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Topological properties of some multiplication operators on $\mathcal{L}(X)$

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Abstract. A pair (u, α) in $X \times X'$, where X is an infinite dimensional Banach space and X' its topological dual space, induces in a natural way two multiplication operators $\mathscr{L}_{\alpha,u}$ and $\mathscr{R}_{\alpha,u}$ on the Banach space $\mathcal{L}(X)$, defined by $\mathscr{L}_{\alpha,u}\Big(T\Big)(x) = \alpha\Big(T(x)\Big)u$, and $\mathscr{R}_{\alpha,u}\Big(T\Big)(x) = \alpha(x)T(u)$, for all T in $\mathcal{L}(X)$ and x in X. In this paper, we present necessary and sufficient conditions for the compactness, demicompactness, stongly demicompactess, power compactness and Riesz property of this family of operators. We also establish sufficient conditions for the quasi-compactness and weak compactness of these operators. Finally, we show that the Dunford-Pettis property fails for the Banach space $\mathcal{L}(X)$ whenever either X or $\mathcal{L}(X)$ is reflexive.

1. Introduction

Let X be an infinite-dimensional Banach space and X' its topological dual space. By $\mathcal{L}(X)$ we denote the set of all bounded and linear operators on X, by $\mathcal{K}(X)$ (resp. $\mathcal{PK}(X)$), we denote the set of all compact (resp. power compact) operators in $\mathcal{L}(X)$ and by $\mathcal{L}(\mathcal{L}(X))$ we denote the set of all bounded and linear operators on $\mathcal{L}(X)$.
Let (u, α, V) in $X \times X' \times \mathcal{L}(X)$. Given the following two problems:

P$_1$. Find a solution T in $\mathcal{L}(X)$ of

$$T(x) - \alpha(T(x))u = V(x), \text{ for every } x \in X. \tag{1}$$

P$_2$. Find a solution T in $\mathcal{L}(X)$ of

$$T(x) - \alpha(x)T(u) = V(x), \text{ for every } x \in X. \tag{2}$$

Let us introduce the two linear operators $\mathscr{L}_{\alpha,u}$ and $\mathscr{R}_{\alpha,u}$ in $\mathcal{L}(\mathcal{L}(X))$, defined by

$$\mathscr{L}_{\alpha,u}\Big(T\Big)(x) = \alpha\Big(T(x)\Big)u,$$
$$\mathscr{R}_{\alpha,u}\Big(T\Big)(x) = \alpha(x)T(u), \text{ for every } T \in \mathcal{L}(X) \text{ and } x \in X.$$

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The problems $\mathbf{P}_i$, $i = 1, 2$ can be rewritten in terms of equations in $\mathcal{L}(X)$,

$\mathbf{P}_1$. Find a solution T in $\mathcal{L}(X)$ of

$$(I - \mathscr{L}_{\alpha,u})(T) = V. \tag{3}$$

$\mathbf{P}_2$. Find a solution T in $\mathcal{L}(X)$ of

$$(I - \mathscr{R}_{\alpha,u})(T) = V. \tag{4}$$

The resolution of these problems requires consideration of two cases:

Case 1. $\alpha(u) \neq 1$. We can solve P_1 by applying the linear functional α to both sides of equation (1). This yields

$$\alpha(T(x)) = \frac{\alpha(V(x))}{1 - \alpha(u)}, \text{ for every } x \in X.$$

Hence, the solution of equation (1) is unique, and it is defined by

$$T = V + \mathscr{L}_{\frac{\alpha}{1-\alpha(u)},u}(V). \tag{5}$$

We can solve $\mathbf{P}_2$ by taking $x = u$ in equation (2), yielding

$$T(u) = \frac{1}{1 - \alpha(u)} V(u).$$

Hence, the solution of equation (2) is unique, and it is defined by

$$T = V + \mathscr{R}_{\frac{\alpha}{1-\alpha(u)},u}(V). \tag{6}$$

Case 2. $\alpha(u) = 1$. The equation (1) (resp. (2)) has a solution if, and only if, $Im(V) \subseteq Ker(\alpha)$, (resp. $u \in Ker(V)$). In this case, $T = V$ is the unique solution of equations (1) and (2) respectively.

The family of linear operators $\mathscr{L}_{\alpha,u}$ and $\mathscr{R}_{\alpha,u}$, indexed with (u, α) in $X \times X'$, constitutes a very interesting subfamily of the family of *multiplication operators*. Indeed, for any fixed (u, α) in $X \times X'$, if we denote by $\mathcal{A}_{u,\alpha}$ the linear operator in $\mathcal{L}(X)$ defined by $\mathcal{A}_{u,\alpha}(x) = \alpha(x)u$ for every x in X, and we use the same notation as in [9], we can recognize the left and the right multiplication operators $L_{\mathcal{A}_{u,\alpha}}$ and $R_{\mathcal{A}_{u,\alpha}}$ respectively.

$$\mathscr{L}_{\alpha,u}(T) = L_{\mathcal{A}_{u,\alpha}}(T) = \mathcal{A}_{u,\alpha}T, \text{ for every } T \in \mathcal{L}(X).$$
$$\mathscr{R}_{\alpha,u}(T) = R_{\mathcal{A}_{u,\alpha}}(T) = T\mathcal{A}_{u,\alpha}, \text{ for every } T \in \mathcal{L}(X).$$

A preliminary investigation allowed us to identify the following property: for any T in $L(X)$, the operators $\mathscr{L}_{\alpha,u}(T)$ and $\mathscr{R}_{\alpha,u}(T)$ are of finite rank. As every finite rank operator in a Banach space is compact, the operators $\mathscr{L}_{\alpha,u}$ and $\mathscr{R}_{\alpha,u}$ satisfy the same property: $\mathscr{L}_{\alpha,u}\big(\mathcal{L}(X)\big) \subseteq \mathcal{K}(X)$ and $\mathscr{R}_{\alpha,u}\big(\mathcal{L}(X)\big) \subseteq \mathcal{K}(X)$. Encouraged by this fact regarding the operators $\mathscr{L}_{\alpha,u}$ and $\mathscr{R}_{\alpha,u}$, our goal is to investigate some topological properties satisfied by them, such as compactness, demicompactness, stongly demicompactess, power compactness and Riesz property. Specifically, we provide necessary and sufficient conditions to establish these properties. In addition, we obtain sufficient conditions for the quasi-compactness and weak-compactness of this class of operators.

The paper is organized in the following way. In section 2, we recall some definitions and results needed in the rest of the paper. Section 3 is entirely devoted to the study of the operator $\mathscr{L}_{\alpha,u}$. After exploring some properties of the operator $\mathscr{L}_{\alpha,u}$, we separately provide necessary and sufficient conditions on its compactness, demicompactness, stongly demicompactess, power compactness, and Riesz property (see Propositions 3.5 and 3.8). We only obtain sufficient conditions for the quasi-compacteness and weakly compactness of the operator $\mathscr{L}_{\alpha,u}$ (see Proposition 3.8, (iii) and Proposition 3.11). As an immediate consequence of the study of the linear operator $\mathscr{L}_{\alpha,u}$ (see Proposition 3.13), we show that $\mathcal{L}(X)$ does not have the Dunford-Pettis property if either X or $\mathcal{L}(X)$ is reflexive. In Section 4, we present several results for the operator $\mathscr{R}_{\alpha,u}$. These results are similar to those obtained for the operator $\mathscr{L}_{\alpha,u}$ in Section 3 and provide further insights into the structure and properties of this subfamily. Finally, we show that the multiplication operator obtained by composing the two operators $\mathscr{L}_{\alpha,u}$ and $\mathscr{R}_{\alpha,u}$ is compact (see Proposition 4.11).

2. Preliminaries

Let X be an infinite-dimensional Banach space. We begin this section by the following definitions:

Definition 2.1. *[10] An operator T in $\mathcal{L}(X)$ is said to be demicompact if for every bounded sequence $(x_n)_{n\geq 0}$ in X such that $x_n - T(x_n)$ converges to $y \in X$, there is a convergent subsequence of $(x_n)_{n\geq 0}$.*

The set of all demicompact operators on X will be denoted by $\mathcal{DC}(X)$.

Proposition 2.2. *[3, 4] For any T in $\mathcal{L}(X)$, the following statements are equivalent.*

(i) *T is a demicompact operator.*
(ii) $\dim \ker(I - T) < +\infty$ *and $\Im(I - T)$ is a closed subset of X.*

Definition 2.3. *An operator T in $\mathcal{L}(X)$ is said to be strongly demicompact if αT is demicompact for every scalar number α.*

The set of all stongly demicompact operators on X will be denoted by $\mathcal{SDC}(X)$. Note that every power compact operator is strongly demicompact. In particular, every compact operator is strongly demicompact.

Definition 2.4. *[2] An operator T in $\mathcal{L}(X)$ is said to be quasi-compact if there exist a positive integer n and a compact operator $K \neq 0$ on X such that $\|T^n - K\| < 1$.*

The set of all quasi-compact operators on X will be denoted by $\mathcal{QK}(X)$.

Definition 2.5. *[1]. A linear operator T in $\mathcal{L}(X)$ is said to be a Riesz operator if $(\lambda I - T)$ is a Fredholm operator of index 0 for all nonzero scalar number λ.*

The set of all Riesz operators on X will be denoted by $\mathcal{R}(X)$.

