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On fixed points of generalized Kannan and Reich type contractive mappings

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Abstract. Kannan or Reich type strict contractive conditions do not ensure the existence of fixed points unless some strong conditions such as compactness of the space and continuity of the mapping are assumed. In this paper, our main aim is to investigate the existence of fixed point of generalized Kannan type contractive mappings in the setting of boundedly compact and T-orbitally compact metric spaces via orbital continuity. In addition to it, asymptotic regularity has been used to prove the Reich type fixed point theorem via altering distance functions. Supporting examples have been given to strengthen the hypotheses of our proved theorems.

1. Introduction and Preliminaries

The Banach contraction principle [3] is considered one of the most celebrated fixed point theorem in Functional Analysis. This theorem paved a way for an important branch of mathematics known as Metric Fixed Point Theory. In 1968, Kannan established a new fixed point theorem which is given as follows:

Theorem 1.1. *[14] Let (X, d) be a complete metric space and $T : X \to X$ satisfy*

$$d(Tx, Ty) \leq \beta[d(x, Tx) + d(y, Ty)] \text{ for all } x, y \in X, \beta \in \left[0, \frac{1}{2}\right). \tag{1}$$

Then T has a unique fixed point in X.

Clearly, the Banach contraction mapping and the Kannan contractive mapping are independent of each other. Also, it is noted that any Banach contraction mapping is continuous but there are several discontinuous mappings which satisfy the contractive condition (1) (see [16]).

Let us consider $\Delta = \{\zeta : (0, \infty) \to [0, 1) : \zeta(t_n) \to 1 \text{ while } t_n \to 0 \text{ as } n \to \infty\}$. There are several functions which are in Δ. One typical example of a function in Δ is $f_m(t) = e^{-mt}$ for all $t > 0$, $m \in \mathbb{N}$. It is to be noted

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that the functions in Δ may not be always continuous. If we take $g(t) = 1 - \log(1 + t)$ for $0 < t < e - 1$ and $= \frac{1}{2}$ for $t \geq e - 1$, then it is clear that $g \in \Delta$ without being continuous. In the year 1973, M. Geraghty proved a fixed point theorem using such functions as a generalization of Banach's fixed point theorem.

Theorem 1.2. *[8] Let (X, d) be a complete metric space and $T : X \to X$ satisfy*

$$d(Tx, Ty) \leq \zeta(d(x, y))d(x, y) \text{ for all } x, y(x \neq y) \in X, \zeta \in \Delta.$$

Then T has a unique fixed point in X.

Subsequently, in this direction of research, several generalizations and extensions of contractive type mapping have been done by many researchers (for this we may refer [1], [9],[12],[18], [21]).

In 2018, Górnicki combined Theorem 1.1 and Theorem 1.2, and proved the following fixed point theorem.

Theorem 1.3. *[11] Let (X, d) be a complete metric space and $T : X \to X$ a self mapping. If T satisfies*

$$d(Tx, Ty) \leq \eta(d(x, y))[d(x, Tx) + d(y, Ty)] \text{ for all } x, y(x \neq y) \in X,$$

where $\eta : (0, \infty) \to \left[0, \frac{1}{2}\right)$ is a function such that $\eta(t_n) \to \frac{1}{2}$ implies $t_n \to 0$ as $n \to \infty$, then T has a unique fixed point in X.

Definition 1.4. *[15] A function $\psi : [0, \infty) \to [0, \infty)$ is called an altering distance function if the following properties are satisfied:*
(i) $\psi(t) = 0$ if and only if $t = 0$;
(ii) ψ is strictly monotonic increasing and continuous.

Let us denote the set of all altering distance functions as Θ.
Several authors have used such functions in their fixed point theorems (see [6, 20]). Recently Haokip et al. [13] proved a generalized Kannan type fixed point theorem in the setting of a b-metric space (for the definition of b-metric see [2]). The theorem is given below.

