Elementary abelian group actions on a product of spaces of cohomology type \((a, b)\)

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Abstract. Let \(X_n\) be a finite CW complex with cohomology type \((a, b)\), characterized by an integer \(n > 1\) [20]. In this paper, we show that if \(G = (\mathbb{Z}_2)^n\) acts freely on the product \(Y = \prod_{i=1}^n X_n\), where \(X_n\) are finite CW complexes with cohomology type \((a, b)\), \(a\) and \(b\) are even for every \(i\), then \(q \leq m\). Moreover, for \(n\) even and \(a = b = 0\), we prove that \(G = (\mathbb{Z}_2)^q\) \((q \leq m)\) is the only finite group which can act freely on \(Y\). These are generalizations of the results which says that the rank of a group acting freely on a space with cohomology type \((0, 0)\) is the only finite group which acts freely on spaces of cohomology type \((0, 0)\) [17].

1. Introduction

Let \(G\) be a finite group and \(p\) be a prime. The rank of \(G\) is defined by \(rk(G) = \max\{q \mid (\mathbb{Z}_p)^q \subset G\text{ as a subgroup}\}\). One of the interesting problems in topological transformation groups is to find \(rk(G)\) when \(G\) acts freely on a space \(X\). P.A. Smith [16] and R.G. Swan [19] showed that if a finite group \(G\) acts freely on a sphere \(S^n\) then \(rk(G) = 1\). Conner [7] proved that if a finite group \(G\) acts freely on \(S^n \times S^m\) then \(rk(G) = 2\). Heller [14] proved the same result for arbitrary product of two spheres \(S^n \times S^m\). In this direction, Benson and Carlson [5] arise the following conjecture: If a finite group \(G\) acts freely on \(X = \prod_{i=1}^m S^{n_i}\) then \(rk(G) \leq m\). So far this conjecture has been proved in the following cases: Carlsson [6] proved the result for \(p = 2\) and \(n_1 = n_2 = \cdots = n_m\) with the condition that the induced action of \(G\) is trivial on the mod 2-cohomology algebra of \(X\). Adem and Browder [1] proved this result for \(n_1 = n_2 = \cdots = n_m\) and \(n_1 \neq 1, 3, 7\). Hanke [13] proved Carlsson conjecture for all primes \(p > 2(3n_1 + \cdots + n_m)\). Okutan and Yalcın [15] proved this result for dimensions \(n_i\) which are higher compared to the differences \(|n_i - n_j|\) among all dimensions.

Cusick [8] proved that if \(G\) acts freely on \(X = \prod_{i=1}^m \mathbb{R}P^{n_i}\) and the induced action of \(G\) on the mod 2 cohomology algebra of \(X\) is trivial then \(rk(G) = v(n_1) + \cdots + v(n_m)\), where

\[
v(n) = \begin{cases} 
0 & \text{if } n \text{ is even} \\
1 & \text{if } n \equiv 1 \pmod{4} \\
2 & \text{if } n \equiv 3 \pmod{4}.
\end{cases}
\]
Adem and Yalcin [2] improved this result without the assumption of trivial induced action on the mod 2 cohomology algebra of $X$. Cusick [9] shown that if $X = \prod_{i=1}^{m} \mathbb{C}P^{n_{i}}$ then $\text{rk}(G) = \nu(n_{1}) + \cdots + \nu(n_{m})$, where

$$
\nu(n) = \begin{cases} 
0 & \text{if } n \text{ is even} \\
1 & \text{if } n \equiv 1 \pmod{4}.
\end{cases}
$$

Allday [4] put the following conjecture: If $G$ acts freely on $X = \prod_{i=1}^{m} \mathbb{C}P^{2n_{i}-1}$, where $p$ an odd prime, then $\text{rk}(G) = m$. Yalcin [18] proved this conjecture when $n_{1} = \cdots = n_{m}$. This conjecture is still open for general case.

In this paper, we have showed that the rank of a finite group $G$ acting freely on the product $\prod_{i=1}^{m} X_{i}$, where $X_{i}$ are the spaces of type $(a, b)$ and $a$ and $b$ are even for each $i$, is almost $m$. Moreover, for $n$ even, we have proved that if a finite $G$ acts freely on $\prod_{i=1}^{m} X_{i}$ where $X_{i}$ are spaces of type $(0, 0)$ then $G = (\mathbb{Z}_{2})^{q}$, where $q \leq m$.

