \mathcal{T}^{\#} & -\mathcal{T}^{\#}\mathcal{B}\mathcal{D}^{\#} \\ -\mathcal{D}^{\#}\mathcal{C}\mathcal{T}^{\#} & \mathcal{D}^{\#} + \mathcal{D}^{\#}\mathcal{C}\mathcal{T}^{\#}\mathcal{B}\mathcal{D}^{\#} \end{bmatrix} \tag{3.3}$$

if and only if

$$\Upsilon_{\mathcal{D}}\mathcal{C} = 0, \quad \mathcal{B}\Gamma_{\mathcal{D}} = 0, \quad \mathcal{C}\Gamma_{\mathcal{T}} = 0, \quad \Upsilon_{\mathcal{T}}\mathcal{B} = 0,$$

where $\mathcal{D}^- = \mathcal{D}^{\#}$, $\mathcal{T}^- = \mathcal{T}^{\#}$, $\mathcal{T} = \mathcal{A} - \mathcal{B}\mathcal{D}^{\#}\mathcal{C}$.

Corollary 3.4. *Let* $\mathcal{M} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \in \mathbb{C}^{(m_1+m_2)\times(m_1+m_2)\times p}$. *If* $\Upsilon_{\mathcal{D}}\mathcal{C} = 0$, $\mathcal{B}\Gamma_{\mathcal{D}} = 0$, *where* $\mathcal{D}^- = \mathcal{D}^{\#}$, *and* $\mathcal{T} = \mathcal{A} - \mathcal{B}\mathcal{D}^{\#}\mathcal{C}$ *is invertible, then*

$$\mathcal{M}^{\#} = \begin{bmatrix} \mathcal{T}^{-1} & -\mathcal{T}^{-1}\mathcal{B}\mathcal{D}^{\#} \\ -\mathcal{D}^{\#}\mathcal{C}\mathcal{T}^{-1} & \mathcal{D}^{\#} + \mathcal{D}^{\#}\mathcal{C}\mathcal{T}^{-1}\mathcal{B}\mathcal{D}^{\#} \end{bmatrix}.$$

If we combine Theorem 3.1 and Theorem 3.3, a new result can be obtained.

Theorem 3.5. *Let* $\mathcal{M} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \in \mathbb{C}^{(m_1+m_2)\times(m_1+m_2)\times p}$. *Then*

$$\mathcal{M}^{\#} = \begin{bmatrix} \mathcal{A}^{\#} + \mathcal{A}^{\#}\mathcal{B}\mathcal{S}^{\#}\mathcal{C}\mathcal{A}^{\#} & -\mathcal{A}^{\#}\mathcal{B}\mathcal{S}^{\#} \\ -\mathcal{S}^{\#}\mathcal{C}\mathcal{A}^{\#} & \mathcal{S}^{\#} \end{bmatrix} = \begin{bmatrix} \mathcal{T}^{\#} & -\mathcal{T}^{\#}\mathcal{B}\mathcal{D}^{\#} \\ -\mathcal{D}^{\#}\mathcal{C}\mathcal{T}^{\#} & \mathcal{D}^{\#} + \mathcal{D}^{\#}\mathcal{C}\mathcal{T}^{\#}\mathcal{B}\mathcal{D}^{\#} \end{bmatrix}$$

if and only if one of the following condition holds.

(*i*) $\Upsilon_{\mathcal{A}}\mathcal{B} = 0,\ \Upsilon_{\mathcal{S}}\mathcal{C} = 0,\ \Upsilon_{\mathcal{D}}\mathcal{C} = 0,\ \mathcal{B}\Gamma_{\mathcal{S}} = 0,\ \mathcal{C}\Gamma_{\mathcal{A}} = 0,\ \mathcal{B}\Gamma_{\mathcal{D}} = 0,$ (3.4)

(*ii*) $\Upsilon_{\mathcal{A}}\mathcal{B} = 0,\ \Upsilon_{\mathcal{T}}\mathcal{B} = 0,\ \Upsilon_{\mathcal{D}}\mathcal{C} = 0,\ \mathcal{B}\Gamma_{\mathcal{D}} = 0,\ \mathcal{C}\Gamma_{\mathcal{A}} = 0,\ \mathcal{C}\Gamma_{\mathcal{T}} = 0,$ (3.5)

where $\mathcal{A}^- = \mathcal{A}^{\#}$, $\mathcal{S}^- = \mathcal{S}^{\#}$, $\mathcal{D}^- = \mathcal{D}^{\#}$, $\mathcal{T}^- = \mathcal{T}^{\#}$, $\mathcal{S} = \mathcal{D} - \mathcal{C}\mathcal{A}^{\#}\mathcal{B}$, $\mathcal{T} = \mathcal{A} - \mathcal{B}\mathcal{D}^{\#}\mathcal{C}$.

Proof. (i) By Theorem 3.1 and Theorem 3.3,

$$\mathcal{M}^{\#} = \begin{bmatrix} \mathcal{A}^{\#} + \mathcal{A}^{\#}\mathcal{B}\mathcal{S}^{\#}\mathcal{C}\mathcal{A}^{\#} & -\mathcal{A}^{\#}\mathcal{B}\mathcal{S}^{\#} \\ -\mathcal{S}^{\#}\mathcal{C}\mathcal{A}^{\#} & \mathcal{S}^{\#} \end{bmatrix} = \begin{bmatrix} \mathcal{T}^{\#} & -\mathcal{T}^{\#}\mathcal{B}\mathcal{D}^{\#} \\ -\mathcal{D}^{\#}\mathcal{C}\mathcal{T}^{\#} & \mathcal{D}^{\#} + \mathcal{D}^{\#}\mathcal{C}\mathcal{T}^{\#}\mathcal{B}\mathcal{D}^{\#} \end{bmatrix}$$

if and only if

$$\Upsilon_{\mathcal{A}}\mathcal{B} = 0,\ \Upsilon_{\mathcal{T}}\mathcal{B} = 0,\ \Upsilon_{\mathcal{S}}\mathcal{C} = 0,\ \Upsilon_{\mathcal{D}}\mathcal{C} = 0,\ \mathcal{B}\Gamma_{\mathcal{S}} = 0,\ \mathcal{C}\Gamma_{\mathcal{A}} = 0,\ \mathcal{B}\Gamma_{\mathcal{D}} = 0,\ \mathcal{C}\Gamma_{\mathcal{T}} = 0. \tag{3.6}$$

Now, we only need to prove (3.6) is equivalent to (3.4). It is clearly (3.6) implies (3.4). Now, we prove the reverse part.

Denoting $\mathcal{T}' = \mathcal{A}^{\#} + \mathcal{A}^{\#}\mathcal{B}\mathcal{S}^{\#}\mathcal{C}\mathcal{A}^{\#}$. By $\Upsilon_{\mathcal{A}}\mathcal{B} = 0, \Upsilon_{\mathcal{S}}\mathcal{C} = 0,\ \mathcal{B}\Gamma_{\mathcal{D}} = 0$, we have

$$\begin{aligned}
\mathcal{T}\mathcal{T}' &= (\mathcal{A} - \mathcal{B}\mathcal{D}^{\#}\mathcal{C})(\mathcal{A}^{\#} + \mathcal{A}^{\#}\mathcal{B}\mathcal{S}^{\#}\mathcal{C}\mathcal{A}^{\#})\\
&= \mathcal{A}\mathcal{A}^{\#} + \mathcal{A}\mathcal{A}^{\#}\mathcal{B}\mathcal{S}^{\#}\mathcal{C}\mathcal{A}^{\#} - \mathcal{B}\mathcal{D}^{\#}\mathcal{C}\mathcal{A}^{\#} - \mathcal{B}\mathcal{D}^{\#}\mathcal{C}\mathcal{A}^{\#}\mathcal{B}\mathcal{S}^{\#}\mathcal{C}\mathcal{A}^{\#}\\
&= \mathcal{A}\mathcal{A}^{\#} + \mathcal{B}\mathcal{S}^{\#}\mathcal{C}\mathcal{A}^{\#} - \mathcal{B}\mathcal{D}^{\#}\mathcal{C}\mathcal{A}^{\#} - \mathcal{B}\mathcal{D}^{\#}(\mathcal{D} - \mathcal{S})\mathcal{S}^{\#}\mathcal{C}\mathcal{A}^{\#}\\
&= \mathcal{A}\mathcal{A}^{\#}
\end{aligned}$$

