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A generalization of the common fixed point theorem for normally 2-generalized hybrid mappings in Hilbert spaces

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Abstract. In this paper, we prove a common fixed point theorem for commutative nonlinear mappings that jointly satisfy a certain condition. A required condition is given as a convex combination of those of a well-known class of nonlinear mappings. From the main theorem of this paper, the common fixed point theorem for nonexpansive mappings, generalized hybrid mappings, and normally 2-generalized hybrid mappings are uniformly derived as corollaries. Our approach expands the applicable range of mappings for existing fixed point theorems to be effective. Using specific mappings, we illustrate the effectiveness.

1. Introduction

Let H be a real Hilbert space equipped with an inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\|\cdot\|$. Let S be a mapping from C into H, where C is a nonempty subset of H. The set of fixed points of S is denoted by

$$F(S) = \{x \in C : Sx = x\}.$$

A mapping $S : C \to H$ is called *nonexpansive* if

$$\|Sx - Sy\| \leq \|x - y\| \quad \text{for all } x, y \in C.$$

Fixed point theorems that assert the existence of fixed points for nonexpansive mappings were established independently in 1965 by three researchers in frameworks of Banach spaces; see Browder [5], Göhde [6], and Kirk [10].

There have also been ongoing works to generalize the class of mappings. In 2010, Kocourek et al. [11] defined the following type of nonlinear mappings. A mapping $S : C \to H$ is called *generalized hybrid* [11], if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha \|Sx - Sy\|^2 + (1 - \alpha) \|x - Sy\|^2 \leq \beta \|Sx - y\|^2 + (1 - \beta) \|x - y\|^2 \tag{1}$$

for all $x, y \in C$. Kocourek et al. [11] established a fixed point theorem for this class of mappings and studied how to approximate fixed points. The class of generalized hybrid mappings contains nonexpansive mappings as the case of $\alpha = 1$ and $\beta = 0$. If $\alpha = 2$ and $\beta = 1$ in (1), a mapping is called *nonspreading*; see

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Kohsaka and Takahashi [13]. If $\alpha = 3/2$ and $\beta = 1/2$ in (1), a mapping is called *hybrid* in the sense of Takahashi [27]. For these classes of mappings, see also Takahashi and Yao [31]. Also, *λ-hybrid mappings* [3] are included in the class of generalized hybrid mappings.

The classes of mappings, for which fixed point theorem can be established, have been furthermore extended. A mapping $S : C \to C$ is called 2-*generalized hybrid* [24] if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$\begin{aligned}&\alpha_1 \left\|S^2x - Sy\right\|^2 + \alpha_2 \left\|Sx - Sy\right\|^2 + (1 - \alpha_1 - \alpha_2) \left\|x - Sy\right\|^2 \\ &\le \beta_1 \left\|S^2x - y\right\|^2 + \beta_2 \left\|Sx - y\right\|^2 + (1 - \beta_1 - \beta_2) \left\|x - y\right\|^2\end{aligned}$$

for all $x, y \in C$. The class of 2-generalized hybrid mappings contains generalized hybrid mappings as a special case of $\alpha_1 = \beta_1 = 0$. Maruyama et al. [24] proved a fixed point theorem for this class of mappings and presented weak convergence theorems for finding fixed points. A mapping $S : C \to C$ is called *normally* 2-*generalized hybrid* [20] if there exist $\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{R}$ such that $\sum_{n=0}^{2} (\alpha_n + \beta_n) \ge 0$, $\alpha_2 + \alpha_1 + \alpha_0 > 0$, and

$$\begin{aligned}&\alpha_2 \left\|S^2x - Sy\right\|^2 + \alpha_1 \left\|Sx - Sy\right\|^2 + \alpha_0 \left\|x - Sy\right\|^2 \\ &+\beta_2 \left\|S^2x - y\right\|^2 + \beta_1 \left\|Sx - y\right\|^2 + \beta_0 \left\|x - y\right\|^2 \le 0\end{aligned} \tag{2}$$

for all $x, y \in C$. The class of normally 2-generalized hybrid mappings contains 2-generalized hybrid mappings as a case of $\alpha_2 + \alpha_1 + \alpha_0 = 1$ and $\beta_2 + \beta_1 + \beta_0 = -1$. Therefore, this class of mappings includes mappings such as nonexpansive, nonspreading, hybrid, and λ-hybrid as special cases. It also includes a class of *normally generalized hybrid mappings* [30] as a special case. It is easy to show that if $\sum_{n=0}^{2} (\alpha_n + \beta_n) > 0$, a normally 2-generalized hybrid mapping has at most one fixed point. For examples of these classes of nonlinear mappings, see Kondo [14, 19] and papers cited therein. For various results concerning 2-generalized hybrid mappings and normally 2-generalized hybrid mappings, see also [2, 7–9, 20–26, 29].

Common fixed point theorems, which assert the existence of common fixed points for multiple nonlinear mappings, have been studied in many papers. The following result was proved by Hojo [7].

