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Minimum degree condition of Berge Hamiltonicity in random 3-uniform hypergraphs

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Abstract. A graph H has Hamiltonicity if it contains a cycle which covers each vertex of H. In graph theory, Hamiltonicity is a classical and worth studying problem. In 1952, Dirac proved that any n-vertex graph H with minimum degree at least $\lceil \frac{n}{2} \rceil$ has Hamiltonicity. In 2012, Lee and Sudakov proved that if $p \gg \frac{\log n}{n}$, then asympotically almost surely each n-vertex subgraph of random graph $G(n,p)$ with minimum degree at least $(1/2 + o(1))np$ has Hamiltonicity. In this paper, we exend Dirac's theorem to random 3-uniform hypergraphs. The random 3-uniform hypergraph model $H^3(n,p)$ consists of all 3-uniform hypergraphs on n vertices and every possible edge appears with probability p randomly and independently. We prove that if $p \gg \frac{\log n}{n^2}$, then asympotically almost surely every n-vertex subgraph of $H^3(n,p)$ with minimum degree at least $(\frac{1}{4} + o(1))\binom{n}{2}p$ has Berge Hamiltonicity. The value $\frac{\log n}{n^2}$ and constant $1/4$ both are best possible.

1. Introduction

Given a graph H, if there is a cycle contains all vertices of H exactly once, then we say the cycle is a Hamilton cycle and the graph H has Hamiltonicity. If the number of edges and vertices of a graph is large enough, then find a Hamilton cycle is NP-complete [1]. So study its sufficient conditions is very important. The one of classic conclusions is Dirac's theorem [2], which stated that any graph on n vertices with minimum degree at least $\lceil n/2 \rceil$ has Hamiltonicity in 1952. We mainly consider the applications of Dirac type in random graphs. And we say that random graph *asympotically almost surely* has property $\mathcal{P}$ if the probability tends to 1 as n goes to infinity. We used $a \gg b$ to indicate $\frac{a}{b} = o(1)$. In 2012, Lee and Sudakov [3] studied the application of Dirac's theorem in random graphs, which stated that if $p \gg \frac{\log n}{n}$, then asympotically almost surely any subgraph of random graph $G(n,p)$ with minimum degree at least $(1/2 + o(1))np$ has Hamiltonicity. And the value $\frac{\log n}{n}$ and $1/2$ both are asymptotically tight.

A *k-uniform hypergraph* is a tuple (V, E), which V is a vertex set, E is an edge set and every edge of E is a set of k distinct vertices. The *random 3-uniform hypergraph* model $H^3(n,p)$ consists of all 3-uniform hypergraphs on n vertices and every possible hyperedge appears with probability p randomly and independently. And Berge cycle is the first cycle defined in different cycle concepts of hypergraph [4]. A cycle

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$v_1e_1v_2e_2\cdots v_te_t(v_{t+1}=v_1)$ is called *Berge cycle* if $v_i \neq v_j$, $e_i \neq e_j$ and $\{v_j, v_{j+1}\} \subset e_j$ for every $i, j \in [t]$ and $i \neq j$. We say a k-uniform hypergraph H has *Berge Hamiltonicity* if it contains a Berge Hamilton cycle which covers all vertices of H.

One of the earlier results of Berge cycles on hypergraphs was obtained by Bermond, Germa and Heydemann [5] in 1976, they proved that for any integer $k \geq 3$ and $n \geq k+1$, if k-uniform hypergraph H has every vertex degree at least $\binom{n-2}{k-1}+k-1$, then H contains a Berge cycle of length at least n. Follows that, Clemens, Ehrenmüller and Person [6] extended the Dirac's theorem to random k-uniform hypergraph $H^k(n,p)$ in 2020, and showed that for every integer $k \geq 3$, if $p \gg \frac{\log^{17k} n}{n^{k-1}}$, then asympotically almost surely every subgraph of $H^k(n,p)$ with minimum degree at least $(\frac{1}{2^{k-1}}+o(1))\binom{n-1}{k-1}p$ has Berge Hamiltonicity. The value $\frac{1}{2^{k-1}}$ is best possible and $\frac{\log^{17k} n}{n^{k-1}}$ is best under some polylogarithmic factor. For other results of Hamiltonicity in hypergraphs see [7, 8], and the results for Hamiltonicity of other types, see [9–14]. In this paper, we give a generalization of Dirac's theorem to Berge Hamiltonicity for random 3-uniform hypergraphs by the similar method of Lee and Sudakov [3]. Furthermore, according to the introduction of Clemens, Ehrenmüller and Person [6], the value $\frac{\log n}{n^2}$ and constant $1/4$ in the following theorem (our main result) are asymptotically tight.

Theorem 1.1. For every $\varepsilon > 0$, there exists a constant $c > 0$ such that if $p \geq \frac{c\log n}{n^2}$, then asympotically almost surely each subgraph $H \subseteq H^3(n,p)$ with minimum degree at least $(\frac{1}{4}+\varepsilon)\binom{n}{2}p$ has Berge Hamiltonicity.

Notation: Given a 3-uniform hypergraph H, denote by $V(H)$ the vertex set, denote by $E(H)$ the edge set and $e(H)$ be the number of edges of H. Especially, given a Berge path $P = a_0e_1a_1\cdots e_la_l$, we define vertex set $V'(P) = \{a_0, a_1, \cdots, a_l\}$ and denote by $|P|$ the length of P. If $V(P) \subset V(H)$, then we say P on vertex set $V(H)$.

For any disjoint subsets Y, M, S of $V(H)$, we denote by $e_H(Y)$ the number of edges in H whose all vertices are both in Y, and denote by $e_H(\binom{Y}{2}, M)$ the number of edges in H, which contains two distinct vertices of Y and one vertex of M, denote by $e_H(Y, M, S)$ the number of edges in H which intersects exactly one vertex with each of Y, M and S.

Given a vertex $a \in V(H)$, we define $d_H(a)$ as its number of edges incident to a in H and define $N_H(a)$ as its number of vertices adjacent to a in H. Define $N_H(Y)$ be the set of all vertices in $V\backslash Y$ whose adjacent to some vertices in Y. We denote by $\delta(H) := \min_{a\in V(H)}\{d_H(a)\}$, and denote by $\Delta(H) := \max_{a\in V(H)}\{d_H(a)\}$. We denote by $\omega(n)$ the arbitrary function which goes to infinity as n goes to infinity.

2. Tools

Now, we introduce a tool (Pósa rotation-extension technique, see [15]) that is important in proving the main theorem.

Let H be a connected 3-uniform hypergraph and let $P = a_0e_1a_1\ldots e_la_l$ be a Berge path on the vertex $V(H)$. If there exists an edge $e_w \in E(H) \setminus E(P)$ satisfies $\{a_0, w\} \subset e_w$ for some $w \in V(H) \setminus V'(P)$, then $P_w = we_wa_0e_1a_1\ldots e_la_l$ is a longer Berge path than P in $H \cup P$. In this case, we say that the path P is *extended*.

