Filomat 37:26 (2023), 8969–8973
https://doi.org/10.2298/FIL2326969S

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Elementary abelian group actions on a product of spaces of cohomology type (a, b)

Hemant Kumar Singh[a], Konthoujam Somorjit Singh[b]

[a] *Department of Mathematics, University of Delhi, Delhi 110007, India*
[b] *Department of Mathematics, National Institute of Technology Manipur, Langol, Imphal 795004, India*

Abstract. Let X_n be a finite CW complex with cohomology type (a, b), characterized by an integer $n > 1$ [20]. In this paper, we show that if $G = (\mathbb{Z}_2)^q$ acts freely on the product $Y = \prod_{i=1}^m X_n^i$, where X_n^i are finite CW complexes with cohomology type (a, b), a and b are even for every i, then $q \leq m$. Moreover, for n even and $a = b = 0$, we prove that $G = (\mathbb{Z}_2)^q$ $(q \leq m)$ is the only finite group which can act freely on Y. These are generalizations of the results which says that the rank of a group acting freely on a space with cohomology type (a, b) where a and b are even, is one and for n even, $G = \mathbb{Z}_2$ is the only finite group which acts freely on spaces of cohomology type $(0, 0)$ [17].

1. Introduction

Let G be a finite group and p be a prime. The rank of G is defined by $\text{rk}(G) = \max \{q \mid (\mathbb{Z}_p)^q \subset G \text{ as a subgroup}\}$. One of the interesting problems in topological transformation groups is to find $\text{rk}(G)$ when G acts freely on a space X. P.A. Smith [16] and R.G. Swan[19] showed that if a finite group G acts freely on a sphere $\mathbb{S}^n$ then $\text{rk}(G) = 1$. Conner [7] proved that if a finite group G acts freely on $\mathbb{S}^n \times \mathbb{S}^n$ then $\text{rk}(G) = 2$. Heller [14] proved the same result for arbitrary product of two spheres $\mathbb{S}^n \times \mathbb{S}^m$. In this direction, Benson and Carlson [5] arise the following conjecture: If a finite group G acts freely on $X = \prod_{i=1}^m \mathbb{S}^{n_i}$ then $\text{rk}(G) \leq m$. So far this conjecture has been proved in the following cases: Carlsson [6] proved the result for $p = 2$ and $n_1 = n_2 = \cdots = n_m$ with the condition that the induced action of G is trivial on the mod 2-cohomology algebra of X. Adem and Browder [1] proved this result for $n_1 = n_2 = \cdots = n_m$ and $n_1 \neq 1, 3, 7$. Hanke [13] proved Carlsson conjecture for all primes $p > 3(n_1 + \cdots + n_k)$. Okutan and Yalcın [15] proved this result for dimensions n_i which are higher compared to the differences $|n_i - n_j|$ among all dimensions.

Cusick [8] proved that if G acts freely on $X = \prod_{i=1}^m \mathbb{R}P^{n_i}$ and the induced action of G on the mod 2 cohomology algebra of X is trivial then $\text{rk(G)} = \nu(n_1) + \cdots + \nu(n_m)$, where

$$\nu(n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \equiv 1 \pmod 4 \\ 2 & \text{if } n \equiv 3 \pmod 4). \end{cases}$$

2020 *Mathematics Subject Classification*. Primary 57S17; Secondary 55T10, 55S10
Keywords. Free action, Leray-Serre spectral sequence, Steenrod square
Received: 25 December 2022; Revised: 26 April 2023; Accepted: 19 May 2023
Communicated by Ljubiša D.R. Kočinac
Research supported by the Science and Engineering Research Board (Department of Science and Technology, Government of India) with reference number MTR/2017/000386
Email addresses: hemantksingh@maths.du.ac.in (Hemant Kumar Singh), ksomorjitmaths@gmail.com (Konthoujam Somorjit Singh)

Adem and Yalcin [2] improved this result without the assumption of trivial induced action on the mod 2 cohomology algebra of X. Cusick [9] shown that if $X = \prod_{i=1}^{m} \mathbb{C}P^{n_i}$ then rk(G) $= \nu(n_1) + \cdots + \nu(n_m)$, where

$$\nu(n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \equiv 1 \text{ (mod 4).} \end{cases}$$

Allday [4] put the following conjecture: If G acts freely on $X = \prod_{i=1}^{m} L_p^{2n_i-1}$, where p an odd prime, then rk$(G) = m$. Yalcin [18] proved this conjecture when $n_1 = \cdots = n_m$. This conjecture is still open for general case.