2. Preliminaries

Given two integers $a$ and $b$, a space $X$ is said to have cohomology type $(a, b)$ if $H^{i}(X; \mathbb{Z}) \cong \mathbb{Z}$ for $j = 0, n, 2n$ and $3n$ only, and the generators $x \in H^{n}(X; \mathbb{Z})$, $y \in H^{2n}(X; \mathbb{Z})$ and $z \in H^{3n}(X; \mathbb{Z})$ satisfies $x^{2} = ay$ and $xy = bz$.

It is denoted by $X_{a, b}$. For example, $S^{n} \times S^{2n}$ has type $(0, 1)$, $CP^{3}$ and $QP^{3}$ have type $(1, 1)$, $CP^{2} \vee S^{6}$ has type $(1, 0)$ and $S^{5} \vee S^{6} \vee S^{3k}$ has type $(0, 0)$. Such spaces were first investigated by James [10] and Toda [20]. Let $Y$ be $\prod_{i=1}^{m} X_{i}$, where $X_{i}$ is a finite CW-complex with cohomology type $(a, b)$. The cohomology algebra of $Y$ is given by

$$
H^{*}(Y; \mathbb{Z}_{2}) \cong \mathbb{Z}_{2}[x_{1}, \ldots, x_{n}, y_{1}, \ldots, y_{m}, z_{1}, \ldots, z_{m}]/I,
$$

where $I$ is a graded ideal generated by a set $A = \{x_{i}^{2} - ay_{i}, y_{i}^{2}, z_{i}^{2}, x_{i}y_{j} - bz_{i}, y_{i}z_{i}, x_{i}z_{i} | 1 \leq i \leq m \}$ and deg $x_{i} = n$, deg $y_{i} = 2n$ and deg $z_{i} = 3n$ for all $1 \leq i \leq m$.

The Borel construction on $X$ is defined as the orbit space $X_{G} = (X \times E_{G})/G$, where the compact Lie group $G$ acts diagonally (and freely) on the product $X \times E_{G}$. The projection $X \times E_{G} \to E_{G}$ gives a fibration $X_{G} \to B_{G}$ with fiber $X$. We will use the Leray-Serre spectral sequence associated to the Borel fibration $X \xrightarrow{i} X_{G} \xrightarrow{\pi} B_{G}$. If $\pi_{1}(B_{G})$ acts trivially on $H^{*}(X; R)$ ($R$ is a field) then the system of local coefficient is simple and $E_{2}$-term of the spectral sequence of the fibration $X \xrightarrow{i} X_{G} \xrightarrow{\pi} B_{G}$ is given by $E_{2}^{i,j} = H^{i}(B_{G}; R) \otimes H^{j}(X; R)$. Note that for $G = (\mathbb{Z}_{2})^{q}$, $H^{*}(B_{G}; \mathbb{Z}_{2}) \cong \mathbb{Z}_{2}[t_{1}, t_{2}, \ldots, t_{q}]$, where deg $t_{i} = 1$ for all $1 \leq i \leq q$.

For the results in spectral sequences, we refer [11]. Throughout this paper, cohomologies are Čech cohomology with coefficient in $\mathbb{Z}_{2}$. Now, we recall some results which were used in this paper.

Proposition 2.1. ([3]) Let $G = (\mathbb{Z}_{2})^{q}$ act on a finitistic space $X$ and $H^{i}(X) = 0$ for all $i > n$. Then $H^{i}(X/G) = 0$ for all $i > n$.

The following results are proved by G. Carlsson [6]:

Proposition 2.2. ([6]) Suppose $\{f_{1}, \ldots, f_{k}\}$ are elements of $H^{n}(B_{2}(Z_{2}); \mathbb{Z}_{2})$, regarded as homogeneous polynomials of degree $n$ in $q$ variables. Then they have a nontrivial common zero in $(\mathbb{Z}_{2})^{q}$ if and only if there exist a inclusion $i : \mathbb{Z}_{2} \hookrightarrow (\mathbb{Z}_{2})^{q}$ such that $i^{*}(f_{j}) = 0$ for all $j$.

Proposition 2.3. ([6]) Let $\langle f_{1}, \ldots, f_{k} \rangle$ be an ideal generated by homogeneous polynomials $f_{j}$ in $\mathbb{Z}_{2}[t_{1}, \ldots, t_{q}]$ which is invariant under the action of the Steenrod algebra. If $q > k$, then there exists nontrivial common zero to $f_{1}, \ldots, f_{k}$.