Theorem 1.5. *[13] Let (X, d) be a complete b-metric space with coefficient $s \geq 1$ and $T : X \to X$ be a self mapping. Also let T satisfy the following condition for some $0 \leq a < \frac{1}{2s+1}$:*

$$\psi(d(Tx, Ty)) \leq a[\psi(d(x, y)) + \psi(d(x, Tx)) + \psi(d(y, Ty))] \text{ for some } \psi \in \Theta,$$

and for all $x, y \in X$. Then T has a unique fixed point in X.

To achieve a fixed point for a contractive (but not contraction) type mapping, sometimes researchers need compactness instead of just completeness of a metric space. Górnicki proved the following result for a Kannan type contractive mapping.

Theorem 1.6. *[10] Let (X, d) be a compact metric space and $T : X \to X$ be a continuous mapping satisfying*

$$d(Tx, Ty) < \frac{1}{2}[d(x, Tx) + d(y, Ty)] \text{ for all } x, y(x \neq y) \in X. \tag{2}$$

Then T has a unique fixed point in X.

Recently, Garai et al. [7] have shown that the assumptions of compactness of the metric space and the continuity of the mapping can be weakened by using the concepts of boundedly compactness and T-orbitally compactness of the metric space and by a weaker version of continuity, orbitally continuity of the given mapping. The related definitions are given as follows.

Definition 1.7. *[7] A metric space (X, d) is called boundedly compact if every bounded sequence in X has a convergent subsequence.*

Definition 1.8. *[7] A metric space (X, d) is said to be T-orbitally compact with respect to a mapping $T : X \to X$ if for all $x \in X$, every sequence in the orbit of T at $x \in X$ given by $O(x, T) = \{x, Tx, T^2x, ...\}$ has a convergent subsequence in X.*

Definition 1.9. *[5] A mapping $T : (X, d) \to (X, d)$ is said to be orbitally continuous if $u \in X$ and such that $u = \lim_{i\to\infty} T^{n_i}x$ for some $x \in X$, then $Tu = \lim_{i\to\infty} TT^{n_i}x$.*

Clearly, boundedly compactness and T-orbitally compactness are weaker versions of compactness. While a compact metric space is always boundedly compact and T-orbitally compact for any self-mapping T, there exist metric spaces that are boundedly compact or T-orbitally compact for some self-mapping T but not compact (for further details, see [7]).

The following theorem is given by Garai et al. [7].

Theorem 1.10. *[7] Let (X, d) be a metric space and T be a self mapping on X. Also let X be either boundedly compact or T-orbitally compact and T satisfy*

$$d(Tx, Ty) < \frac{1}{2}[d(x, Tx) + d(y, Ty)] \text{ for all } x, y (x \neq y) \in X.$$

Then T has a unique fixed point in X, provided T is orbitally continuous in X.

In [10], Górnicki raised the follwing open question:

Question: Does there exist a complete but noncompact metric space (X, d) and a continuous mapping $T : X \to X$ such that

$$d(Tx, Ty) < \frac{1}{2}[d(x, Tx) + d(y, Ty)] \text{ for all } x, y (x \neq y) \in X.$$

and T is fixed point free?

Górnicki [11] and Garai et al. [7] have already answered the above open question (see Example 1.5 of [11] and Example 2.5 of [7]). Here we give another answer to this open question.

Example 1.11. *Let $X = \{1 + \frac{1}{2^n} : n \geq 1\}$ and $\sigma : X \times X \to [0, \infty)$ be given by*

$$\sigma(x, y) = \begin{cases} 0, & \text{if } x = y \\ 1 + |x - y|, & \text{if } x \neq y. \end{cases}$$

Then σ is a metric on X and X is complete with respect to σ. But X is not compact since the sequence $\{1 + \frac{1}{2^n}\}_{n\in\mathbb{N}}$ has no convergent subsequence. Let $T : X \to X$ be defined by

$$T\left(1 + \frac{1}{2^n}\right) = 1 + \frac{1}{2^{n+2}} \text{ for all } n \geq 1.$$

Then T satisfies contractive condition (2) and also T is continuous in X. It is clear that T has no fixed point in X.

In the next section, we prove some fixed point theorems for several generalized Kannan type contractive mappings in complete and also in weaker compact metric spaces. Supporting examples have been given to strengthen the hypothesis of our theorems.