In this paper, we have showed that the rank of a finite group G acting freely on the product $\prod_{i=1}^{m} X_n^i$, where X_n^i are the spaces of type (a, b), a and b are even for each i, is atmost m. Moreover, for n even, we have proved that if a finite G acts freely on $\prod_{i=1}^{m} X_n^i$ where X_n^i are spaces of type $(0, 0)$ then $G = (\mathbb{Z}_2)^q$, where $q \leq m$.

2. Preliminaries

Given two integers a and b, a space X is said to have cohomology type (a, b) if $H^j(X;\mathbb{Z}) \cong \mathbb{Z}$ for $j = 0, n, 2n$ and $3n$ only, and the generators $x \in H^n(X;\mathbb{Z})$, $y \in H^{2n}(X;\mathbb{Z})$ and $z \in H^{3n}(X;\mathbb{Z})$ satisfies $x^2 = ay$ and $xy = bz$. It is denoted by X_n. For example, $\mathbb{S}^n \times \mathbb{S}^{2n}$ has type (0, 1), $\mathbb{C}P^3$ and $\mathbb{Q}P^3$ have type (1, 1), $\mathbb{C}P^2 \vee \mathbb{S}^6$ has type (1, 0) and $\mathbb{S}^n \vee \mathbb{S}^n \vee \mathbb{S}^{3n}$ has type (0, 0). Such spaces were first investigated by James [10] and Toda [20]. Let Y be $\prod_{i=1}^{m} X_n^i$, where X_n^i is a finite CW-complex with cohomology type (a, b). The cohomology algebra of Y is given by

$$H^*(Y;\mathbb{Z}_2) \cong \mathbb{Z}_2\left[x_1, \ldots, x_m, y_1, \ldots, y_m, z_1, \ldots, z_m\right]/I,$$

where I is a graded ideal generated by a set $A = \{x_i^2 - ay_i, y_i^2, z_i^2, x_iy_i - bz_i, y_iz_i, x_iz_i | 1 \leq i \leq m\}$ and deg $x_i = n$, deg $y_i = 2n$ and deg $z_i = 3n$ for all $1 \leq i \leq m$.

The Borel construction on X is defined as the orbit space $X_G = (X \times E_G)/G$, where the compact Lie group G acts diagonally (and freely) on the product $X \times E_G$. The projection $X \times E_G \to E_G$ gives a fibration $X_G \to B_G$ with fiber X. We will use the Leray-Serre spectral sequence associated to the Borel fibration $X \overset{i}{\hookrightarrow} X_G \overset{\pi}{\to} B_G$. If $\pi_1(B_G)$ acts trivially on $H^*(X;R)$ (R is a field) then the system of local coefficient is simple and E_2-term of the spectral sequence of the fibration $X \overset{i}{\hookrightarrow} X_G \overset{\pi}{\to} B_G$ is given by $E_2^{k,l} = H^k(B_G;R) \otimes H^l(X;R)$. Note that for $G = (\mathbb{Z}_2)^q$, $H^*(B_G;\mathbb{Z}_2) \cong \mathbb{Z}_2[t_1, t_2 \ldots, t_q]$, where deg $t_i = 1$ for all $1 \leq i \leq q$.

For the results in spectral sequences, we refer [11]. Throughout this paper, cohomologies are Čech cohomology with coefficient in $\mathbb{Z}_2$. Now, we recall some results which were used in this paper.