Theorem 1.1 ([7]). *Let C be a nonempty, closed, and convex subset of H. Let $S, T : C \to C$ be normally 2-generalized hybrid mappings that satisfy $ST = TS$. Suppose that there exists $z \in C$ such that $\{S^kT^lz : k, l \in \mathbb{N} \cup \{0\}\}$ is bounded. Then, $F(S) \cap F(T)$ is not empty.*

See also Kohsaka [12] and Hojo et al. [8]. Hojo et al. [8] and Hojo [7] also proved common *attractive point* theorems without assuming that C is closed or convex. For the concept of an attractive point, see Takahashi and Takeuchi [28]. Very recently, Kondo [16] proved a common fixed point theorem by exploiting a convex combination of conditions (1) and extended the common fixed point theorem for generalized hybrid mappings; see Corollary 4.3 in this paper.

In this paper, following Hojo [7] and Kondo [16], we prove a common fixed point theorem for commutative nonlinear mappings. The mappings are required to jointly satisfy a condition represented as a convex combination of those of normally 2-generalized hybrid mappings (2). From the main theorem of this paper, Theorem 1.1 is derived as a corollary. In this sense, our main theorem is a generalization of a common fixed point theorem for normally 2-generalized hybrid mappings. Our approach significantly expands the applicable range of mappings for existing well-known common fixed point theorems to be effective. This paper is organized as follows. Section 2 demonstrates the main theorem. Sections 3 and 4 present corollaries deduced from the main theorem with examples of the mappings. In Section 3, some remarks are also given. Section 5 provides a concise conclusion.

2. Main Result

In this section, we establish the main theorem of this paper, which generalizes a common fixed point theorem (Theorem 1.1) for normally 2-generalized hybrid mappings. A condition required for mappings is given as a convex combination of (2).

Theorem 2.1. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $S, T : C \to C$ that satisfy $ST = TS$. Suppose that there exist $\lambda \in (0, 1)$ and $\alpha_i, \beta_i, \alpha'_i, \beta'_i \in \mathbb{R}$ $(i = 0, 1, 2)$ such that*

$$\begin{aligned}
&\lambda(\alpha_2 \left\|S^2x - Sy\right\|^2 + \alpha_1 \left\|Sx - Sy\right\|^2 + \alpha_0 \left\|x - Sy\right\|^2 \\
&+\beta_2 \left\|S^2x - y\right\|^2 + \beta_1 \left\|Sx - y\right\|^2 + \beta_0 \left\|x - y\right\|^2) \\
&+ (1 - \lambda) (\alpha'_2 \left\|T^2x - Ty\right\|^2 + \alpha'_1 \left\|Tx - Ty\right\|^2 + \alpha'_0 \left\|x - Ty\right\|^2 \\
&+\beta'_2 \left\|T^2x - y\right\|^2 + \beta'_1 \left\|Tx - y\right\|^2 + \beta'_0 \left\|x - y\right\|^2) \le 0
\end{aligned} \tag{3}$$

for all $x, y \in C$, where

$$\sum_{n=0}^{2} (\alpha_n + \beta_n) \ge 0, \quad \alpha_2 + \alpha_1 + \alpha_0 > 0, \tag{4}$$

$$\sum_{n=0}^{2} (\alpha'_n + \beta'_n) \ge 0, \quad \alpha'_2 + \alpha'_1 + \alpha'_0 > 0. \tag{5}$$

Suppose that there exists $z \in C$ such that $\{S^kT^lz : k, l \in \mathbb{N} \cup \{0\}\}$ is bounded. Then, $F(S) \cap F(T)$ is not empty.

Proof. Define

$$A_n = \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} S^kT^lz$$

for all $n \in \mathbb{N}$. From the hypothesis that $\left\{S^kT^lz : k, l \in \mathbb{N} \cup \{0\}\right\}$ is bounded, $\{A_n\}$ is also bounded. Thus, there exists a subsequence $\{A_{n_i}\}$ of $\{A_n\}$ such that $A_{n_i} \rightharpoonup v$ for some $v \in H$. We show that $v \in F(S) \cap F(T)$.

As $ST = TS$, substituting $x = S^kT^lz$ and $y = v$ in (3), we have that

$$\begin{aligned}
&\lambda(\alpha_2 \left\|S^{k+2}T^lz - Sv\right\|^2 + \alpha_1 \left\|S^{k+1}T^lz - Sv\right\|^2 + \alpha_0 \left\|S^kT^lz - Sv\right\|^2 \\
&+\beta_2 \left\|S^{k+2}T^lz - v\right\|^2 + \beta_1 \left\|S^{k+1}T^lz - v\right\|^2 + \beta_0 \left\|S^kT^lz - v\right\|^2) \\
&+ (1 - \lambda) (\alpha'_2 \left\|S^kT^{l+2}z - Tv\right\|^2 + \alpha'_1 \left\|S^kT^{l+1}z - Tv\right\|^2 + \alpha'_0 \left\|S^kT^lz - Tv\right\|^2 \\
&+\beta'_2 \left\|S^kT^{l+2}z - v\right\|^2 + \beta'_1 \left\|S^kT^{l+1}z - v\right\|^2 + \beta'_0 \left\|S^kT^lz - v\right\|^2) \le 0
\end{aligned}$$

for all $k, l \in \mathbb{N} \cup \{0\}$. For simplicity, define

$$\begin{aligned}
\alpha &\equiv \alpha_2 + \alpha_1 + \alpha_0, \quad \beta \equiv \beta_2 + \beta_1 + \beta_0, \\
\alpha' &\equiv \alpha'_2 + \alpha'_1 + \alpha'_0, \quad \beta' \equiv \beta'_2 + \beta'_1 + \beta'_0.
\end{aligned}$$