2. Main Result

This section is divided into three subsections. In Section 2.1, we prove some fixed point theorems for generalized Kannan type strict contractive mappings over boundedly compact metric space, assuming orbital continuity of the mapping. In Section 2.2, we introduce a new type of contractive mapping, namely generalized Geraghty-Kannan type contractive mapping and prove some fixed point theorems over complete metric space. In Section 2.3, we prove the Reich type fixed point theorem via altering distance function using asymptotic regularity.

2.1. Boundedly compactness and T-orbital compactness

Theorem 2.1. *Let (X, d) be a boundedly compact metric space and $T : X \to X$ satisfy*

$$\psi(d(Tx, Ty)) < \frac{1}{2}[\psi(d(x, Tx)) + \psi(d(y, Ty))] \tag{3}$$

for all $x, y \in X$ with $x \neq y$ and for some $\psi \in \Theta$. If T is orbitally continuous in X then T has a unique fixed point in X.

Proof. Let $x_0 \in X$ be arbitrarily chosen and we construct the Picard iterating sequence $\{x_n\}$, where $x_n = T^n x_0$ for all $n \geq 1$. If for some $n \in \mathbb{N} \cup \{0\}$, $x_n = x_{n+1}$ then clearly T has a fixed point in X. So let $x_n \neq x_{n+1}$ for all $n \geq 0$. Now from the contractive condition (3) we get,

$$\begin{aligned} &\psi(d(x_n, x_{n+1})) < \frac{1}{2}[\psi(d(x_{n-1}, x_n)) + \psi(d(x_n, x_{n+1}))] \\ \Rightarrow\ &\psi(d(x_n, x_{n+1})) < \psi(d(x_{n-1}, x_n)) \text{ for all } n \geq 1. \end{aligned}$$

This shows that $\{\psi(d(x_{n-1}, x_n))\}_{n\in\mathbb{N}}$ is a monotonic decreasing sequence which is bounded below. Thus $\{\psi(d(x_{n-1}, x_n))\}_{n\in\mathbb{N}}$ is convergent. Now we see that $\psi(d(x_n, x_{n+1}))$
$< \psi(d(x_0, x_1)) = M$(say) for all $n \geq 1$. Then for any $n, m \in \mathbb{N}$ we have,

$$\psi(d(x_n, x_m)) < \frac{1}{2}[\psi(d(x_{n-1}, x_n)) + \psi(d(x_{m-1}, x_m))] < M.$$

Let $A = \{d(x_n, x_m) : n, m \geq 1\}$. If A is not bounded then $\limsup_{n,m\to\infty} d(x_n, x_m) = \infty$ and since ψ is strictly monotonic increasing it follows that $\limsup_{n,m\to\infty} \psi(d(x_n, x_m)) = \infty$, a contradiction. Thus A is bounded and so $\{x_n\}_{n\in\mathbb{N}}$ is a bounded sequence in X. Since X is boundedly compact, $\{x_n\}_{n\in\mathbb{N}}$ has a convergent subsequence say $\{x_{n_k}\}_{k\in\mathbb{N}}$ which converges to z. By orbitally continuity of T we have $\lim_{k\to\infty} x_{n_k+1} = Tz$ and $\lim_{k\to\infty} x_{n_k+2} = T^2 z$. Now if $z \neq Tz$ then due to the contractive condition (3) we see that

$$\begin{aligned} &\psi(d(Tz, T^2 z)) < \frac{1}{2}[\psi(d(z, Tz)) + \psi(d(Tz, T^2 z))] \\ \Rightarrow\ &\psi(d(Tz, T^2 z)) < \psi(d(z, Tz)). \end{aligned}$$