and by $\Upsilon_{\mathcal{D}}\mathcal{C} = 0,\ \mathcal{B}\Gamma_{\mathcal{S}} = 0,\ \mathcal{C}\Gamma_{\mathcal{A}} = 0$, we have

$$\begin{aligned}
\mathcal{T}'\mathcal{T} &= (\mathcal{A}^{\#} + \mathcal{A}^{\#}\mathcal{B}\mathcal{S}^{\#}\mathcal{C}\mathcal{A}^{\#})(\mathcal{A} - \mathcal{B}\mathcal{D}^{\#}\mathcal{C})\\
&= \mathcal{A}^{\#}\mathcal{A} - \mathcal{A}^{\#}\mathcal{B}\mathcal{D}^{\#}\mathcal{C} + \mathcal{A}^{\#}\mathcal{B}\mathcal{S}^{\#}\mathcal{C}\mathcal{A}^{\#}\mathcal{A} - \mathcal{A}^{\#}\mathcal{B}\mathcal{S}^{\#}\mathcal{C}\mathcal{A}^{\#}\mathcal{B}\mathcal{D}^{\#}\mathcal{C}\\
&= \mathcal{A}^{\#}\mathcal{A} - \mathcal{A}^{\#}\mathcal{B}\mathcal{D}^{\#}\mathcal{C} + \mathcal{A}^{\#}\mathcal{B}\mathcal{S}^{\#}\mathcal{C} - \mathcal{A}^{\#}\mathcal{B}\mathcal{S}^{\#}(\mathcal{D} - \mathcal{S})\mathcal{D}^{\#}\mathcal{C}\\
&= \mathcal{A}^{\#}\mathcal{A}.
\end{aligned}$$

Now, it is easy to get

$$\mathcal{T}\mathcal{T}'\mathcal{T} = \mathcal{A}\mathcal{A}^{\#}(\mathcal{A} - \mathcal{B}\mathcal{D}^{\#}\mathcal{C}) = \mathcal{A} - \mathcal{B}\mathcal{D}^{\#}\mathcal{C} = \mathcal{T},$$

$$\mathcal{T}'\mathcal{T}\mathcal{T}' = \mathcal{A}^{\#}\mathcal{A}(\mathcal{A}^{\#} + \mathcal{A}^{\#}\mathcal{B}\mathcal{S}^{\#}\mathcal{C}\mathcal{A}^{\#}) = \mathcal{A}^{\#} + \mathcal{A}^{\#}\mathcal{B}\mathcal{S}^{\#}\mathcal{C}\mathcal{A}^{\#} = \mathcal{T}'.$$

Thus, $\mathcal{T}' = \mathcal{T}^{\#}$. Hence, $\mathcal{T}^{\#}\mathcal{T} = \mathcal{A}^{\#}\mathcal{A}$ and $\mathcal{T}\mathcal{T}^{\#} = \mathcal{A}\mathcal{A}^{\#}$. Now, we get $\Upsilon_{\mathcal{A}}\mathcal{B} = \Upsilon_{\mathcal{T}}\mathcal{B} = 0$ and $\mathcal{C}\Gamma_{\mathcal{A}} = \mathcal{C}\Gamma_{\mathcal{T}} = 0$, which means (3.4) implying (3.6). Thus, (3.4) is equivalent to (3.6).

(ii) The proof is similar to the proof of (*i*).

□

In the following, we give some expressions related to the Drazin inverse of the block tensors. Before that, we denote $\Pi_{\alpha} = \mathcal{I} - \alpha^{D}\alpha = \mathcal{I} - \alpha\alpha^{D}$, where $\alpha = \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{S}, \mathcal{T}$.

Theorem 3.6. *Let* $\mathcal{M} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \in \mathbb{C}^{(m_1+m_2)\times(m_1+m_2)\times p}$. *If*

$$\Pi_{\mathcal{S}}\mathcal{C}\mathcal{A}^{D} = 0,\ \Pi_{\mathcal{A}}\mathcal{B}\mathcal{S}^{D} = 0,\ \mathcal{C}\Pi_{\mathcal{A}} = 0,\ \mathcal{B}\Pi_{\mathcal{S}} = 0,\ \Pi_{\mathcal{D}} = \Pi_{\mathcal{S}},$$

then

$$\mathcal{M}^{D} = \begin{bmatrix} \mathcal{A}^{D} + \mathcal{A}^{D}\mathcal{B}\mathcal{S}^{D}\mathcal{C}\mathcal{A}^{D} & -\mathcal{A}^{D}\mathcal{B}\mathcal{S}^{D} \\ -\mathcal{S}^{D}\mathcal{C}\mathcal{A}^{D} & \mathcal{S}^{D} \end{bmatrix}, \tag{3.7}$$

where $\mathcal{S} = \mathcal{D} - \mathcal{C}\mathcal{A}^{D}\mathcal{B}$.

Proof. Let

$$\mathcal{X} = \begin{bmatrix} \mathcal{A}^{D} + \mathcal{A}^{D}\mathcal{B}\mathcal{S}^{D}\mathcal{C}\mathcal{A}^{D} & -\mathcal{A}^{D}\mathcal{B}\mathcal{S}^{D} \\ -\mathcal{S}^{D}\mathcal{C}\mathcal{A}^{D} & \mathcal{S}^{D} \end{bmatrix}.$$

By $\Pi_{\mathcal{S}}\mathcal{C}\mathcal{A}^{D} = 0,\ \Pi_{\mathcal{A}}\mathcal{B}\mathcal{S}^{D} = 0$, we have

$$\mathcal{M}\mathcal{X} = \mathcal{X}\mathcal{M} = \begin{bmatrix} \mathcal{A}\mathcal{A}^{D} & \mathcal{O} \\ \mathcal{O} & \mathcal{S}\mathcal{S}^{D} \end{bmatrix}.$$

Moreover,

$$\begin{aligned}
\mathcal{X}\mathcal{M}\mathcal{X} &= \begin{bmatrix} \mathcal{A}^{D} + \mathcal{A}^{D}\mathcal{B}\mathcal{S}^{D}\mathcal{C}\mathcal{A}^{D} & -\mathcal{A}^{D}\mathcal{B}\mathcal{S}^{D} \\ -\mathcal{S}^{D}\mathcal{C}\mathcal{A}^{D} & \mathcal{S}^{D} \end{bmatrix}\begin{bmatrix} \mathcal{A}\mathcal{A}^{D} & \mathcal{O} \\ \mathcal{O} & \mathcal{S}\mathcal{S}^{D} \end{bmatrix}\\
&= \begin{bmatrix} \mathcal{A}^{D} + \mathcal{A}^{D}\mathcal{B}\mathcal{S}^{D}\mathcal{C}\mathcal{A}^{D} & -\mathcal{A}^{D}\mathcal{B}\mathcal{S}^{D} \\ -\mathcal{S}^{D}\mathcal{C}\mathcal{A}^{D} & \mathcal{S}^{D} \end{bmatrix} = \mathcal{X}.
\end{aligned}$$

Next, we will prove $\mathcal{M}^{k+1}\mathcal{X} = \mathcal{M}^k$. By $\mathcal{C}\Pi_{\mathcal{A}} = 0$, $\mathcal{B}\Pi_{\mathcal{S}} = 0$, $\Pi_{\mathcal{D}} = \Pi_{\mathcal{S}}$, we have

$$\begin{aligned}\mathcal{M}^2\mathcal{X} &= \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}\begin{bmatrix} \mathcal{A}\mathcal{A}^D & \mathcal{O} \\ \mathcal{O} & \mathcal{S}\mathcal{S}^D \end{bmatrix} = \begin{bmatrix} \mathcal{A}^2\mathcal{A}^D & \mathcal{B}\mathcal{S}\mathcal{S}^D \\ \mathcal{C}\mathcal{A}\mathcal{A}^D & \mathcal{D}\mathcal{S}\mathcal{S}^D \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} - \begin{bmatrix} \mathcal{A}(\mathcal{I}-\mathcal{A}\mathcal{A}^D) & \mathcal{O} \\ \mathcal{O} & \mathcal{D}(\mathcal{I}-\mathcal{D}\mathcal{D}^D) \end{bmatrix}.\end{aligned}$$

Furthermore,

$$\mathcal{M}^{k+1}\mathcal{X} = \mathcal{M}^k - \begin{bmatrix} \mathcal{A}^k(\mathcal{I}-\mathcal{A}\mathcal{A}^D) & \mathcal{O} \\ \mathcal{O} & \mathcal{D}^k(\mathcal{I}-\mathcal{D}\mathcal{D}^D) \end{bmatrix},$$

which implies that $\mathcal{M}^{k+1}\mathcal{X} = \mathcal{M}^k$, when $k \geq \max\{ind(A), ind(D)\}$. □

Similarly, we have the following result.