3. Main theorems

In this section, our aim is to determine an elementary abelian 2-group, which can act freely on a finite product of spaces of type $(a, b)$. We show that the rank of elementary 2-abelian groups which acts freely on $Y$ will not exceed $m$. This generalizes Theorem 3.2 [17].
Theorem 3.1. Let $G = (\mathbb{Z}_2)^m$ act freely on a space $Y = \prod_{i=1}^{m} X_i^n$, where $X_i^n$ is a finite CW-complex with cohomology type $(a, b)$, and $a$ and $b$ are even for every $i$. If $G$ acts trivially on $H^*(Y)$ then $q \leq m$.

Proof. As $G$ acts trivially on $H^*(Y)$, $E^2_{ij} = H^n(BG) \otimes H^i(Y)$. Let $x_i \in H^i(Y)$, $y_i \in H^{2n}(Y)$ and $z_i \in H^{3n}(Y)$ be generators of the cohomology algebra of $H^*(Y)$.

First, we prove that $d_{n+1}(1 \otimes x_i) = 0$ for all $1 \leq i \leq m$. Let $d_{n+1}(1 \otimes x_i) = v_i \otimes 1$ for some $i$. Consider $d_{n+1}(1 \otimes x_i) = \sum_j w_{ij} \otimes x_j$, where $w_{ij} \in \{0, 1\}$. By the multiplicative property of spectral sequence, we have $0 = d_{n+1}(1 \otimes x_i y_j) = v_i \otimes y_j + \sum_{j \neq i} w_{ij} \otimes x_j x_i$, a contradiction. This implies that $d_{n+1}(1 \otimes x_i) = 0$ for all $1 \leq i \leq m$. Therefore, $d_i(1 \otimes x_i) = 0$ for all $i$ and $r \geq 2$.

Next, we have observed that both $d_{n+1}(1 \otimes y_i)$ and $d_{n+1}(1 \otimes z_i)$ can’t be trivial simultaneously for all $i$. For that, if $d_{n+1}(1 \otimes y_i) = d_{n+1}(1 \otimes z_i) = 0$ for all $i$ then, $E^{3n}_{2n+1} = E^\infty_{2n+1}$. Clearly, $d_{n+1}(1 \otimes y_i) = 0$ for all $i$. So, $d_i(1 \otimes y_i) = 0$ for all $i$ and for all $r \geq 2$. If $d_{n+1}(1 \otimes z_i) \neq 0$ for some $i$ then $d_{n+1}(1 \otimes z_i) = \sum_{j=1} w_{ij} \otimes x_j$, where $w_{ij} \in \{0, 1\}$. Then $0 = d_{n+1}(1 \otimes x_j) = \sum_{j=1} w_{ij} \otimes x_j x_i$. Thus, $x_j = 0$ for all $j \neq i$ and $x_j = 1$ for $j = i$. Let $d_{n+1}(1 \otimes z_i) = v_i \otimes x_i$ for all $1 \leq i \leq k \leq m$. Then, we get $E^{3n+1}_{2n+2} \cong E_{2n+2}^\infty \cong (E^{2n+1}_{2n+1} - S)/Q$, where $S$ is a graded ideal generated by $\{1 \otimes x_1, \ldots, 1 \otimes x_k\}$ and graded ideal $Q$ is generated by $[v_1 \otimes x_1, \ldots, v_k \otimes x_k]$ for all $1 \leq i \leq k$ and $\beta \in A$. Clearly, $d_{n+1}(1 \otimes z_i) = 0$ for all $k + 1 \leq i \leq m$ and so $d_i(1 \otimes z_i) = 0$ for all $k + 1 \leq i \leq m$ and for all $r \geq 3n + 1$. Therefore, $d_i = 0$ for all $r \geq 3n + 1$. So, $E^{3n+1}_{2n+2} \cong E^{\infty}_{2n+2}$, which contradicts Proposition 2.1. Thus, the following three cases are possible:

(i) $d_{n+1}(1 \otimes y_i) \neq 0$ for some $i$ and $d_{n+1}(1 \otimes z_i) = 0$ for all $i$.

(ii) $d_{n+1}(1 \otimes y_i) = 0$ for all $i$ and $d_{n+1}(1 \otimes z_i) \neq 0$ for some $i$.