Now by the continuity of the metric we get $\lim_{k\to\infty} d(x_{n_k}, x_{n_k+1}) = d(z, Tz)$ and also $\lim_{k\to\infty} d(x_{n_k}, x_{n_k+1}) = d(Tz, T^2 z)$. Therefore $\lim_{k\to\infty} \psi(d(x_{n_k}, x_{n_k+1})) = \psi(d(z, Tz))$ and also $\lim_{k\to\infty} \psi(d(x_{n_k}, x_{n_k+1})) = \psi(d(Tz, T^2 z))$. Since $\{\psi(d(x_{n-1}, x_n))\}_{n\in\mathbb{N}}$ is convergent, implies that $\psi(d(Tz, T^2 z)) = \psi(d(z, Tz))$, a contradiction. Hence $Tz = z$ and z is a fixed point of T. To prove the uniqueness of the fixed point we take two different fixed points u and v of T. Then we see that

$$\psi(d(u, v)) = \psi(d(Tu, Tv)) < \frac{1}{2}[\psi(d(u, Tu)) + \psi(d(v, Tv))] = 0.$$

which contradicts that $\psi(t) \geq 0$ for all $t \geq 0$.

This shows that the fixed point of T is unique in X. □

Example 2.2. *Let $X = \mathbb{R}$ with the usual metric and $T : X \to X$ be given by*

$$T(x) = \begin{cases} \frac{1}{2}, & \text{if } x \le 1 \\ 0, & \text{if } x > 1. \end{cases}$$

Then T satisfies the contractive condition (3) for any $\psi_k(t) = kt$ for all $t \ge 0$; $k > 0$. Also X is boundedly compact, T is orbitally continuous and T has a unique fixed point $\frac{1}{2}$ in X. Here it is to be noted that X is not compact and also T is not continuous, so Theorem 1.6 is not applicable herein.

Corollary 2.3. *If we take $\psi(t) = t$ for all $t \ge 0$ in the contractive condition (3) then Theorem 2.1 of [7] follows from Theorem 2.1.*

Theorem 2.4. *Let (X, d) be a metric space and $T : X \to X$ satisfy*

$$\psi(d(Tx, Ty)) < \frac{1}{2}[\psi(d(x, Tx)) + \psi(d(y, Ty))] \tag{4}$$

for all $x, y \in X$ with $x \neq y$ and for some $\psi \in \Theta$. If X is T-orbitally compact and T is orbitally continuous in X then T has a unique fixed point in X.

Proof. Let us choose $x_0 \in X$ as arbitrary and construct the Picard iterating sequence $\{x_n\}$, where $x_n = T^n x_0$ for all $n \ge 1$. If for some $n \in \mathbb{N} \cup \{0\}$, $x_n = x_{n+1}$ then clearly T has a fixed point in X. So let $x_n \neq x_{n+1}$ for all $n \ge 0$. Since X is T-orbitally compact then $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ which converges to some $z \in X$. Since T is orbitally continuous, we have $\lim_{k\to\infty} x_{n_k+1} = Tz$ and $\lim_{k\to\infty} x_{n_k+2} = T^2 z$. The remaining portion of the proof is similar to Theorem 2.1. □

Example 2.5. *Let $X = \mathbb{N}$ endowed with the metric*

$$D(x, y) = \begin{cases} 0, & \text{if } x = y \\ \min\{x + y, 4\}, & \text{if } x \neq y. \end{cases}$$

Let us define $T : X \to X$ by

$$T(x) = \begin{cases} 2, & \text{if } x \in \{1, 2, 3\} \\ 1, & \text{if } x \ge 4. \end{cases}$$

Then T satisfies the contractive condition (4) for any $\psi_k(t) = kt$ for all $t \ge 0$; $k > 0$. Here, it is notable that X is neither compact nor boundedly compact, yet it is T-orbitally compact. Additionally, T is orbitally continuous, and it possesses a unique fixed point 2 in X.

Example 2.6. *Let $X = \{0, 1\} \cup (2, 3]$ equipped with the usual metric and $T : X \to X$ be given by*

$$T(x) = \begin{cases} 1, & \text{if } x \in X \setminus \{3\} \\ 0, & \text{if } x = 3. \end{cases}$$

Then T satisfies the contractive condition (4) for any $\psi_k(t) = kt$ for all $t \ge 0$; $k > 0$. Here, it is notable that X is neither compact nor boundedly compact, yet it is T-orbitally compact. Additionally, T is orbitally continuous, and it possesses a unique fixed point 1 in X.