Theorem 3.7. *Let* $\mathcal{M} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \in \mathbb{C}^{(m_1+m_2)\times(m_1+m_2)\times p}$. *If*

$$\Pi_{\mathcal{T}}\mathcal{B}\mathcal{D}^D = 0,\ \Pi_{\mathcal{D}}\mathcal{C}\mathcal{T}^D = 0,\ \mathcal{B}\Pi_{\mathcal{D}} = 0,\ \mathcal{C}\Pi_{\mathcal{T}} = 0,\ \Pi_{\mathcal{A}} = \Pi_{\mathcal{T}},$$

then

$$\mathcal{M}^D = \begin{bmatrix} \mathcal{T}^D & -\mathcal{T}^D\mathcal{B}\mathcal{D}^D \\ -\mathcal{D}^D\mathcal{C}\mathcal{T}^D & \mathcal{D}^D + \mathcal{D}^D\mathcal{C}\mathcal{T}^D\mathcal{B}\mathcal{D}^D \end{bmatrix}, \tag{3.8}$$

where $\mathcal{T} = \mathcal{A} - \mathcal{B}\mathcal{D}^D\mathcal{C}$.

Combining the above two theorems, we can get a new theorem as follows.

Theorem 3.8. *Let* $\mathcal{M} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \in \mathbb{C}^{(m_1+m_2)\times(m_1+m_2)\times p}$. *If*

$$\Pi_{\mathcal{D}}\mathcal{C} = 0,\ \Pi_{\mathcal{A}}\mathcal{B} = 0,\ \mathcal{C}\Pi_{\mathcal{A}} = 0,\ \mathcal{B}\Pi_{\mathcal{D}} = 0,\ \Pi_{\mathcal{D}} = \Pi_{\mathcal{S}},\ \Pi_{\mathcal{A}} = \Pi_{\mathcal{T}},$$

then

$$\mathcal{M}^D = \begin{bmatrix} \mathcal{A}^D + \mathcal{A}^D\mathcal{B}\mathcal{S}^D\mathcal{C}\mathcal{A}^D & -\mathcal{A}^D\mathcal{B}\mathcal{S}^D \\ -\mathcal{S}^D\mathcal{C}\mathcal{A}^D & \mathcal{S}^D \end{bmatrix} = \begin{bmatrix} \mathcal{T}^D & -\mathcal{T}^D\mathcal{B}\mathcal{D}^D \\ -\mathcal{D}^D\mathcal{C}\mathcal{T}^D & \mathcal{D}^D + \mathcal{D}^D\mathcal{C}\mathcal{T}^D\mathcal{B}\mathcal{D}^D \end{bmatrix},$$

where $\mathcal{S} = \mathcal{D} - \mathcal{C}\mathcal{A}^D\mathcal{B}$, $\mathcal{T} = \mathcal{A} - \mathcal{B}\mathcal{D}^D\mathcal{C}$.

4. The Core Inverse of the 2 × 2 Block Tensor

In this part, we will establish several expressions of the core inverse of the partitioned tensor.

Theorem 4.1. *Let* $\mathcal{M} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \in \mathbb{C}^{(m_1+m_2)\times(m_1+m_2)\times p}$. *Then,*

$$\mathcal{M}^{\#} = \begin{bmatrix} \mathcal{A}^{\#} + \mathcal{A}^{\#}\mathcal{B}\mathcal{S}^{\#}\mathcal{C}\mathcal{A}^{\#} & -\mathcal{A}^{\#}\mathcal{B}\mathcal{S}^{\#} \\ -\mathcal{S}^{\#}\mathcal{C}\mathcal{A}^{\#} & \mathcal{S}^{\#} \end{bmatrix}$$

if and only if

$$\Upsilon_{\mathcal{A}}\mathcal{B}\mathcal{S}^{\#} = 0,\ \Upsilon_{\mathcal{S}}\mathcal{C}\mathcal{A}^{\#} = 0,\ \Gamma_{\mathcal{A}}\mathcal{B} = 0,\ \Gamma_{\mathcal{S}}\mathcal{C} = 0, \tag{4.1}$$

where $\mathcal{A}^- = \mathcal{A}^{\#}$, $\mathcal{S}^- = \mathcal{S}^{\#}$, $\mathcal{S} = \mathcal{D} - \mathcal{C}\mathcal{A}^{\#}\mathcal{B}$.

Proof. Let

$$\mathcal{X}=\begin{bmatrix}\mathcal{A}^{\textcircled{\#}}+\mathcal{A}^{\textcircled{\#}}\mathcal{B}\mathcal{S}^{\textcircled{\#}}\mathcal{C}\mathcal{A}^{\textcircled{\#}} & -\mathcal{A}^{\textcircled{\#}}\mathcal{B}\mathcal{S}^{\textcircled{\#}}\\ -\mathcal{S}^{\textcircled{\#}}\mathcal{C}\mathcal{A}^{\textcircled{\#}} & \mathcal{S}^{\textcircled{\#}}\end{bmatrix}.$$

If $\mathcal{X}$ is the core inverse of $\mathcal{M}$, then by the definition, we have

$$\mathcal{X}=\mathcal{M}^{\textcircled{\#}}\Leftrightarrow(\mathcal{M}\mathcal{X})^H=\mathcal{M}\mathcal{X},\ \mathcal{M}\mathcal{X}^2=\mathcal{X},\ \mathcal{X}\mathcal{M}^2=\mathcal{M}.$$

According to (2.2), we have

$$(\mathcal{M}\mathcal{X})^H=\mathcal{M}\mathcal{X}\Leftrightarrow\Upsilon_{\mathcal{S}}\mathcal{C}\mathcal{A}^{\textcircled{\#}}=0,\ \Upsilon_{\mathcal{A}}\mathcal{B}\mathcal{S}^{\textcircled{\#}}=0.$$

Since $\mathcal{X}\mathcal{M}^2=\mathcal{M}$, that is,

$$\begin{bmatrix}\mathcal{A}^{\textcircled{\#}}+\mathcal{A}^{\textcircled{\#}}\mathcal{B}\mathcal{S}^{\textcircled{\#}}\mathcal{C}\mathcal{A}^{\textcircled{\#}} & -\mathcal{A}^{\textcircled{\#}}\mathcal{B}\mathcal{S}^{\textcircled{\#}}\\ -\mathcal{S}^{\textcircled{\#}}\mathcal{C}\mathcal{A}^{\textcircled{\#}} & \mathcal{S}^{\textcircled{\#}}\end{bmatrix}\begin{bmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{C} & \mathcal{D}\end{bmatrix}^2=\begin{bmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{C} & \mathcal{D}\end{bmatrix},$$

we have

$$\begin{cases}\mathcal{A}=\mathcal{A}^{\textcircled{\#}}\mathcal{A}^2-\mathcal{A}^{\textcircled{\#}}\mathcal{B}\mathcal{S}^{\textcircled{\#}}\mathcal{C}(\mathcal{I}-\mathcal{A}^{\textcircled{\#}}\mathcal{A})\mathcal{A}+\mathcal{A}^{\textcircled{\#}}\mathcal{B}(\mathcal{I}-\mathcal{S}^{\textcircled{\#}}\mathcal{S})\mathcal{C},\\ \mathcal{B}=\mathcal{A}^{\textcircled{\#}}\mathcal{A}\mathcal{B}-\mathcal{A}^{\textcircled{\#}}\mathcal{B}\mathcal{S}^{\textcircled{\#}}\mathcal{C}(\mathcal{I}-\mathcal{A}^{\textcircled{\#}}\mathcal{A})\mathcal{B}+\mathcal{A}^{\textcircled{\#}}\mathcal{B}(\mathcal{I}-\mathcal{S}^{\textcircled{\#}}\mathcal{S})\mathcal{D},\\ \mathcal{C}=\mathcal{S}^{\textcircled{\#}}\mathcal{C}(\mathcal{I}-\mathcal{A}^{\textcircled{\#}}\mathcal{A})\mathcal{A}+\mathcal{S}^{\textcircled{\#}}\mathcal{S}\mathcal{C},\\ \mathcal{D}=\mathcal{S}^{\textcircled{\#}}\mathcal{C}(\mathcal{I}-\mathcal{A}^{\textcircled{\#}}\mathcal{A})\mathcal{B}+\mathcal{S}^{\textcircled{\#}}\mathcal{S}\mathcal{D}.\end{cases}\tag{4.2}$$

The third equality of (4.2) gives $\mathcal{C}=\mathcal{S}^{\textcircled{\#}}\mathcal{S}\mathcal{C}$. Substituting the fourth equality of (4.2) into the second equality, we will get $\mathcal{B}=\mathcal{A}^{\textcircled{\#}}\mathcal{A}\mathcal{B}$. Hence, we have $\Gamma_{\mathcal{A}}\mathcal{B}=0,\ \Gamma_{\mathcal{S}}\mathcal{C}=0$.