These different subsets of linear operators satisfy the following inclusions:

$$\begin{aligned} &\mathcal{K}(X) \subset \mathcal{PK}(X) \subset \mathcal{R}(X) \subset \mathcal{QK}(X) \\ \text{and } &\mathcal{K}(X) \subset \mathcal{PK}(X) \subset \mathcal{SDC}(X) \subset \mathcal{DC}(X). \end{aligned}$$

Lemma 2.6. *If $\mathcal{L}(X)$ is reflexive, then X is reflexive.*

Proof. Assume that u in X with $u \neq 0_X$. Recall that the linear operator M_u defined by

$$\begin{array}{rcccc} M_u & : & \mathcal{L}(X) & \to & X \\ & & T & \mapsto & T(u) \end{array}$$

is in $\mathcal{L}(\mathcal{L}(X), X)$, where $\|M_u\| = \|u\|$ and the image set $\Im(M_u) = X$. Thus, X is reflexive, from (9) p.p 198 in [11]. □

Definition 2.7. *[5, 6] A Banach space X has the Dunford-Pettis property if every continuous weakly compact operator T from X into another Banach space Y transforms weakly compact sets in X into norm-compact sets in Y (such operators are called completely continuous).*

Proposition 2.8. *[8] Let X be a Banach space having the Dunford-Pettis property. If T_1 and T_2 in $\mathcal{L}(X)$ are weakly compact operators, then the composition operator T_1T_2 is compact.*

3. Some properties of the operator $\mathscr{L}_{\alpha,u}$

Let X be a Banach space and X' its dual space. For any (u, α) in $X \times X'$, let $\mathscr{L}_{\alpha,u}$ be the linear operator on $\mathcal{L}(X)$ defined by,

$$\begin{array}{rccl} \mathscr{L}_{\alpha,u} & : & \mathcal{L}(X) & \to & \mathcal{L}(X) \\ & & T & \mapsto & \mathscr{L}_{\alpha,u}(T) \end{array}$$

where $\mathscr{L}_{\alpha,u}\Big(T\Big)(x) = \alpha\Big(T(x)\Big)u$, for every $T \in \mathcal{L}(X)$ and $x \in X$.

The following properties are easily derived and require no proof. While the proofs of these properties are straightforward and can be readily verified, we omit them here for the sake of brevity and to focus on the more complex aspects of the analysis.

Proposition 3.1. *For any α, β in X' and u, v in X, the following properties hold.*

- (a) $\mathscr{L}_{\alpha,u} \in \mathcal{L}\Big(\mathcal{L}(X)\Big)$ *where* $\|\mathscr{L}_{\alpha,u}\| = \|\alpha\|\|u\|$.
- (b) $\mathscr{L}_{\alpha,u} \circ \mathscr{L}_{\beta,v} = \alpha(v)\mathscr{L}_{\beta,u}$.
- (c) $\mathscr{L}^{n+1}_{\alpha,u} = (\alpha(u))^n \mathscr{L}_{\alpha,u}$, *for all integer* $n \geq 0$.
- (d) $\mathscr{L}_{\alpha,u} + \mathscr{L}_{\beta,u} = \mathscr{L}_{\alpha+\beta,u}$.
- (e) $\mathscr{L}_{\alpha,u} + \mathscr{L}_{\alpha,v} = \mathscr{L}_{\alpha,u+v}$.
- (f) $\lambda\mathscr{L}_{\alpha,u} = \mathscr{L}_{\lambda.\alpha,u} = \mathscr{L}_{\alpha,\lambda u}$, *for every scalar number* λ.
- (g) $\mathscr{L}_{\alpha,u} = 0$ *if, and only if,* $\alpha = 0_{X'}$ *or* $u = 0_X$.

To investigate the compactness of the operator $\mathscr{L}_{\alpha,u}$, we rely on some auxiliary results.

Lemma 3.2. *For any non-zero α in X', the following properties hold.*

- (i) *For any β in X', there exists T_β in $\mathcal{L}(X)$ such that $\beta = \alpha \circ T_\beta$.*
- (ii) *For any bounded sequence $(\beta_n)_{n\geq 0}$ in X', there exists a bounded sequence $(T_n)_{n\geq 0}$ in $\mathcal{L}(X)$ such that $\beta_n = \alpha \circ T_n$, for all integer $n \geq 0$.*

Proof. Let α in X' where $\alpha \neq 0_{X'}$ be fixed. Since $\alpha \neq 0_{X'}$, there exists u in X such that $\alpha(u) = 1$. Given $\beta \in X'$ and let's consider the bounded linear operator T_β on X defined by $T_\beta(x) = \beta(x)u$ for every $x \in X$. Clearly, the operator T_β is linear from X to itself and satisfies $\|T_\beta\| = \|\beta\|\|u\|$. Furthermore, we have $\alpha \circ T_\beta(x) = \alpha\Big(\beta(x)u\Big) = \beta(x)\alpha(u) = \beta(x)$ for every $x \in X$. This implies that $\beta = \alpha \circ T_\beta$. Therefore, (i) holds.

Assume that (u, α) in $X \times X'$ where $\alpha(u) = 1$. Let $(\beta_n)_{n\geq 0}$ be a bounded sequence in X' and consider the sequence $(T_n)_{n\geq 0}$ in $\mathcal{L}(X)$ defined by $T_n(x) = \beta_n(x)u$ for every $x \in X$ and every integer $n \geq 0$. Since $(\beta_n)_{n\geq 0}$ is a bounded sequence in X', there exists $M > 0$ such that $\|\beta_n\| \leq M$ for every $n \geq 0$. It follows that $\|T_n\| = \|u\|\|\beta_n\| \leq \|u\|M$ for every $n \geq 0$. Moreover, we have $\beta_n = \alpha \circ T_n$ for every $n \geq 0$. Therefore, (ii) holds. □

Lemma 3.3. *For any non-zero α in X', the operator $\mathbf{L}_\alpha$ defined by,*

$$\begin{array}{rccl} \mathbf{L}_\alpha & : & \mathcal{L}(X) & \to & X' \\ & & T & \mapsto & \mathbf{L}_\alpha(T) = \alpha \circ T \end{array}$$

is surjective linear and where $\|\mathbf{L}_\alpha\| = \|\alpha\|$.

Proof. Assume that α in X' where $\alpha \neq 0_{X'}$. Let T_i in $\mathcal{L}(X)$, $i = 1, 2$ and λ be a scalar number, we can write $\mathbf{L}_\alpha(T_1 + \lambda T_2) = \alpha \circ (T_1 + \lambda T_2) = \alpha \circ T_1 + \lambda\alpha \circ T_2 = \mathbf{L}_\alpha(T_1) + \lambda\mathbf{L}_\alpha(T_2)$. Since $|\alpha(x)| \leq \|\alpha\|\|x\|$ for every $x \in X$, it comes that $\|\mathbf{L}_\alpha\| \leq \|\alpha\|$. Indeed, we have

$$\begin{aligned}
\|\mathbf{L}_\alpha\| &= \sup_{\|T\|=1} \|\mathbf{L}_\alpha(T)\| \\
&= \sup_{\|T\|=1}\big(\sup_{\|x\|=1} |\alpha(T(x))|\big) \\
&\leq \sup_{\|T\|=1}\big(\sup_{\|x\|=1} \|\alpha\|\|T(x)\|\big) \\
&\leq \|\alpha\| \sup_{\|T\|=1}\big(\sup_{\|x\|=1} \|T(x)\|\big) \\
&\leq \|\alpha\| \sup_{\|T\|=1} \|T\| \\
&\leq \|\alpha\|.
\end{aligned}$$

Since $\|I\| = 1$, then $\|\mathbf{L}_\alpha\| \geq \|\mathbf{L}_\alpha(I)\| = \sup_{\|x\|=1} |\alpha(x)| = \|\alpha\|$. Thus, $\|\mathbf{L}_\alpha\| = \|\alpha\|$. Finally, by Lemma 3.2, (i), we can conclude that $\mathbf{L}_\alpha$ is surjective. □

Lemma 3.4. *For any (u, α) in $X \times X'$ where $u \neq 0_X$ and $\alpha \neq 0_{X'}$, the following properties hold.*

(i) $\ker(\mathscr{L}_{\alpha,u}) = \{T \in \mathcal{L}(X) |\ \Im(T) \subseteq \ker(\alpha)\} = \{T \in \mathcal{L}(X) |\ \overline{\Im(T)} \subseteq \ker(\alpha)\}$.

(ii) $\Im(\mathscr{L}_{\alpha,u}) = \{L \in \mathcal{L}(X) |\ \Im(L) \subseteq \mathrm{Vect}\{u\}\}$.

Proof. Assume that (u, α) in $X \times X'$ where $\alpha \neq 0_{X'}$ and $u \neq 0_X$. We have

$$\begin{aligned}
\ker(\mathscr{L}_{\alpha,u}) &= \{T \in \mathcal{L}(X) |\ \mathscr{L}_{\alpha,u}(T) = 0\} \\
&= \{T \in \mathcal{L}(X) |\ \mathscr{L}_{\alpha,u}(T)(x) = 0, \ \text{for all} \ \ x \in X\} \\
&= \{T \in \mathcal{L}(X) |\ \alpha(T(x))u = 0, \ \text{for all} \ \ x \in X\} \\
&= \{T \in \mathcal{L}(X) |\ T(x) \in \ker(\alpha), \ \text{for all} \ \ x \in X\} \\
&= \{T \in \mathcal{L}(X) |\ \Im(T) \subseteq \ker(\alpha)\}.
\end{aligned}$$

By the continuity of $\alpha \in X'$ and since $\ker(\alpha) = \alpha^{-1}\big(\{0\}\big)$ is a closed subset of X and $\Im(T) \subseteq \overline{\Im(T)}$ for all $T \in \mathcal{L}(X)$, we deduce that $\ker(\mathscr{L}_{\alpha,u}) = \{T \in \mathcal{L}(X) |\ \overline{\Im(T)} \subseteq \ker(\alpha)\}$. Hence, (i) holds.