Corollary 2.7. *If in Theorem 2.4, we put $\psi(t) = t$ for all $t \ge 0$ in the contractive condition (4) then Theorem 2.2 of [7] follows immediately.*

Just completeness of a metric space can not ensure the existence of a fixed point for a mapping T satisfying the contractive condition (3).

Example 2.8. *Let* $X = \{1 + \frac{1}{2^n} : n \geq 1\}$ *be a metric space endowed with the metric*

$$\rho(x, y) = \begin{cases} 0, & \text{if } x = y \\ x + y, & \text{if } x \neq y. \end{cases}$$

Let us consider the mapping $T : X \to X$ *defined by* $T(1 + \frac{1}{2^n}) = 1 + \frac{1}{2^{n+1}}$ *for all* $n \in \mathbb{N}$. *Then T satisfies the contractive condition (3) for any* $\psi_k(t) = kt$ *for all* $t \geq 0$; $k > 0$. *Here X is neither compact nor boundedly compact nor T-orbitally compact and we see that T has no fixed point in X. But here we see that X is complete.*

Corollary 2.9. *Theorem* 2.2 *of [10] follows from both of our theorems, Theorem* 2.1 *and Theorem* 2.4*, since for* $\psi(t) = t$ *for all* $t \geq 0$ *the contractive condition* (3) *reduces to the contractive condition* (2)*, any compact metric space is boundedly compact and also T-orbitally compact for any mapping T and any continuous map is clearly orbitally continuous.*

2.2. On generalized Geraghty-Kannan type contractive mapping

Definition 2.10. *Let* (X, d) *be a metric space and* $T : X \to X$ *be a mapping. Then T is said to be generalized Geraghty-Kannan type contractive mapping if there exist some* $\psi \in \Theta$ *and* $\zeta \in \Delta$ *such that*

$$\psi(d(Tx, Ty)) \leq \zeta(d(x, y))[\alpha_1\psi(d(x, y)) + \alpha_2\psi(d(x, Tx)) + \alpha_3\psi(d(y, Ty))+ \alpha_4\psi\left(\frac{d(x, Ty) + d(Tx, y)}{2}\right)] \text{ for all } x, y \in X \text{ with } x \neq y, \quad (5)$$

where $\alpha_i \in [0, 1]$ *for all* $i = 1(1)4$ *with* $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \leq 1$.

We use the following lemma to prove our main result.

Lemma 2.11. *[17] Let* $\{x_n\}$ *be a sequence in a metric space* (X, d) *such that* $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. *If* $\{x_n\}$ *is not a Cauchy sequence in X then there exists* $\epsilon > 0$ *and two sequences* $\{m_k\}$, $\{n_k\}$ *of positive integers such that* $n_k > m_k > k$ *and such that the following sequences*

$$\{d(x_{m_k}, x_{n_k})\}, \{d(x_{m_k}, x_{n_k+1})\}, \{d(x_{m_k-1}, x_{n_k})\}, \{d(x_{m_k-1}, x_{n_k+1})\}, \{d(x_{m_k+1}, x_{n_k+1})\}$$

converges to ϵ *as* $k \to \infty$.

Theorem 2.12. *Let* (X, d) *be a complete metric space and* $T : X \to X$ *be a generalized Geraghty-Kannan type contractive mapping for* $\alpha_3 + \alpha_4 < 1$. *Then T has a unique fixed point in X.*

Proof. Let $x_0 \in X$ be chosen as arbitrary and let us construct the Picard iterating sequence $\{x_n\}$, where $x_n = T^n x_0$ for all $n \geq 1$. If for some $n \in \mathbb{N} \cup \{0\}$, $x_n = x_{n+1}$ then clearly T has a fixed point in X. So let $x_n \neq x_{n+1}$ for all $n \geq 0$. Then