On the other hand, if there exists an edge $e \in E(H) \setminus E(P)$ satisfies $\{a_0, a_i\} \subset e$ for same $i \in [l-1]$, then there is another Berge path $P' = a_{i-1}e_{i-1}a_{i-2}\ldots a_0ea_i\ldots e_la_l$ of length $|P|$ in $H \cup P$ (see figure 2). In this case, we say that P' is obtained from P by a *rotation*. We call a_l the *fixed endpoint*, a_i the *pivot* and e_i the *broken edge* of the rotation.

Based on these, there are some new definitions. Let Y be the set of endpoints obtained by some rotations of P. For each $y \in Y$, let P_y be the path obtained from P by some rotations. Denote by $N_H(v_1|P) = \{v|(v_1, v) \subset e \ for\ some\ e \in E(H) \setminus E(P)\}$. $N_H(Y|P) = \cup_{y\in Y}N_H(y|P) \setminus Y$. Let $X \subset V \setminus Y$, denote by $E_H(Y, X|P) = \{e \in E(H) \setminus E(P_y)|y \in Y, y \in e, e \cap X \neq \phi\}$, and denote by $e_H(Y, X|P) = |E_H(Y, X|P)|$.

The proof of Theorem 1.1 mainly depends on the following results, which will be proven in detail later.

Definition 2.1. *Let $\eta > 0$. A connected 3-uniform hypergraph H on n vertices is called has property $RE(\eta)$ if for every Berge path P on $V(H)$, one of the following holds in 3-uniform hypergraph $H \cup P$:*

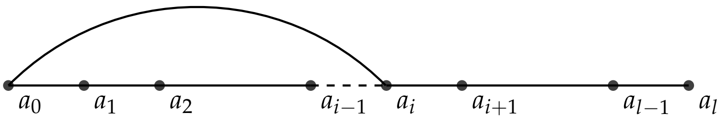

Figure 1: P'

(i) there is a Berge path longer than P,

(ii) there is a subset $A \subseteq V(H)$ with $|A| \geq \eta n$ and for each vertex $a \in A$, there exists a set $B_a \subseteq V(H)$ with $|B_a| \geq \eta n$ such that for all $b \in B_a$, $H \cup P$ contains a Berge path T_{ab} between a and b with $|T_{ab}| = |P|$.

Theorem 2.2. *For every $0 < \varepsilon < 1$, there are constants $c > 0$ and $\lambda > 0$ such that if $p \geq \frac{c \log n}{n^2}$, then the random 3-uniform hypergraph $H = H^3(n,p)$ asympotically almost surely satisfies the following property. For all $H_1 \subseteq H$ with $\Delta(H_1) \leq (\frac{3}{4} - 3\varepsilon)\binom{n}{2}p$, the hypergraph $H_2 := H - H_1$ contains a subgraph which has property $RE(\frac{1}{2} + \frac{2}{3}\varepsilon)$ and at most $\lambda n^3 p$ edges.*

Definition 2.3. *Let constant $\eta > 0$ and let H_0 be a n-vertex 3-uniform hypergraph with property $RE(\eta)$. A 3-uniform hypergraph H_2 on $V(H_0)$ is called complements H_0 if for every Berge path P in 3-uniform hypergraph H_0, one of the following holds:*

(i) there is a Berge path longer than P in $H_0 \cup P$,

(ii) there are two vertex sets A and B_a of $V(H_0)$ as in Definition 2.1 and exists vertices $a \in A$, $b \in B_a$ and edge $e \notin E(P_a)$ such that $\{a, b\} \subseteq e$ in $H_0 \cup H_2$.

Theorem 2.4. *For every $0 < \varepsilon < 1$, there are constants $c > 0$ and $\lambda > 0$ such that if $p \geq \frac{c \log n}{n^2}$, then the random 3-uniform hypergraph $H = H^3(n,p)$ asympotically almost surely satisfies the following property. For each subgraph $H_1 \subseteq H$ with $\Delta(H_1) \leq (\frac{3}{4} - 2\varepsilon)\binom{n}{2}p$, let $H_2 := H - H_1$, then the hypergraph H_2 complements all subgraphs $H' \subseteq H$ which has property $RE(\frac{1}{2} + \frac{2}{3}\varepsilon)$ and at most $\lambda n^3 p$ edges.*

Next, we introduce a modification of Proposition 3.4 in [3], and the proof is very similar to the original one.

Proposition 2.5. *(Proposition 3.4 [3]) Let constant $\eta > 0$. For every 3-uniform hypergraph H_0 with $RE(\eta)$, if 3-uniform hypergraph H_2 on $V(H_0)$ complementing H_0, then the 3-uniform hypergraph $H_0 \cup H_2$ has Berge Hamiltonicity.*

2.1. Properties of $H^3(n,p)$

Theorem 2.6. *(Chernoff's inequatily, see [16][17]) Let $0 < \varepsilon < 1$. Suppose that $Y \sim Bi(n,p)$ is a binomial random variable with parameters n and p, then*

$$\Pr(|Y - np| > \varepsilon np) < e^{-\frac{\varepsilon^2}{3}np}.$$

And if $t > 2np$, then

$$\Pr(Y \geq t) < e^{-\frac{3}{16}t}.$$

Proposition 2.7. *For every $0 < \varepsilon < 1$, there exists a constant $c > 0$ such that if $p \geq \frac{c \log n}{n^2}$, then the random 3-uniform hypergraph $H = H^3(n,p)$ asympotically almost surely has the following properties:*

(i) $(1-\varepsilon)\binom{n}{3}p \leq e(H) \leq (1+\varepsilon)\binom{n}{3}p$;

(ii) for each $v \in V(H)$, $(1-\varepsilon)\binom{n}{2}p \leq d_H(v) \leq (1+\varepsilon)\binom{n}{2}p$;

(iii) for any disjoint subsets $Y, M, S \subseteq V(H)$ with $|Y| \leq \frac{n}{4}$, $|M| \leq \frac{n}{4}$ and $|S| \leq \frac{n}{\log n}(\log\log n)^{1/2} + 1$,

$$e_H(Y, M, S) = |Y||M||S|p + o\left(|Y||M||S|p + \omega(n)n\right),$$

and

$$e_H\left(\binom{Y}{2}, S\right) = \frac{|Y|^2}{2}|S|p + o\left(\frac{|Y|^2}{2}|S|p + \omega(n)n\right).$$

Proof. (i) For $\mathrm{E}(e(H)) = \binom{n}{3}p$ is sufficiently large, by Theorem 2.6 we have

$$\Pr\left[\left|e(H) - \binom{n}{3}p\right| > \varepsilon\binom{n}{3}p\right] \le e^{-\frac{\varepsilon^2}{3}\binom{n}{3}p} = o(1).$$

(ii) Since $\mathrm{E}(d_H(v)) = \binom{n-1}{2}p$, by Theorem 2.6 we have

$$\sum_{v\in V(H)} \Pr\left(\left|d_H(v) - \binom{n}{2}p\right| > \varepsilon\binom{n}{2}p\right) \le n\cdot e^{-\frac{\varepsilon^2}{3}\binom{n}{2}p} = o(1),$$

in which the inequality holds for $c\varepsilon^2 > 7$.