We have

$$
\begin{aligned}
&\lambda\{\alpha_2\left(\left\|S^{k+2}T^lz-Sv\right\|^2-\left\|S^kT^lz-Sv\right\|^2\right)\\
&+\alpha_1\left(\left\|S^{k+1}T^lz-Sv\right\|^2-\left\|S^kT^lz-Sv\right\|^2\right)\\
&+\alpha\left\|S^kT^lz-Sv\right\|^2\\
&+\beta_2\left(\left\|S^{k+2}T^lz-v\right\|^2-\left\|S^kT^lz-v\right\|^2\right)\\
&+\beta_1\left(\left\|S^{k+1}T^lz-v\right\|^2-\left\|S^kT^lz-v\right\|^2\right)\\
&+\beta\left\|S^kT^lz-v\right\|^2\}\\
&+(1-\lambda)\{\alpha_2'\left(\left\|S^kT^{l+2}z-Tv\right\|^2-\left\|S^kT^lz-Tv\right\|^2\right)\\
&+\alpha_1'\left(\left\|S^kT^{l+1}z-Tv\right\|^2-\left\|S^kT^lz-Tv\right\|^2\right)\\
&+\alpha'\left\|S^kT^lz-Tv\right\|^2\\
&+\beta_2'\left(\left\|S^kT^{l+2}z-v\right\|^2-\left\|S^kT^lz-v\right\|^2\right)\\
&+\beta_1'\left(\left\|S^kT^{l+1}z-v\right\|^2-\left\|S^kT^lz-v\right\|^2\right)\\
&+\beta'\left\|S^kT^lz-v\right\|^2\}\leq 0.
\end{aligned}
$$

It holds true that

$$
\begin{aligned}
&\lambda\{\alpha_2\left(\left\|S^{k+2}T^lz-Sv\right\|^2-\left\|S^kT^lz-Sv\right\|^2\right)\\
&+\alpha_1\left(\left\|S^{k+1}T^lz-Sv\right\|^2-\left\|S^kT^lz-Sv\right\|^2\right)\\
&+\alpha\left(\left\|S^kT^lz-v\right\|^2+2\left\langle S^kT^lz-v,\ v-Sv\right\rangle+\|v-Sv\|^2\right)\\
&+\beta_2\left(\left\|S^{k+2}T^lz-v\right\|^2-\left\|S^kT^lz-v\right\|^2\right)\\
&+\beta_1\left(\left\|S^{k+1}T^lz-v\right\|^2-\left\|S^kT^lz-v\right\|^2\right)\\
&+\beta\left\|S^kT^lz-v\right\|^2\}\\
&+(1-\lambda)\{\alpha_2'\left(\left\|S^kT^{l+2}z-Tv\right\|^2-\left\|S^kT^lz-Tv\right\|^2\right)\\
&+\alpha_1'\left(\left\|S^kT^{l+1}z-Tv\right\|^2-\left\|S^kT^lz-Tv\right\|^2\right)\\
&+\alpha'\left(\left\|S^kT^lz-v\right\|^2+2\left\langle S^kT^lz-v,\ v-Tv\right\rangle+\|v-Tv\|^2\right)\\
&+\beta_2'\left(\left\|S^kT^{l+2}z-v\right\|^2-\left\|S^kT^lz-v\right\|^2\right)\\
&+\beta_1'\left(\left\|S^kT^{l+1}z-v\right\|^2-\left\|S^kT^lz-v\right\|^2\right)\\
&+\beta'\left\|S^kT^lz-v\right\|^2\}\leq 0.
\end{aligned}
$$

From (4), it holds that $\alpha+\beta\geq 0$ and $\alpha'+\beta'\geq 0$. Therefore, subtracting $(\alpha+\beta)\left\|S^kT^lz-v\right\|^2(\geq 0)$ and

$(\alpha'+\beta')\left\|S^kT^lz-v\right\|^2$ (≥ 0) from the left-hand side (LHS) yields

$$
\begin{aligned}
&\lambda\{\alpha_2\left(\left\|S^{k+2}T^lz-Sv\right\|^2-\left\|S^kT^lz-Sv\right\|^2\right)\\
&+\alpha_1\left(\left\|S^{k+1}T^lz-Sv\right\|^2-\left\|S^kT^lz-Sv\right\|^2\right)\\
&+\alpha\left(2\left\langle S^kT^lz-v,\ v-Sv\right\rangle+\|v-Sv\|^2\right)\\
&+\beta_2\left(\left\|S^{k+2}T^lz-v\right\|^2-\left\|S^kT^lz-v\right\|^2\right)\\
&+\beta_1\left(\left\|S^{k+1}T^lz-v\right\|^2-\left\|S^kT^lz-v\right\|^2\right)\}\\
&+(1-\lambda)\{\alpha_2'\left(\left\|S^kT^{l+2}z-Tv\right\|^2-\left\|S^kT^lz-Tv\right\|^2\right)\\
&+\alpha_1'\left(\left\|S^kT^{l+1}z-Tv\right\|^2-\left\|S^kT^lz-Tv\right\|^2\right)\\
&+\alpha'\left(2\left\langle S^kT^lz-v,\ v-Tv\right\rangle+\|v-Tv\|^2\right)\\
&+\beta_2'\left(\left\|S^kT^{l+2}z-v\right\|^2-\left\|S^kT^lz-v\right\|^2\right)\\
&+\beta_1'\left(\left\|S^kT^{l+1}z-v\right\|^2-\left\|S^kT^lz-v\right\|^2\right)\}\leq 0
\end{aligned}
$$