$$\begin{aligned}\psi(d(x_n, x_{n+1})) &= \psi(d(Tx_{n-1}, Tx_n)) \\ &\leq \zeta(d(x_{n-1}, x_n))[\alpha_1\psi(d(x_{n-1}, x_n)) + \alpha_2\psi(d(x_{n-1}, Tx_{n-1}))+ \\ &\quad \alpha_3\psi(d(x_n, Tx_n)) + \alpha_4\psi\left(\frac{d(x_{n-1}, Tx_n) + d(Tx_{n-1}, x_n)}{2}\right)] \\ &\leq \zeta(d(x_{n-1}, x_n))[\alpha_1\psi(d(x_{n-1}, x_n)) + \alpha_2\psi(d(x_{n-1}, x_n))+ \\ &\quad \alpha_3\psi(d(x_n, x_{n+1})) + \alpha_4\psi\left(\frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2}\right)] \ \ \forall n \geq 1. \end{aligned} \quad (6)$$

Now if for some $n \in \mathbb{N}$, $d(x_{n-1}, x_n) \leq d(x_n, x_{n+1})$ then from (6) we have

$$\begin{aligned}\psi(d(x_n, x_{n+1})) &\leq \zeta(d(x_{n-1}, x_n))(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)\psi(d(x_n, x_{n+1})) \\ &< \psi(d(x_n, x_{n+1})), \text{ a contracdiction.}\end{aligned}$$

Thus $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$ for all $n \geq 1$. So $\{d(x_{n-1}, x_n)\}_{n\in\mathbb{N}}$ is a monotonic decreasing sequence which is bounded below. Therefore there exists some $r \geq 0$ such that $d(x_n, x_{n+1}) \to r$ as $n \to \infty$. If $r > 0$ then,

$$\begin{aligned}\psi(d(x_n, x_{n+1})) &\leq \zeta(d(x_{n-1}, x_n))(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)\psi(d(x_{n-1}, x_n))\\ &\leq \zeta(d(x_{n-1}, x_n))\psi(d(x_{n-1}, x_n)) \text{ for all } n \geq 1.\end{aligned}$$

That is, $0 < \frac{\psi(d(x_n,x_{n+1}))}{\psi(d(x_{n-1},x_n))} \leq \zeta(d(x_{n-1}, x_n)) < 1$ for all $n \geq 1$. Now $\psi(d(x_{n-1}, x_n)) \to \psi(r) > 0$, so by taking $n \to \infty$ we get $\lim_{n\to\infty} \zeta(d(x_{n-1}, x_n)) = 1$, leads us to a contradiction. Thus $r = 0$ and we see that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. Now we show that $\{x_n\}$ is Cauchy in X. If $\{x_n\}$ is not Cauchy then by Lemma 2.11 there exists an $\epsilon > 0$ and two sequences $\{m_k\}$, $\{n_k\}$ of positive integers such that $n_k > m_k > k$ and the sequences $\{d(x_{m_k}, x_{n_k})\}, \{d(x_{m_k}, x_{n_k+1})\}, \{d(x_{m_k-1}, x_{n_k})\}, \{d(x_{m_k-1}, x_{n_k+1})\}, \{d(x_{m_k+1}, x_{n_k+1})\}$, $\{d(x_{m_k+1}, x_{n_k})\}$ converges to ϵ as $k \to \infty$. Then using the contractive condition (5) we get

$$\begin{aligned}&\psi(d(x_{m_k+1}, x_{n_k+1}))\\ &= \psi(d(Tx_{m_k}, Tx_{n_k}))\\ &\leq \zeta(d(x_{m_k}, x_{n_k}))[\alpha_1\psi(d(x_{m_k}, x_{n_k})) + \alpha_2\psi(d(x_{m_k}, Tx_{m_k}))+\\ &\quad \alpha_3\psi(d(x_{n_k}, Tx_{n_k})) + \alpha_4\psi\left(\frac{d(x_{m_k}, Tx_{n_k}) + d(Tx_{m_k}, x_{n_k})}{2}\right)]\\ &= \zeta(d(x_{m_k}, x_{n_k}))[\alpha_1\psi(d(x_{m_k}, x_{n_k})) + \alpha_2\psi(d(x_{m_k}, x_{m_k+1}))+\\ &\quad \alpha_3\psi(d(x_{n_k}, x_{n_k+1})) + \alpha_4\psi\left(\frac{d(x_{m_k}, x_{n_k+1}) + d(x_{m_k+1}, x_{n_k})}{2}\right)] \text{ for all } k \geq 1. \quad (7)\end{aligned}$$