Conversely, if

$$\Upsilon_{\mathcal{A}}\mathcal{B}\mathcal{S}^{\textcircled{\#}}=0,\ \Upsilon_{\mathcal{S}}\mathcal{C}\mathcal{A}^{\textcircled{\#}}=0,\ \Gamma_{\mathcal{A}}\mathcal{B}=0,\ \Gamma_{\mathcal{S}}\mathcal{C}=0,$$

we will prove that

$$\mathcal{X}=\begin{bmatrix}\mathcal{A}^{\textcircled{\#}}+\mathcal{A}^{\textcircled{\#}}\mathcal{B}\mathcal{S}^{\textcircled{\#}}\mathcal{C}\mathcal{A}^{\textcircled{\#}} & -\mathcal{A}^{\textcircled{\#}}\mathcal{B}\mathcal{S}^{\textcircled{\#}}\\ -\mathcal{S}^{\textcircled{\#}}\mathcal{C}\mathcal{A}^{\textcircled{\#}} & \mathcal{S}^{\textcircled{\#}}\end{bmatrix}$$

is the core inverse of $\mathcal{M}$.

Since $\Upsilon_{\mathcal{A}}\mathcal{B}\mathcal{S}^{\textcircled{\#}}=0$ and $\Upsilon_{\mathcal{S}}\mathcal{C}\mathcal{A}^{\textcircled{\#}}=0$, we have $(\mathcal{M}\mathcal{X})^H=\mathcal{M}\mathcal{X}$ and

$$\mathcal{M}\mathcal{X}^2=\begin{bmatrix}\mathcal{A}\mathcal{A}^{\textcircled{\#}} & \mathcal{O}\\ \mathcal{O} & \mathcal{S}\mathcal{S}^{\textcircled{\#}}\end{bmatrix}\begin{bmatrix}\mathcal{A}^{\textcircled{\#}}+\mathcal{A}^{\textcircled{\#}}\mathcal{B}\mathcal{S}^{\textcircled{\#}}\mathcal{C}\mathcal{A}^{\textcircled{\#}} & -\mathcal{A}^{\textcircled{\#}}\mathcal{B}\mathcal{S}^{\textcircled{\#}}\\ -\mathcal{S}^{\textcircled{\#}}\mathcal{C}\mathcal{A}^{\textcircled{\#}} & \mathcal{S}^{\textcircled{\#}}\end{bmatrix}=\mathcal{X}.$$

Since $\Gamma_{\mathcal{A}}\mathcal{B}=0$, $\Gamma_{\mathcal{S}}\mathcal{C}=0$ and $\Gamma_{\mathcal{S}}\mathcal{D}=(\mathcal{I}-\mathcal{S}^{\textcircled{\#}}\mathcal{S})(\mathcal{S}+\mathcal{C}\mathcal{A}^{\textcircled{\#}}\mathcal{B})=0$, we have

$$\begin{aligned}\mathcal{X}\mathcal{M}^2 &= \begin{bmatrix}\mathcal{A}^{\textcircled{\#}}\mathcal{A}-\mathcal{A}^{\textcircled{\#}}\mathcal{B}\mathcal{S}^{\textcircled{\#}}\mathcal{C}(\mathcal{I}-\mathcal{A}^{\textcircled{\#}}\mathcal{A}) & \mathcal{A}^{\textcircled{\#}}\mathcal{B}(\mathcal{I}-\mathcal{S}^{\textcircled{\#}}\mathcal{S})\\ \mathcal{S}^{\textcircled{\#}}\mathcal{C}(\mathcal{I}-\mathcal{A}^{\textcircled{\#}}\mathcal{A}) & \mathcal{S}^{\textcircled{\#}}\mathcal{S}\end{bmatrix}\begin{bmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{C} & \mathcal{D}\end{bmatrix}\\ &= \begin{bmatrix}\mathcal{A}+\mathcal{A}^{\textcircled{\#}}\mathcal{B}(\mathcal{I}-\mathcal{S}^{\textcircled{\#}}\mathcal{S})\mathcal{C} & \mathcal{A}^{\textcircled{\#}}\mathcal{A}\mathcal{B}-\mathcal{A}^{\textcircled{\#}}\mathcal{B}\mathcal{S}^{\textcircled{\#}}\mathcal{C}(\mathcal{I}-\mathcal{A}^{\textcircled{\#}}\mathcal{A})\mathcal{B}+\mathcal{A}^{\textcircled{\#}}\mathcal{B}(\mathcal{I}-\mathcal{S}^{\textcircled{\#}}\mathcal{S})\mathcal{D}\\ \mathcal{S}^{\textcircled{\#}}\mathcal{S}\mathcal{C} & \mathcal{S}^{\textcircled{\#}}\mathcal{C}(\mathcal{I}-\mathcal{A}^{\textcircled{\#}}\mathcal{A})\mathcal{B}+\mathcal{S}^{\textcircled{\#}}\mathcal{S}\mathcal{D}\end{bmatrix}\\ &= \begin{bmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{C} & \mathcal{D}\end{bmatrix}=\mathcal{M}.\end{aligned}$$

Above all, $(\mathcal{M}\mathcal{X})^H=\mathcal{M}\mathcal{X}$, $\mathcal{M}\mathcal{X}^2=\mathcal{X}$ and $\mathcal{X}\mathcal{M}^2=\mathcal{M}$. Then, we have $\mathcal{X}=\mathcal{M}^{\textcircled{\#}}$. □

If $\mathcal{S}$ is a nonsingular tensor, then $\Upsilon_{\mathcal{S}}\mathcal{C}\mathcal{A}^{\textcircled{\#}}=0,\ \Gamma_{\mathcal{S}}\mathcal{C}=0$ are always true. Hence, we can get the following corollary.

Corollary 4.2. *Let* $\mathcal{M} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \in \mathbb{C}^{(m_1+m_2)\times(m_1+m_2)\times p}$ *and* $\mathcal{S} = \mathcal{D} - \mathcal{C}\mathcal{A}^{\#}\mathcal{B}$ *be invertible. If* $\Upsilon_{\mathcal{A}}\mathcal{B}\mathcal{S}^{-1} = 0$, $\Gamma_{\mathcal{A}}\mathcal{B} = 0$, *where* $\mathcal{A}^{-} = \mathcal{A}^{\#}$, *then*

$$\mathcal{M}^{\#} = \begin{bmatrix} \mathcal{A}^{\#} + \mathcal{A}^{\#}\mathcal{B}\mathcal{S}^{-1}\mathcal{C}\mathcal{A}^{\#} & -\mathcal{A}^{\#}\mathcal{B}\mathcal{S}^{-1} \\ -\mathcal{S}^{-1}\mathcal{C}\mathcal{A}^{\#} & \mathcal{S}^{-1} \end{bmatrix}.$$

Dually, we have the following results.