Proposition 2.1. ([3]) *Let $G = (\mathbb{Z}_2)^q$ act on a finitistic space X and $H^i(X) = 0$ for all $i > n$. Then $H^i(X/G) = 0$ for all $i > n$.*

The following results are proved by G. Carlsson [6]:

Proposition 2.2. ([6]) *Suppose $\{f_1, \ldots, f_k\}$ are elements of $H^n(B_{(\mathbb{Z}_2)^q};\mathbb{Z}_2)$, regarded as homogeneous polynomials of degree n in q variables. Then they have a nontrivial common zero in $(\mathbb{Z}_2)^q$ if and only if there exist a inclusion $i : \mathbb{Z}_2 \hookrightarrow (\mathbb{Z}_2)^q$ such that $i^*(f_j) = 0$ for all j.*

Proposition 2.3. ([6]) *Let $\langle f_1, \ldots, f_k\rangle$ be an ideal generated by homogeneous polynomials f_j in $\mathbb{Z}_2\left[t_1, \ldots, t_q\right]$ which is invariant under the action of the Steenrod algebra. If $q > k$, then there exists nontrivial common zero to $f_1, \ldots, f_k$.*

3. Main theorems

In this section, our aim is to determine an elementary abelian 2-group, which can act freely on a finite product of spaces of type (a, b). We show that the rank of elementary 2-abelian groups which acts freely on Y will not exceed m. This generalizes Theorem 3.2 [17].

Theorem 3.1. *Let $G = (\mathbb{Z}_2)^q$ act freely on a space $Y = \prod_{i=1}^{m} X_n^i$, where X_n^i is a finite CW-complex with cohomology type (a, b), a and b are even for every i. If G acts trivially on $H^*(Y)$ then $q \leq m$.*

Proof. As G acts trivially on $H^*(Y)$, $E_2^{k,l} = H^k(B_G) \otimes H^l(Y)$. Let $x_i \in H^n(Y)$, $y_i \in H^{2n}(Y)$ and $z_i \in H^{3n}(Y)$ be generators of the cohomology algebra of $H^*(Y)$.

First, we prove that $d_{n+1}(1 \otimes x_i) = 0$ for all $1 \leq i \leq m$. Let $d_{n+1}(1 \otimes x_i) = v_i \otimes 1$ for some i. Consider $d_{n+1}(1 \otimes y_i) = \sum_j w_{i,j} \otimes \alpha_{i,j}x_j$, where $\alpha_{i,j} \in \{0, 1\}$. By the multiplicative property of spectral sequence, we have $0 = d_{n+1}(1 \otimes x_i y_i) = v_i \otimes y_i + \sum_{j\neq i} w_{i,j} \otimes \alpha_{i,j}x_j x_i$, a contradiction. This implies that $d_{n+1}(1 \otimes x_i) = 0$ for all $1 \leq i \leq m$. Therefore, $d_r(1 \otimes x_i) = 0$ for all i and $r \geq 2$.

Next, we have observed that both $d_{n+1}(1 \otimes y_i)$ and $d_{n+1}(1 \otimes z_i)$ can't be trivial simultaneously for all i. For that, if $d_{n+1}(1 \otimes y_i) = d_{n+1}(1 \otimes z_i) = 0$ for all i then, $E_{2n+1}^{*,*} = E_2^{*,*}$. Clearly, $d_{2n+1}(1 \otimes y_i) = 0$ for all i. So, $d_r(1 \otimes y_i) = 0$ for all i and for all $r \geq 2$. If $d_{2n+1}(1 \otimes z_i) \neq 0$ for some i then $d_{2n+1}(1 \otimes z_i) = \sum_{j=1} u_{i,j} \otimes \alpha_{i,j}x_j$ where $\alpha_{i,j} \in \{0, 1\}$. Then $0 = d_{2n+1}(1 \otimes z_i x_i) = \sum_{j\neq i} u_{i,j} \otimes \alpha_{i,j}x_j$. Thus, $\alpha_{i,j} = 0$ for all $j \neq i$ and $\alpha_{i,j} = 1$ for $j = i$. Let $d_{2n+1}(1 \otimes z_i) = v_i \otimes x_i$ for all $1 \leq i \leq k$ $(\leq m)$. Then, we get $E_{3n+1}^{*,*} \cong E_{2n+2}^{*,*} \cong (E_2^{*,*} - S)/Q$, where S is a graded ideal generated by $\{1 \otimes z_1, \ldots, 1 \otimes z_k\}$ and graded ideal Q is generated by $\{v_i \otimes x_i, \beta |$ for all $1 \leq i \leq k$ and $\beta \in A\}$. Clearly, $d_{3n+1}(1 \otimes z_i) = 0$ for all $k + 1 \leq i \leq m$ and so $d_r(1 \otimes z_i) = 0$ for all $k + 1 \leq i \leq m$ and for all $r \geq 3n + 1$. Therefore, $d_r = 0$ for all $r \geq 3n + 1$. So, $E_{3n+1}^{*,*} \cong E_\infty^{*,*}$, which contradicts Proposition 2.1. Thus, the following three cases are possible:

(i) $d_{n+1}(1 \otimes y_i) \neq 0$ for some i and $d_{n+1}(1 \otimes z_i) = 0$ for all i.

(ii) $d_{n+1}(1 \otimes y_i) = 0$ for all i and $d_{n+1}(1 \otimes z_i) \neq 0$ for some i.

(iii) $d_{n+1}(1 \otimes y_i) \neq 0$ for some i and $d_{n+1}(1 \otimes z_j) \neq 0$ for some j.

Case (i). Let $d_{n+1}(1 \otimes y_i) \neq 0$ for some i and $d_{n+1}(1 \otimes z_i) = 0$ for all i. If $d_{n+1}(1 \otimes y_i) \neq 0$ then $d_{n+1}(1 \otimes y_i) = \sum_j u_{i,j} \otimes \alpha_{i,j}x_j$, where $u_{i,j} \in H^{n+1}(B_G)$ and $\alpha_{i,j} \in \{0, 1\}$. We have $0 = d_{n+1}(1 \otimes x_i y_i) = \sum_{j\neq i} u_{i,j} \otimes \alpha_{i,j}x_j x_i$. This implies $\alpha_{i,j} = 0$ for all $j \neq i$ and $\alpha_{i,i} \neq 0$. Therefore, $d_{n+1}(1 \otimes y_i) = u_{i,i} \otimes x_i$. Let $d_{n+1}(1 \otimes y_i) = u_{i,i} \otimes x_i$ for all $1 \leq i \leq m$, where $\{u_{i,i}\}$ is linearly independent subset of $H^{n+1}(B_G)$, and $d_{n+1}(1 \otimes z_i) = 0$ for all i. Then $d_{n+1}(1 \otimes x_1 \ldots x_{i-1} y_i x_{i+1} \ldots x_m) = u_{i,i} \otimes x_1 \ldots x_{i-1} x_i x_{i+1} \ldots x_m$ for all i. Now, consider the submodule Q generated by $\{u_{i,i} \otimes x_1 \ldots x_{i-1} x_i x_{i+1} \ldots x_m |$ for all $1 \leq i \leq m\}$ of acyclic module $\{\alpha \otimes x_1 \ldots x_{i-1} x_i x_{i+1} \ldots x_m | \ \alpha \in H^*(B_G)\} \cong H^*(B_G)$ (as a $H^*(B_G)$-module). By definition of Steenrod square in E_2, we have, $Sq^i(u_{i,i} \otimes x_1 \ldots x_{i-1} x_i x_{i+1} \ldots x_m) = u_{i,i} \otimes Sq_1^i(x_1 \ldots x_{i-1} x_i x_{i+1} \ldots x_m)$, where Sq_1 is an Steenrod square defined on $H^*(Y)$. By Cartan's formula of Steenrod square, we have $Sq_1^i(x_1 \ldots x_{i-1} x_i x_{i+1} \ldots x_m) = 0$ for all $i > 0$. Therefore, Q is invariant under the action of the Steenrod algebra. By Propositions 2.2 and 2.3, we get $q \leq m$.