(iii)Suppose that $|Y| \le \frac{n}{4}$, $|M| \le \frac{n}{4}$ and $|S| \le \frac{n}{\log n}(\log\log n)^{1/2} + 1$, then $\mathrm{E}(e_H(Y, M, S)) = |Y||M||S|p$. Theorem 2.6 states that if $\mathrm{E}[e_H(Y, M, S)] = o(\omega(n)n)$, then

$$2^n \cdot 2^n \cdot 2^n \cdot \Pr\left[|e_H(Y, M, S) - |Y||M||S|p| \ge \varepsilon(|Y||M||S|p + \omega(n)n)\right] \le 2^{3n}\cdot e^{-\frac{3}{16}\omega(n)n} = o(1),$$

otherwise,

$$\begin{aligned} &2^n \cdot 2^n \cdot 2^n \cdot \Pr\left[|e_H(Y, M, S) - |Y||M||S|p| \ge \varepsilon(|Y||M||S|p + \omega(n)n)\right] \\ &\le 2^{3n}\cdot e^{-\frac{\varepsilon^2}{3}\mathrm{E}[e_H(Y,M,S)]} \le 2^{3n}\cdot e^{-\frac{\varepsilon^2}{3}\omega(n)n} = o(1). \end{aligned}$$

Also, for $\mathrm{E}\left(e_H\left(\binom{Y}{2}, S\right)\right) = \binom{|Y|}{2}|S|p$, Theorem 2.6 states that if $\mathrm{E}\left[e_H\left(\binom{Y}{2}, S\right)\right] = o(\omega(n)n)$, then

$$2^n\cdot 2^n\cdot \Pr\left[\left|e_H\left(\binom{Y}{2}, S\right) - \frac{|Y|^2}{2}|S|p\right| \ge \varepsilon\left(\frac{|Y|^2}{2}|S|p + \omega(n)n\right)\right] \le 2^{2n}\cdot e^{-\frac{3}{16}\omega(n)n} = o(1),$$

otherwise,

$$\begin{aligned} &2^n\cdot 2^n\cdot \Pr\left[\left|e_H\left(\binom{Y}{2}, S\right) - \frac{|Y|^2}{2}|S|p\right| \ge \varepsilon\left(\frac{|Y|^2}{2}|S|p + \omega(n)n\right)\right] \\ &\le 2^{2n}\cdot e^{-\frac{\varepsilon^2}{3}\mathrm{E}\left[e_H\left(\binom{Y}{2}, S\right)\right]} \le 2^{2n}\cdot e^{-\frac{\varepsilon^2}{3}\omega(n)n} = o(1). \quad \square \end{aligned}$$

Proposition 2.8. *For every $0 < \varepsilon < 1$, there exists a constant $c > 0$ such that if $p \ge \frac{c\log n}{n^2}$, then the random 3-uniform hypergraph $H = H^3(n, p)$ asympotically almost surely has the following properties: for every Berge path P on $V(H)$, and suppose that Y is the set of endpoints obtained by taking some rotations of P in H, and let $S \subset V(P)\setminus Y$,*

(i) if $|Y| \le (\log n)^{-\frac{1}{4}}(np)^{-1}$, then $(1-\varepsilon)|Y|\binom{n}{2}p \le e_H(Y, V\setminus Y|P)$ and $|N_H(Y|P)| \ge (2-3\varepsilon)|Y|\binom{n}{2}p$;

(ii) if $n(\log n)^{-1/2} \le |Y| \le \frac{\varepsilon}{6}n$, $|S| \ge (\frac{1}{2} - \frac{\varepsilon}{3})n$, then

$$e_H(Y, S|P) > |Y|\left(\frac{3}{4} - \varepsilon\right)\binom{n}{2}p;$$

(iii) if $|Y| \le \frac{n}{4}$, $|S| \le \frac{n}{4}$, then

$$e_H(Y, S|P) = |Y||S|\left(n - \frac{|Y|}{2} - \frac{|S|}{2}\right)p + o\left(|Y||S|\left(n - \frac{|Y|}{2} - \frac{|S|}{2}\right)p + \omega(n)n\right).$$

Proof. For each $y \in Y$, let P_y be the path obtained from P by some rotations, in which y is one of the endpoints.

(i) Let $s_1 = (1-\varepsilon)|Y|\binom{n}{2}p$ and $s_2 = (2-3\varepsilon)|Y|\binom{n}{2}p \leq n(\log n)^{-1/4}$. Assume that $e_H(Y, V\setminus Y|P) \geq S_1$ and $|N_H(Y|P)| < s_2$, then there exists a subgraph of H induced by $Y \cup N_H(Y|P)$ has at least s_1 edges adjacent to Y. Therefore

$$\begin{aligned}
&\Pr(\{s_1 \leq e_H(Y, V\setminus Y|P)\} \cap \{|N_H(Y|P)| < s_2\}) \\
\leq & \binom{n-|Y|}{s_2}\binom{\binom{|Y|+s_2}{3}-\binom{s_2}{3}}{s_1}p^{s_1} \leq \binom{n-|Y|}{s_2}\binom{|Y|s_2^2}{s_1}p^{s_1} \\
\leq & \left(\frac{en}{s_2}\right)^{s_2}\left(\frac{e|Y|s_2^2p}{s_1}\right)^{s_1} = \left(\frac{en}{s_2}\right)^{s_2}\left(\frac{es_2^2}{(1-\varepsilon)\binom{n}{2}}\right)^{s_1} \leq \left(\frac{en}{s_2}\right)^{s_2}\left(\frac{es_2}{n}\right)^{2s_1} \\
= & e^{s_2(1+\log\frac{n}{s_2})+2s_1(1+\log\frac{s_2}{n})} = e^{(1+o(1))(2s_1-s_2)\log\frac{s_2}{n}} \\
\leq & e^{(1+o(1))\varepsilon|Y|\binom{n}{2}p(-1/4)\log\log n}.
\end{aligned}$$

Since $|Y| = o(n)$, on the other hand, there is

$$\mathrm{E}[e_H(Y, V\setminus Y|P)] \geq |Y|\left[\binom{n-|Y|}{2}-3\right]p = (1-o(1))|Y|\binom{n}{2}p.$$

