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ΨDOs with radial symbols and spaces of type G

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Abstract. We investigate various G type spaces on $\mathbb{R}^d_+$ and their relations with the Gelfand-Shilov S type spaces on $\mathbb{R}^{2d}$ through the mapping $w : \mathbb{R}^{2d} \to \mathbb{R}^d_+$, $w(x,\xi) = (2x_1^2 + 2\xi_1^2, \dots, 2x_d^2 + 2\xi_d^2)$. Sufficient conditions for the hypoellipticity of symbols originating from the coordinante radial symbols in G type spaces are also given. Two open problems explained in the introduction are posed.

1. Introduction

When a solution of a certain partial differential equation of global type is smooth but not analytic, we look for a space where we can describe its decay for $|x| \to \infty$ and its regularity in $\mathbb{R}^d$. For these needs, Gelfand-Shilov type spaces [11] i.e. S type spaces are significantly useful. Moreover, they are used as a framework for the time frequency analysis of pseudo-differential operators with symbols being of global type. Recall that Gelfand and Shilov introduced such spaces in order to find solutions of certain parabolic initial-value problems. There is a vast literature dealing with pseudo-differential calculus in Gelfand-Shilov setting, see e.g. monographs [6, 19, 25] as well as articles [1, 2, 4, 22–24, 26] and references therein. However, there are only a few papers related to similar type of spaces defined over $\mathbb{R}^d_+ = (0,\infty)^d$. Here, we mention [8, 9], where Duran introduced G type spaces and papers [13–15]. In one dimensional case, the space of tempered distributions supported by $(0,\infty)$ is determined by the corresponding space of rapidly decreasing functions in $(0,\infty)$ (cf. [10, 29]). It should be underlined that the transfer of the space of tempered distributions over $\mathbb{R}^d$ to the one with the domain $\mathbb{R}^d_+$ is not a trivial task, see [12].

In [16], we considered G type spaces $G^\alpha_\alpha(\mathbb{R}^d_+)$ and their duals for $\alpha \geq 1$ and gave their full topological characterization. Furthermore, we analyzed the Weyl pseudo-differential operators on the Gelfand-Shilov spaces

$$\widetilde{a}^W(x,D)f(x) = (2\pi)^{-d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi}\widetilde{a}\Big(\frac{x+y}{2},\xi\Big)f(y)\,dy\,d\xi, \qquad f \in \mathcal{S}(\mathbb{R}^d), \tag{1}$$

with symbols $\widetilde{a}(x,\xi)$ such that $\widetilde{a}(x,\xi) = a(r_1,...,r_d)$, $r_i = 2x_i^2 + 2\xi_i^2$, $i = 1,\dots,d$, where $a(r)$ belongs to a dual space of a G type space. Symbol $\widetilde{a}(x,\xi)$ is called a symbol with coordinate radial arguments. Essential

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relations between G type spaces and Gelfand-Shilov type spaces are determined through the mapping

$$\mathbb{R}^d \ni (x_1, ..., x_n) \mapsto v(x) = r = (r_1, ..., r_d) = (x_1^2, ..., x_n^2) \in \mathbb{R}^d_+. \tag{2}$$

and

$$(x, \xi) = (x_1, \xi_1, \dots, x_d, \xi_d) \mapsto w(x, \xi) = (2x_1^2 + 2\xi_1^2, \dots, 2x_d^2 + 2\xi_d^2) = (r_1, \dots, r_d). \tag{3}$$

Wong [28, Theorem 24.5] used the Laguerre expansions of symbols with coordinate radial arguments which are tempered functions on $\mathbb{R}^2$ in order to obtain a sufficient condition for the boundedness of the Weyl pseudo-differential operators in the $L^2(\mathbb{R})$ setting. Especially, this expansion is useful for the analysis of the global hypoellipticity. This was well analyzed in [16] through the Laguerre expansion of symbols with radial arguments belonging to the projective limit space

$$\mathcal{R}^{\{s\};p}(\mathbb{R}_+) := \varprojlim_{h\to 0} L^p_{\exp(-h|\cdot|^{1/s})}(\mathbb{R}^d_+),$$

and the Hermite expansions in $\mathcal{S}'^{\alpha/2}_{\alpha/2}(\mathbb{R}^{2d}), \alpha \geq 1$. This space is a subspace of $G'^{\alpha}_{\alpha}(\mathbb{R}^d_+)$. (In [16] we also considered the Beurling case.) In this paper we show by an example that we can still have symbols, with the Laguerre expansions as in [16] which are not elements of the spaces introduced there.