We always have

$$\begin{aligned}
\Im(\mathscr{L}_{\alpha,u}) &= \{L \in \mathcal{L}(X) |\ \exists\, T \in \mathcal{L}(X), \ \mathscr{L}_{\alpha,u}(T) = L\} \\
&= \{L \in \mathcal{L}(X) |\ \exists\, T \in \mathcal{L}(X), \ \mathscr{L}_{\alpha,u}(T)(x) = L(x), \ \text{for all} \ \ x \in X\} \\
&= \{L \in \mathcal{L}(X) |\ \exists\, T \in \mathcal{L}(X), \ \alpha(T(x))u = L(x), \ \text{for all} \ \ x \in X\} \\
&\subseteq \{L \in \mathcal{L}(X) |\ \Im(L) \subseteq \mathrm{Vect}\{u\}\}.
\end{aligned}$$

Conversely, let L in $\{L \in \mathcal{L}(X) |\ \Im(L) \subseteq \mathrm{Vect}\{u\}\}$. There exists β in X' where $\|\beta\| = \|L\|\|u\|^{-1}$ such that $L(x) = \beta(x)u$ for all $x \in X$. By lemma 3.2 (i), since $\alpha \neq 0_{X'}$, there exists $T \in \mathcal{L}(X)$ such that $\beta = \alpha \circ T$, and hence, $L(x) = \big(\alpha \circ T\big)(x)u$ for all $x \in X$, i.e., $L = \mathscr{L}_{\alpha,u}(T) \in \Im(\mathscr{L}_{\alpha,u})$. Accordingly, we have $\Im(\mathscr{L}_{\alpha,u}) = \{L \in \mathcal{L}(X) | \Im(L) \subseteq \mathrm{Vect}\{u\}\}$. Therefore, (ii) holds. □

Proposition 3.5. *For any (u, α) in $X \times X'$, the following statements are equivalents.*

(i) *$\mathscr{L}_{\alpha,u}$ is a compact operator.*

(ii) *$\alpha = 0_{X'}$ or $u = 0_X$.*

Proof. $(i) \Rightarrow (ii)$. Suppose that $\mathscr{L}_{\alpha,u}$ is compact where $u \neq 0_X$ and $\alpha \neq 0_{X'}$. Let $(\beta_n)_{n\geq 0}$ be any bounded sequence in X'. By lemma 3.2 (ii), there exists a bounded sequence $(T_n)_{n\geq 0}$ in $\mathcal{L}(X)$ such that $\beta_n = \alpha \circ T_n$ for every integer $n \geq 0$. By the assumption $\mathscr{L}_{\alpha,u}$ is compact and by the fact that the sequence $(T_n)_{n\geq 0}$ is bounded in $\mathcal{L}(X)$, the sequence $(\mathscr{L}_{\alpha,u}(T_n))_{n\geq 0}$ has a convergent subsequence $(\mathscr{L}_{\alpha,u}(T_{\varphi(n)}))_{n\geq 0}$ in $\mathcal{L}(X)$. Since $\mathcal{L}(X)$ is a Banach space, the sequence $(\mathscr{L}_{\alpha,u}(T_{\varphi(n)}))_{n\geq 0}$ is a Cauchy sequence, i.e., for all $\varepsilon > 0$, there exists an integer $\exists \geq 0$ such that for all integers $n > m > N_\varepsilon$, we have $\|\mathscr{L}_{\alpha,u}(T_{\varphi(n)}) - \mathscr{L}_{\alpha,u}(T_{\varphi(m)})\| < \varepsilon\|u\|$. Notice that

$$\begin{aligned}\|\mathscr{L}_{\alpha,u}(T_{\varphi(n)}) - \mathscr{L}_{\alpha,u}(T_{\varphi(m)})\| &= \|u\| \sup_{\|x\|=1} \|\alpha \circ T_{\varphi(n)}(x) - \alpha \circ T_{\varphi(m)}(x)\| \\ &= \|u\| \sup_{\|x\|=1} \|\beta_{\varphi(n)}(x) - \beta_{\varphi(m)}(x)\| \\ &= \|u\|\|\beta_{\varphi(n)} - \beta_{\varphi(m)}\|.\end{aligned}$$

Then, for all $\varepsilon > 0$, there exists an integer $N_\varepsilon \geq 0$ such that for all integers $n > m > N_\varepsilon$, we have $\|\beta_{\varphi(n)} - \beta_{\varphi(m)}\| < \varepsilon$. Hence, $(\beta_{\varphi(n)})_{n\geq 0}$ is a Cauchy sequence in the Banach space X'. Thus, the sequence $(\beta_{\varphi(n)})_{n\geq 0}$ converges in X'. As a conclusion, any bounded sequence $(\beta_n)_{n\geq 0}$ in X' has a convergent subsequence. This requires that the space X' is of finite dimension and then the space X is also of finite dimension. This is a contradiction.

(ii) $\Rightarrow$ (i). When $\alpha = 0_{X'}$ or $u = 0_X$, it is clear that $\mathscr{L}_{\alpha,u} = 0_{\mathcal{L}(X)}$ the null operator which is a compact operator. □

To analyze the demicompactness of the linear operator $\mathscr{L}_{\alpha,u}$, we must establish some technical Lemmas.

Lemma 3.6. *For any (u, α) in $X \times X'$ where $\alpha \neq 0_{X'}$ and $u \neq 0_X$, the following statements hold.*

(i) *If $\alpha(u) \neq 1$, then* $\ker(I - \mathscr{L}_{\alpha,u}) = \{0_{\mathcal{L}(X)}\}$.

(ii) *If $\alpha(u) = 1$, then* $\ker(I - \mathscr{L}_{\alpha,u})$ *is an infinite and closed subspace of the Banach space* $\mathcal{L}(X)$, *isomorphic to X' and defined by*

$$\begin{aligned}\ker(I - \mathscr{L}_{\alpha,u}) &= \{T \in \mathcal{L}(X) | \ \Im(T) \subseteq \text{Vect}\{u\}\} \\ &= \{T \in \mathcal{L}(X) | \ \overline{\Im(T)} \subseteq \text{Vect}\{u\}\} \\ &= \{T \in \mathcal{L}(X) | \ \exists! \ \beta \in X', \ T(x) = \beta(x)u, \ \text{for all } x \in X\}.\end{aligned}$$

Proof. Assume that (u, α) in $X \times X'$ where $\alpha \neq 0_{X'}$ and $u \neq 0_X$. Notice that we have $\ker(I - \mathscr{L}_{\alpha,u}) = (I - \mathscr{L}_{\alpha,u})^{-1}\big(\{0\}\big)$ and $I - \mathscr{L}_{\alpha,u}$ is continuous, then $\ker(I - \mathscr{L}_{\alpha,u})$ is a closed subspace of the Banach space $\mathcal{L}(X)$, and hence, $\ker(I - \mathscr{L}_{\alpha,u})$ is also a Banach space. Moreover, we have

$$\begin{aligned}\ker(I - \mathscr{L}_{\alpha,u}) &= \{T \in \mathcal{L}(X) | \ (I - \mathscr{L}_{\alpha,u})(T) = 0\} \\ &= \{T \in \mathcal{L}(X) | \ \mathscr{L}_{\alpha,u}(T) = T\} \\ &= \{T \in \mathcal{L}(X) | \ T(x) = \alpha(T(x))u, \ \text{ for all } \ x \in X\}.\end{aligned}$$

We have to distinguish two cases.

Case $\alpha(u) \neq 1$. Let T in $\ker(I - \mathscr{L}_{\alpha,u})$, then $T(x) = \alpha(T(x))u$ for all $x \in X$. By applying the linear functional α on both sides of the last equation, we get $(1 - \alpha(u))\alpha(T(x)) = 0$ for all $x \in X$. Since $\alpha(u) \neq 1$, then $\alpha(T(x)) = 0$ for all $x \in X$. This yields, $T(x) = \alpha(T(x))u = 0$ for all $x \in X$. Hence, $\ker(I - \mathscr{L}_{\alpha,u}) = \{0_{\mathcal{L}(X)}\}$.

Case $\alpha(u) = 1$. We have $\ker(I - \mathscr{L}_{\alpha,u}) \subseteq \{T \in \mathcal{L}(X) | \ \Im(T) \subseteq \text{Vect}\{u\}\}$. Conversely, let T in $\{T \in \mathcal{L}(X) | \ \Im(T) \subseteq \text{Vect}\{u\}\}$. There exists β in X' where $\|\beta\| = (\|T\|/\|u\|)$ such that $T(x) = \beta(x)u$ for all $x \in X$. If we apply α on both sides of the last equation and we use the assumption $\alpha(u) = 1$, we obtain $\alpha(T(x)) = \beta(x)$ for all $x \in X$. It follows that, $T(x) = \alpha(T(x))u$ for all $x \in X$, and hence, $T \in \ker(I - \mathscr{L}_{\alpha,u})$. Consequently, $\ker(I - \mathscr{L}_{\alpha,u}) = \{T \in \mathcal{L}(X) | \ \Im(T) \subseteq \text{Vect}\{u\}\}$, if $\alpha(u) = 1$. Moreover, since $\text{Vect}\{u\}$ is a closed subset of X, i.e., $\text{Vect}\{u\} = \overline{\text{Vect}\{u\}}$ and since we have $\Im(T) \subseteq \overline{\Im(T)}$ for all $T \in \mathcal{L}(X)$, we conclude that $\ker(I - \mathscr{L}_{\alpha,u}) = \{T \in \mathcal{L}(X) | \overline{\Im(T)} \subseteq \text{Vect}\{u\}\}$. It's clear that $\{T \in \mathcal{L}(X) | \ \exists! \ \beta \in X', \ T(x) = \beta(x)u \text{ for all } x \in X\} \subseteq \{T \in \mathcal{L}(X) | \ \Im(T) \subseteq \text{Vect}\{u\}\}$.

Conversely, let T in $\ker(I - \mathscr{L}_{\alpha,u}) = \{T \in \mathcal{L}(X) | \ \Im(T) \subseteq \text{Vect}\{u\}\}$ and assume that there exist β and γ in X' such that $T(x) = \beta(x)u = \gamma(x)u$ for all $x \in X$. Since we have $u \neq 0_X$, then $\beta = \gamma$. Hence, $T \in \{T \in \mathcal{L}(X) | \ \exists! \ \beta \in$

$X', T(x) = \beta(x)u$ for all $x \in X\}$. Therefore, $\ker(I - \mathscr{L}_{\alpha,u}) \subseteq \{T \in \mathcal{L}(X) | \exists! \beta \in X', T(x) = \beta(x)u$ for all $x \in X\}$. Thus, $\ker(I - \mathscr{L}_{\alpha,u}) = \{T \in \mathcal{L}(X) | \exists! \beta \in X', T(x) = \beta(x)u$ for all $x \in X\}$.
Based on the previous statement and in a structural and constructive way, let's consider the linear operator $\mathcal{M}$ that maps the Banach space X' to the Banach space $\ker(I - \mathscr{L}_{\alpha,u})$ defined by $\mathcal{M}(\beta) = T_\beta$ where T_β is in $\mathcal{L}(X)$ and defined by $T_\beta(x) = \beta(x)u$ for all $x \in X$. We have

$$\begin{aligned}\|\mathcal{M}\| &= \sup_{\|\beta\|=1} \|\mathcal{M}(\beta)\| \\ &= \sup_{\|\beta\|=1} \|T_\beta\| \\ &= \sup_{\|\beta\|=1} \sup_{\|x\|=1} \|T_\beta(x)\| \\ &= \sup_{\|\beta\|=1} \sup_{\|x\|=1} \|\beta(x)u\| \\ &= \|u\| \sup_{\|\beta\|=1} \sup_{\|x\|=1} \|\beta(x)\| \\ &= \|u\| \sup_{\|\beta\|=1} \|\beta\| \\ &= \|u\|.\end{aligned}$$