(iii) $d_{n+1}(1 \otimes y_i) \neq 0$ for some $i$ and $d_{n+1}(1 \otimes z_i) \neq 0$ for some $j$.

Case (i). Let $d_{n+1}(1 \otimes y_i) \neq 0$ for some $i$ and $d_{n+1}(1 \otimes z_i) = 0$ for all $i$. If $d_{n+1}(1 \otimes y_i) \neq 0$ then $d_{n+1}(1 \otimes y_i) = \sum_j u_{ij} \otimes x_j$, where $u_{ij} \in H^n(BG)$ and $x_j \in \{0, 1\}$. We have $0 = d_{n+1}(1 \otimes y_i) = \sum_{j=1} u_{ij} \otimes x_j x_i$. This implies $x_j = 0$ for all $j \neq i$ and $x_i \neq 0$. Therefore, $d_{n+1}(1 \otimes y_i) = u_{ij} \otimes x_i$ for all $1 \leq i \leq m$, where $u_{ij}$ is linearly independent subset of $H^n(BG)$, and $d_{n+1}(1 \otimes z_i) = 0$ for all $i$. Then $d_{n+1}(1 \otimes x_i) = v_i \otimes x_i$ for all $1 \leq i \leq k \leq m$, where $v_i$ is a graded ideal generated by $\{1 \otimes x_1, \ldots, 1 \otimes x_k\}$ and graded ideal $Q$ is generated by $[v_1 \otimes x_1, \ldots, v_k \otimes x_k]$ for all $1 \leq i \leq k$ and $\beta \in A$. Clearly, $d_i(1 \otimes y_i) = 0$ for all $i$ and $r \geq 2$. If $d_{n+1}(1 \otimes z_i) = 0$ for all $k + 1 \leq i \leq m$ then as above, we get $d_i = 0$ for all $r \geq n + 2$, which contradicts Proposition 2.1. If $d_{n+1}(1 \otimes z_i) \neq 0$ for all $k + 1 \leq i \leq m$ then $d_{n+1}(1 \otimes z_i) = v_i \otimes x_i$. Thus, we have $E^{3n+1}_{2n+2} \cong (E^{2n+1}_{2n+1} - S)/Q$, where $S$ and $Q$ are graded ideal generated by $\{1 \otimes x_1, \ldots, 1 \otimes x_k\}$ and $\{w_{i} \otimes x_i \}, \ldots, \{w_{m} \otimes x_m \}$, respectively. Clearly, $d_i = 0$ for all $r \geq 2n + 2$, which is a contradiction. So, we let $d_{n+1}(1 \otimes z_i) = \sum_{j=1} u_{ij} \otimes x_j x_i$, where for all $1 \leq i \leq m$ and $x_i \neq 0$. If $x_i \neq 0$ is not equal to zero for some $j = j_1$, then $d_{n+1}(1 \otimes x_{j_1} \otimes \cdots \otimes x_{j_k}) = u_{ij} \otimes x_{j_1} \otimes \cdots \otimes x_{j_k}$ for all $1 \leq i \leq k$ and $d_{n+1}(1 \otimes x_{j_1} \otimes \cdots \otimes x_{j_k}) = u_{ij} \otimes x_{j_1} \otimes \cdots \otimes x_{j_k}$ for $k + 1 \leq i \leq m$, if $y_i = 0$ and $\beta_j \neq 0$ for some $j = j_i$. Consider the graded ideal $Q$ generated by $[u_i \otimes w_{ij}, \otimes x_{j_1} \otimes \cdots \otimes x_{j_k}]$ for all $1 \leq i \leq k$ and $[u_i \otimes w_{ij}, \otimes x_{j_1} \otimes \cdots \otimes x_{j_k}]$ for $k + 1 \leq i \leq m$. As in Case (1), $Q$ is invariant.
under the action of the Steenrod algebra. We have chosen $u_i + w_{i,j}$ for all $1 \leq i \leq k$ and $w_{i,j}$ for $k + 1 \leq i \leq m$ such that they are linearly independent. Consequently, we have $q \leq m$. 