Except in the case explained above, we are not able to characterize pseudo differential operators which originate from symbols in $G'^{\alpha}_{\alpha}(\mathbb{R}^d_+)$ even in the case when such symbols are compactly supported in $\mathbb{R}^d_+$. This is an open problem. More precisely, the characterization of $\int_{\mathbb{R}^d_+} e^{-ix^2\tau}\phi(x)dx$, $\phi \in G^{\alpha}_{\alpha}(\mathbb{R}^d_+)$ is not obtained.

Our results relates subspaces of $G^{\alpha}_{\alpha}(\mathbb{R}^d_+)$, the corresponding subspaces of $\mathcal{S}^{\alpha/2}_{\alpha/2}(\mathbb{R}^{2d}), \alpha \geq 1$ and their duals. We introduce $G^{\alpha}_{0,\alpha}(\mathbb{R}^d_+)$ which is a subspace of $G^{\alpha}_{\alpha}(\mathbb{R}^d_+)$. From the point of view of microlocal analyses, it seems that $G^{\alpha}_{0,\alpha}(\mathbb{R}^d_+)$ offers a new field of investigations concerning the behaviour of functions at the corner point zero. For example, let $[0, \infty] \ni t \mapsto f(t) = \phi(t)e^{-1/t}(f(0) = 0)$, where $\phi \in G^5_5(\mathbb{R}_+)$. Then, $f \in G^6_{0,6}(\mathbb{R}_+) \subset G^6_6(\mathbb{R}_+)$ and $f \in G^5_5(\mathbb{R}_+)$. We know that $G^5_5(\mathbb{R}_+) \neq G^6_{0,6}(\mathbb{R}_+)$ but we do not know whether $f \in G^5_{0,5}(\mathbb{R}_+)$. This is also an open problem.

Assuming $\alpha > 2$, we introduce $G^{\alpha}_{c,\alpha}(\mathbb{R}^d_+)$ as a completion of $\mathcal{D}^{\alpha/2}_{\alpha/2}(\mathbb{R}^d_+)$ in the topology of $G^{\alpha}_{\alpha}(\mathbb{R}^d_+)$ and analyse the corresponding dual pairing; $G^{\alpha}_{c,\alpha}(\mathbb{R}^d_+)$ is a subspace of $G^{\alpha}_{\alpha}(\mathbb{R}^d_+)$, but it is not dense in $G^{\alpha}_{0,\alpha}(\mathbb{R}^d_+)$.

We finish the paper with a result on global hypoellipticity as a consequence of the fact that a symbol is a pull-back of an $a \in G'^{\alpha}_{0,\alpha}(\mathbb{R}^d_+)$ with additional assumptions on symbols.

2. Notations and preliminaries

Sets of positive integers, non-negative integers, integers, real and complex numbers are denoted by $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$, respectively. Let $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ and $n = (n_1, \dots, n_d) \in \mathbb{N}^d_0$. We use the notation: $x^n := \prod_{j=1}^d x_j^{n_j}$; $\langle x\rangle = (1+|x|^2)^{1/2}$, where $|x| := (x_1^2 + \ldots + x_d^2)^{1/2}$; $D^n := D^n_x := D_1^{n_1} \cdots D_d^{n_d}$, where $D_j^{n_j} := (-i\partial/\partial x_j)^{n_j}$ and $j = 1, \dots, d$. Given $p, q \in \mathbb{N}^d$, we set $p! := p_1! \dots p_d!$; $|p| := p_1 + \ldots + p_d$; $a^p = a^{|p|}, a > 0$, $q \leq p$ means $q_i \leq p_i$ for $i = 1, \dots d$; $p - q$ is the multi-index $(p_1 - q_1, \dots p_d - q_d)$, whenever $q \leq p$.