Through the carefully designed construction, we can demonstrate that the linear operator $\mathcal{M}$ is bijective, mapping from the dual space of X to the kernel of $(I - \mathscr{L}_{\alpha,u})$. □

Lemma 3.7. *For any (u, α) in $X \times X'$ where $\alpha \neq 0_{X'}$ and $u \neq 0_X$, the following statements hold.*

(*i*) *If $\alpha(u) \neq 1$, then $\Im(I - \mathscr{L}_{\alpha,u}) = \mathcal{L}(X)$.*
(*ii*) *If $\alpha(u) = 1$, then $\Im(I - \mathscr{L}_{\alpha,u})$ is a closed subset of $\mathcal{L}(X)$ and defined by,*

$$\begin{aligned}\Im(I - \mathscr{L}_{\alpha,u}) &= \{L \in \mathcal{L}(X) | \Im(L) \subseteq \ker(\alpha)\} \\ &= \{L \in \mathcal{L}(X) | \overline{\Im(L)} \subseteq \ker(\alpha)\} \\ &= \ker(\mathscr{L}_{\alpha,u}).\end{aligned}$$

Proof. Let(u, α) in $X \times X'$ where $\alpha \neq 0_{X'}$ and $u \neq 0_X$. We have

$$\Im(I - \mathscr{L}_{\alpha,u}) = \{L \in \mathcal{L}(X) | \exists\, T \in \mathcal{L}(X),\ L = (I - \mathscr{L}_{\alpha,u})(T)\}.$$

Assume that $\alpha(u) \neq 1$. Given L in $\mathcal{L}(X)$ and let's introduce T in $\mathcal{L}(X)$ defined by $T(x) = L(x) + (1 - \alpha(u))^{-1}\alpha(L(x))u$ for all $x \in X$. For all $x \in X$, we have

$$\begin{aligned}(I - \mathscr{L}_{\alpha,u})(T)(x) &= T(x) - \mathscr{L}_{\alpha,u}(T)(x) \\ &= L(x) + \frac{\alpha\Big(L(x)\Big)}{1 - \alpha(u)}u - \alpha\Big(L(x) + \frac{\alpha(L(x))u}{1 - \alpha(u)}\Big)u \\ &= L(x) + \frac{\alpha\Big(L(x)\Big)}{1 - \alpha(u)}u - \alpha\Big(L(x))u - \frac{\alpha(L(x)\Big)\alpha(u)}{1 - \alpha(u)}u \\ &= L(x) + \Big(\frac{1}{1 - \alpha(u)} - 1 - \frac{\alpha(u)}{1 - \alpha(u)}\Big)\alpha\Big(L(x)\Big)u \\ &= L(x).\end{aligned}$$

Equivalently, $L = (I - \mathscr{L}_{\alpha,u})(T)$ where $T \in \mathcal{L}(X)$. This implies that $\Im(I - \mathscr{L}_{\alpha,u}) = \mathcal{L}(X)$. Hence, (*i*) holds.

Assume that $\alpha(u) = 1$. Let L in $\Im(I - \mathscr{L}_{\alpha,u})$, then there exists $T \in \mathcal{L}(X)$ such that $L = (I - \mathscr{L}_{\alpha,u})(T)$, i.e., $L(x) = T(x) - \alpha(T(x))u$ for all $x \in X$. By applying α to both hand sides of the last equation, we get $\alpha(L(x)) = 0$

for all $x \in X$. This implies that $\Im(L) \subseteq \ker(\alpha)$. Therefore, $\Im(I-\mathscr{L}_{\alpha,u}) \subseteq \{L \in \mathcal{L}(X) | \ \Im(L) \subseteq \ker(\alpha)\}$. Conversely, let $L \in \{L \in \mathcal{L}(X) | \ \Im(L) \subseteq \ker(\alpha)\}$. Clearly, $\Im(L) \subseteq \ker(\alpha)$, and hence, $(I - \mathscr{L}_{\alpha,u})(L)(x) = L(x) - \alpha(L(x))u = L(x)$ for all $x \in X$, i.e., $L = (I - \mathscr{L}_{\alpha,u})(L) \in \Im(I - \mathscr{L}_{\alpha,u})$.Thus, $\Im(I - \mathscr{L}_{\alpha,u}) = \{L \in \mathcal{L}(X) | \ \Im(L) \subseteq \ker(\alpha)\}$. Finally, by Lemma 3.4, (i), we have $\Im(I - \mathscr{L}_{\alpha,u}) = \ker(\mathscr{L}_{\alpha,u})$. Therefore,$(ii)$ holds. □

Proposition 3.8. *For any (u, α) in $X \times X'$ where $\alpha \neq 0_{X'}$ and $u \neq 0_X$, the following statements hold.*

- (i) *$\mathscr{L}_{\alpha,u}$ is demicompact if, and only if, $\alpha(u) \neq 1$.*
- (ii) *$\mathscr{L}_{\alpha,u}$ is strongly demicompact if, and only if, $\alpha(u) = 0$.*
- (iii) *$\mathscr{L}_{\alpha,u}$ is quasi-compact if $|\alpha(u)|^n \|\alpha\|\|u\| < 1$, for some integer $n \geq 0$.*
- (iv) *$\mathscr{L}_{\alpha,u}$ is power compact if, and only if, $\alpha(u) = 0$.*
- (v) *$\mathscr{L}_{\alpha,u}$ is a Riesz operator if, and only if, $\alpha(u) = 0$.*

Proof. Let (u, α) in $X \times X'$ where $\alpha \neq 0_{X'}$ and $u \neq 0_X$. By the Proposition 2.2, $\mathscr{L}_{\alpha,u}$ is demicompact if, and only if, $\dim \ker(I - \mathscr{L}_{\alpha,u}) < +\infty$ and $\Im(I - \mathscr{L}_{\alpha,u})$ is a closed subset of $\mathcal{L}(X)$. In view of Lemmas 3.6 and 3.7, it comes that $\mathscr{L}_{\alpha,u}$ is demicompact if, and only if, $\alpha(u) \neq 1$. Therefore, (i) holds.

Let λ be a scalar number. By Proposition 3.1, (f), we may write $I - \lambda\mathscr{L}_{\alpha,u} = I - \mathscr{L}_{\lambda\alpha,u}$. Next, we have to treat the two following cases:

Case $\alpha(u) = 0$. Then, $\lambda\alpha(u) = 0 \neq 1$. By the last property (i), the operator $I - \lambda\mathscr{L}_{\alpha,u}$ is demicompact for all scalar number λ. Equivalently, $\mathscr{L}_{\alpha,u}$ is strongly demicompact.

Case $\alpha(u) \neq 0$. Then, $I - \lambda\mathscr{L}_{\alpha,u} = I - \mathscr{L}_{\lambda\alpha,u}$ is demicompact if, and only if,, $\lambda \neq (\alpha(u))^{-1}$. Thus, $\mathscr{L}_{\alpha,u}$ is not strongly demicompact. Therefore, (ii) holds.

Assume that there exists an integer $n \geq 0$ such that $|\alpha(u)|^n \|\alpha\|\|u\| < 1$. By Proposition 3.1, (a) and (c), we have $\|\mathscr{L}^{n+1}_{\alpha,u}\| = |\alpha(u)|^n \|\alpha\|\|u\| < 1$. Hence, there exist an integer $N = n+1 \geq 1$ and $K = 0$ in $\mathcal{K}(\mathcal{L}(X))$ such that $\|\mathscr{L}^N_{\alpha,u} - K\| < 1$. Thus, the operator $\mathscr{L}_{\alpha,u}$ is quasi-compact. Hence, (iii) holds.

Assume that (u, α) in $X \times X'$ such that $u \neq 0_X$ and $\alpha \neq 0_{X'}$. Suppose that $\mathscr{L}_{\alpha,u}$ is power compact, i.e., there exists an integer N such that $\mathscr{L}^N_{\alpha,u}$ is compact. By Proposition 3.1, (a) and (c), we have $\mathscr{L}^N_{\alpha,u} = (\alpha(u))^{N-1}\mathscr{L}_{\alpha,u}$. But, by virtue of the Proposition 3.5, and by the assumptions $u \neq 0_X$ and $\alpha \neq 0_{X'}$, this requires that $N \geq 2$ and $\alpha(u) = 0$. Conversely, assume that $\alpha(u) = 0$. From Proposition 3.1, (c), observe that $\mathscr{L}^2_{\alpha,u} = 0$ which is a compact operator. Therefore, (iv) holds.

Assume that (u, α) in $X \times X'$ are such that $u \neq 0_X$ and $\alpha \neq 0_{X'}$. Suppose that $\mathscr{L}_{\alpha,u}$ is a Riesz operator. Then, the operator $\mathscr{L}_{\alpha,u}$ is strongly quasi-compact, and hence, $\mathscr{L}_{\alpha,u}$ is strongly demicompact. By the previous property (ii), this requires that $\alpha(u) = 0$. Conversely, if $\alpha(u) = 0$, then $\mathscr{L}_{\alpha,u}$ is a power compact operator, and hence, it is a Riesz operator. Therefore, (v) holds. □

Proposition 3.9. *For any α in X' where $\alpha \neq 0_{X'}$, it does not exist β in X' and γ in X' such that $\alpha\Big(T(x)\Big) = \beta(T)\gamma(x)$, for every T in $\mathcal{L}(X)$ and x in X.*