Taking $k \to \infty$ in (7) we have $\psi(\epsilon) < (\alpha_1 + \alpha_4)\psi(\epsilon) \leq \psi(\epsilon)$, a contradiction. Hence $\{x_n\}$ is Cauchy in X and due to the completeness of X there exists some $z \in X$ such that $x_n \to z$ as $n \to \infty$. Now,

$$\begin{aligned}\psi(d(x_{n+1}, Tz)) &\leq \zeta(d(x_n, z))[\alpha_1\psi(d(x_n, z)) + \alpha_2\psi(d(x_n, x_{n+1}))+\\ &\quad \alpha_3\psi(d(z, Tz)) + \alpha_4\psi\left(\frac{d(x_n, Tz) + d(x_{n+1}, z)}{2}\right)] \text{ for all } n \geq 1.\end{aligned}$$

Since $\{x_n\}$ converges to z and d is continuous it follows that $\psi(d(z, Tz)) \leq (\alpha_3 + \alpha_4)\psi(d(z, Tz))$. By our assumed condition $\alpha_3 + \alpha_4 < 1$ and therefore $\psi(d(z, Tz)) = 0$ implies that $d(z, Tz) = 0$. So $Tz = z$ and z is a fixed point of T. Now to prove the uniqueness of the fixed point of T we assume that u and v be two fixed points of T such that $u \neq v$. Then we have,

$$\begin{aligned}\psi(d(u, v)) &= \psi(d(Tu, Tv))\\ &\leq \zeta(d(u, v))[\alpha_1\psi(d(u, v)) + \alpha_2\psi(d(u, Tu))+\\ &\quad \alpha_3\psi(d(v, Tv)) + \alpha_4\psi\left(\frac{d(u, Tv) + d(v, Tu)}{2}\right)]\\ &< (\alpha_1 + \alpha_4)\psi(d(u, v)) \leq \psi(d(u, v)), \text{ a contradiction.}\end{aligned}$$

Hence T has a unique fixed point in X. □

Example 2.13. *Let us consider $X = [0, 1]$ equipped with the usual metric and let $T : X \to X$ be defined by*

$$T(x) = \begin{cases} \frac{x^2}{16}, & \text{if } 0 \leq x < 1\\ 0, & \text{if } x = 1.\end{cases}$$

Then T satisfies the contractive condition (5) for $\psi(t) = \sqrt{t}$ for all $t \geq 0$, $\zeta(t) = 1 - \frac{t}{4}$ for $0 < t \leq 1$; $= 0$ for $t > 1$ and $\alpha_1 = \frac{1}{2} = \alpha_3$, $\alpha_2 = 0 = \alpha_4$. Also all the conditions of Theorem 2.12 are satisfied and we see that 0 is the unique fixed point of T in X.

Several corollaries can be obtained from our established fixed point theorem.

Corollary 2.14. *In the following we set* $\psi(t) = t$ *for all* $t \geq 0$ *in the contractive condition* (5). *Then*

(i) for $\alpha_1 = 1$ *and* $\alpha_2 = 0 = \alpha_3 = \alpha_4$ *we see that* T *satisfies* $d(Tx, Ty) \leq \zeta(d(x, y))d(x, y)$ *for all* $x, y \in X$ *with* $x \neq y$ *and from Theorem* 2.12 *we get Theorem* 1.2*,*

(ii) for $\alpha_1 = 0 = \alpha_4$ *and* $\alpha_2 = \alpha_3 = \frac{1}{2}$ *we see that* T *satisfies* $d(Tx, Ty) \leq \frac{\zeta(d(x,y))}{2}[d(x, Tx) + d(y, Ty)]$ *for all* $x, y \in X$ *with* $x \neq y$ *and from Theorem* 2.12 *we get Theorem* 5.1 *of [11],*

(ii) for $\alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{3}$ *and* $\alpha_4 = 0$ *we see that* T *satisfies* $d(Tx, Ty) \leq \frac{\zeta(d(x,y))}{3}[d(x, y) + d(x, Tx) + d(y, Ty)]$ *for all* $x, y \in X$ *with* $x \neq y$ *and from Theorem* 2.12 *we get Theorem* 5.2 *of [11].*

2.3. Fixed point theorem via asymptotic regularity

Let (X, d) be a metric space. A mapping $T : X \to X$ satisfying the condition $\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0$ for all $x \in X$ is called asymptotically regular. Asymptotic regularity has been used often to prove fixed point theorems (see [19], [4]).