Case (ii). Let $d_{n+1}(1 \otimes y_i) = 0$ for all i and $d_{n+1}(1 \otimes z_i) \neq 0$ for some i. Let $d_{n+1}(1 \otimes y_i) = 0$ for all i, $d_{n+1}(1 \otimes z_i) = v_i \otimes y_i + \sum_{j\neq i} u_{i,j} \otimes \alpha_{i,j}x_i x_j$ for all $1 \leq i \leq k$ $(\leq m)$, where $v_i, u_{i,j} \in H^{n+1}(B_G)$ and $\alpha_{i,j} \in \{0, 1\}$ and $d_{n+1}(1 \otimes z_i) = 0$ for all $k + 1 \leq i \leq m$ $(k < m)$. Then, we get $E_{2n+1}^{*,*} \cong E_{n+2}^{*,*} \cong (E_2^{*,*} - S)/Q$, where S is a graded ideal generated by $\{1 \otimes z_1, \ldots, 1 \otimes z_k\}$ and Q is generated by $\{v_i \otimes y_i + \sum_{j\neq i} u_{i,j} \otimes \alpha_{i,j}x_i x_j, \beta | \ 1 \leq i \leq k$ and $\beta \in A\}$. Clearly, $d_r(1 \otimes y_i) = 0$ for all i and $r \geq 2$. If $d_{2n+1}(1 \otimes z_i) = 0$ for all $k + 1 \leq i \leq m$ then as above, we get $d_r = 0$ for all $r \geq n + 2$, which contradicts Proposition 2.1. If $d_{2n+1}(1 \otimes z_i) \neq 0$ for all $k + 1 \leq i \leq m$ then $d_{2n+1}(1 \otimes z_i) = w_i \otimes x_i$. Thus, we have $E_{2n+2} \cong (E_{2n+1} - S')/I'$, where S' and I' are graded ideals generated by $\{1 \otimes z_{k+1}, \ldots, 1 \otimes z_m\}$ and $\{w_{k+1} \otimes x_{k+1}, \ldots, w_m \otimes x_m\}$, respectively. Clearly, $d_r = 0$ for all $r \geq 2n + 2$, which is a contradiction. So, we let $d_{n+1}(1 \otimes z_i) = \sum_{j\neq i} u_{i,j} \otimes \alpha_{i,j}x_i x_j$, where for all $1 \leq i \leq m$ and $\alpha_{i,j}$ are not all zero. If $\alpha_{1,j}$ is not equal to zero for some $j = j_1$, then $d_{n+1}(1 \otimes z_1 x_2 \ldots x_{j_1-1} x_{j_1+1} \ldots x_m) = u_{1,j_1} \otimes x_1 x_2 \ldots x_m$. Similarly as above, Q is generated by $\{u_{i,j_i} \otimes x_1 \ldots x_{i-1} x_i x_{i+1} \ldots x_m |$ for all $1 \leq i \leq m\}$ which is invariant under the action of the Steenrod algebra. Note that $\{u_{i,j_i}\}$ are linearly independent. This implies that $q \leq m$.

Case (iii). Let $d_{n+1}(1 \otimes y_i) \neq 0$ for some i and $d_{n+1}(1 \otimes z_j) \neq 0$ for some j. Consider $d_{n+1}(1 \otimes y_i) = u_i \otimes \alpha_i x_i$ and $d_{n+1}(1 \otimes z_i) = v_i \otimes \gamma_i y_i + \sum_{j\neq i} w_{i,j} \otimes \beta_{i,j} x_i x_j$ for all i, where $u_i, v_i, w_{i,j} \in H^{n+1}(B_G)$ and $\alpha_i, \gamma_i, \beta_{i,j} \in \{0, 1\}$. Note that if $\alpha_i \neq 0$ for some i then $\gamma_i = 0$. Suppose $\alpha_i \neq 0$ for all $1 \leq i \leq k$ and $\alpha_i = 0$ for all $k + 1 \leq i \leq m$. Then $\gamma_i = 0$ for all $1 \leq i \leq k$.