Theorem 2.6 implies $\Pr\left[s_1 > e_H(Y, V\setminus Y|P)\right] \leq e^{-\frac{\varepsilon^2}{3}\left(|Y|\binom{n}{2}p\right)}$. Therefore

$$\begin{aligned}
&\sum_{|Y|=1}^{(\log n)^{-\frac{1}{4}}(np)^{-1}} \Pr(\{s_1 > e_H(Y, V\setminus Y|P)\} \cup \{|N_H(Y|P)| < s_2\}) \\
= & \sum_{|Y|=1}^{(\log n)^{-\frac{1}{4}}(np)^{-1}} \Pr(s_1 > e_H(Y, V\setminus Y|P)) + \Pr(\{s_1 \leq e_H(Y, V\setminus Y|P)\} \cap \{|N_H(Y|P)| < s_2\}) \\
\leq & \sum_{|Y|=1}^{(\log n)^{-\frac{1}{4}}(np)^{-1}} \binom{n}{|Y|} e^{-\frac{\varepsilon^2}{3}\left(|Y|\binom{n}{2}p\right)} e^{(1+o(1))\varepsilon|Y|\binom{n}{2}p(-1/4)\log\log n} \leq \sum_{|Y|=1}^{(\log n)^{-\frac{1}{4}}(np)^{-1}} \binom{n}{|Y|} n^{-c_1|Y|} = o(1),
\end{aligned}$$

in which the inequality holds for $c_1 = c_1(c, \varepsilon) \geq 2$ by choosing the appropriate constant c.

(ii) Suppose that $n(\log n)^{-1/2} \leq |Y| \leq \frac{\varepsilon}{6}n$ and $|S| \geq (\frac{1}{2}-\frac{\varepsilon}{3})$. For every $y \in Y$ and $s \in S$, there are at most three edges contains $\{y, s\}$ in P_y, since we have

$$\begin{aligned}
\mathrm{E}[e_H(Y, S|P)] &\geq |Y|\left[\binom{n-|Y|}{2} - \binom{n-|Y|-|S|}{2} - 3\right]p \\
&\geq |Y|\left[\binom{n-\frac{\varepsilon}{6}n}{2} - \binom{n-n(\log n)^{-1/2}-(\frac{1}{2}-\frac{\varepsilon}{3})n}{2} - 3\right]p \\
&= (1-o(1))|Y|\left[\left(1-\frac{\varepsilon}{6}\right)^2 - \left(\frac{1}{2}+\frac{\varepsilon}{3}-(\log n)^{-1/2}\right)^2\right]\binom{n}{2}p \\
&= (1-o(1))|Y|\left(\frac{3}{4}-\frac{5}{6}\varepsilon\right)\binom{n}{2}p,
\end{aligned}$$

by Theorem 2.6 there is

$$2^n \cdot 2^n \cdot \Pr\left(e_H(Y, S|P) \leq |Y|\left(\frac{3}{4}-\varepsilon\right)\binom{n}{2}p\right) \leq 2^{2n} \cdot \Pr\left(e_H(Y, S|P) \leq |Y|\left(1-\frac{\varepsilon}{5}\right)\mathrm{E}[e_H(Y, S|P)]\right)$$

$$\leq 2^{2n} \cdot e^{-\frac{\varepsilon^2}{75}\mathrm{E}[e_H(Y,S|P)]} \leq 2^{2n} \cdot e^{-\frac{\varepsilon^2}{75}cn(\log n)^{1/2}} = o(1).$$

(iii) Suppose that $|Y| \leq \frac{n}{4}$ and $|S| \leq \frac{n}{4}$. For every $y \in Y$ and $s \in S$, there are at most three edges contains $\{y,s\}$ in P_y, thus

$$\begin{aligned}\mathrm{E}[e_H(Y,S|P)] &\geq \left(|Y||S|n - \binom{|Y|}{2}|S| - \binom{|S|}{2}|Y| - 3|Y|\right)\\ &= (1-o(1))|Y||S|\left(n - \frac{|Y|}{2} - \frac{|S|}{2}\right)p.\end{aligned}$$

Define $\tau := |Y||S|\left(n - \frac{|Y|}{2} - \frac{|S|}{2}\right)$. For $e_H(Y,S|P)$ is a binomial random variable, Theorem 2.6 implies that if $\mathrm{E}[e_H(Y,S)] = o(\omega(n)n)$, then

$$2^n \cdot 2^n \cdot \Pr\left[|e_H(Y,S) - \tau| \geq \varepsilon\left(\tau + \omega(n)n\right)\right] \leq 2^{2n} \cdot e^{-\frac{3}{16}\omega(n)n} = o(1),$$

otherwise,

$$2^n \cdot 2^n \cdot \Pr\left[|e_H(Y,S) - \tau| \geq \varepsilon\left(\tau + \omega(n)n\right)\right] \leq e^{-\frac{\varepsilon^2}{3}\mathrm{E}[e_H(Y,S|P)]} \leq 2^{2n} \cdot e^{-\frac{\varepsilon^2}{3}\omega(n)n} = o(1). \quad \square$$

Proposition 2.9. *For every $0 < \varepsilon < 1$, there exists a constant $c > 0$ such that if $p \geq \frac{c\log n}{n^2}$, then the random 3-uniform hypergraph $H = H^3(n,p)$ asympotically almost surely satisfies the following properties. For each $H_1 \subseteq H$ with $\Delta(H_1) \leq (\frac{3}{4} - 2\varepsilon)\binom{n}{2}p$, let $H_2 := H - H_1$. Let P be a Berge path on $V(H_2)$ and Y be the set of the endpoints obtained by taking some rotations of P in H_2,*

(i) if $|Y| \leq (\log n)^{-\frac{1}{4}}(np)^{-1}$, then $|N_{H_2}(Y|P)| \geq (\frac{1}{2} + \varepsilon)|Y|\binom{n}{2}p$;

(ii) if $n(\log n)^{-1/2} \leq |Y| \leq \frac{\varepsilon}{6}n$, then $|N_{H_2}(Y|P) \geq (\frac{1}{2} + \frac{\varepsilon}{6})n$;

(iii) H_2 is connected.

Proof. (i) Let $|Y| \leq (\log n)^{-\frac{1}{4}}(np)^{-1}$. By Proposition 2.8, we can get $(1-\varepsilon)|Y|\binom{n}{2}p \leq e_H(Y, V \setminus Y|P)$ and $|N_H(Y|P)| \geq (2 - 3\varepsilon)|Y|\binom{n}{2}p$. Hence

$$\begin{aligned}|N_{H_2}(Y|P)| &\geq |N_H(Y|P)| - |Y| \cdot 2\Delta(H_1)\\ &\geq (2-3\varepsilon)|Y|\binom{n}{2}p - |Y| \cdot 2\left(\frac{3}{4} - 2\varepsilon\right)\binom{n}{2}p\\ &\geq \left(\frac{1}{2} + \varepsilon\right)|Y|\binom{n}{2}p.\end{aligned}$$

(ii) If not, assume that $|N_{H_2}(Y|P)| < (\frac{1}{2} + \frac{\varepsilon}{6})n$, then $|V(H) \setminus (Y \cup N_{H_2}(Y))| \geq (\frac{1}{2} - \frac{\varepsilon}{3})n$ and $e_{H_2}(Y, V(H) \setminus (Y \cup N_{H_2}(Y))|P) = 0$. Thus

$$e_H(Y, V(H) \setminus (Y \cup N_{H_2}(Y))|P) \leq |Y|\Delta(H_1) = |Y|\left(\frac{3}{4} - 2\varepsilon\right)\binom{n}{2}p,$$

which contradicts Proposition 2.8.