Theorem 2.15. *Let* (X, d) *be a complete metric space and* $T : X \to X$ *be an asymptotically regular mapping. Suppose that there exist* $\psi \in \Theta$ *and* $\zeta \in \Delta$ *such that*

$$\psi(d(Tx, Ty)) \leq \zeta(d(x, y))[\psi(d(x, y)) + \psi(d(x, Tx)) + \psi(d(y, Ty))] \tag{8}$$

for all $x, y \in X$ *with* $x \neq y$*. Then* T *has a unique fixed point in* X*, provided* T *is orbitally continuous in* X*.*

Proof. Let $x_0 \in X$ be taken as arbitrary and let us consider the Picard iterating sequence $\{x_n\}$, where $x_n = T^n x_0$ for all $n \geq 1$. If for some $n \in \mathbb{N} \cup \{0\}$, $x_n = x_{n+1}$ then clearly T has a fixed point in X. So let $x_n \neq x_{n+1}$ for all $n \geq 0$. Since T is asymptotically regular then $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Now we prove that the sequence $\{x_n\}$ is Cauchy in X. If not then by proceeding similar to the Theorem 2.12 we arrive at a contradiction. Therefore sequence $\{x_n\}$ is Cauchy and by the completeness of X there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$. Now since T is orbitally continuous we have $Tx_n \to Tu$ as $n \to \infty$. Hence $Tu = u$ and u is a fixed point of T. If possible let there exits another fixed point of T, $v(\neq u) \in X$. Then

$$\begin{aligned}
\psi(d(u, v)) &= \psi(d(Tu, Tv)) \\
&\leq \zeta(d(u, v))[\psi(d(u, v)) + \psi(d(u, Tu)) + \psi(d(v, Tv))] \\
&< \psi(d(u, v)), \text{ a contradiction.}
\end{aligned}$$

Therefore T has a unique fixed point in X. □

Example 2.16. *Let us consider* $X = [0, 1]$ *endowed with the usual metric and let* $T : X \to X$ *be defined by*

$$T(x) = \begin{cases} \frac{x^2}{4}, & \text{if } 0 \leq x < 1 \\ \frac{1}{2}, & \text{if } x = 1. \end{cases}$$

Then T *satisfies the contractive condition (8) for any* $\psi_k(t) = kt$ *for all* $t \geq 0$*;* $k > 0$ *and* $\zeta(t) = 1 - \frac{t}{3}$ *for* $0 < t \leq 1$*;* $= 0$ *for* $t > 1$*. Also all the conditions of Theorem 2.15 are satisfied and we see that* 0 *is the unique fixed point of* T *in* X*.*

Corollary 2.17. *If we take* $\psi(t) = t$ *for all* $t \geq 0$ *in the contractive condition* (8) *then Theorem* 6.2 *and Corollary* 6.3 *of [11] follow from Theorem* 2.15*.*

3. Conclusion

Completeness alone is often insufficient for ensuring the existence of fixed points for certain strict contractive mappings. As a result, we require strong conditions on either the mappings or the underlying spaces, or both. While compactness can sometimes guarantee the existence of fixed points for continuous contractive mappings, the required conditions are often too strong to be assumed. In this manuscript, we establish fixed point theorems by imposing weaker conditions on both the mappings and the underlying spaces, which are weaker than compactness and continuity, respectively. Furthermore, the contractive conditions (3), (5) and (8) can be generalized, and weaker conditions than boundedly compactness and T-orbital compactness can be introduced to prove new fixed point theorems.

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