Theorem 4.3. *Let* $\mathcal{M} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \in \mathbb{C}^{(m_1+m_2)\times(m_1+m_2)\times p}$. *Then,*

$$\mathcal{M}^{\#} = \begin{bmatrix} \mathcal{T}^{\#} & -\mathcal{T}^{\#}\mathcal{B}\mathcal{D}^{\#} \\ -\mathcal{D}^{\#}\mathcal{C}\mathcal{T}^{\#} & \mathcal{D}^{\#} + \mathcal{D}^{\#}\mathcal{C}\mathcal{T}^{\#}\mathcal{B}\mathcal{D}^{\#} \end{bmatrix}$$

if and only if

$$\Upsilon_{\mathcal{D}}\mathcal{C}\mathcal{T}^{\#} = 0,\ \Upsilon_{\mathcal{T}}\mathcal{B}\mathcal{D}^{\#} = 0,\ \Gamma_{\mathcal{D}}\mathcal{C} = 0,\ \Gamma_{\mathcal{T}}\mathcal{B} = 0,$$

where $\mathcal{D}^{-} = \mathcal{D}^{\#}$, $\mathcal{T}^{-} = \mathcal{T}^{\#}$, $\mathcal{T} = \mathcal{A} - \mathcal{B}\mathcal{D}^{\#}\mathcal{C}$.

Corollary 4.4. *Let* $\mathcal{M} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \in \mathbb{C}^{(m_1+m_2)\times(m_1+m_2)\times p}$ *and* $\mathcal{T} = \mathcal{A} - \mathcal{B}\mathcal{D}^{\#}\mathcal{C}$ *be invertible. If* $\Upsilon_{\mathcal{D}}\mathcal{C}\mathcal{T}^{-1} = 0$, $\Gamma_{\mathcal{D}}\mathcal{C} = 0$, *where* $\mathcal{D}^{-} = \mathcal{D}^{\#}$, *then*

$$\mathcal{M}^{\#} = \begin{bmatrix} \mathcal{T}^{-1} & -\mathcal{T}^{-1}\mathcal{B}\mathcal{D}^{\#} \\ -\mathcal{D}^{\#}\mathcal{C}\mathcal{T}^{-1} & \mathcal{D}^{\#} + \mathcal{D}^{\#}\mathcal{C}\mathcal{T}^{-1}\mathcal{B}\mathcal{D}^{\#} \end{bmatrix}.$$

If we combine Theorem 4.1 with Theorem 4.3, we can get a corollary as follows. In the following theorem, we denote $\Gamma_{\alpha} = \mathcal{I} - \alpha^{\#}\alpha$ and $\Upsilon_{\alpha} = \mathcal{I} - \alpha\alpha^{\#}$, where $\alpha = \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{S}, \mathcal{T}$.

Corollary 4.5. *Let* $\mathcal{M} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \in \mathbb{C}^{(m_1+m_2)\times(m_1+m_2)\times p}$, $\mathcal{S} = \mathcal{D} - \mathcal{C}\mathcal{A}^{\#}\mathcal{B}$ *and* $\mathcal{T} = \mathcal{A} - \mathcal{B}\mathcal{D}^{\#}\mathcal{C}$. *Then,*

$$\mathcal{M}^{\#} = \begin{bmatrix} \mathcal{A}^{\#} + \mathcal{A}^{\#}\mathcal{B}\mathcal{S}^{\#}\mathcal{C}\mathcal{A}^{\#} & -\mathcal{A}^{\#}\mathcal{B}\mathcal{S}^{\#} \\ -\mathcal{S}^{\#}\mathcal{C}\mathcal{A}^{\#} & \mathcal{S}^{\#} \end{bmatrix} = \begin{bmatrix} \mathcal{T}^{\#} & -\mathcal{T}^{\#}\mathcal{B}\mathcal{D}^{\#} \\ -\mathcal{D}^{\#}\mathcal{C}\mathcal{T}^{\#} & \mathcal{D}^{\#} + \mathcal{D}^{\#}\mathcal{C}\mathcal{T}^{\#}\mathcal{B}\mathcal{D}^{\#} \end{bmatrix} \tag{4.3}$$

if and only if

$$\Upsilon_{\mathcal{A}}\mathcal{B}\mathcal{S}^{\#} = 0,\ \Gamma_{\mathcal{A}}\mathcal{B} = 0,\ \Upsilon_{\mathcal{S}}\mathcal{C}\mathcal{A}^{\#} = 0,\ \Gamma_{\mathcal{S}}\mathcal{C} = 0,\ \Upsilon_{\mathcal{D}}\mathcal{C}\mathcal{T}^{\#} = 0,\ \Gamma_{\mathcal{D}}\mathcal{C} = 0,\ \Upsilon_{\mathcal{T}}\mathcal{B}\mathcal{D}^{\#} = 0,\ \Gamma_{\mathcal{T}}\mathcal{B} = 0. \tag{4.4}$$

By using Corollary 4.5, we can get the following corollary.

Corollary 4.6. *Let* $\mathcal{M} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \in \mathbb{C}^{(m_1+m_2)\times(m_1+m_2)\times p}$. *If*

$$\Upsilon_{\mathcal{A}}\mathcal{B}\mathcal{S}^{\#} = 0,\ \Gamma_{\mathcal{A}}\mathcal{B} = 0,\ \Upsilon_{\mathcal{S}}\mathcal{C}\mathcal{A}^{\#} = 0,\ \Gamma_{\mathcal{S}}\mathcal{C} = 0,\ \Upsilon_{\mathcal{D}}\mathcal{C}\mathcal{T}^{\#} = 0,\ \Gamma_{\mathcal{D}}\mathcal{C} = 0,\ \Upsilon_{\mathcal{T}}\mathcal{B}\mathcal{D}^{\#} = 0,\ \Gamma_{\mathcal{T}}\mathcal{B} = 0,$$

then

$$\mathcal{T}^{\#} = \mathcal{A}^{\#} + \mathcal{A}^{\#}\mathcal{B}\mathcal{S}^{\#}\mathcal{C}\mathcal{A}^{\#},\ \mathcal{S}^{\#} = \mathcal{D}^{\#} + \mathcal{D}^{\#}\mathcal{C}\mathcal{T}^{\#}\mathcal{B}\mathcal{D}^{\#},\ \mathcal{T}\mathcal{T}^{\#} = \mathcal{A}\mathcal{A}^{\#},\ \mathcal{S}\mathcal{S}^{\#} = \mathcal{D}\mathcal{D}^{\#},$$

where $\mathcal{S} = \mathcal{D} - \mathcal{C}\mathcal{A}^{\#}\mathcal{B}$ *and* $\mathcal{T} = \mathcal{A} - \mathcal{B}\mathcal{D}^{\#}\mathcal{C}$.

Proof. If

$$\Upsilon_{\mathcal{A}}\mathcal{B}\mathcal{S}^{\#} = 0,\ \Gamma_{\mathcal{A}}\mathcal{B} = 0,\ \Upsilon_{\mathcal{S}}\mathcal{C}\mathcal{A}^{\#} = 0,\ \Gamma_{\mathcal{S}}\mathcal{C} = 0,\ \Upsilon_{\mathcal{D}}\mathcal{C}\mathcal{T}^{\#} = 0,\ \Gamma_{\mathcal{D}}\mathcal{C} = 0,\ \Upsilon_{\mathcal{T}}\mathcal{B}\mathcal{D}^{\#} = 0,\ \Gamma_{\mathcal{T}}\mathcal{B} = 0,$$

then the core inverse of $\mathcal{M}$ possesses the expression of (4.3). Hence, we have $\mathcal{T}^{\text{\#}} = \mathcal{A}^{\text{\#}} + \mathcal{A}^{\text{\#}}\mathcal{B}\mathcal{S}^{\text{\#}}\mathcal{C}\mathcal{A}^{\text{\#}}$, $\mathcal{S}^{\text{\#}} = \mathcal{D}^{\text{\#}} + \mathcal{D}^{\text{\#}}\mathcal{C}\mathcal{T}^{\text{\#}}\mathcal{B}\mathcal{D}^{\text{\#}}$. Since $\Gamma_{\mathcal{D}}\mathcal{C} = 0$, one has

$$\Gamma_{\mathcal{D}}\mathcal{S}^{\text{\#}} = \Gamma_{\mathcal{D}}\mathcal{S}(\mathcal{S}^{\text{\#}})^2 = \Gamma_{\mathcal{D}}(\mathcal{D} - \mathcal{C}\mathcal{A}^{\text{\#}}\mathcal{B})(\mathcal{S}^{\text{\#}})^2 = 0.$$