(iii) If H_2 is not connected. Let H' be the minimum connected component of H_2, which implies $|N_{H_2}(V(H'))| = |V(H')|$. Since $\left(\frac{1}{2} + \varepsilon\right)\binom{n}{2}p > 1$, following the result of (i), we have

$$|V(H')| \geq \left(\frac{1}{2} + \varepsilon\right)(\log n)^{-1/4}(np)^{-1}\binom{n}{2}p > n(\log n)^{-1/2}.$$

By (ii), we can get $|V(H')| \geq (\frac{1}{2} + \frac{\varepsilon}{6})n$ that contradicts the facts. $\square$

2.2. Proof of Theorem 2.2.

Before proving Theorem 2.2, we prove the following Lemma 2.10.

Lemma 2.10. *For every real $0 < \varepsilon < 1$, there exists a constant $c > 0$ such that if $p \geq \frac{c\log n}{n^2}$, then the random 3-uniform hypergraph $H = H^3(n,p)$ asympotically almost surely has the following properties: for each $H_1 \subset H$ with $\Delta(H_1) \leq (\frac{3}{4} - 2\varepsilon)\binom{n}{2}p$, the 3-uniform hypergraph $H_2 := H - H_1$ has $RE(\frac{1}{2} + \frac{2}{3}\varepsilon)$.*

Proof. Let $P := a_0e_1a_1 \cdots a_l$ be a Berge path on $V(H_2)$. If there is a Berge path longer than P in $H_2 \cup P$, then we are done.

So we suppose that P is the longest Berge path in $H_2 \cup P$. In the following, we will consider an endpoint set obtained by taking some rotations of P with fixed endpoint a_l in H_2, and give a lower bound $(\frac{1}{2} + \frac{2}{3}\varepsilon)n$ on the number of those endpoints. The endpoint set will be constructed by iterative method. We use Y_t to denote the endpoint set obtained by the tth rotation of P with fixed enpoint v_l in H_2, especially, $Y_0 = \{a_0\}$. Since P is the longest Berge path in $H_2 \cup P$, note that for every $t \in [n]$ we must have $N_{H_2}(Y_t|P) \subseteq V(P)$.

Claim 1. $|Y_{t+1}| \geq \frac{1}{2}(|N_{H_2}(Y_t|P)| - 3|Y_t|)$.

Proof. For any $a \in Y_t$, if $w \in N_{H_2}(a|P)$, then there exists an endpoint by a rotation of P_a using v as pivot point. Let $Y_t^+ = \{a_{i+1}|a_i \in Y_t\}$, $Y_t^- = \{a_{i-1}|a_i \in Y_t\}$. Hence, if a vertex $v \in N_{H_2}(Y_t|P)$ does not belong to $Y_t \cup Y_t^- \cup Y_t^+$, then the edges in P incident with v were not broken in the previous rotations. We can get a new endpoint v^- or v^+ (see Figure 2 and Figure 3), and at most two such pivot points can obtain the same endpoint since the order for unbroken interval either the same as or reverse to P. Therefore, $|Y_{t+1}| \geq \frac{1}{2}\left(|N_{H_2}(Y_t|P) - 3|Y_t|\right)$. This completes the proof of Claim 1.

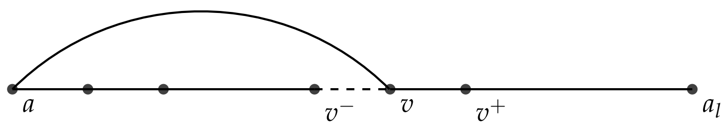

Figure 2: same order

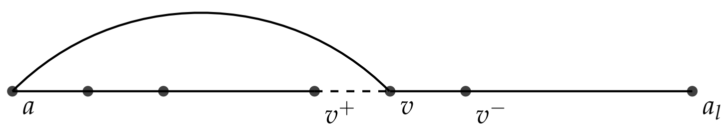

Figure 3: reverse order

Suppose that $|Y_t| = \left(\frac{n^2p}{2^5}\right)^t \geq 1$ for some integer $t \geq 0$. Since

$$\delta(H_2) \geq (1-\varepsilon)\binom{n}{2}p - \Delta(H_1) \geq \left(\frac{1}{4} + \varepsilon\right)\binom{n}{2}p,$$

by Claim 2 together with Proposition 2.9 (i), we have

$$\begin{aligned}|Y_{t+1}| &\geq \frac{1}{2}(|N_{H_2}(Y_t|P)| - 3|Y_t|) \geq \frac{1}{2}\left(\left(\frac{1}{4}+\varepsilon\right)|Y_t|\binom{n}{2}p - 3|Y_t|\right)\\ &\geq \frac{1}{2}\left(\left(\frac{1}{4}+\varepsilon\right)\left(\frac{n^2p}{2^5}\right)^t\binom{n}{2}p - 3|Y_t|\right) \geq \left(\frac{n^2p}{2^5}\right)^{t+1}.\end{aligned}$$

Let $\left(\frac{n^2p}{2^5}\right)^s = (\log n)^{-1/4}(np)^{-1}$, it follows that there is an integer $s \leq \frac{\log n}{\log\log n}$ such that $|Y_s| = (\log n)^{-1/4}(np)^{-1}$ if $c \geq 2^5$. Repeat the same argument as above to $|Y_s|$, there is

$$|Y_{s+1}| \geq \left(\frac{n^2p}{2^5}\right)(\log n)^{-1/4}(np)^{-1} \geq \frac{n}{2^5(\log n)^{1/4}} > \frac{n}{(\log n)^{1/2}}.$$

Agian, repeat the same argument as above to subset with size $\dfrac{n}{(\log n)^{1/2}}$ of Y_{s+1} , and combine with Proposition 2.9, there is

$$|Y_{s+2}| \geq \frac{1}{2}(|N_H(Y_{s+1}|P)| - 3|Y_{s+1}|) \geq \frac{1}{2}\left(\left(\frac{1}{2}+\frac{1}{6}\varepsilon\right)n - 3\frac{n}{(\log n)^{1/2}}\right) \geq \frac{n}{4}.$$

Finally, we give a proof of $|Y_{s+3}| \geq (\frac{1}{2}+\frac{2}{3}\varepsilon)n$. Let $S := Y_{s+3}$ and $Y \subseteq Y_{s+2}$ be any subset with size $\frac{n}{4}$. We partition P into $r := \frac{\log n}{(\log\log n)^{1/2}}$ vertex disjoint intervals, such that the length of each interval are either $\lfloor\frac{|P|}{r}\rfloor$ or $\lceil\frac{|P|}{r}\rceil$. For each $i \in [r]$, let $\tilde{Y}_i \subseteq Y$ be a vertex subset, in which all those vertices are obtained by some rotations with some broken edges of P_i in the previous rotations. Let $Y_{i,+}$ and $Y_{i,-}$ be the collections of all those vertices of Y obtained by some rotations such that P_i is unbroken in the previous rotations, and the path from every vertex of $Y_{i,+}$ and $Y_{i,-}$ to v_l traverses P_i in the same and reverse order as P, expectively. Thus $Y = \tilde{Y}_i \cup Y_{i,+} \cup Y_{i,-}$ for all $i \in [r]$.