If $\beta_{i,j}$ is not equal to zero for some $j = j_i$, then $d_{n+1}(1 \otimes (x_1 \ldots x_{i-1} y_i x_{i+1} \ldots x_m + x_1 \ldots z_i \ldots x_{j_i-1} x_{j_i+1} \ldots x_m)) = (u_i + w_{i,j_i}) \otimes x_1 x_2 \ldots x_m$, for all $1 \leq i \leq k$ and $d_{n+1}(1 \otimes (x_1 \ldots x_{i-1} z_i x_{i+1} \ldots x_{j_{i-1}} x_{j_{i+1}} \ldots x_m) = w_{i,j_i} \otimes x_1 x_2 \ldots x_m$ for $k + 1 \leq i \leq m$, if $\gamma_i = 0$ and $\beta_{i,j} \neq 0$ for some $j = j_i$. Consider the graded ideal Q generated by $(u_i + w_{i,j_i}) \otimes x_1 x_2 \ldots x_m$ for all $1 \leq i \leq k$ and $w_{i,j_i} \otimes x_1 x_2 \ldots x_m$ for $k + 1 \leq i \leq m$. As in Case (1), Q is invariant under the action of the Steenrod algebra. We have chosen $u_i + w_{i,j_i}$ for all $1 \le i \le k$ and w_{i,j_i} for $k+1 \le i \le m$ such that they are linearly independent. Consequently, we have $q \le m$. □

From the above theorem, it is clear that if G acts freely on $Y = \prod_{i=1}^{m} X_n^i$ and trivially on $H^*(Y)$, then $\text{rk}(G) \le m$. For spaces of cohomology type $(0,0)$, we have following result:

Theorem 3.2. *Let X_n^i be a finite CW complex with cohomology type $(0,0)$, for every i. Let G be a finite group acting freely on a space $Y = \prod_{i=1}^{m} X_n^i$. If n is even and G acts trivially on $H^*(Y)$ then $G = (\mathbb{Z}_2)^q$ and $q \le m$.*

Proof. Let p be an odd prime and $p||G|$ then by the Flyod's formula, $\chi(X^G) \equiv 2^{2m} (\text{mod } p)$. This gives that the fixed point set is nonempty. Therefore, the order of G is a power of 2. Suppose H is a cyclic subgroup of G of order 4 and K is a subgroup of H of order 2. Note that $H^*(B_H) \cong \mathbb{Z}_2[t] \otimes \wedge(s)$, where deg $t = 2$ and deg $s = 1$, and $H^*(B_K) \cong \mathbb{Z}_2[t']$, where deg $t' = 1$. Let $i^* : H^*(B_H) \to H^*(B_K)$ be the homomorphism induced by the inclusion map $i : K \hookrightarrow H$. Then $i^*(s) = 0$ and $i^*(t) = t'^2$. Since fibrations $X \hookrightarrow X_H \to B_H$ and $X \hookrightarrow X_K \to B_K$ have simple system of local coefficient, so $E_2^{k,l} = H^k(B_H) \otimes H^l(X)$ and $\bar{E}_2^{k,l} = H^k(B_K) \otimes H^l(X)$. By the naturality of the spectral sequence, we have the following commutative diagram

$$\begin{array}{ccc} E_r^{k,\,l} & \xrightarrow{d_r} & E_r^{k+r,\,l+1-r} \\ \downarrow \alpha & & \downarrow \alpha \\ \bar{E}_r^{k,\,l} & \xrightarrow{\bar{d}_r} & \bar{E}_r^{k+r,\,l+1-r} \end{array}$$

where $\alpha = i^* \otimes 1$ in E_2-term. As $i^*(s) = 0$, we have $\bar{d}_{n+1}(\bar{1} \otimes y_i) = \bar{d}_{n+1}(\alpha(1 \otimes y_i)) = \alpha(d_{n+1}(1 \otimes y_i)) = 0$. Therefore, $\bar{d}_{n+1}(\bar{1} \otimes y_i) = 0$ for all i. Similarly, $\bar{d}_{n+1}(\bar{1} \otimes z_i) = 0$ for all i. By the proof of above theorem, we know that $\bar{d}_{n+1}(\bar{1} \otimes y_i)$ and $\bar{d}_{n+1}(\bar{1} \otimes z_i)$ can't be trivial simultaneously for all i. Thus G contains no element of order 4. The result follows from Theorem 3.1. □

The above result generalizes Theorem 3.8 [17].