Proof. Let $\alpha \in X'$ where $\alpha \neq 0_{X'}$. Assume that there exist two non-zero linear functional β and γ in X' such that $\alpha(T(x)) = \beta(T)\gamma(x)$ for all $T \in \mathcal{L}(X)$ and $x \in X$. Since $\alpha \neq 0_{X'}$, then α is surjective and so that there exists $u \neq 0_X$ such that $\alpha(u) = 1$. By Proposition 3.5, the operator $\mathscr{L}_{\alpha,u}$ is not compact . On the other hand, putting $L_{\gamma,u}(x) = \gamma(x)u$ for all $x \in X$. Clearly, $L_{\gamma,u} \in \mathcal{L}(X)$ and $\|L_{\gamma,u}\| = \|\gamma\|\|u\|$. Consider the operator $M_{\beta,\gamma,u}$ defined by $M_{\beta,\gamma,u}(T) = \beta(T)L_{\gamma,u}$ for all $T \in \mathcal{L}(X)$. We can show that $M_{\beta,\gamma,u} \in \mathcal{L}(\mathcal{L}(X))$, where $\|M_{\beta,\gamma,u}\| = \|\gamma\|\|\beta\|\|u\|$. The operator $M_{\beta,\gamma,u}$ is compact because it has finite rank. Indeed, we have $\Im(M_{\beta,\gamma,u}) = \text{Vect}\{L_{\gamma,u}\}$ and so that $\dim \Im(M_{\beta,\gamma,u}) = 1$. This leads to a contradiction because we have $M_{\beta,\gamma,u} = \mathscr{L}_{\alpha,u}$, where $\mathscr{L}_{\alpha,u}$ is not compact, while $M_{\beta,\gamma,u}$ is compact. □

Proposition 3.10. *Let X be an infinite separable Banach space and (u, α) in $X \times X'$ where $u \neq 0_X$ and $\alpha \neq 0_{X'}$. For any bounded sequence $(T_n)_{n\geq 0}$ in $\mathcal{L}(X)$, the sequence $\Big(\mathscr{L}_{\alpha,u}(T_n)\Big)_{n\geq 0}$ has a pointwise convergent subsequence $\Big(\mathscr{L}_{\alpha,u}(T_{\varphi(n)})\Big)_{n\geq 0}$ that converges to $\mathscr{L}_{\alpha,u}(T)$ where T in $\mathcal{L}(X)$.*

Proof. Assume that X is an infinite separable Banach space and (u, α) in $X \times X'$ where $u \neq 0_X$ and $\alpha \neq 0_{X'}$. Let $(T_n)_{n\geq 0}$ be a bounded sequence in $\mathcal{L}(X)$, such that there exists some $M > 0$ such that $\|T_n\| < M$ for all $n \geq 0$. Consider the sequence $(\beta_n)_{n\geq 0}$ in X' defined by $\beta_n = \alpha \circ T_n$ for all integer $n \geq 0$. The sequence $(\beta_n)_{n\geq 0}$ is bounded in X', since we have $\|\beta_n\| \leq M\|\alpha\|$ for all $n \geq 0$. By the assumption X is a separable space and by referring to [11], Theorem 8.13, the bounded sequence $(\beta_n)_{n\geq 0}$ has a weak* convergent subsequence $(\beta_{\varphi(n)})_{n\geq 0}$ in X'. This means that there exists a unique β in X' such that $(\beta_{\varphi(n)}(x))_{n\geq 0}$ converges to $\beta(x)$ for all $x \in X$. Immediately, $(\mathscr{L}_{\alpha,u}(T_{\varphi(n)})(x))_{n\geq 0}$ converges to $\beta(x)u$ for all $x \in X$. By Lemma 3.2 (*i*), there exists $T_\beta \in \mathcal{L}(X)$ such that $\beta = \alpha \circ T_\beta$. As a consequence, for all $x \in X$, $(\mathscr{L}_{\alpha,u}(T_{\varphi(n)})(x))_{n\geq 0}$ converges to $\mathscr{L}_{\alpha,u}(T_\beta)(x)$. □

Proposition 3.11. *When X is a reflexive Banach space, the linear operator $\mathscr{L}_{\alpha,u}$ is weakly compact for every (u, α) in $X \times X'$.*

Proof. Assume that X is reflexive and let (u, α) in $X \times X'$. If either $\alpha = 0_{X'}$ or $u = 0_X$, the operator $\mathscr{L}_{\alpha,u} = 0$ is compact and therefore weakly compact.
Suppose that $\alpha \neq 0_{X'}$ and $u \neq 0_X$ and denote by $\mathcal{A}_{u,\alpha}$ the linear operator in $\mathcal{L}(X)$ defined by $\mathcal{A}_{u,\alpha}(x) = \alpha(x)u$ for all x in X. Since $u \neq 0_X$ and $\alpha \neq 0_{X'}$, it follows that $\mathfrak{I}(\mathcal{A}_{u,\alpha}) = \text{Vect}\{u\}$. The operator $\mathcal{A}_{u,\alpha}$ is compact, since it is of finite rank. Moreover, we can express $\mathscr{L}_{\alpha,u}(T)(x) = \alpha(T(x))u = \mathcal{A}_{u,\alpha}\big(T(x)\big)$ for all x in X and T in $\mathcal{L}(X)$. Then $\mathscr{L}_{\alpha,u}(T) = \mathcal{A}_{u,\alpha}T$ for all T in $\mathcal{L}(X)$, which implies that $\mathscr{L}_{\alpha,u} = L_{\mathcal{A}_{u,\alpha}}$ where $L_{\mathcal{A}_{u,\alpha}}(T) = \mathcal{A}_{u,\alpha}T$ for all T in $\mathcal{L}(X)$. Finally, by applying Appendix 6, Theorem 6 in [9], we conclude that $\mathscr{L}_{\alpha,u} = L_{\mathcal{A}_{u,\alpha}}$ is a weakly compact operator on $\mathcal{L}(X)$, if X is reflexive. □

Corollary 3.12. *If $\mathcal{L}(X)$ is reflexive, then $\mathscr{L}_{\alpha,u}$ is weakly compact, for every (u, α) in $X \times X'$.*

Proof. It is a straightforward consequence of Proposition 3.11 and Lemma 2.6. □

Proposition 3.13. *If X is reflexive or $\mathcal{L}(X)$ is reflexive, then $\mathcal{L}(X)$ does not posses the Dunford-Pettis property.*

Proof. Assume that X or $\mathcal{L}(X)$ is reflexive. By Lemma 2.6, it follows that X is reflexive. Suppose that $\mathcal{L}(X)$ has the Dunford-Pettis Property. In view of the Proposition 3.11, we infer that $\mathscr{L}_{\alpha,u}$ is weakly compact for every (u, α) in $X \times X'$. In particular, $\mathscr{L}_{\alpha,u}$ is weakly compact, if $\alpha(u) \neq 0$. By Proposition 2.8 and the assumption $\mathcal{L}(X)$ has the Dunford-Pettis Property, we have $\mathscr{L}^2_{\alpha,u} = \alpha(u)\mathscr{L}_{\alpha,u}$ is compact. This is a contradiction, since $\mathscr{L}_{\alpha,u}$ where $\alpha \neq 0_{X'}$ and $u \neq 0_X$, is not compact by the Proposition 3.5. As a consequence, $\mathcal{L}(X)$ does not posses the Dunford-Pettis property. □

4. Some properties of the operator $\mathscr{R}_{\alpha,u}$.

Let X be an infinite Banach space and X' its dual space. For every (u, α) in $X \times X'$, let us introduce the linear operator $\mathscr{R}_{\alpha,u}$ on $\mathcal{L}(X)$ defined by

$$\begin{array}{rccc} \mathscr{R}_{\alpha,u} & : & \mathcal{L}(X) & \to & \mathcal{L}(X) \\ & & T & \mapsto & \mathscr{R}_{\alpha,u}(T) \end{array}$$

where $\mathscr{R}_{\alpha,u}(T)(x) = \alpha(x)T(u)$ for every T in $\mathcal{L}(X)$ and x in X.

The following properties are immediate and the proof are omitted.

Proposition 4.1. *Let α, β in X' and u, v in X, the following properties hold.*

(*a*) $\mathscr{R}_{\alpha,u} \in \mathcal{L}(\mathcal{L}(X))$ *where* $\|\mathscr{R}_{\alpha,u}\| = \|\alpha\|\|u\|$.
(*b*) $\mathscr{R}_{\alpha,u} \circ \mathscr{R}_{\beta,v} = \beta(u)\mathscr{R}_{\alpha,v}$.
(*c*) $\mathscr{R}^{n+1}_{\alpha,u} = \big(\alpha(u)\big)^n \mathscr{R}_{\alpha,u}$, *for all integer* $n \geq 0$.
(*d*) $\mathscr{R}_{\alpha,u} + \mathscr{R}_{\beta,u} = \mathscr{R}_{\alpha+\beta,u}$.
(*e*) $\mathscr{R}_{\alpha,u} + \mathscr{R}_{\alpha,v} = \mathscr{R}_{\alpha,u+v}$.
(*f*) $\lambda\mathscr{R}_{\alpha,u} = \mathscr{R}_{\lambda\alpha,u} = \mathscr{R}_{\alpha,\lambda u}$, *for all scalar number* λ.

(*g*) $\mathscr{R}_{\alpha,u} = 0$ *if, and only if,* $\alpha = 0_{X'}$ *or* $u = 0_X$.

To investigate the compactness of the operator $\mathscr{R}_{\alpha,u}$ operator, we require the application of some specialized and intricate lemmas that provide crucial insights into the behavior of the operator.

Lemma 4.2. *Let u in X where $u \neq 0_X$. The following properties hold.*

(*i*) *For every v in X, there exists T_v in $\mathcal{L}(X)$ such that $v = T_v(u)$.*

(*ii*) *For every bounded sequence $(v_n)_{n\geq 0}$ in X, there exists a bounded sequence $(T_n)_{n\geq 0}$ in $\mathcal{L}(X)$ such that $v_n = T_n(u)$, for all integer $n \geq 0$.*

Proof. Assuming that u is in X where $u \neq 0_X$. Clearly, there exists α in X' such that $\alpha(u) = 1$. The operator T_v defined by $T_v(x) = \alpha(x)v$ for all $x \in X$ is a bounded linear operator from X to itself, since α is linear and $\|T_v\| = \|\alpha\|\|v\|$. In addition, $T_v(u) = v$ for all $v \in X$. Hence, (*i*) holds.

Assuming that u is in X where $u \neq 0_X$ and let $(v_n)_{n\geq 0}$ be a bounded sequence in X, i.e., there exists $M > 0$ such that $\|v_n\| \leq M$ for all $n \geq 0$. By the last property (*i*), there exists a sequence $(T_n)_{n\geq 0}$ in $\mathcal{L}(X)$ defined by $T_n(x) = \alpha(x)v_n$ for all $x \in X$ and $n \geq 0$. Such sequence satisfies $\|T_n\| = \|\alpha\|\|v_n\| \leq \|\alpha\|M$ and $T_n(u) = v_n$ for all $n \geq 0$. Hence, (*ii*) holds. □

Lemma 4.3. *For every u in X where $u \neq 0_X$, the operator R_u defined by*

$$R_u \;:\; \begin{array}{ccc} \mathcal{L}(X) & \to & X \\ T & \mapsto & R_u(T) = T(u) \end{array}$$

is surjective linear and where $\|R_u\| = \|u\|$.