And by $\Upsilon_{\mathcal{A}}\mathcal{B}\mathcal{S}^{\text{\#}} = 0$, $\Upsilon_{\mathcal{S}}\mathcal{C}\mathcal{A}^{\text{\#}} = 0$, $\Gamma_{\mathcal{D}}\mathcal{S}^{\text{\#}} = 0$, we have that

$$\begin{aligned}
\mathcal{T}\mathcal{T}^{\text{\#}} &= (\mathcal{A} - \mathcal{B}\mathcal{D}^{\text{\#}}\mathcal{C})(\mathcal{A}^{\text{\#}} + \mathcal{A}^{\text{\#}}\mathcal{B}\mathcal{S}^{\text{\#}}\mathcal{C}\mathcal{A}^{\text{\#}})\\
&= \mathcal{A}\mathcal{A}^{\text{\#}} - \mathcal{B}\mathcal{D}^{\text{\#}}\mathcal{C}\mathcal{A}^{\text{\#}} + \mathcal{A}\mathcal{A}^{\text{\#}}\mathcal{B}\mathcal{S}^{\text{\#}}\mathcal{C}\mathcal{A}^{\text{\#}} - \mathcal{B}\mathcal{D}^{\text{\#}}\mathcal{C}\mathcal{A}^{\text{\#}}\mathcal{B}\mathcal{S}^{\text{\#}}\mathcal{C}\mathcal{A}^{\text{\#}}\\
&= \mathcal{A}\mathcal{A}^{\text{\#}} + \mathcal{B}\mathcal{S}^{\text{\#}}\mathcal{C}\mathcal{A}^{\text{\#}} - \mathcal{B}\mathcal{D}^{\text{\#}}\mathcal{C}\mathcal{A}^{\text{\#}} - \mathcal{B}\mathcal{D}^{\text{\#}}(\mathcal{D} - \mathcal{S})\mathcal{S}^{\text{\#}}\mathcal{C}\mathcal{A}^{\text{\#}}\\
&= \mathcal{A}\mathcal{A}^{\text{\#}} + \mathcal{B}\mathcal{S}^{\text{\#}}\mathcal{C}\mathcal{A}^{\text{\#}} - \mathcal{B}\mathcal{D}^{\text{\#}}\mathcal{C}\mathcal{A}^{\text{\#}} - \mathcal{B}\mathcal{S}^{\text{\#}}\mathcal{C}\mathcal{A}^{\text{\#}} + \mathcal{B}\mathcal{D}^{\text{\#}}\mathcal{C}\mathcal{A}^{\text{\#}}\\
&= \mathcal{A}\mathcal{A}^{\text{\#}}.
\end{aligned}$$

Similarly, $\mathcal{S}\mathcal{S}^{\text{\#}} = \mathcal{D}\mathcal{D}^{\text{\#}}$. □

Furthermore, we can get the new sufficient conditions for the expression (4.3) being true.

Theorem 4.7. *Let* $\mathcal{M} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \in \mathbb{C}^{(m_1+m_2)\times(m_1+m_2)\times p}$, $\mathcal{S} = \mathcal{D} - \mathcal{C}\mathcal{A}^{\text{\#}}\mathcal{B}$ *and* $\mathcal{T} = \mathcal{A} - \mathcal{B}\mathcal{D}^{\text{\#}}\mathcal{C}$. *If one of the following condition holds*

$$(i)\ \Upsilon_{\mathcal{A}}\mathcal{B} = 0,\ \Gamma_{\mathcal{A}}\mathcal{B} = 0,\ \Upsilon_{\mathcal{S}}\mathcal{C} = 0,\ \Upsilon_{\mathcal{D}}\mathcal{C} = 0,\ \Gamma_{\mathcal{S}}\mathcal{C} = 0,\ \Gamma_{\mathcal{D}}\mathcal{C} = 0,\ \mathcal{C}\Gamma_{\mathcal{A}} = 0, \tag{4.5}$$

$$(ii)\ \Upsilon_{\mathcal{A}}\mathcal{B} = 0,\ \Gamma_{\mathcal{A}}\mathcal{B} = 0,\ \Upsilon_{\mathcal{D}}\mathcal{C} = 0,\ \Gamma_{\mathcal{D}}\mathcal{C} = 0,\ \Upsilon_{\mathcal{T}}\mathcal{B} = 0,\ \Gamma_{\mathcal{T}}\mathcal{B} = 0,\ \mathcal{B}\Gamma_{\mathcal{D}} = 0, \tag{4.6}$$

then

$$\mathcal{M}^{\text{\#}} = \begin{bmatrix} \mathcal{A}^{\text{\#}} + \mathcal{A}^{\text{\#}}\mathcal{B}\mathcal{S}^{\text{\#}}\mathcal{C}\mathcal{A}^{\text{\#}} & -\mathcal{A}^{\text{\#}}\mathcal{B}\mathcal{S}^{\text{\#}} \\ -\mathcal{S}^{\text{\#}}\mathcal{C}\mathcal{A}^{\text{\#}} & \mathcal{S}^{\text{\#}} \end{bmatrix} = \begin{bmatrix} \mathcal{T}^{\text{\#}} & -\mathcal{T}^{\text{\#}}\mathcal{B}\mathcal{D}^{\text{\#}} \\ -\mathcal{D}^{\text{\#}}\mathcal{C}\mathcal{T}^{\text{\#}} & \mathcal{D}^{\text{\#}} + \mathcal{D}^{\text{\#}}\mathcal{C}\mathcal{T}^{\text{\#}}\mathcal{B}\mathcal{D}^{\text{\#}} \end{bmatrix}.$$

Proof. (i) We will prove that (4.5) implies (4.4). Denote $\mathcal{T}' = \mathcal{A}^{\text{\#}} + \mathcal{A}^{\text{\#}}\mathcal{B}\mathcal{S}^{\text{\#}}\mathcal{C}\mathcal{A}^{\text{\#}}$. Notice that $\Gamma_{\mathcal{D}}\mathcal{C} = 0$ implies

$$\Gamma_{\mathcal{D}}\mathcal{S}^{\text{\#}} = \Gamma_{\mathcal{D}}\mathcal{S}(\mathcal{S}^{\text{\#}})^2 = \Gamma_{\mathcal{D}}(\mathcal{D} - \mathcal{C}\mathcal{A}^{\text{\#}}\mathcal{B})(\mathcal{S}^{\text{\#}})^2 = 0.$$

By $\Upsilon_{\mathcal{A}}\mathcal{B} = 0$, $\Upsilon_{\mathcal{S}}\mathcal{C} = 0$, $\Gamma_{\mathcal{D}}\mathcal{S}^{\text{\#}} = 0$, we have that

$$\begin{aligned}
\mathcal{T}\mathcal{T}' &= (\mathcal{A} - \mathcal{B}\mathcal{D}^{\text{\#}}\mathcal{C})(\mathcal{A}^{\text{\#}} + \mathcal{A}^{\text{\#}}\mathcal{B}\mathcal{S}^{\text{\#}}\mathcal{C}\mathcal{A}^{\text{\#}})\\
&= \mathcal{A}\mathcal{A}^{\text{\#}} - \mathcal{B}\mathcal{D}^{\text{\#}}\mathcal{C}\mathcal{A}^{\text{\#}} + \mathcal{A}\mathcal{A}^{\text{\#}}\mathcal{B}\mathcal{S}^{\text{\#}}\mathcal{C}\mathcal{A}^{\text{\#}} - \mathcal{B}\mathcal{D}^{\text{\#}}\mathcal{C}\mathcal{A}^{\text{\#}}\mathcal{B}\mathcal{S}^{\text{\#}}\mathcal{C}\mathcal{A}^{\text{\#}}\\
&= \mathcal{A}\mathcal{A}^{\text{\#}} + \mathcal{B}\mathcal{S}^{\text{\#}}\mathcal{C}\mathcal{A}^{\text{\#}} - \mathcal{B}\mathcal{D}^{\text{\#}}\mathcal{C}\mathcal{A}^{\text{\#}} - \mathcal{B}\mathcal{D}^{\text{\#}}(\mathcal{D} - \mathcal{S})\mathcal{S}^{\text{\#}}\mathcal{C}\mathcal{A}^{\text{\#}}\\
&= \mathcal{A}\mathcal{A}^{\text{\#}} + \mathcal{B}\mathcal{S}^{\text{\#}}\mathcal{C}\mathcal{A}^{\text{\#}} - \mathcal{B}\mathcal{D}^{\text{\#}}\mathcal{C}\mathcal{A}^{\text{\#}} - \mathcal{B}\mathcal{S}^{\text{\#}}\mathcal{C}\mathcal{A}^{\text{\#}} + \mathcal{B}\mathcal{D}^{\text{\#}}\mathcal{C}\mathcal{A}^{\text{\#}}\\
&= \mathcal{A}\mathcal{A}^{\text{\#}}.
\end{aligned}$$