From the above theorem, it is clear that if $G$ acts freely on $Y = \coprod_{i=1}^{m} X_{n}$ and trivially on $H^{r}(Y)$, then $\text{rk}(G) \leq m$. For spaces of cohomology type $(0,0)$, we have the following result:

**Theorem 3.2.** Let $X_n$ be a finite CW complex with cohomology type $(0,0)$, for every $i$. Let $G$ be a finite group acting freely on a space $Y = \coprod_{i=1}^{m} X_n$. If $n$ is even and $G$ acts trivially on $H^{r}(Y)$ then $G = (\mathbb{Z}_{2})^n$ and $q \leq m$.

**Proof.** Let $p$ be an odd prime and $p|\lvert G\rvert$ then by the Flyod’s formula, $\chi(X^{S}) \equiv 2^{m}(\text{mod } p)$. This gives that the fixed point set is nonempty. Therefore, the order of $G$ is a power of 2. Suppose $H$ is a cyclic subgroup of $G$ of order 4 and $K$ is a subgroup of $H$ of order 2. Note that $H^{r}(B_{H}) \cong \mathbb{Z}_{2}[t] \otimes \wedge(s)$, where $\deg t = 2$ and $s = 1$, and $H^{r}(B_{K}) \cong \mathbb{Z}_{2}[l]$, where $\deg l = 1$. Let $\iota : H^{r}(B_{H}) \to H^{r}(B_{K})$ be the homomorphism induced by the inclusion map $i : K \hookrightarrow H$. Then $\iota(s) = 0$ and $\iota(l) = l^{2}$. Since fibrations $X \hookrightarrow X_{n} \to B_{H}$ and $X \hookrightarrow X_{n} \to B_{K}$ have simple system of local coefficient, so $H^{2r}(B_{H}) \otimes H^{r}(X)$ and $H^{2r}(B_{K}) \otimes H^{r}(X)$. By the naturality of the spectral sequence, we have the following commutative diagram

$$
\begin{array}{ccc}
E^{r+1}_{p,q} & \xrightarrow{d} & E^{r+1}_{p+q,r+1-q} \\
\downarrow \alpha & & \downarrow \alpha \\
E^{r+1}_{p,q} & \xrightarrow{d} & E^{r+1}_{p+q,r+1-q}
\end{array}
$$

where $\alpha = \iota \otimes 1$ in $E_{2}$-term. As $\iota(s) = 0$, we have $d_{s+1}(1 \otimes y_{i}) = d_{s+1}(a(1 \otimes y_{i})) = a(d_{s+1}(1 \otimes y_{i})) = 0$. Therefore, $d_{s+1}(1 \otimes y_{i}) = 0$ for all $i$. Similarly, $d_{s+1}(1 \otimes z_{i}) = 0$ for all $i$. By the proof of above theorem, we know that $d_{s+1}(1 \otimes y_{i})$ and $d_{s+1}(1 \otimes z_{i})$ can’t be trivial simultaneously for all $i$. Thus $G$ contains no element of order 4. The result follows from Theorem 3.1. 

The above result generalizes Theorem 3.8 [17].

An example of free action of $G = \mathbb{Z}_{2}$ on spaces of cohomology type $(0,0)$, and the cohomological structure of orbit space has been discussed in [12]. Here, we give an example of orbit space of free involution on spaces of cohomology type $(0,0)$.