for all $k,l\in\mathbb{N}\cup\{0\}$. Summing these inequalities with respect to $k=0,1,\cdots,n-1$, we have

$$
\begin{aligned}
&\lambda\{\alpha_2\left(\left\|S^{n+1}T^lz-Sv\right\|^2+\left\|S^nT^lz-Sv\right\|^2-\left\|ST^lz-Sv\right\|^2-\left\|T^lz-Sv\right\|^2\right)\\
&+\alpha_1\left(\left\|S^nT^lz-Sv\right\|^2-\left\|T^lz-Sv\right\|^2\right)\\
&+\alpha\left(2\left\langle\sum_{k=0}^{n-1}S^kT^lz-nv,\ v-Sv\right\rangle+n\|v-Sv\|^2\right)\\
&+\beta_2\left(\left\|S^{n+1}T^lz-v\right\|^2+\left\|S^nT^lz-v\right\|^2-\left\|ST^lz-v\right\|^2-\left\|T^lz-v\right\|^2\right)\\
&+\beta_1\left(\left\|S^nT^lz-v\right\|^2-\left\|T^lz-v\right\|^2\right)\}\\
&+(1-\lambda)\{\alpha_2'\left(\sum_{k=0}^{n-1}\left\|S^kT^{l+2}z-Tv\right\|^2-\sum_{k=0}^{n-1}\left\|S^kT^lz-Tv\right\|^2\right)\\
&+\alpha_1'\left(\sum_{k=0}^{n-1}\left\|S^kT^{l+1}z-Tv\right\|^2-\sum_{k=0}^{n-1}\left\|S^kT^lz-Tv\right\|^2\right)\\
&+\alpha'\left(2\left\langle\sum_{k=0}^{n-1}S^kT^lz-nv,\ v-Tv\right\rangle+n\|v-Tv\|^2\right)\\
&+\beta_2'\left(\sum_{k=0}^{n-1}\left\|S^kT^{l+2}z-v\right\|^2-\sum_{k=0}^{n-1}\left\|S^kT^lz-v\right\|^2\right)\\
&+\beta_1'\left(\sum_{k=0}^{n-1}\left\|S^kT^{l+1}z-v\right\|^2-\sum_{k=0}^{n-1}\left\|S^kT^lz-v\right\|^2\right)\}\leq 0.
\end{aligned}
$$

for all $l\in\mathbb{N}\cup\{0\}$. Summing these inequalities with respect to $l=0,1,\cdots,n-1$ and dividing by n^2, we obtain

$$\begin{aligned}
&\frac{\lambda}{n^2}\{\alpha_2(\sum_{l=0}^{n-1}\left\|S^{n+1}T^l z - Sv\right\|^2 + \sum_{l=0}^{n-1}\left\|S^n T^l z - Sv\right\|^2 \\
&-\sum_{l=0}^{n-1}\left\|ST^l z - Sv\right\|^2 - \sum_{l=0}^{n-1}\left\|T^l z - Sv\right\|^2) \\
&+\alpha_1\left(\sum_{l=0}^{n-1}\left\|S^n T^l z - Sv\right\|^2 - \sum_{l=0}^{n-1}\left\|T^l z - Sv\right\|^2\right) \\
&+\beta_2(\sum_{l=0}^{n-1}\left\|S^{n+1}T^l z - v\right\|^2 + \sum_{l=0}^{n-1}\left\|S^n T^l z - v\right\|^2 \\
&-\sum_{l=0}^{n-1}\left\|ST^l z - v\right\|^2 - \sum_{l=0}^{n-1}\left\|T^l z - v\right\|^2) \\
&+\beta_1\left(\sum_{l=0}^{n-1}\left\|S^n T^l z - v\right\|^2 - \sum_{l=0}^{n-1}\left\|T^l z - v\right\|^2\right)\} \\
&+\lambda\alpha\left(2\langle A_n - v,\ v - Sv\rangle + \|v - Sv\|^2\right) \\
&+\frac{1-\lambda}{n^2}\{\alpha_2'(\sum_{k=0}^{n-1}\left\|S^k T^{n+1} z - Tv\right\|^2 + \sum_{k=0}^{n-1}\left\|S^k T^n z - Tv\right\|^2 \\
&-\sum_{k=0}^{n-1}\left\|S^k T z - Tv\right\|^2 - \sum_{k=0}^{n-1}\left\|S^k z - Tv\right\|^2) \\
&+\alpha_1'\left(\sum_{k=0}^{n-1}\left\|S^k T^n z - Tv\right\|^2 - \sum_{k=0}^{n-1}\left\|S^k z - Tv\right\|^2\right) \\
&+\beta_2'(\sum_{k=0}^{n-1}\left\|S^k T^{n+1} z - v\right\|^2 + \sum_{k=0}^{n-1}\left\|S^k T^n z - v\right\|^2 \\
&-\sum_{k=0}^{n-1}\left\|S^k T z - v\right\|^2 - \sum_{k=0}^{n-1}\left\|S^k z - v\right\|^2) \\
&+\beta_1'\left(\sum_{k=0}^{n-1}\left\|S^k T^n z - v\right\|^2 - \sum_{k=0}^{n-1}\left\|S^k z - v\right\|^2\right)\} \\
&+(1-\lambda)\,\alpha'\left(2\langle A_n - v,\ v - Tv\rangle + \|v - Tv\|^2\right) \le 0.
\end{aligned}$$