On the other hand, $\Gamma_{\mathcal{S}}\mathcal{C} = 0$ implies $\Gamma_{\mathcal{S}}\mathcal{D}^{\text{\#}} = 0$ and by $\Upsilon_{\mathcal{D}}\mathcal{C} = 0$, $\mathcal{C}\Gamma_{\mathcal{A}} = 0$, $\Gamma_{\mathcal{S}}\mathcal{D}^{\text{\#}} = 0$, we have

$$\begin{aligned}
\mathcal{T}'\mathcal{T} &= (\mathcal{A}^{\text{\#}} + \mathcal{A}^{\text{\#}}\mathcal{B}\mathcal{S}^{\text{\#}}\mathcal{C}\mathcal{A}^{\text{\#}})(\mathcal{A} - \mathcal{B}\mathcal{D}^{\text{\#}}\mathcal{C})\\
&= \mathcal{A}^{\text{\#}}\mathcal{A} - \mathcal{A}^{\text{\#}}\mathcal{B}\mathcal{D}^{\text{\#}}\mathcal{C} + \mathcal{A}^{\text{\#}}\mathcal{B}\mathcal{S}^{\text{\#}}\mathcal{C}\mathcal{A}^{\text{\#}}\mathcal{A} - \mathcal{A}^{\text{\#}}\mathcal{B}\mathcal{S}^{\text{\#}}\mathcal{C}\mathcal{A}^{\text{\#}}\mathcal{B}\mathcal{D}^{\text{\#}}\mathcal{C}\\
&= \mathcal{A}^{\text{\#}}\mathcal{A} - \mathcal{A}^{\text{\#}}\mathcal{B}\mathcal{D}^{\text{\#}}\mathcal{C} + \mathcal{A}^{\text{\#}}\mathcal{B}\mathcal{S}^{\text{\#}}\mathcal{C} - \mathcal{A}^{\text{\#}}\mathcal{B}\mathcal{S}^{\text{\#}}(\mathcal{D} - \mathcal{S})\mathcal{D}^{\text{\#}}\mathcal{C}\\
&= \mathcal{A}^{\text{\#}}\mathcal{A} - \mathcal{A}^{\text{\#}}\mathcal{B}\mathcal{D}^{\text{\#}}\mathcal{C} + \mathcal{A}^{\text{\#}}\mathcal{B}\mathcal{S}^{\text{\#}}\mathcal{C} - \mathcal{A}^{\text{\#}}\mathcal{B}\mathcal{S}^{\text{\#}}\mathcal{C} + \mathcal{A}^{\text{\#}}\mathcal{B}\mathcal{D}^{\text{\#}}\mathcal{C}\\
&= \mathcal{A}^{\text{\#}}\mathcal{A}.
\end{aligned}$$

Using $\Gamma_{\mathcal{A}}\mathcal{B} = 0$, one has

$$\mathcal{T}'\mathcal{T}\mathcal{T} = \mathcal{A}^{\text{\#}}\mathcal{A}(\mathcal{A} - \mathcal{B}\mathcal{D}^{\text{\#}}\mathcal{C}) = \mathcal{A} - \mathcal{B}\mathcal{D}^{\text{\#}}\mathcal{C} = \mathcal{T}.$$

Finally, it is easy to see $\mathcal{T}\mathcal{T}'\mathcal{T}' = \mathcal{T}'$. Thus, $\mathcal{T}' = \mathcal{T}^{\#}$. Then, $\mathcal{T}\mathcal{T}^{\#} = \mathcal{A}\mathcal{A}^{\#}$ and $\mathcal{T}^{\#}\mathcal{T} = \mathcal{A}^{\#}\mathcal{A}$. Therefore,

$$\Upsilon_{\mathcal{A}}\mathcal{B} = 0,\ \Gamma_{\mathcal{A}}\mathcal{B} = 0 \Rightarrow \Upsilon_{\mathcal{T}}\mathcal{B} = 0,\ \Gamma_{\mathcal{T}}\mathcal{B} = 0.$$

Hence, (4.5) implies (4.4). By Corollary 4.5, we get

$$\mathcal{M}^{\#} = \begin{bmatrix} \mathcal{A}^{\#} + \mathcal{A}^{\#}\mathcal{B}\mathcal{S}^{\#}\mathcal{C}\mathcal{A}^{\#} & -\mathcal{A}^{\#}\mathcal{B}\mathcal{S}^{\#} \\ -\mathcal{S}^{\#}\mathcal{C}\mathcal{A}^{\#} & \mathcal{S}^{\#} \end{bmatrix} = \begin{bmatrix} \mathcal{T}^{\#} & -\mathcal{T}^{\#}\mathcal{B}\mathcal{D}^{\#} \\ -\mathcal{D}^{\#}\mathcal{C}\mathcal{T}^{\#} & \mathcal{D}^{\#} + \mathcal{D}^{\#}\mathcal{C}\mathcal{T}^{\#}\mathcal{B}\mathcal{D}^{\#} \end{bmatrix}.$$

(ii) Similarly. □

5. The Application to the Quotient Property of the Block Tensor

In 1969, Crabtree and Haynsworth [2] showed a quotient formula for Schur complement of a matrix. The formula was reproven twice by Ostrowski in 1971 [16] and in 1973 [17], respectively. In this part, we will have a further study on the quotient property for the block tensor.

In the following, we denote

$$\mathcal{Z} = \begin{bmatrix} \mathcal{A} & \mathcal{B} & \mathcal{E} \\ \mathcal{C} & \mathcal{D} & \mathcal{F} \\ \mathcal{G} & \mathcal{H} & \mathcal{L} \end{bmatrix}, \mathcal{M}_1 = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}, \mathcal{M}_2 = \begin{bmatrix} \mathcal{B} & \mathcal{E} \\ \mathcal{D} & \mathcal{F} \end{bmatrix}, \mathcal{M}_3 = \begin{bmatrix} \mathcal{D} & \mathcal{F} \\ \mathcal{H} & \mathcal{L} \end{bmatrix}, \mathcal{M}_4 = \begin{bmatrix} \mathcal{C} & \mathcal{D} \\ \mathcal{G} & \mathcal{H} \end{bmatrix}. \tag{5.1}$$

Moreover, let $\Delta = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix}$, where $\Delta = \mathcal{Z}, \mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4$, then

$$(\Delta/\alpha_1)_c = \alpha_4 - \alpha_3\alpha_1^{\#}\alpha_2,\ (\Delta/\alpha_4)_c = \alpha_1 - \alpha_2\alpha_4^{\#}\alpha_3. \tag{5.2}$$

The next theorem presents the quotient property of the generalized Schur complement based on the core inverse.