Let $\alpha \geq 1$ and $a > 1$. The sequence space $s^{\alpha,a}$ is defined as a space of all complex sequences $\{a_n\}_{n\in\mathbb{N}^d_0}$ for which $\|\{a_n\}_{n\in\mathbb{N}^d_0}\|_{s^{\alpha,a}} = \sup_{n\in\mathbb{N}^d_0} |a_n| a^{|n|^{1/\alpha}} < \infty$. We define $s^{\alpha} = \varinjlim_{a\to 1^+} s^{\alpha,a}$. The strong dual $(s^{\alpha})'$ of s^{α} is the space of all complex valued sequences $\{b_n\}_{n\in\mathbb{N}^d_0}$ such that, for each $a > 1$, $\|\{b_n\}_{n\in\mathbb{N}^d_0}\|_{(s^{\alpha})',a} = \sum_{n\in\mathbb{N}^d_0} |b_n| a^{-|n|^{1/\alpha}} < \infty$. Its topology is generated by the system of seminorms $\|\cdot\|_{(s^{\alpha})',a}$. We also point out that s^{α} is a nuclear (DFS) space (therefore, (DFN) space) while $(s^{\alpha})'$ is an (FN) space (see [27] for these notions and consequences).

Remark concerning the notation. *Contrary to the usual notation, we use notation $\mathcal{S}^{\alpha/2}_{\alpha/2}(\mathbb{R}^d)$, $\alpha \geq 1$, $\mathcal{E}^{\alpha/2}(\mathbb{R}^d), \alpha \geq 0$, and for $\alpha > 2, \mathcal{D}^{\alpha/2}(\mathbb{R}^d)$, since we will compare these spaces with the G-type spaces given below which will have symbol α in the definition. Thus, all the spaces over $\mathbb{R}^d$ or $\mathbb{R}^{2d}$ will have super- (and sub-) exponent $\alpha/2$ while those ones which are defined over $\mathbb{R}^d_+$ will be signed with α.*

Let $\alpha \geq 0, h > 0$. Recall [17],

$$\mathcal{E}^{\alpha/2}(\mathbb{R}^d_+) = \varprojlim_{K\subset\subset\mathbb{R}^d_+} \varinjlim_{h\to\infty} \mathcal{E}^{\alpha/2}_{K,h}, \text{ where } \phi \in \mathcal{E}^{\alpha/2}_{K,h} \text{ if } \|\phi\|_{K,h} = \sup_{x\in K, q\in\mathbb{N}^d_0} \frac{|\phi^{(q)}(x)|}{h^q q!^{\alpha/2}} < \infty,$$

while, for $\alpha > 2$, the Roumieu type space $\mathcal{D}^{\alpha/2}(\mathbb{R}^d_+)$ of smooth functions equals

$$\mathcal{D}^{\alpha/2}(\mathbb{R}^d_+) = \varinjlim_{K\subset\subset\mathbb{R}^d_+} \varinjlim_{h\to\infty} \mathcal{D}^{\alpha/2}_{K,h}.$$

$\mathcal{D}^{\alpha/2}_{K,h}$ is defined by the seminorms $\|\phi\|_{K,h}$ given above, but ϕ is supported by K (supp $\phi \subset\subset K$).

Let $\alpha \geq 1$. We recall the definition of Gelfand-Shilov type spaces [11] (see also [3, 5, 18, 21]). Let $A > 0$. Then $\mathcal{S}^{\alpha/2,A}_{\alpha/2,A}(\mathbb{R}^d)$ is the Banach space of all $\varphi \in C^\infty(\mathbb{R}^d)$ with the norm

$$\|\varphi\|^{\alpha/2}_{\mathcal{S},A,\alpha/2} = \sup_{p,q\in\mathbb{N}^d_0} \frac{\left\|x^p\partial^q\varphi(x)\right\|_{L^2(\mathbb{R}^d)}}{(A^{p+q}p!^{\alpha/2}q!^{\alpha/2})} < \infty. \tag{4}$$

The Gelfand-Shilov space $\mathcal{S}^{\alpha/2}_{\alpha/2}(\mathbb{R}^d)$ is the inductive limit of $\mathcal{S}^{\alpha/2,A}_{\alpha/2,A}(\mathbb{R}^d)$ i.e. $\mathcal{S}^{\alpha/2}_{\alpha/2}(\mathbb{R}^d) = \varinjlim_{A\to\infty} \mathcal{S}^{\alpha/2,A}_{\alpha/2,A}(\mathbb{R}^d)$. The corresponding dual space of $S^{\alpha/2}_{\alpha/2}(\mathbb{R}^d)$ is the space of ultradistributions of Roumieu type $\mathcal{S}'^{\alpha/2}_{\alpha/2}(\mathbb{R}^d) = (\mathcal{S}^{\alpha/2}_{\alpha/2}(\mathbb{R}^d))' = \varprojlim_{A\to 0}(\mathcal{S}^{\alpha/2,A}_{\alpha/2,A}(\mathbb{R}^d))'$ (cf. [17]).