Proof. Assume that $u \in X$ where $u \neq 0_X$. For every $T_i \in \mathcal{L}(X)$, $i = 1, 2$ and λ a scalar number, we have $R_u(T_1 + \lambda T_2) = R_u(T_1) + \lambda R_u(T_2)$. Moreover, since we have $\|T(x)\| \leq \|T\|\|x\|$ for all $x \in X$ and $T \in \mathcal{L}(X)$, we may write $\|R_u\| = \sup\limits_{\|T\|=1} \|T(u)\| \leq \|u\|$. Since, $\|I\| = 1$, then $\|R_u\| = \sup\limits_{\|T\|=1} \|T(u)\| \geq \|I(u)\| = \|u\|$. Accordingly, $R_u \in \mathcal{L}(\mathcal{L}(X), X)$ where $\|R_u\| = \|u\|$. In view of Lemma 4.2 (*i*), we infer that the operator R_u is surjective. □

Lemma 4.4. *For every $(u, \alpha) \in X \times X'$ where $u \neq 0_X$ and $\alpha \neq 0_{X'}$, the following statements hold.*

(*i*) $\ker(\mathscr{R}_{\alpha,u}) = \{T \in \mathcal{L}(X) |\ u \in \ker(T)\}$.

(*ii*) $\Im(\mathscr{R}_{\alpha,u}) = \{L_w \in \mathcal{L}(X) |\ w \in X\}$ *where $L_w(x) = \alpha(x)w$, for every $x \in X$.*

Proof. Assume that $(u, \alpha) \in X \times X'$ where $\alpha \neq 0_{X'}$ and $u \neq 0_X$. We have

$$\begin{aligned} \ker(\mathscr{R}_{\alpha,u}) &= \{T \in \mathcal{L}(X) |\ \mathscr{R}_{\alpha,u}(T) = 0\} \\ &= \{T \in \mathcal{L}(X) |\ \mathscr{R}_{\alpha,u}(T)(x) = 0, \text{ for all } x \in X\} \\ &= \{T \in \mathcal{L}(X) |\ \alpha(x)T(u) = 0, \text{ for all } x \in X\} \\ &= \{T \in \mathcal{L}(X) |\ T(u) = 0\} \\ &= \{T \in \mathcal{L}(X) |\ u \in \ker(T)\}. \end{aligned}$$

Hence, (*i*) holds.

We have

$$\begin{aligned} \Im(\mathscr{R}_{\alpha,u}) &= \{L \in \mathcal{L}(X) |\ \exists\, T \in \mathcal{L}(X) \text{ such that } \mathscr{R}_{\alpha,u}(T) = L\} \\ &= \{L \in \mathcal{L}(X) |\ \exists\, T \in \mathcal{L}(X) \text{ such that } \mathscr{R}_{\alpha,u}(T)(x) = L(x), \text{ for all } x \in X\} \\ &= \{L \in \mathcal{L}(X) |\ \exists\, T \in \mathcal{L}(X) \text{ such that } \alpha(x)T(u) = L(x), \text{ for all } x \in X\} \\ &\subseteq \{L_w \in \mathcal{L}(X) |\ w \in X\}, \text{ where } L_w(x) = \alpha(x)w, \text{ for all } x \in X. \end{aligned}$$

Clearly, $L_w \in \mathcal{L}(X)$ and $\|L_w\| = \|\alpha\|\|w\|$ for all $w \in X$. Conversely, for every $w \in X$, there exists $T_w \in \mathcal{L}(X)$ such that $w = T_w(u)$, on account of Lemma 4.3, (*i*). We may write $L_w(x) = \alpha(x)T_w(u) = \mathscr{R}_{\alpha,u}(T_w)(x)$ for all $x \in X$. Thus, $L_w = \mathscr{R}_{\alpha,u}(T_w) \in \Im(\mathscr{R}_{\alpha,u})$. Hence, (*ii*) holds. □

Proposition 4.5. *For every $(u, \alpha) \in X \times X'$, the following statements are equivalent.*

(*i*) $\mathscr{R}_{\alpha,u}$ *is a compact operator.*

(*ii*) $\alpha = 0_{X'}$ *or* $u = 0_X$.

Proof. (*i*) $\Rightarrow$ (*ii*). Suppose that $\mathscr{R}_{\alpha,u}$ is compact for some (u, α) in $X \times X'$ where $u \neq 0_X$ and $\alpha \neq 0_{X'}$. Let $(v_n)_{n\geq 0}$ be a bounded sequence in X. By lemma 4.2, (*ii*), there exists a bounded sequence $(T_n)_{n\geq 0}$ in $\mathcal{L}(X)$ such that $v_n = T_n(u)$ for all $n \geq 0$. By the assumption $\mathscr{R}_{\alpha,u}$ is compact, and since $(T_n)_{n\geq 0}$ is a bounded sequence in $\mathcal{L}(X)$, the sequence $(\mathscr{R}_{\alpha,u}(T_n))_{n\geq 0}$ has a convergent subsequence $(\mathscr{R}_{\alpha,u}(T_{\varphi(n)}))_{n\geq 0}$ in $\mathcal{L}(X)$. Since $\mathcal{L}(X)$ is a Banach space, $(\mathscr{R}_{\alpha,u}(T_{\varphi(n)}))_{n\geq 0}$ is a Cauchy sequence, i.e., for all $\varepsilon > 0$, there exists an integer $N_\varepsilon \geq 0$, for all integers $n > m \geq N_\varepsilon$ we have $\|\mathscr{R}_{\alpha,u}(T_{\varphi(n)}) - \mathscr{R}_{\alpha,u}(T_{\varphi(m)})\| < \varepsilon\|\alpha\|$. Observe that for every non-negative integers n and m, we have $\sup\limits_{\|x\|=1} \|\alpha(x)T_{\varphi(n)}(u) - \alpha(x)T_{\varphi(m)}(u)\| = \|T_{\varphi(n)}(u) - T_{\varphi(m)}(u)\| \sup\limits_{\|x\|=1} \|\alpha(x)\|$ $= \|v_{\varphi(n)} - v_{\varphi(m)}\|\|\alpha\|$. Therefore, for any $\varepsilon > 0$ there exists an integer $N_\varepsilon \geq 0$ for all integers $n > m \geq N_\varepsilon$, we have $\|v_{\varphi(n)} - v_{\varphi(m)}\| < \varepsilon$. Since X is a Banach space, the sequence $(v_n)_{n\geq 0}$ has a convergent subsequence $(v_{\varphi(n)})_{n\geq 0}$. As a consequence, any bounded sequence $(v_n)_{n\geq 0}$ in X has a convergent subsequence. This requires that the vector space X has a finite dimension. This is a contradiction.

(*ii*) $\Rightarrow$ (*i*). If either $\alpha = 0_{X'}$ or $u = 0_X$, then $\mathscr{R}_{\alpha,u} = 0$ is compact. □

Further, we will study the demicompactness of the linear operator $\mathscr{R}_{\alpha,u}$. We need first to establish the following Lemmas.

Lemma 4.6. *For any $(u, \alpha) \in X \times X'$ such that $\alpha \neq 0_{X'}$ and $u \neq 0_X$, the following statements hold.*

(*i*) *If* $\alpha(u) \neq 1$, *then* $\ker(I - \mathscr{R}_{\alpha,u}) = \{0_{\mathcal{L}(X)}\}$.

(*ii*) *If* $\alpha(u) = 1$, *then*

a) $\ker(I - \mathscr{R}_{\alpha,u}) = \{T_w \in \mathcal{L}(X) | \ w \in X\}$ *where* $T_w(x) = \alpha(x)w$ *for all* $x \in X$.

b) $\ker(I - \mathscr{R}_{\alpha,u})$ *is isomorphic to X and it is an infinite closed subspace of the Banach space* $\mathcal{L}(X)$.

Proof. Assume that $(u, \alpha) \in X \times X'$ where $\alpha \neq 0_{X'}$ and $u \neq 0_X$. Since, $\ker(I - \mathscr{R}_{\alpha,u}) = (I - \mathscr{R}_{\alpha,u})^{-1}\big(\{0_{\mathcal{L}(X)}\}\big)$ and $I - \mathscr{R}_{\alpha,u}$ in $\mathcal{L}(X)$, then $\ker(I - \mathscr{R}_{\alpha,u})$ is a closed subspace of the Banach space $\mathcal{L}(X)$, and hence, $\ker(I - \mathscr{R}_{\alpha,u})$ is also a Banach space. Notice that,

$$\begin{aligned}\ker(I - \mathscr{R}_{\alpha,u}) &= \{T \in \mathcal{L}(X) | \ (I - \mathscr{R}_{\alpha,u})(T) = 0_{\mathcal{L}(X)}\} \\ &= \{T \in \mathcal{L}(X) | \ T = \mathscr{R}_{\alpha,u}(T)\} \\ &= \{T \in \mathcal{L}(X) | \ T(x) = \alpha(x)T(u), \ \text{ for all } \ x \in X\}.\end{aligned}$$

We have to address two cases:

Case $\alpha(u) \neq 1$. Let $T \in \ker(I - \mathscr{R}_{\alpha,u})$. This means that $T(x) = \alpha(x)T(u)$ for all $x \in X$. For $x = u$, we get $(1 - \alpha(u))T(u) = 0$. Under the assumption $\alpha(u) \neq 1$, it follows that $T(u) = 0$. This yields, $T(x) = 0_X$ for all $x \in X$, i.e., $T = 0_{\mathcal{L}(X)}$. Hence, $\ker(I - \mathscr{R}_{\alpha,u}) = \{0_{\mathcal{L}(X)}\}$. Therefore, (*i*) holds.

Case $\alpha(u) = 1$. We have

$$\begin{aligned}\ker(I - \mathscr{R}_{\alpha,u}) &= \{T \in \mathcal{L}(X) | \ T(x) = \alpha(x)T(u), \ \text{ for all } \ x \in X\} \\ &\subseteq \{T_w \in \mathcal{L}(X) | \ w \in X\},\end{aligned}$$

where $T_w(x) = \alpha(x)w$ for all $x \in X$.