Let $J = \{i \in [r] : |\tilde{Y}_i| \geq (\log\log n)^{-1/4}|Y|\}$. We claim the first fact that $|J| = o(r)$. Indeed, since every vertex in Y is obtained by at most $\frac{2\log n}{\log\log n}$ rotations of P. Let $\tilde{E}$ denote the number of edges that broken in the previous rotations for obtain Y. There is

$$|Y|\frac{2\log n}{(\log\log n)} \geq \tilde{E} \geq (\log\log n)^{-1/4}|Y||J|,$$

which implies $|J| = o(r)$.

Next, we define $V := V(H)$, $P := V'(P)$, $P_i := V'(P_i)$ for any $i \in [r]$, and show the second fact that

$$\begin{aligned} e_H(Y, V\setminus P|P) + \sum_{i\in[r]}\sum_{\pm\in\{+,-\}}\{e_H(Y_{i,\pm}, (P_i\cap P_i^{\pm})\setminus S^{\pm}|P) - e_H(Y_{i,\pm}, (P_i\cap P_i^{\pm})\setminus S^{\pm}, V\setminus P|P)\} \\ \leq e_{H_1}(Y, V). \qquad (1) \end{aligned}$$

For P is the longest Berge path and by applying Proposition 2.9 to Y, we can get $e_{H_2}(Y, V\setminus P|P) = 0$ and $|V\setminus P| \leq \frac{n}{4}$. For each $i \in [r]$, if there exists vertices $x \in Y_{i,+}$, $a_j \in P_i$ and an edge $e \notin E(P_y)$ contains $\{a_j, x\}$ in H_2, then $a_{j-1} \in S$ and $a_j \in S^+$, therefore $e_{H_2}(Y_{i,+}, (P_i\cap P_i^+)\setminus S^+|P) = 0$(similarly, $e_{H_2}(Y_{i,-}, (P_i\cap P_i^-)\setminus S^-|P) = 0$). On the other hand, for every edge $e \in E_H(Y_{i,+}, (P_i\cap P_i^+)\setminus S^+|P) \cup E_H(Y_{i,-}, (P_i\cap P_i^-)\setminus S^-|P)$, if $e\cap(V\setminus P) \neq \phi$, then e will be counted repeatedly. This completes the proof of (1), Now we estimate the left inequality of (1).

By definition, we know $|P_i| \leq \lceil\frac{|P|}{r}\rceil \leq \frac{n}{\log n}(\log\log n)^{1/2} + 1$. Thus by Proposition 2.7 and Proposition 2.8, we can get the lower bound of the left inequality of (1) is as follows

$$\begin{aligned} &|Y||V\setminus P|\left[n - \frac{|Y|}{2} - \frac{|V\setminus P|}{2}\right]p + o(w(n)n) \\ &+ \sum_{i\in[r]}\sum_{\pm\in\{+,-\}}\left(|Y_{i,\pm}||(P_i\cap P_i^{\pm})\setminus S^{\pm}|\left[n - \frac{|Y_{i,\pm}|}{2} - \frac{|(P_i\cap P_i^{\pm})\setminus S^{\pm}|}{2}\right]p + o\left(n^2\frac{|P|}{r}p\right)\right) \\ &- \sum_{i\in[r]}\sum_{\pm\in\{+,-\}}[(1+o(1))|Y_{i,\pm}||V\setminus P||(P_i\cap P_i^{\pm})\setminus S^{\pm}|p + o(w(n)n)]. \qquad (2) \end{aligned}$$

Due to $||(P_i \cap P_i^+) \setminus S^+| - |P_i \setminus S|| \leq 2$(similar for $(P_i \cap P_i^-) \setminus S^-$), the second line in the inequality (2) is at least

$$\sum_{i\in[r]} \sum_{\pm\in\{+,-\}} \left(|Y_{i,\pm}||P_i \setminus S| \left[n - \frac{|Y_{i,\pm}|}{2} - \frac{|(P_i \cap P_i^{\pm}) \setminus S^{\pm}|}{2} \right] p + o\left(\frac{n^3}{r} p\right) - o\left(2\frac{n^2}{r} p\right) \right)$$

$$\geq \sum_{i\in[r]} \sum_{\pm\in\{+,-\}} |Y_{i,\pm}||P_i \setminus S| \left[n - \frac{|Y_{i,\pm}|}{2} - \frac{|P_i \setminus S|}{2} \right] p - o(n^3 p)$$

$$\geq \sum_{i\in[r]} \left(\sum_{\pm\in\{+,-\}} |Y_{i,\pm}||P_i \setminus S| \left(n - \frac{|P_i \setminus S|}{2} \right) - \frac{|Y_{i,+}|^2 + |Y_{i,-}|^2}{2} |P_i \setminus S| \right) p - o(n^3 p)$$

$$\geq \sum_{i\in[r]} \left(|Y \setminus \tilde{Y}_i||P_i \setminus S| \left(n - \frac{|P_i \setminus S|}{2} \right) - \frac{(|Y_{i,+}| + |Y_{i,-}|)^2}{2} |P_i \setminus S| \right) p - o(n^3 p)$$

$$\geq \sum_{i\in[r]} |Y \setminus \tilde{Y}_i||P_i \setminus S| \left(n - \frac{|P_i \setminus S|}{2} \right) p - \frac{|Y|^2}{2} |P \setminus S| p - o(n^3 p).$$