Denote by $h_n, n \in \mathbb{N}^d$ the Hermite functions which form an orthonormal basis of $L^2(\mathbb{R}^d)$. Then $\iota : \mathcal{S}^{\alpha/2}_{\alpha/2}(\mathbb{R}^d) \to s^\alpha$, defined by $\iota(f) = \{\langle f, h_n\rangle\}_{n\in\mathbb{N}^d_0}$ is a continuous linear bijection between $\mathcal{S}^{\alpha/2}_{\alpha/2}(\mathbb{R}^d)$ and s^α, cf. [18], for example. Furthermore, for each $f \in \mathcal{S}^{\alpha/2}_{\alpha/2}(\mathbb{R}^d)$, $\sum_{n\in\mathbb{N}^d_0}\langle f, h_n\rangle h_n$ converges absolutely to f in $\mathcal{S}^{\alpha/2}_{\alpha/2}(\mathbb{R}^d)$.

Let $\alpha \geq 1$. We recall the definition of our basic space from [13]. For $A > 0$, we denote by $G^{\alpha,A}_{\alpha,A}(\mathbb{R}^d_+)$ the space of all $\psi \in C^\infty(\mathbb{R}^d_+)$ for which the norms

$$\|\psi\|^\alpha_{G,A,j,\alpha} = \sup_{p,q\in\mathbb{N}^d_0} \frac{\|t^{(p+q)/2}\partial^q\psi(t)\|_{L^2(\mathbb{R}^d_+)}}{A^{p+q}p!^{\alpha/2}q!^{\alpha/2}} + \sup_{\substack{|q|\leq j\\|p|\leq j}}\sup_{t\in\mathbb{R}^d_+}|t^p\partial^q\psi(t)|, \;\; j\in\mathbb{N}_0. \tag{5}$$

are finite. Then, the G type space $G^\alpha_\alpha(\mathbb{R}^d_+)$ is the inductive limit of the Fréchet spaces $G^{\alpha,A}_{\alpha,A}(\mathbb{R}^d)$ i.e. $G^\alpha_\alpha(\mathbb{R}^d_+) = \varinjlim_{A\to\infty} G^{\alpha,A}_{\alpha,A}(\mathbb{R}^d)$. The corresponding dual space of $G^\alpha_\alpha(\mathbb{R}^d_+)$ is the spaces of ultradistributions of Roumieu type $G'^\alpha_\alpha(\mathbb{R}^d_+) = (G^\alpha_\alpha(\mathbb{R}^d_+))' = \varprojlim_{A\to 0}(G^{\alpha,A}_{\alpha,A}(\mathbb{R}^d_+))'$.

Using the Laguerre orthonormal basis of $L^2(\mathbb{R}^d_+)$, by $\ell_n, n \in \mathbb{N}^d_0$ Laguerre functions are denoted, it is proved in [13, Theorem 6.1], that the mapping $\iota : G^\alpha_\alpha(\mathbb{R}^d_+) \to s^\alpha$, $\iota(f) = \{\langle f, \ell_n\rangle\}_{n\in\mathbb{N}^d_0}$ is a topological isomorphism between $G^\alpha_\alpha(\mathbb{R}^d_+)$ and s^α. Hence, the G type space inherits topological properties of the sequence space s^α. Furthermore, for each $f \in G^\alpha_\alpha(\mathbb{R}^d_+)$, $\sum_{n\in\mathbb{N}^d_0}\langle f, \ell_n\rangle\ell_n = f$ in $G^\alpha_\alpha(\mathbb{R}^d_+)$, see [8, 13].