Conversely, let T in $\{T_w \in \mathcal{L}(X) | \ w \in X\}$. This means that there exists w in X such that $T(x) = \alpha(x)w$ for all $x \in X$. For $x = u$, we get $T(u) = w$ and so that $T(x) = \alpha(x)T(u)$ for all $x \in X$, i.e., $\mathscr{R}_{\alpha,u}(T)(x) = T(x)$ for all $x \in X$. Equivalently, $\mathscr{R}_{\alpha,u}(T) = T$, and hence, $T \in \ker(I - \mathscr{R}_{\alpha,u})$. Accordingly, if $\alpha(u) = 1$, we have $\ker(I - \mathscr{R}_{\alpha,u}) = \{T_w \in \mathcal{L}(X) | \ w \in X\}$ where $T_w(x) = \alpha(x)w$ for all $x \in X$. Therefore, *a*) holds.

From the previous statement and in a natural way, we can consider the linear operator $\mathcal{M}$ from the Banach space X to the Banach space $\ker(I-\mathscr{R}_{\alpha,u})$ defined by $\mathcal{M}(w)=T_w$, where $T_w\in\mathcal{L}(X)$ and defined by $T_w(x)=\alpha(x)w$ for all $x\in X$. The operator $\mathcal{M}$ satisfies

$$\begin{aligned}\|\mathcal{M}\| &= \sup_{\|w\|=1}\|\mathcal{M}(w)\|\\ &= \sup_{\|w\|=1}\|T_w\|\\ &= \sup_{\|w\|=1}\sup_{\|x\|=1}\|T_w(x)\|\\ &= \sup_{\|w\|=1}\sup_{\|x\|=1}\|\alpha(x)w\|\\ &= \sup_{\|w\|=1}\|w\|\sup_{\|x\|=1}\|\alpha(x)\|\\ &= \|\alpha\|.\end{aligned}$$

Through a carefully designed construction and under the assumption $\alpha\neq 0_{X'}$, the linear operator $\mathcal{M}$ is a bijection between the space X and the kernel of the operator $I-\mathscr{R}_{\alpha,u}$. Therefore, $b)$ holds. □

Lemma 4.7. *For any (u,α) in $X\times X'$ where $\alpha\neq 0_{X'}$ and $u\neq 0_X$, the set $\Im(I-\mathscr{R}_{\alpha,u})$ is a closed subset of $\mathcal{L}(X)$ and satisfies*

(*i*) *If $\alpha(u)\neq 1$, then $\Im(I-\mathscr{R}_{\alpha,u})=\mathcal{L}(X)$.*

(*ii*) *If $\alpha(u)=1$, then $\Im(I-\mathscr{R}_{\alpha,u})=\ker(\mathscr{R}_{\alpha,u})=\{L\in\mathcal{L}(X)|\ u\in\ker(L)\}$.*

Proof. Assume that (u,α) in $X\times X'$ such that $\alpha\neq 0_{X'}$ and $u\neq 0_X$. Clearly, we have $\Im(I-\mathscr{R}_{\alpha,u})=\{L=(I-\mathscr{R}_{\alpha,u})(T)|\ T\in\mathcal{L}(X)\}\subseteq\mathcal{L}(X)$. In what follows, we are dealing with two cases:

Case $\alpha(u)\neq 1$. Given $L\in\mathcal{L}(X)$, let's take $T(x)=L(x)+\dfrac{\alpha(x)}{1-\alpha(u)}L(u)$ for all $x\in X$. Observe that $T\in\mathcal{L}(X)$. Moreover, for every $x\in X$, we have

$$\begin{aligned}(I-\mathscr{R}_{\alpha,u})(T)(x) &= T(x)-\mathscr{R}_{\alpha,u}(T)(x)\\ &= T(x)-\alpha(x)T(u)\\ &= L(x)+\frac{\alpha(x)}{1-\alpha(u)}L(u)-\alpha(x)\Big(L(u)+\frac{\alpha(u)}{1-\alpha(u)})L(u)\Big)\\ &= L(x).\end{aligned}$$

i.e., $L=(I-\mathscr{R}_{\alpha,u})(T)$. This implies that $\mathcal{L}(X)\subseteq\Im(I-\mathscr{R}_{\alpha,u})$. Hence, (*i*) holds.

Case $\alpha(u)=1$. Given $L\in\Im(I-\mathscr{R}_{\alpha,u})$, this means that there exists $T\in\mathcal{L}(X)$ such that $L=(I-\mathscr{R}_{\alpha,u})(T)$, i.e., $L(x)=T(x)-\alpha(x)T(u)$ for all $x\in X$. For $x=u$, we get $L(u)=0$. This implies that $\Im(I-\mathscr{R}_{\alpha,u})\subseteq\{L\in\mathcal{L}(X)|\ u\in\ker(L)\}$. Conversely, let $L\in\{L\in\mathcal{L}(X)|\ u\in\ker(L)\}$. Clearly, we have $L(u)=0$, and hence, $(I-\mathscr{R}_{\alpha,u})(L)(x)=L(x)-\alpha(x)L(u)=L(x)$ for all $x\in X$, i.e., $L=(I-\mathscr{R}_{\alpha,u})(L)\in\Im(I-\mathscr{R}_{\alpha,u})$. Thus, $\Im(I-\mathscr{R}_{\alpha,u})=\{L\in\mathcal{L}(X)|\ u\in\ker(L)\}$. In view of Lemma 4.4, (*i*), we conclude that $\Im(I-\mathscr{R}_{\alpha,u})=\ker(\mathscr{R}_{\alpha,u})$. Therefore, (*ii*) holds. □

Proposition 4.8. *For any $(u,\alpha)\in X\times X'$ where $u\neq 0_X$ and $\alpha\neq 0_{X'}$, the following statements hold.*

(*i*) *$\mathscr{R}_{\alpha,u}$ is demicompact if, and only if, $\alpha(u)\neq 1$.*
(*ii*) *$\mathscr{R}_{\alpha,u}$ is stongly demicompact if, and only if, $\alpha(u)=0$.*
(*iii*) *$\mathscr{R}_{\alpha,u}$ is quasi-compact, if $|\alpha(u)|^n\|\alpha\|\|u\|<1$, for some integer $n\geq 0$.*
(*iv*) *$\mathscr{R}_{\alpha,u}$ is power compact if, and only if, $\alpha(u)=0$.*
(*v*) *$\mathscr{R}_{\alpha,u}$ is a Riesz operator if, and only if, $\alpha(u)=0$.*

Proof. Assume that $(u, \alpha) \in X \times X'$ where $u \neq 0_X$ and $\alpha \neq 0_{X'}$. By Proposition 2.2, the operator $\mathscr{R}_{\alpha,u}$ is demicompact if, and only if, $\dim \ker(I - \mathscr{R}_{\alpha,u}) < +\infty$ and $\Im(I - \mathscr{R}_{\alpha,u})$ is a closed subset of $\mathcal{L}(X)$. By Lemmas 4.6 and 4.7, it follows that $\mathscr{R}_{\alpha,u}$ is demicompact if, and only if, $\alpha(u) \neq 1$. Therefore, (i) holds.

By Proposition 4.1, (f), we have $I - \lambda\mathscr{R}_{\alpha,u} = I - \mathscr{R}_{\lambda\alpha,u}$ for all scalar number λ. When $\alpha(u) = 0$, then $\lambda\alpha(u) = 0 \neq 1$ and by virtue of the previous property (i), it comes that $I - \lambda\mathscr{R}_{\alpha,u}$ is demicompact, for every scalar number λ. Therefore, $\mathscr{R}_{\alpha,u}$ is strongly demicompact.
When $\alpha(u) \neq 0$, then $I - \lambda\mathscr{R}_{\alpha,u}$ is demicompact if, and only if, $\lambda \neq (\alpha(u))^{-1}$. This allows us to say that $\mathscr{R}_{\alpha,u}$ is not strongly demicompact. Hence, (ii) holds.

Assume that there exists an integer $n \geq 0$ such that $|\alpha(u)|^n.\|\alpha\|.\|u\| < 1$. By Proposition 4.1, (a) and (c), we get $\|\mathscr{R}_{\alpha,u}^{n+1}\| = |\alpha(u)|^n\|\alpha\|\|u\| < 1$. By the assumption and the definition of the quasi-compactness, there exist an integer $N = n + 1 \geq 1$ and $K = 0 \in \mathcal{K}(\mathcal{L}(X))$ such that $\|\mathscr{R}_{\alpha,u}^N - K\| < 1$. Hence, (iii) holds.

Suppose that the operator $\mathscr{R}_{\alpha,u}$ is power compact, i.e., there exists an integer $N \geq 1$ such that $\mathscr{R}_{\alpha,u}^N$ is compact. By Proposition 4.1, (a) and (c), we have $\mathscr{R}_{\alpha,u}^N = (\alpha(u))^{N-1}\mathscr{R}_{\alpha,u}$. By Proposition 4.5, and by the assumption $u \neq 0_X$ and $\alpha \neq 0_{X'}$, this requires that $N \geq 2$ and $\alpha(u) = 0$. Conversely, assume that $\alpha(u) = 0$. From Proposition 4.1 (c), we have $\mathscr{R}_{\alpha,u}^2 = 0 \in \mathcal{K}(\mathcal{L}(X))$. Hence, (iv) holds.