Since $J = o(r)$, there is $|Y \setminus \tilde{Y}_i| = (1 - o(1))|Y|$ for $i \in [r] \setminus J$. The inequality above is

$$\geq \sum_{i\in[r]\setminus J} (1 - o(1))|Y||P_i \setminus S| \left(n - \frac{|P_i \setminus S|}{2} \right) p - \frac{|Y|^2}{2} |P \setminus S| p - o(n^3 p)$$

$$\geq \sum_{i\in[r]} (1 - o(1))|Y||P_i \setminus S| \left(n - \frac{|P_i \setminus S|}{2} \right) p - o(r)(1 - o(1))|Y| \frac{|P|}{r} np - \frac{|Y|^2}{2} |P \setminus S| p - o(n^3 p)$$

$$\geq \sum_{i\in[r]} |Y||P_i \setminus S| \left(n - \frac{|P_i \setminus S|}{2} \right) p - \frac{|Y|^2}{2} |P \setminus S| p - o(n^3 p)$$

$$\geq |Y||P \setminus S| n - \frac{|Y|^2}{2} |P \setminus S| p - o(n^3 p).$$

The inequality (2) can be expressed as

$$\geq |Y||V \setminus P| \left[n - \frac{|Y|}{2} - \frac{|V \setminus P|}{2} \right] p + o(w(n)n) + |Y||P \setminus S| np - \frac{|Y|^2}{2} |P \setminus S| p - o(n^3 p)$$
$$- \sum_{i\in[r]} |Y \setminus \tilde{Y}_i||V \setminus P||P_i \setminus S| p - o(w(n)n)$$

$$\geq |Y||V \setminus P| \left[n - \frac{|Y|}{2} - \frac{|V \setminus P|}{2} \right] p + |Y||P \setminus S| np - \frac{|Y|^2}{2} |P \setminus S| p - |Y||V \setminus P||P \setminus S| p - o(n^3 p)$$

$$\geq |Y|(|V \setminus P| + |P \setminus S|) np - \frac{|Y|^2}{2} (|V \setminus P| + |P \setminus S|) p - |Y||V \setminus P| \left(\frac{|V \setminus P|}{2} + |P \setminus S| \right) p - o(n^3 p)$$

$$\geq |Y||V \setminus S| np - \frac{|Y|^2}{2} |V \setminus S| p - |Y||V \setminus P||V \setminus S| p - o(n^3 p)$$

$$\geq \frac{3}{4} n|Y||V \setminus S| p - \frac{|Y|^2}{2} |V \setminus S| p - o(n^3 p).$$

Therefore inequality (1) implies $\frac{3}{4} n|Y||V \setminus S|p - \frac{|Y|^2}{2}|V \setminus S|p - o(n^3 p) \leq e_{H_1}(Y, V)$.

On the other hand, $e_{H_1}(Y, V) + e_{H_1}(\binom{Y}{2}, V) \leq |Y|\Delta_{H_1}(Y)$, thus following Proposition 2.7 we can get

$$\frac{3}{4} n|Y||V \setminus S| p - o(n^3 p) \leq |Y| \left(\frac{3}{4} - 2\varepsilon \right) \binom{n}{2} p \leq |Y| \left(\frac{3}{8} - \frac{\varepsilon}{2} \right) n^2 p.$$

It's easy to check $|S| \geq \left(\frac{1}{2} + \frac{2}{3}\varepsilon\right) n$ by $|V \setminus S| \leq \left(\left(\frac{3}{8} - \frac{\varepsilon}{2}\right) n + o(n^2)\right) \frac{4}{3} \leq \left(\frac{1}{2} - \frac{2}{3}\varepsilon\right) n$.

Hence, we can get an endpoint set S of size at least $(\frac{1}{2} + \frac{2}{3}\varepsilon)n$, in which for every $y \in S$, there is a Berge path of length $|P|$ in $H_2 \cup P$ with endpoints a_l and y. Similarly, for such Berge path we can fixed y to obtain an endpoint set $S_y \subseteq V(H)$ of size at least $(\frac{1}{2} + \frac{2}{3}\varepsilon)n$ such that, for every $s \in S_y$, there is a Berge path of length $|P|$ in $H_2 \cup P$ from s to y. □

Proof of Theorem 2.2. Let $p' = \lambda p$ and let H' be the 3-uniform hypergraph obtained by taking each edge of $H^3(n,p)$ independently with probability λ. Thus H' has the same distribution with $H^3(n,p')$. By Lemma 2.10 and Proposition 2.7, we can get

$$\Pr\left[H' \in RE\left(\frac{1}{2} + \frac{2}{3}\varepsilon\right) \textit{ with at most } \lambda n^3 p \textit{ edges}\right] = 1 - o(1). \tag{3}$$

Now, we define that a 3-uniform hypergraph H is *good*: if for each $H_1 \subseteq H$ with $\Delta(H_1) \leq (\frac{3}{4} - 2\varepsilon)\binom{n}{2}p'$, the 3-uniform hypergraph $H - H_1$ has at most $n^3 p' = \lambda n^3 p$ edges and is $RE(\frac{1}{2} + \frac{2}{3}\varepsilon)$. Otherwise call H is *bad*. Under this definition, (3) means $\Pr[H' \text{ is good}] = 1 - o(1)$.

Let $\mathcal{H}$ be the collection of all 3-uniform hypergraphs H satifies $\Pr(H' \text{ is good} | H = H^3(n,p)) \geq \frac{5}{6}$, in other words, $\Pr(H' \text{ is good} | H \notin \mathcal{H}) < \frac{5}{6}$, following this, there is

$$o(1) = \Pr(H' \text{ is bad}) \geq \Pr(H' \text{ is bad} | H^3(n,p) \notin \mathcal{H}) \cdot \Pr(H^3(n,p) \notin \mathcal{H}) \geq \frac{1}{6}\Pr(H^3(n,p) \notin \mathcal{H}).$$

Hence $\Pr(H^3(n,p) \notin \mathcal{H}) = o(1)$, that means $\Pr(H^3(n,p) \in \mathcal{H}) = 1 - o(1)$.

Next, let $H_1 \subseteq H$ be any subgraph with $\Delta(H_1) \leq (\frac{3}{4} - 3\varepsilon)\binom{n}{2}p$. By Theorem 2.6, there is

$$\sum_{v \in V(H')} \Pr\left(d_{H' \cap H_1}(v) \geq \left(\frac{3}{4} - 2\varepsilon\right)\binom{n}{2}p'\right) \leq \sum_{v \in V(H')} \Pr\left(d_{H' \cap H_1}(v) \geq (1+\varepsilon)\left(\frac{3}{4} - 3\varepsilon\right)\binom{n}{2}p'\right) = o(1).$$

Hence, there exists a subgraph $H' \subset H$ that is good and the maximum degree of $H' \cap H_1$ at most $(\frac{3}{4} - 2\varepsilon)\binom{n}{2}p'$. For such H', by the definition of good, the hypergraph $H' - (H' \cap H_1) \subseteq H - H_1$ which has $RE(\frac{1}{2} + \frac{2}{3}\varepsilon)$ and $|E(H' - (H' \cap H_1))| \leq \lambda n^3 p$. □

2.3. *Proof of Theorem 2.4.*

Proof of Theorem 2.4. Let $\mathcal{H}_2$ be the collection of all subgraphs $H - H_1$, which satisfy $\Delta(H_1) \leq (\frac{3}{4} - 2\varepsilon)\binom{n}{2}p$.

$$\begin{aligned}
&\Pr\left[\bigcup_{H' \in RE(\frac{1}{2}+\frac{2}{3}\varepsilon), |E(H')| \leq \lambda n^3 p} (\{H' \subseteq H\} \cap \{\text{exists } H_2 \in \mathcal{H}_2 \text{ does not complement } H'\})\right] \\
&\leq \sum_{H' \in RE(\frac{1}{2}+\frac{2}{3}\varepsilon), |E(H')| \leq \lambda n^3 p} \Pr(H' \subseteq H) \cdot \Pr(\text{exists } H_2 \in \mathcal{H}_2 \text{ does not complement } H' | H' \subseteq H) \\
&\leq \sum_{m=1}^{\lambda n^3 p} \binom{\binom{n}{3}}{m} p^m \cdot \Pr(\text{exists } H_2 \in \mathcal{H}_2 \text{ does not complement } H' | H' \subseteq H),
\end{aligned} \tag{4}$$

where the $H' \subseteq H$ of last line of inequality (4) are taken over all labeled 3-uniform hypergraphs with $RE(\frac{1}{2} + \frac{2}{3}\varepsilon)$ and m edges.