Recall that $A_{n_i} \rightharpoonup v$. Replacing n with n_i and taking the limit as $i \to \infty$, we have

$$\lambda\alpha\,\|v - Sv\|^2 + (1-\lambda)\,\alpha'\,\|v - Tv\|^2 \le 0.$$

From (4), $\alpha, \alpha' > 0$. Subtracting the second term $(1-\lambda)\,\alpha'\,\|v - Tv\|^2\ (\ge 0)$ of the LHS, we obtain $\lambda\alpha\,\|v - Sv\|^2 \le 0$. Dividing by $\lambda\alpha\ (> 0)$ yields $v \in F(S)$. Similarly, we obtain $v \in F(T)$. Therefore, $F(S) \cap F(T)$ is not empty. This completes the proof. □

3. Derivative Results and Remarks

In this section, from Theorem 2.1, we derive some corollaries to show the theorem's applicability. An example is presented to highlight the difference from existing research. Some remarks are also provided.

3.1. Corollaries for normally 2-generalized hybrid mappings.

First, note that from Theorem 2.1, **Theorem 1.1** ([7]) is derived. Theorem 1.1 asserts the existence of a common fixed point of two commutative normally 2-generalized hybrid mappings. This fact is verified as follows: Let S and T be commutative normally 2-generalized hybrid mappings (2) with parameters $(\alpha_i, \beta_i; i = 0, 1, 2)$ and $\left(\alpha'_i, \beta'_i; i = 0, 1, 2\right)$, respectively. Our aim is to show that $F(S) \cap F(T)$ is not empty. We have

$$\begin{aligned}&\alpha_2 \left\|S^2x - Sy\right\|^2 + \alpha_1 \left\|Sx - Sy\right\|^2 + \alpha_0 \left\|x - Sy\right\|^2 \\ &+\beta_2 \left\|S^2x - y\right\|^2 + \beta_1 \left\|Sx - y\right\|^2 + \beta_0 \left\|x - y\right\|^2 \leq 0\end{aligned} \tag{6}$$

and

$$\begin{aligned}&\alpha'_2 \left\|T^2x - Ty\right\|^2 + \alpha'_1 \left\|Tx - Ty\right\|^2 + \alpha'_0 \left\|x - Ty\right\|^2 \\ &+\beta'_2 \left\|T^2x - y\right\|^2 + \beta'_1 \left\|Tx - y\right\|^2 + \beta'_0 \left\|x - y\right\|^2 \leq 0\end{aligned} \tag{7}$$

for all $x, y \in C$, where the conditions (4) and (5) are required for the parameters $(\alpha_i, \beta_i; i = 0, 1, 2)$ and $\left(\alpha'_i, \beta'_i; i = 0, 1, 2\right)$, respectively. From (6) and (7), the condition (3) is obtained with a coefficient λ of the convex combination. Consequently, the existence of a common fixed point of S and T is guaranteed from Theorem 2.1 and Theorem 1.1 is obtained.

As mentioned in the Introduction, the class of normally 2-generalized hybrid mappings contains various classes of nonlinear mappings including nonexpansive mappings, generalized hybrid mappings, and 2-generalized hybrid mappings as special cases. Hence, common fixed point theorems for these types of mappings are simultaneously obtained.

A fixed point theorem for a normally 2-generalized hybrid mapping is also derived from Theorem 2.1. Indeed, set $S = T$, $\alpha_i = \alpha'_i$, and $\beta_i = \beta'_i$ $(i = 0, 1, 2)$ in Theorem 2.1. From this operation, the following fixed point theorem is obtained:

Theorem 3.1 ([20]). *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $S : C \to C$ be a normally 2-generalized hybrid mapping. Suppose that there exists $z \in C$ such that the sequence $\{S^n z\}$ is bounded. Then, S has a fixed point.*

Theorem 3.1 was proved in Kondo and Takahashi [20]. A mapping $S : C \to H$ with $F(S) \neq \emptyset$ is called *quasi-nonexpansive* if

$$\left\|Sx - q\right\| \leq \left\|x - q\right\| \quad \text{for all } x \in C \text{ and } q \in F(S). \tag{8}$$

Kondo and Takahashi [20] showed the following:

Claim 3.2 ([20]). *Let $S : C \to C$ be a normally 2-generalized hybrid mapping such that $F(S) \neq \emptyset$, where C is a nonempty subset of H. Then, S is quasi-nonexpansive.*

Using this fact, Kondo and Takahashi [20] demonstrated weak convergence theorems for finding fixed points. They also showed weak convergence to attractive points; for the concept called *attractive point*, see Takahashi and Takeuchi [28] and other studies [1, 4, 7, 8, 20–23, 30].