Theorem 5.1. *Let $\mathcal{Z}$ and $\mathcal{M}_1$ be the form of (5.1). If*

$$\Upsilon_{\mathcal{A}}\mathcal{B} = 0,\ \Gamma_{\mathcal{A}}\mathcal{B} = 0,\ \Upsilon_{\mathcal{S}}\mathcal{C} = 0,\ \Gamma_{\mathcal{S}}\mathcal{C} = 0, \tag{5.3}$$

where $\mathcal{A}^- = \mathcal{A}^{\#}$, $\mathcal{S}^- = \mathcal{S}^{\#}$, $\mathcal{S} = \mathcal{D} - \mathcal{C}\mathcal{A}^{\#}\mathcal{B}$, then

$$(\mathcal{Z}/\mathcal{M}_1)_c = ((\mathcal{Z}/\mathcal{A})_c/(\mathcal{M}_1/\mathcal{A})_c)_c.$$

Proof. According to the definition of the generalized Schur complement based on the core inverse, we have

$$(\mathcal{Z}/\mathcal{M}_1)_c = \mathcal{L} - \begin{bmatrix} \mathcal{G} & \mathcal{H} \end{bmatrix} \mathcal{M}_1^{\#} \begin{bmatrix} \mathcal{E} \\ \mathcal{F} \end{bmatrix},\ (\mathcal{M}_1/\mathcal{A})_c = \mathcal{D} - \mathcal{C}\mathcal{A}^{\#}\mathcal{B},$$

$$(\mathcal{Z}/\mathcal{A})_c = \begin{bmatrix} \mathcal{D} & \mathcal{F} \\ \mathcal{H} & \mathcal{L} \end{bmatrix} - \begin{bmatrix} \mathcal{C} \\ \mathcal{G} \end{bmatrix} \mathcal{A}^{\#} \begin{bmatrix} \mathcal{B} & \mathcal{E} \end{bmatrix} = \begin{bmatrix} \mathcal{D} - \mathcal{C}\mathcal{A}^{\#}\mathcal{B} & \mathcal{F} - \mathcal{C}\mathcal{A}^{\#}\mathcal{E} \\ \mathcal{H} - \mathcal{G}\mathcal{A}^{\#}\mathcal{B} & \mathcal{L} - \mathcal{G}\mathcal{A}^{\#}\mathcal{E} \end{bmatrix}.$$

From which we have

$$\begin{aligned} ((\mathcal{Z}/\mathcal{A})_c/(\mathcal{M}_1/\mathcal{A})_c)_c &= \mathcal{L} - \mathcal{G}\mathcal{A}^{\#}\mathcal{E} - (\mathcal{H} - \mathcal{G}\mathcal{A}^{\#}\mathcal{B})(\mathcal{D} - \mathcal{C}\mathcal{A}^{\#}\mathcal{B})^{\#}(\mathcal{F} - \mathcal{C}\mathcal{A}^{\#}\mathcal{E}) \\ &= \mathcal{L} - \begin{bmatrix} \mathcal{G} & \mathcal{H} \end{bmatrix} \begin{bmatrix} \mathcal{A}^{\#} + \mathcal{A}^{\#}\mathcal{B}(\mathcal{D} - \mathcal{C}\mathcal{A}^{\#}\mathcal{B})^{\#}\mathcal{C}\mathcal{A}^{\#} & -\mathcal{A}^{\#}\mathcal{B}(\mathcal{D} - \mathcal{C}\mathcal{A}^{\#}\mathcal{B})^{\#} \\ -(\mathcal{D} - \mathcal{C}\mathcal{A}^{\#}\mathcal{B})^{\#}\mathcal{C}\mathcal{A}^{\#} & (\mathcal{D} - \mathcal{C}\mathcal{A}^{\#}\mathcal{B})^{\#} \end{bmatrix} \begin{bmatrix} \mathcal{E} \\ \mathcal{F} \end{bmatrix}. \end{aligned}$$

By Theorem 4.1, we have

$$((\mathcal{Z}/\mathcal{A})_c/(\mathcal{M}_1/\mathcal{A})_c)_c = \mathcal{L} - \begin{bmatrix} \mathcal{G} & \mathcal{H} \end{bmatrix} \mathcal{M}_1^{\#} \begin{bmatrix} \mathcal{E} \\ \mathcal{F} \end{bmatrix} = (\mathcal{Z}/\mathcal{M}_1)_c.$$

□

The next theorem is the first Sylvester identity based on the core inverse.

Theorem 5.2. *Let $\mathcal{Z}, \mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$ and $\mathcal{M}_4$ be the form of (5.1). If*

$$\Upsilon_{\mathcal{D}}\mathcal{F} = 0,\ \Gamma_{\mathcal{D}}\mathcal{F} = 0,\ \Upsilon_{\mathcal{W}}\mathcal{H} = 0,\ \Gamma_{\mathcal{W}}\mathcal{H} = 0. \tag{5.4}$$

where $\mathcal{D}^- = \mathcal{A}^{\#}$, $\mathcal{W}^- = \mathcal{S}^{\#}$, $\mathcal{W} = \mathcal{L} - \mathcal{H}\mathcal{D}^{\#}\mathcal{F}$, then

$$(\mathcal{Z}/\mathcal{M}_3)_c = ((\mathcal{Z}/\mathcal{D})_c/(\mathcal{M}_3/\mathcal{D})_c)_c = (\mathcal{M}_1/\mathcal{D})_c - (\mathcal{M}_2/\mathcal{D})_c(\mathcal{M}_3/\mathcal{D})_c^{\#}(\mathcal{M}_4/\mathcal{D})_c.$$

Proof. According to the definition of the generalized Schur complement based on the core inverse, we have

$$(\mathcal{Z}/\mathcal{M}_3)_c = \mathcal{A} - \begin{bmatrix}\mathcal{B} & \mathcal{E}\end{bmatrix}\mathcal{M}_3^{\#}\begin{bmatrix}\mathcal{G}\\ \mathcal{H}\end{bmatrix},\ (\mathcal{M}_3/\mathcal{D})_c = \mathcal{L} - \mathcal{H}\mathcal{D}^{\#}\mathcal{F},$$

$$(\mathcal{Z}/\mathcal{D})_c = \begin{bmatrix}\mathcal{A} & \mathcal{E}\\ \mathcal{G} & \mathcal{L}\end{bmatrix} - \begin{bmatrix}\mathcal{B}\\ \mathcal{H}\end{bmatrix}\mathcal{D}^{\#}\begin{bmatrix}\mathcal{C} & \mathcal{F}\end{bmatrix} = \begin{bmatrix}\mathcal{A} - \mathcal{B}\mathcal{D}^{\#}\mathcal{C} & \mathcal{E} - \mathcal{B}\mathcal{D}^{\#}\mathcal{F}\\ \mathcal{G} - \mathcal{H}\mathcal{D}^{\#}\mathcal{C} & \mathcal{L} - \mathcal{H}\mathcal{D}^{\#}\mathcal{F}\end{bmatrix}.$$

Thus, we have

$$\begin{aligned}
&((\mathcal{Z}/\mathcal{D})_c/(\mathcal{M}_3/\mathcal{D})_c)_c = \mathcal{A} - \mathcal{B}\mathcal{D}^{\#}\mathcal{C} - (\mathcal{E} - \mathcal{B}\mathcal{D}^{\#}\mathcal{F})(\mathcal{L} - \mathcal{H}\mathcal{D}^{\#}\mathcal{F})^{\#}(\mathcal{G} - \mathcal{H}\mathcal{D}^{\#}\mathcal{C})\\
&= (\mathcal{M}_1/\mathcal{D})_c - (\mathcal{M}_2/\mathcal{D})_c(\mathcal{M}_3/\mathcal{D})_c^{\#}(\mathcal{M}_4/\mathcal{D})_c\\
&= \mathcal{A} - \begin{bmatrix}\mathcal{B} & \mathcal{E}\end{bmatrix}\begin{bmatrix}\mathcal{D}^{\#} + \mathcal{D}^{\#}\mathcal{F}(\mathcal{L} - \mathcal{H}\mathcal{D}^{\#}\mathcal{F})^{\#}\mathcal{H}\mathcal{D}^{\#}\mathcal{C} & -\mathcal{D}^{\#}\mathcal{F}(\mathcal{L} - \mathcal{H}\mathcal{D}^{\#}\mathcal{F})^{\#}\\ -(\mathcal{L} - \mathcal{H}\mathcal{D}^{\#}\mathcal{F})^{\#}\mathcal{H}\mathcal{D}^{\#} & (\mathcal{L} - \mathcal{H}\mathcal{D}^{\#}\mathcal{F})^{\#}\end{bmatrix}\begin{bmatrix}\mathcal{G}\\ \mathcal{H}\end{bmatrix}.
\end{aligned}$$

By Theorem 4.1, we have

$$((\mathcal{Z}/\mathcal{D})_c/(\mathcal{M}_3/\mathcal{D})_c)_c = \mathcal{A} - \begin{bmatrix}\mathcal{B} & \mathcal{E}\end{bmatrix}\mathcal{M}_3^{\#}\begin{bmatrix}\mathcal{G}\\ \mathcal{H}\end{bmatrix} = (\mathcal{Z}/\mathcal{M}_3)_c.$$

□

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