An example of free action of $G = \mathbb{Z}_2$ on spaces of cohomology type $(0,0)$, and the cohomological structure of orbit space has been discussed in [12]. Here, we give an example of orbit space of free involution on spaces of cohomology type (0,0).

Example 3.3. Consider the antipodal action of $\mathbb{Z}_2$ on $\mathbb{S}^{2n}$ and $\mathbb{S}^{3n}$, where $n > 1$. Then, $\mathbb{S}^{n-1} \subset \mathbb{S}^{2n} \cap \mathbb{S}^{3n}$ is invariant under this action. So, we have a free $\mathbb{Z}_2$-action on $X_n = \mathbb{S}^{2n} \cup_{\mathbb{S}^{n-1}} \mathbb{S}^{3n}$ which is obtained by attaching the spheres $\mathbb{S}^{2n}$ and $\mathbb{S}^{3n}$ along $\mathbb{S}^{n-1}$. We have shown that X_n is a space of type $(0,0)$ [17]. It is easy to show that $X_n/\mathbb{Z}_2$ is homeomorphic to $Y = \mathbb{R}P^{2n} \cup_{\mathbb{R}P^{(n-1)}} \mathbb{R}P^{3n}$, which is obtained by attaching the real projective spaces $\mathbb{R}P^{2n}$ and $\mathbb{R}P^{3n}$ along $\mathbb{R}P^{n-1}$. Now, we determine the cohomology structure of $X_n/\mathbb{Z}_2$. As X_n is connected, $H^0(X_n/\mathbb{Z}_2) \cong \mathbb{Z}_2$. Note that $i : \mathbb{S}^{3n} \hookrightarrow X_n$ is a $\mathbb{Z}_2$-equivariant map, therefore, $u^i \neq 0$ for all $i \le 3n$ and $u \in H^1(X_n/\mathbb{Z}_2)$ is the characteristic class of the principal $\mathbb{Z}_2$-bundle $X_n \to X_n/\mathbb{Z}_2$. By the Gysin-sequence of the principal $\mathbb{Z}_2$-bundle $X_n \to X_n/\mathbb{Z}_2$, we have $H^i(X_n/\mathbb{Z}_2) \cong \mathbb{Z}_2$, for all $i \le n-1$ and $H^i(X_n/\mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ for all $n \le i \le 2n-1$. Also, $H^i(X_n/\mathbb{Z}_2)$ is generated by u^i for all $1 \le i \le n-1$ and $H^{n+i}(X_n/\mathbb{Z}_2)$ is generated by u^{n+i} and $u^i v$ for all $0 \le i \le n-1$, where $v \in H^n(X_n/\mathbb{Z}_2)$ such that $\pi^*(v) = x$. If $\pi^* : H^{2n}(X_n/\mathbb{Z}_2) \to H^{2n}(X_n)$ is nontrivial, then $H^i(X_n/\mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ for all $2n \le i \le 3n$. This implies that $H^{3n+1}(X_n/\mathbb{Z}_2) \neq 0$, a contradiction. Therefore, $\pi^* : H^{2n}(X_n/\mathbb{Z}_2) \to H^{2n}(X_n)$ must be trivial. We have $H^{2n}(X_n/\mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $H^i(X_n/\mathbb{Z}_2) \cong \mathbb{Z}_2$ for all $2n+1 \le i \le 3n$ and $H^i(X_n/\mathbb{Z}_2) = 0$ for all $i > 3n$. Also, for $2n \le i \le 3n$, $H^i(X_n/\mathbb{Z}_2)$ is generated by u^i. Clearly, $u^{3n+1} = v^2 + \alpha u^{2n} + \beta u^n v = u^{n+1} v + \gamma u^{2n+1} = 0$, where $\alpha, \beta, \gamma \in \{0, 1\}$. Therefore, $H^*(X_n/\mathbb{Z}_2) \cong \mathbb{Z}_2[u, v]/\langle u^{3n+1}, v^2 + \beta u^n v, u^{n+1} v + \gamma u^{2n+1} \rangle$, where deg $u = 1$ and deg $v = n$. This realizes Theorem 4.1 [12].