3.2. Another corollary and an example

Next, we present a corollary which shows another usage of Theorem 2.1. Furthermore, we illustrate the corollary using an example, which highlights the difference of our result from the existing ones. As S and S^2 are commutative, setting $T = S^2$ in Theorem 2.1, we obtain the following:

Corollary 3.3. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let S be a mapping from C into itself. Suppose that there exist $\lambda \in (0,1)$ and $\alpha_i, \beta_i, \alpha_i', \beta_i' \in \mathbb{R}$ $(i = 0,1,2)$ that satisfy* (4), (5), *and*

$$\begin{aligned}
&\lambda(\alpha_2 \left\|S^2x - Sy\right\|^2 + \alpha_1 \left\|Sx - Sy\right\|^2 + \alpha_0 \left\|x - Sy\right\|^2 \\
&+\beta_2 \left\|S^2x - y\right\|^2 + \beta_1 \left\|Sx - y\right\|^2 + \beta_0 \left\|x - y\right\|^2) \\
&+ (1-\lambda)\,(\alpha_2' \left\|S^4x - S^2y\right\|^2 + \alpha_1' \left\|S^2x - S^2y\right\|^2 + \alpha_0' \left\|x - S^2y\right\|^2 \\
&+\beta_2' \left\|S^4x - y\right\|^2 + \beta_1' \left\|S^2x - y\right\|^2 + \beta_0' \left\|x - y\right\|^2) \le 0
\end{aligned} \tag{9}$$

for all $x, y \in C$. Suppose that there exists $z \in C$ such that $\{S^nz\}$ is bounded. Then, $F(S)$ is not empty.

To demonstrate the effectiveness of this result, the following operations are performed. Set $\alpha_2 = \alpha_2' = 1$, $\beta_2 = \beta_2' = -1$, and the other parameters are 0 in (9). This parameter constellation satisfies (4) and (5). Then, we have

$$\lambda \left\|S^2x - Sy\right\|^2 + (1-\lambda) \left\|S^4x - S^2y\right\|^2 \le \lambda \left\|S^2x - y\right\|^2 + (1-\lambda) \left\|S^4x - y\right\|^2. \tag{10}$$

A mapping S in the next example, which is a slightly modified version of that in Kondo [15], satisfies the condition (10):

Example 3.1 ([15]). Let H be a Hilbert space. Given $\lambda \in (0,1)$, define $S : H \to H$ as follows:

$$Sx = \begin{cases} \dfrac{x}{\|x\|} & \text{if } \|x\| > \sqrt{\lambda} \text{ and } \|x\| \neq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

It holds that $S^2x = 0$ for all $x \in H$ and hence, $\{S^nx\}$ is bounded. Setting $S^2x = S^4x = S^2y = 0$ in (10) deduces

$$\lambda \left\|Sy\right\|^2 \le \left\|y\right\|^2 \quad \text{for all } y \in H. \tag{12}$$

The mapping S defined in (11) satisfies the condition (12). Thus, S exists within the class addressed in Corollary 3.3. In this case, $F(S) = \{0\}$. □

The mapping S in this example is not quasi-nonexpansive (8), although it has a fixed point. From Claim 3.2, S is not a normally 2-generalized hybrid mapping. This fact illustrates that the mapping S does not reside within the class exhibited in Theorems 1.1 or 3.1. As mentioned in the Introduction, the class of normally 2-generalized hybrid mappings (2) contains nonexpansive mappings, generalized hybrid mappings, and 2-generalized hybrid mappings as special cases, the mapping S in Example 3.1 is not within these classes. Hence, this example reveals that Theorem 2.1 does expand the range of mappings for which fixed points are guaranteed to exist, compared to existing works, e.g., Theorem 1.1.

3.3. Remarks.

In this subsection, two remarks are provided. The first concerns convergence results for finding (common) fixed points and the second concerns the boundedness assumption.

Many previous studies deal with convergence theorems that approximate (common) fixed points; see, e.g., [11, 17, 18, 30, 31]. The proofs of the convergence theorems are based on the hypothesis that the mappings are quasi-nonexpansive (8). As shown in Example 3.1, mappings addressed in this study are not necessarily quasi-nonexpansive even if they have fixed points. Therefore, convergence theorems are difficult to prove along with the elements developed in the literature.

All results concerning common fixed point theorems (Theorem 1.1 and 2.1) presented in this paper are based on the boundedness of $\{S^kT^lz : k, l \in \mathbb{N} \cup \{0\}\}$ for some $z \in C$. Similarly, fixed point theorems (Theorem 3.1 and Corollary 3.3) are derived from the boundedness of $\{S^nz\}$ for some $z \in C$. The boundedness of $\{S^kT^lz\}$ and $\{S^nz\}$ is guaranteed if the set C is bounded. Concerning the boundedness, from Theorem 1.1, the following result is established:

Claim 3.4. *Let C be a nonempty, closed, and convex subset of H. Let $S, T : C \to C$ be normally 2-generalized hybrid mappings such that $ST = TS$. Then, the following three statements are equivalent:*
(i) *For all $x \in C$, $\{S^k T^l x : k, l \in \mathbb{N} \cup \{0\}\}$ is bounded.*
(ii) *There exists $z \in C$ such that $\{S^k T^l z : k, l \in \mathbb{N} \cup \{0\}\}$ is bounded.*
(iii) *$F(S) \cap F(T)$ is not empty.*