**Example 3.3.** Consider the antipodal action of $\mathbb{Z}_{2}$ on $S^{2n}$ and $S^{3n}$, where $n > 1$. Then, $S^{2n-1} \subset S^{2n} \cap S^{3n}$ is invariant under this action. So, we have a free $\mathbb{Z}_{2}$-action on $X_{n} = S^{2n} \cup_{S^{2n}} S^{3n}$ which is obtained by attaching the spheres $S^{2n}$ and $S^{3n}$ along $S^{2n-1}$. We have shown that $X_{n}$ is a space of type $(0,0)$ [17]. It is easy to show that $X_{n}/\mathbb{Z}_{2}$ is homeomorphic to $Y = \mathbb{R}P^{2n} \cup_{\mathbb{R}P^{n-1}} \mathbb{R}P^{3n}$, which is obtained by attaching the real projective spaces $\mathbb{R}P^{2n}$ and $\mathbb{R}P^{3n}$ along $\mathbb{R}P^{n-1}$. Now, we determine the cohomology structure of $X_{n}/\mathbb{Z}_{2}$. As $X_{n}$ is connected, $H^{r}(X_{n}/\mathbb{Z}_{2}) \cong \mathbb{Z}_{2}$. Note that $i : S^{2n} \hookrightarrow X_{n}$ is a $\mathbb{Z}_{2}$-equivariant map, therefore, $u^{\prime} \neq 0$ for all $i \leq 3n$ and $u \in H^{r}(X_{n}/\mathbb{Z}_{2})$ is the characteristic class of the principal $\mathbb{Z}_{2}$-bundle $X_{n} \to X_{n}/\mathbb{Z}_{2}$. By the Gysin-sequence of the principal $\mathbb{Z}_{2}$-bundle $X_{n} \to X_{n}/\mathbb{Z}_{2}$, we have $H^{r}(X_{n}/\mathbb{Z}_{2}) \cong \mathbb{Z}_{2}$, for all $i \leq n - 1$ and $H^{r}(X_{n}/\mathbb{Z}_{2}) \cong \mathbb{Z}_{2} \oplus \mathbb{Z}_{2}$ for all $i \leq n - 2$. It is easy to show that $H^{r}(X_{n}/\mathbb{Z}_{2})$ is generated by $u^{\prime}$ for all $i \leq n - 1$ and $H^{r+1}(X_{n}/\mathbb{Z}_{2})$ is generated by $u^{\prime}v^{\prime}$ and $u^{\prime}v$ for all $0 \leq i \leq n - 1$, where $v \in H^{s}(X_{n}/\mathbb{Z}_{2})$ such that $\pi^{\ast}(v) = x$. If $\pi^{\ast} : H^{r}(X_{n}/\mathbb{Z}_{2}) \to H^{r}(X_{n})$ is nontrivial, then $H^{r}(X_{n}/\mathbb{Z}_{2}) \cong \mathbb{Z}_{2} \oplus \mathbb{Z}_{2} \oplus \mathbb{Z}_{2}$ for all $2n \leq i \leq n$. This implies that $H^{2n+1}(X_{n}/\mathbb{Z}_{2}) \neq 0$, a contradiction. Therefore, $\pi^{\ast} : H^{2n}(X_{n}/\mathbb{Z}_{2}) \to H^{2n}(X_{n})$ must be trivial. We have $H^{2n}(X_{n}/\mathbb{Z}_{2}) \cong \mathbb{Z}_{2} \oplus \mathbb{Z}_{2}$, $H^{3n}(X_{n}/\mathbb{Z}_{2}) \cong \mathbb{Z}_{2}$ for all $2n + 1 \leq i \leq 3n$ and $H^{3n}(X_{n}/\mathbb{Z}_{2}) = 0$ for all $i > 3n$. Also, for $2n \leq i \leq 3n$, $H^{r}(X_{n}/\mathbb{Z}_{2}) \neq 0$. Clearly, $u^{2n+1} = v^{2} + au^{2n} + \beta u v^{\prime} = u^{2n+1} v + \gamma u^{2n+1} = 0$, where $\alpha, \beta, \gamma \in [0, 1]$. Therefore, $H^{r}(X_{n}/\mathbb{Z}_{2}) \cong \mathbb{Z}_{2}[u, v]/(u^{2n+1}, v^{2} + \beta u v^{\prime}, u^{2n+1} v + \gamma u^{2n+1})$, where $u, v = 1$ and $\deg v = n$. This realizes Theorem 4.1 [12].

Next, we observe that $G$ acts trivially on $H^{r}(Y)$. Let $Y = X_{n} \times \cdots \times X_{n}$ ($m$ times) and $g$ be a generator of $G$. Note that the diagonal action of above action on $Y$ gives free $G = \mathbb{Z}_{2}$ action. Let $\sigma = (\sigma_{1}, \ldots, \sigma_{m}) : \Delta^{m} \to Y$ be a n-simplex. Then, $g \sigma = (g \sigma_{1}, \ldots, g \sigma_{m}) = - (\sigma_{1}, \ldots, \sigma_{m})$. Thus $g^{\prime}$ acts trivially on $H^{r}(Y)$. Similarly, $g^{\prime}$ acts trivially on $H^{2n}(Y)$ and $H^{3n}(Y)$. Therefore, $g^{\prime}$ acts trivially on $H^{r}(Y)$.

We conclude with the following generalizations:
Conjecture 3.4. Let $G$ act freely on a space $Y = \prod_{i=1}^{m} X_n$, where $X_n$ is a finite CW complex with cohomology type $(a, b)$, characterized by an integer $n_i$ and $a$ and $b$ are even for every $i$. Then $\text{rk}(G) \leq m$.

References