Next we consider $\Pr(\text{exists } H_2 \in \mathcal{H}_2 \text{ does not complement } H' | H' \subseteq H)$. Let P be a fixed Berge path on $V(H')$. Recall the definition of complement. If H_2 does not complement H', then there is not Berge path longer than P in $H' \cup P$. On the one hand, since $H' \in RE(\frac{1}{2} + \frac{2}{3}\varepsilon)$, we can find an endpoint set $A \subseteq V(H)$ which $|A| \geq (\frac{1}{2} + \frac{2}{3}\varepsilon)n$ and for every $a \in A$, there is an endpoint set B_a of $V(H')$ with $|B_a| \geq (\frac{1}{2} + \frac{2}{3}\varepsilon)n$ satisfies for all $b \in B_a$, the 3-uniform hypergraph $H' \cup P$ contains a Berge path T_{ab} between a and b with

$|T_{ab}| = |P|$. On the other hand, since H_2 does not complement H', for every such T_{ab}, there is not edge $e \notin T_{ab}$ contains $\{a, b\}$ in H_2. By the maximum degree of H_1 at most $(\frac{3}{4} - 2\varepsilon)\binom{n}{2}p$, there is

$$e_H(a, B_a|P) \leq \left(\frac{3}{4} - 2\varepsilon\right)\binom{n}{2}p.$$

Since

$$\begin{aligned}
\mathrm{E}[e_H(a, B_a|P)] &\geq \left[\binom{n-1}{2} - \binom{n-1-|B_a|}{2} - 3\right]p \\
&\geq \left(\binom{n-1}{2} - \binom{n-1-(\frac{1}{2}-\frac{\varepsilon}{3})n}{2} - 3\right)p \\
&= (1-o(1))\left(1 - \left(\frac{1}{2} + \frac{\varepsilon}{3}\right)^2\right)\binom{n}{2}p \\
&\geq \left(\frac{3}{4} - \frac{\varepsilon}{2}\right)\binom{n}{2}p.
\end{aligned}$$

Theorem 2.6 implies that the probability of $e_H(a, B_a|P) \leq \left(\frac{3}{4} - 2\varepsilon\right)\binom{n}{2}p$ is at most $e^{-\frac{\varepsilon^2}{3}(n^2p)}$. Therefore,

$$\Pr\left[\bigcap_{a \in A}\left(e_H(a, B_a|P) \leq \left(\frac{3}{4} - 2\varepsilon\right)\binom{n}{2}p\right)\right] \leq e^{-\frac{\varepsilon^2}{3}(n^3p)}.$$

Note that there are at most n choices for the length of Berge path P. For each $j \in [n]$, there are at most $n-2$ choices for the edges of any two vertices of $V'(P)$ in H, thus there are at most $\frac{n}{(j-1)!}(n-2)^j$ Berge path of length j. Based on this, the third line of inequality (4)

$$\begin{aligned}
&\sum_{m=1}^{\lambda n^3 p}\binom{\binom{n}{3}}{m}p^m \cdot \Pr(\text{exists } H_2 \in \mathcal{H}_2 \text{ does not complement } H'|H' \subseteq H) \\
\leq &\sum_{m=1}^{\lambda n^3 p}\binom{\binom{n}{3}}{m}p^m \cdot n \cdot \frac{n!}{(j-1)!}(n-2)^j \cdot e^{-\frac{\varepsilon^2}{3}(n^3p)} \\
\leq &e^{-\frac{\varepsilon^2}{3}(n^3p)}\sum_{m=1}^{\lambda n^3 p}\binom{\binom{n}{3}}{m}p^m \\
\leq &e^{-\frac{\varepsilon^2}{3}(n^3p)}\sum_{m=1}^{\lambda n^3 p}\left(\frac{en^3p}{m}\right)^m \\
\leq &e^{-\frac{\varepsilon^2}{3}(n^3p)}(\lambda n^3 p)\left(\frac{e}{\lambda}\right)^{\lambda n^3 p} \\
= &e^{-\frac{\varepsilon^2}{3}(n^3p)}e^{O(\lambda \log(\frac{1}{\lambda})n^3p)} = o(1),
\end{aligned}$$

which the inequality holds for $\left(\frac{en^3p}{m}\right)^m$ is monotone increasing in the range $1 \leq m \leq \lambda n^3 p$ and $\lambda = \lambda(\varepsilon)$ is sufficiently small. □

3. Proof of Theorem 1.1.

Proof of Theorem 1.1. Let $0 < \varepsilon < 1$. Let $c = c(\varepsilon) > 0$ and $\lambda = \lambda(\varepsilon) > 0$ be constants such that for $p \geq \frac{c \log n}{n^2}$, the 3-uniform hypergraph $H = H^3(n, p)$ asympotically almost surely holds for Proposition 2.7, Theorem

2.2 and Theorem 2.4, especially, Proposition 2.7 hold with 2ε instead of ε. That means, $d_H(v) \geq (1-2\varepsilon)\binom{n}{2}p$ for every vertex $v \in V(H)$. Therefore, for any 3-uniform hypergraph H_2 with minimum degree at least $(\frac{1}{4}+\varepsilon)\binom{n}{2}p$ can be obtained by the following way: exists a subgraph $H_1 \subseteq H$ with maximum degree at most $(\frac{3}{4}-3\varepsilon)\binom{n}{2}p$ such that $H_2 = H - H_1$. Next we will show that $H - H_1$ is Berge Hamiltonian.

On the one hand, by Theorem 2.2, there exists a subgraph $H^* \subseteq H - H_1$ which has property $RE(\frac{1}{2}+\frac{2}{3}\varepsilon)$ and $|E(H^*)| \leq \lambda n^3 p$. On the other hand, by Theorem 2.4 and the fact that $(\frac{3}{4}-3\varepsilon)\binom{n}{2}p \leq (\frac{3}{4}-2\varepsilon)\binom{n}{2}p$, there is $H - H_1$ complement H^*. Therefore, Proposition 2.5 implies that $H - H_1$ is Berge Hamiltonian.

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