Proof. (i) implies (ii). The part (ii)$\Longrightarrow$(iii) is obtained from Theorem 1.1. We show that (iii)$\Longrightarrow$(i). Assume (iii) and choose $q \in F(S) \cap F(T)$ arbitrarily. Let $x \in C$. As S and T are normally 2-generalized hybrid mappings (2) such that $F(S) \cap F(T) \neq \emptyset$, from Claim 3.2, S and T are quasi-nonexpansive (8). As $q \in F(S) \cap F(T)$, it follows that

$$\begin{aligned}
\left\|S^k T^l x\right\| &\leq \left\|S^k T^l x - q\right\| + \|q\| \\
&\leq \left\|S^{k-1} T^l x - q\right\| + \|q\| \\
&\leq \cdots \\
&\leq \left\|T^l x - q\right\| + \|q\| \\
&\leq \left\|T^{l-1} x - q\right\| + \|q\| \\
&\leq \cdots \\
&\leq \|x - q\| + \|q\|
\end{aligned}$$

for all $k, l \in \mathbb{N} \cup \{0\}$. This indicates that $\{S^k T^l x\}$ is bounded for all $x \in C$. The proof is completed. □

Note that the proof for the part (iii)$\Longrightarrow$(i) requires the fact that a normally 2-generalized hybrid mapping that has a fixed point is quasi-nonexpansive (Claim 3.2). As demonstrated in Example 3.1, a mapping S in Corollary 3.3 (or mappings S, T in Theorem 2.1) is not necessarily quasi-nonexpansive even if it has a fixed point. Hence, for mappings S and T with the condition (3), it is not possible to show that $\{S^k T^l x\}$ is bounded for all $x \in C$ based on the reasoning in the proof of Claim 3.4. If C is bounded, so is $\{S^k T^l x\}$, however.

The part (iii)$\Longrightarrow$(ii) in Claim 3.4 is true without any assumption on mappings S or T. Indeed, letting $z \in F(S) \cap F(T)$, we obtain $S^k T^l z = z$ for all $k, l \in \mathbb{N} \cup \{0\}$ and then, $\{S^k T^l z\}$ is bounded.

4. Additional corollaries and examples

In this section, we present some more corollaries and examples to delve into Theorem 2.1, although some overlap exists with the previous work Kondo [16]. Set $\alpha_1 = \alpha_1' = 1$, $\beta_0 = \beta_0' = -1$, and the other parameters are 0 in the condition (3) of mappings S and T. Note that this parameter constellation satisfies the conditions (4) and (5). We obtain the following:

Corollary 4.1 ([16]). *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $S, T : C \to C$ with $ST = TS$. Suppose that there exists $\lambda \in (0, 1)$ such that*

$$\lambda \|Sx - Sy\|^2 + (1 - \lambda) \|Tx - Ty\|^2 \leq \|x - y\|^2 \tag{13}$$

for all $x, y \in C$. Suppose that there exists $z \in C$ such that $\{S^k T^l z : k, l \in \mathbb{N} \cup \{0\}\}$ is bounded. Then, $F(S) \cap F(T)$ is not empty.

This corollary was included in a recent study by Kondo [16]. Some remarks concerning Corollary 4.1 are given below. First, setting $S = T$ in Corollary 4.1, we obtain a fixed point theorem for a nonexpansive mapping. Second, assume that S and T are nonexpansive mappings. Then, the condition (13) is fulfilled and therefore, a common fixed point theorem for nonexpansive mappings is derived from this corollary. We present an example of two commutative nonexpansive mappings and their common fixed point below (Example 4.1). The next claim is provided by Kohsaka; see Example 5.1 in [12].

Claim 4.2 ([12]). *Let H be a Hilbert space and define $B_r = \{x \in H : \|x\| \leq r\}$, where $r > 0$. Let $S : H \to H$ be a linear and isometric mapping and let $T : H \to B_r$ be the metric projection from H onto B_r. Then, S and T are commutative.*

Proof. Choose $x \in H$. First, assume that $\|x\| > r\,(> 0)$. Then, $Tx = rx/\|x\|$. As S is isometric, it follows that $\|Sx\|\,(= \|x\|) > r$. Consequently, $TSx = rSx/\|Sx\|$. As S is linear, we have

$$STx = S\left(\frac{rx}{\|x\|}\right) = \frac{r}{\|x\|}Sx = \frac{r}{\|Sx\|}Sx = TSx.$$

Next, let us assume that $\|x\| \leq r$. As S is isometric, $\|Sx\| = \|x\| \leq r$. In this case, $Tx = x$ and $TSx = Sx$. Thus, we obtain $STx = Sx = TSx$. This ends the proof. □

Given the analysis in Claim 4.2, we present the following example:

Example 4.1. Let $H = C = \mathbb{R}^2$. Consider a matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear mapping associated with A. Then, S is isometric. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the metric projection from $\mathbb{R}^2$ onto the unit sphere $U = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$. Then, S and T are nonexpansive. From Claim 4.2, they are commutative. It holds that $\{S^kT^lx\} \subset U$ if $l \geq 1$ for all $x \in \mathbb{R}^2$. Therefore, $\{S^kT^lx : k, l \in \mathbb{N} \cup \{0\}\}$ is bounded. In this case, $F(S) \cap F(T) = \{0\}$. □

In addition to the case of two nonexpansive mappings, Corollary 4.1 applies to the next example:

Example 4.2. Let $H = C = \mathbb{R}$. Define $S, T : \mathbb{R} \to \mathbb{R}$ as follows:

$$Sx = \begin{cases} -\sqrt{\dfrac{3}{2}}x & \text{if } x < 0, \\ 0 & \text{if } x \geq 0, \end{cases}$$

$$Tx = \begin{cases} -\dfrac{\sqrt{3}}{2}x & \text{if } x < 0, \\ 0 & \text{if } x \geq 0. \end{cases}$$

As $STx = TSx = 0$ for all $x \in \mathbb{R}$, S and T are commutative. Furthermore, $\{S^kT^lx\} = \{0\}$ if $k + l \geq 2$ for all $x \in \mathbb{R}$ and thus, $\{S^kT^lx : k, l \in \mathbb{N} \cup \{0\}\}$ is bounded. The mappings jointly satisfy (13) with $\lambda = 1/3$, namely,

$$\frac{1}{3}\left\|Sx - Sy\right\|^2 + \frac{2}{3}\left\|Tx - Ty\right\|^2 \leq \left\|x - y\right\|^2 \tag{14}$$

for all $x, y \in \mathbb{R}$. The inequality (14) can be verified as follows. (i) First, let $x, y \geq 0$. In this case, $Sx = Sy = Tx = Ty = 0$ and therefore, (14) holds true. (ii) Let $x, y < 0$. Then,

$$\begin{aligned} & LHS \text{ of (14)} - RHS \\ = \;& \frac{1}{3}\left(-\sqrt{\frac{3}{2}}x + \sqrt{\frac{3}{2}}y\right)^2 + \frac{2}{3}\left(-\frac{\sqrt{3}}{2}x + \frac{\sqrt{3}}{2}y\right)^2 - (x-y)^2 \\ = \;& \frac{1}{3}\cdot\frac{3}{2}(x-y)^2 + \frac{2}{3}\cdot\frac{3}{4}(x-y)^2 - (x-y)^2 = 0, \end{aligned}$$

which shows that (14) is satisfied. (iii) Assume, without loss of generality, that $x < 0 \leq y$. In this case, as $Sy = Ty = 0$, it follows that

$$\begin{aligned} & LHS \text{ of (14)} - RHS \\ = \;& \frac{1}{3}\left(-\sqrt{\frac{3}{2}}x\right)^2 + \frac{2}{3}\left(-\frac{\sqrt{3}}{2}x\right)^2 - (x-y)^2 \\ = \;& x^2 - (x-y)^2 = y(2x - y) \leq 0. \end{aligned}$$

Hence, the condition (14) is met for all $x, y \in \mathbb{R}$ as claimed. Therefore, the mappings S and T are within the class that is targeted by Corollary 4.1 even though S is not nonexpansive. In this case, $F(S) \cap F(T) = \{0\}$. □

Reviewing Theorem 2.1 again. Set $\alpha_1 = \alpha$, $\alpha_0 = 1-\alpha$, $\beta_1 = -\beta$, $\beta_0 = -(1-\beta)$, $\alpha_1' = \alpha'$, $\alpha_0' = 1-\alpha'$, $\beta_1' = -\beta'$, $\beta_0' = -(1-\beta')$, and the other parameters are 0 in (3). This parameter constellation satisfies the conditions (4) and (5). We obtain the following:

Corollary 4.3 ([16]). *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $S, T : C \to C$ with $ST = TS$. Suppose that there exist $\lambda \in (0,1)$ and $\alpha, \beta \in \mathbb{R}$ such that*

$$\begin{aligned} & \lambda\left(\alpha\left\|Sx - Sy\right\|^2 + (1-\alpha)\left\|x - Sy\right\|^2\right) \\ & + (1-\lambda)\left(\alpha'\left\|Tx - Ty\right\|^2 + (1-\alpha')\left\|x - Ty\right\|^2\right) \\ \leq\ & \lambda\left(\beta\left\|Sx - y\right\|^2 + (1-\beta)\left\|x - y\right\|^2\right) \\ & + (1-\lambda)\left(\beta'\left\|Tx - y\right\|^2 + (1-\beta')\left\|x - y\right\|^2\right) \end{aligned}$$

for all $x, y \in C$. Suppose that there exists $z \in C$ such that $\left\{S^k T^l z : k, l \in \mathbb{N} \cup \{0\}\right\}$ is bounded. Then, $F(S) \cap F(T)$ is not empty.

Corollary 4.3 is established in a recent study by Kondo [16]. From this corollary, a common fixed point theorem for generalized hybrid mappings (1) is obtained as are those for nonexpansive, nonspreading, hybrid, and λ-hybrid mappings. A fixed point theorem for a generalized hybrid mapping is also obtained.

5. Conclusion

In this paper, we proved a common fixed point theorem for nonlinear mappings. A required condition for such mappings is given as a convex combination of conditions for normally 2-generalized hybrid mappings. To demonstrate the effectiveness of our approach, some corollaries and concrete examples of mappings are provided. As implied by some examples, the mappings addressed in this paper are not necessarily quasi-nonexpansive, and therefore, convergence theorems that approximate common fixed points are not easy to prove. As a final remark, it also seems difficult to use our method to guarantee the existence of a common attractive point of nonlinear mappings.

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