Suppose that $\mathscr{R}_{\alpha,u}$ is a Riesz operator. From the characterization of Riesz operators, $\mathscr{R}_{\alpha,u}$ is strongly quasi-compact and then $\mathscr{R}_{\alpha,u}$ is strongly demicompact. From the previous property (ii), we get $\alpha(u) = 0$. Conversely, if $\alpha(u) = 0$, then $\mathscr{R}_{\alpha,u}$ is power compact and then it is a Riesz operator. Therefore, (v) holds. □

Proposition 4.9. *When X is a reflexive Banach space, the linear operator $\mathscr{R}_{\alpha,u}$ is weakly compact for every (u, α) in $X \times X'$.*

Proof. Assume that X is reflexive and let $(u, \alpha) \in X \times X'$. When $\alpha = 0_{X'}$ or $u = 0_X$, we have $\mathscr{R}_{\alpha,u} = 0_{\mathcal{L}(\mathcal{L}(X))}$ is weakly compact.
Suppose that $\alpha \neq 0_{X'}$ and $u \neq 0_X$. Recall that the linear operator $\mathcal{A}_{u,\alpha}$ defined by $\mathcal{A}_{u,\alpha}(x) = \alpha(x)u$ for every $x \in X$, is compact. On the other hand, $\mathscr{R}_{\alpha,u}(T)(x) = \alpha(x)T(u) = T\mathcal{A}_{u,\alpha}(x)$ for every $x \in X$ and $T \in \mathcal{L}(X)$. Then $\mathscr{R}_{\alpha,u}(T) = T\mathcal{A}_{u,\alpha}$ for every $T \in \mathcal{L}(X)$. Therefore, $\mathscr{R}_{\alpha,u} = R_{\mathcal{A}_{u,\alpha}}$ where $R_{\mathcal{A}_{u,\alpha}}(T) = T\mathcal{A}_{u,\alpha}$ for every $T \in \mathcal{L}(X)$. In view of Appendix 6, Theorem 6 in [9], $\mathscr{R}_{\alpha,u} = R_{\mathcal{A}_{u,\alpha}}$ is a weakly compact operator on $\mathcal{L}(X)$, if X is reflexive. □

Corollary 4.10. *When the Banach space $\mathcal{L}(X)$ is reflexive, then the operator $\mathscr{R}_{\alpha,u}$ is weakly compact for all $(u, \alpha) \in X \times X'$.*

Proposition 4.11. *For any α, β in X' and u, v in X, the following statements hold.*

- (i) $\mathscr{L}_{\alpha,u} \circ \mathscr{R}_{\beta,v} = \mathscr{R}_{\beta,v} \circ \mathscr{L}_{\alpha,u}$.
- (ii) *If $\alpha \neq 0_{X'}$, $\beta \neq 0_{X'}$, $u \neq 0_X$ and $v \neq 0_X$, then* $\Im\left(\mathscr{L}_{\alpha,u} \circ \mathscr{L}_{\beta,v}\right) = \mathrm{Vect}\{\mathcal{A}_{u,\beta}\}$ *where $\mathcal{A}_{u,\beta} \in \mathcal{L}(X)$ is defined by $\mathcal{A}_{u,\beta}(x) = \beta(x)u$, for all $x \in X$.*
- (iii) *$\mathscr{L}_{\alpha,u} \circ \mathscr{R}_{\beta,v}$ is a compact operator.*
- (iv) *Suppose that X' is of infinite dimension, the linear operator $\mathscr{L}_{\alpha,u} + \mathscr{R}_{\beta,v}$ is compact if, and only if, ($\alpha = 0_{X'}$ or $u = 0_X$) and ($\beta = 0_{X'}$ or $v = 0_X$).*

Proof. Assume that $\alpha, \beta \in X'$ and $u, v \in X$. For any $T \in \mathcal{L}(X)$ and $x \in X$, we may write

$$\begin{aligned}
\mathscr{L}_{\alpha,u} \circ \mathscr{R}_{\beta,v}(T)(x) &= \mathscr{L}_{\alpha,u}(\mathscr{R}_{\beta,v}(T))(x)\\
&= \alpha(\mathscr{R}_{\beta,v}(T)(x))u\\
&= \alpha(\beta(x)T(v))u\\
&= \beta(x)\alpha(T(v))u\\
&= \beta(x)\mathscr{L}_{\alpha,u}(T)(v)\\
&= \mathscr{R}_{\beta,v}(\mathscr{L}_{\alpha,u}(T))(x)\\
&= \mathscr{R}_{\beta,v} \circ \mathscr{L}_{\alpha,u}(T)(x).
\end{aligned}$$

This yields, $\mathscr{L}_{\alpha,u} \circ \mathscr{R}_{\beta,v} = \mathscr{R}_{\beta,v} \circ \mathscr{L}_{\alpha,u}$. Therefore, (*i*) holds.

Assume that $\alpha, \beta \in X'$ and $u, v \in X$ where $\alpha \neq 0_{X'}$, $\beta \neq 0_{X'}$, $u \neq 0_X$ and $v \neq 0_X$. We have

$$\begin{aligned}
\Im(\mathscr{L}_{\alpha,u} \circ \mathscr{R}_{\beta,v}) &= \{L = \mathscr{L}_{\alpha,u} \circ \mathscr{R}_{\beta,v}(T) | \ T \in \mathcal{L}(X)\} \\
&= \{L \in \mathcal{L}(X) | \ \exists \ T \in \mathcal{L}(X) \text{ such that } \ L(x) = \beta(x)\alpha(T(v))u, \text{ for all } \ x \in X\} \\
&= \{L \in \mathcal{L}(X) | \ \exists \ T \in \mathcal{L}(X) \text{ such that } \ L(x) = \alpha(T(v))\mathcal{A}_{u,\beta}(x), \text{ for all } \ x \in X\} \\
&= \{L \in \mathcal{L}(X) | \ \exists \ T \ \in \mathcal{L}(X) \text{ such that } \ L = \alpha(T(v))\mathcal{A}_{u,\beta}\} \\
&\subseteq \text{Vect}\{\mathcal{A}_{u,\beta}\}.
\end{aligned}$$

where $\mathcal{A}_{u,\beta} \in \mathcal{L}(X)$ is defined by $\mathcal{A}_{u,\beta}(x) = \beta(x)u$ for all $x \in X$. Conversely, let L in Vect$\{\mathcal{A}_{u,\beta}\}$, then there exists a scalar number γ such that $L = \gamma\mathcal{A}_{u,\beta}$. Consider the linear functional e_v defined by $e_v(T) = \alpha(T(v))$ for all $T \in \mathcal{L}(X)$. We have $\|e_v\| = \sup\limits_{\|T\|=1} |\alpha(T(v))| \leq \|\alpha\|\|v\|$. Notice that $e_v \neq 0_{\mathcal{L}(X)'}$. Indeed, by the assumption $\alpha \neq 0_{X'}$, there exists $w \in X$ such that $\alpha(w) \neq 0$. By Lemma 4.2, (*i*) and the assumption $u \neq 0_X$, there exists $T \in \mathcal{L}(X)$ such that $w = T(v)$ and so that $e_v(T) = \alpha(T(v)) = \alpha(w) \neq 0$. Since, $e_v \neq 0_{\mathcal{L}(X)'}$ and $e_v \in \mathcal{L}(X)'$, the linear functional e_v is surjective. Thus, for any scalar number δ, there exists $T \in \mathcal{L}(X)$ such that $\delta = e_v(T)$. This implies that $L = \delta\mathcal{A}_{u,\beta} = e_v(T)\mathcal{A}_{u,\beta} = \alpha(T(v))\mathcal{A}_{u,\beta} = \mathscr{L}_{\alpha,u} \circ \mathscr{R}_{\beta,v}(T)$. Hence, $\Im(\mathscr{L}_{\alpha,u} \circ \mathscr{R}_{\beta,v}) = \text{Vect}\{\mathcal{A}_{u,\beta}\}$. Hence, (*ii*) holds.

Assume that $\alpha, \beta \in X'$ and $u, v \in X$. When $\alpha = 0_{X'}$ or $\beta = 0_{X'}$ or $u = 0_X$ or $v = 0_X$, we have $\mathscr{L}_{\alpha,u} \circ \mathscr{R}_{\beta,v} = 0$ is a compact operator.

Suppose that $\alpha \neq 0_{X'}$, $\beta \neq 0_{X'}$, $u \neq 0_X$ and $v \neq 0_X$. From the previous property (*ii*), since $\Im(\mathscr{L}_{\alpha,u} \circ \mathscr{R}_{\beta,v}) =$ Vect$\{\mathcal{A}_{u,\beta}\}$ then, $\mathscr{L}_{\alpha,u} \circ \mathscr{R}_{\beta,v}$ has a finite rank, and hence, it is a compact operator. Therefore, (*iii*) holds.

Assume that $\alpha, \beta \in X'$ and $u, v \in X$. If ($\alpha = 0_{X'}$ or $u = 0_X$) and ($\beta = 0_{X'}$ or $v = 0_X$), we can see that $\mathscr{L}_{\alpha,u} + \mathscr{R}_{\beta,v} = 0$, which is a compact operator.

Suppose that there exist $\alpha, \beta \in X'$ and $u, v \in X$ such that

$$M = \mathscr{L}_{\alpha,u} + \mathscr{R}_{\beta,v} \tag{7}$$

is a compact operator.

For any (w, δ) in $X \times X'$, while applying the operator $\mathscr{L}_{\delta,w}$ from the left hand sides of equation (7), we obtain

$$\begin{aligned}
\mathscr{L}_{\delta,w} \circ M &= \mathscr{L}_{\delta,w} \circ \mathscr{L}_{\alpha,u} + \mathscr{L}_{\delta,w} \circ \mathscr{R}_{\beta,v} \\
&= \delta(u)\mathscr{L}_{\alpha,w} + \mathscr{L}_{\delta,w} \circ \mathscr{R}_{\beta,v}.
\end{aligned}$$

This yields, $\delta(u)\mathscr{L}_{\alpha,w} = \mathscr{L}_{\delta,w} \circ M - \mathscr{L}_{\delta,w} \circ \mathscr{R}_{\beta,v}$ is a compact operator, for every (w, δ) in $X \times X'$. This requires that $u = 0_X$ or $\alpha = 0'_X$, under the assumption X is of infinite dimension and due to the Proposition 3.5.

For every (w, δ) in $X \times X'$, while applying the operator $\mathscr{R}_{\delta,w}$, from the right hand sides of equation (7), we get

$$\begin{aligned}
M \circ \mathscr{L}_{\delta,w} &= \mathscr{L}_{\alpha,u} \circ \mathscr{R}_{\delta,w} + \mathscr{R}_{\beta,v} \circ \mathscr{R}_{\delta,w} \\
&= \mathscr{L}_{\alpha,u} \circ \mathscr{R}_{\delta,w} + \delta(v)\mathscr{R}_{\beta,w}.
\end{aligned}$$

This gives, $\delta(v)\mathscr{R}_{\beta,w} = M \circ \mathscr{L}_{\delta,w} - \mathscr{L}_{\alpha,u} \circ \mathscr{R}_{\delta,w}$ is a compact operator for every (w, δ) in $X \times X'$. This obliges that $v = 0_X$ or $\beta = 0'_X$, by Proposition 4.5. Hence, (*iv*) holds. □

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