Next, we observe that G acts trivially on $H^*(Y)$. Let $Y = X_n \times \cdots \times X_n$ (m times) and g be a generator of G. Note that the diagonal action of above action on Y gives free $G = \mathbb{Z}_2$ action. Let $\sigma = (\sigma_1, \ldots, \sigma_m) : \Delta^n \to Y$ be a n-simplex. Then, $g\sigma = (g\sigma_1, \ldots, g\sigma_m) = -(\sigma_1, \ldots, \sigma_m)$. Thus g^* acts trivially on $H^n(Y)$. Similarly, g^* acts trivially on $H^{2n}(Y)$ and $H^{3n}(Y)$. Therefore, g^* acts trivially on $H^*(Y)$.

We conclude with the following generalizations:

Conjecture 3.4. *Let G act freely on a space $Y = \prod_{i=1}^{m} X_{n_i}$, where X_{n_i} is a finite CW complex with cohomology type (a, b), characterized by an integer n_i and a and b are even for every i. Then $rk(G) \leq m$.*

References

[1] A. Adem, W. Browder, *The free rank of symmetry of* $(\mathbb{S}^n)^k$, Invent. Math. **92** (1988), 431–440.
[2] A. Adem, E. Yalcin, *On some examples of group actions and group extensions*, J. Group Theory **2** (1999), 69–79.
[3] C. Allday, V. Puppe, *Cohomological methods in transformation groups*, Cambridge Studies in Advanced Mathematics 32, Cambridge University Press, Cambridge, 1993.
[4] C. Allday, *Elementary abelian p-group actions on lens spaces*, Topology Hawaii (Honolulu, HI, 1990), 1–11, World Sci. Publishing, River Edge, NJ, 1992.
[5] D. J. Benson, J. F. Carlson, *Complexity and multiple complexes*, Math. Z. **195** (1987), 221–238.
[6] G. Carlsson, *On the non-existence of free actions of elementary abelian groups on products of spheres*, Amer. J. Math. **102** (1980), 1147–1157.
[7] P. E. Conner, *On the action of a finite group on* $\mathbb{S}^n \times \mathbb{S}^n$, Ann. of Math. **66** (1957), 586–588.
[8] L. W. Cusick, *Elementary abelian 2-groups that act freely on products of real projective spaces*, Proc. Amer. Math. Soc. **87** (1983), 728–730.
[9] L. W. Cusick, *Free actions on spaces with nonzero Euler characteristic*, Topology Appl. **33** (1989), 185–196.
[10] I. James, *Note on cup products*, Amer. Math. Soc. **14** (1957), 374–383.
[11] J. McCleary, *A user's guide to spectral sequences*, (IInd edition), Cambridge University Press, Cambridge, 2001.
[12] P. L. Q. Pergher, H. K. Singh, T. B. Singh, *On* $\mathbb{Z}_2$ *and* $\mathbb{S}^1$ *free actions on spaces of cohomology type (a, b)*, Houston J. Math. **36** (2010), 137–146.
[13] B. Hanke, *The stable free rank of symmetry of products of spheres*, Invent. Math. **178** (2009), 265–298.
[14] A. Heller, *A note on spaces with operators*, Illinois J. Math. **3** (1959), 98–100.
[15] O. B. Okutan, E. Yalcın, *Actions on products of spheres at high dimensions*, Algebr. Geom. Topol. **13** (1959), 2087–2099.
[16] P. A. Smith, *Permutable periodic transformations*, Proc. Nat. Acad. Sci. USA. **30** (1944), 105–108.
[17] K. S. Singh, H. K. Singh, T. B. Singh, *Free Action of Finite Groups on Spaces of Cohomology Type* $(0, b)$, Glasgow Math. J. **60** (2018), 673–680.
[18] E. Yalcın, *Free actions of p-groups on products of lens spaces*, Proc. Amer. Math. Soc. **129** (2002), 887–898.
[19] R. G. Swan, *Periodic resolutions for finite groups*, Ann. of Math. **94** (1960), 267–291.
[20] H. Toda, *Note on cohomology ring of certain spaces*, Proc. Amer. Math. Soc. **14** (1963), 89–95.