Let $\alpha > 2$. We recall the definitions from [24] (with $\rho = 1, \sigma = s$ in [24], and $s = \alpha/2$ here). In the Roumieu case, for any fixed $h > 0$, $m > 0$, by $\Gamma^{\alpha/2,\infty}_{\alpha/2}(\mathbb{R}^{2d}; h; m)$ is denoted the space of all functions $a(x, \xi) \in C^\infty(\mathbb{R}^{2d})$ such that

$$\sup_{q,p\in\mathbb{N}^d_0}\sup_{(x,\xi)\in\mathbb{R}^{2d}} \frac{|D^q_\xi D^p_x a(x,\xi)|\langle(x,\xi)\rangle^{p+q}e^{-m(|x|^{2/\alpha}+|\xi|^{2/\alpha})}}{h^{p+q}p!^{\alpha/2}q!^{\alpha/2}} < \infty.$$

Then

$$\Gamma^{\alpha/2,\infty}_{\alpha/2}(\mathbb{R}^{2d}) = \varinjlim_{h\to\infty}\varprojlim_{m\to 0}\Gamma^{\alpha/2,\infty}_{\alpha/2}(\mathbb{R}^{2d}; h; m).$$

Let $a \in \Gamma^{\alpha/2,\infty}_{\alpha/2}(\mathbb{R}^{2d})$. It is said that a symbol a is $\Gamma^{\alpha/2,\infty}_{\alpha/2}$-hypoelliptic if

(i) There exist $R > 0$ and $C, m > 0$ such that

$$|a(x,\xi)| \geq Ce^{-m(|x|^{2/\alpha}+|\xi|^{2/\alpha})}, \qquad (x,\xi) \in Q_R^c = \{(x,\xi) : \langle x\rangle \geq R \text{ or } \langle \xi\rangle \geq R\}. \tag{6}$$

(ii) There exists $R > 0$ such that for every $h > 0$ there exists $C > 0$ such that

$$|D_\xi^q D_x^p a(x,\xi)| \leq \frac{h^{p+q}|a(x,\xi)|p!^{\alpha/2}q!^{\alpha/2}}{\langle (x,\xi)\rangle^{p+q}}, \qquad p,q \in \mathbb{N}^d,\ (x,\xi) \in Q_R^c. \tag{7}$$

3. Relations between G-type spaces

Let $\alpha \geq 1$. The subspace $\mathcal{S}^{\alpha/2}_{\alpha/2,even}(\mathbb{R}^d)$ of $\mathcal{S}^{\alpha/2}_{\alpha/2}(\mathbb{R}^d)$ consists of all even functions in $\mathcal{S}^{\alpha/2}_{\alpha/2}(\mathbb{R}^d)$: $\psi(x_1,\dots,x_{j-1}, -x_j, x_{j+1},\dots,x_d) = \psi(x), x \in \mathbb{R}^d$, $j = 1,\dots,d$. A topological isomorphism between $G^\alpha_\alpha(\mathbb{R}^d_+)$ and $S^{\alpha/2}_{\alpha/2,even}(\mathbb{R}^d)$,

$$G^\alpha_\alpha(\mathbb{R}^d_+) = \mathcal{S}^{\alpha/2}_{\alpha/2,even}(\mathbb{R}^d), \tag{8}$$

is established in [15] through the push forward $G^\alpha_\alpha(\mathbb{R}^d_+) \to S^{\alpha/2}_{\alpha/2}(\mathbb{R}^d)$ by the mapping v given in (2). In relation to $G^\alpha_\alpha(\mathbb{R}^d_+)$, the elements of the next space have essentially different behaviour at zero.

Let $A > 0$ and $\alpha \geq 1$; $G^{\alpha,A}_{0,\alpha,A}$ is defined as an F- space of smooth functions ϕ on $\mathbb{R}^d_+$ such that

$$||\phi||^\alpha_{G,0,A,j,\alpha} = \sup_{\substack{p,q\in\mathbb{N}^d_0\\ x\in\mathbb{R}^d_+}} \frac{||\langle t\rangle^{(p+q)/2}\partial^q\phi(t)||_{L^2(\mathbb{R}^d_+)}}{A^{p+q}p!^{\alpha/2}q!^{\alpha/2}} + \sup_{\substack{|q|\leq j\\ |p|\leq j}}\sup_{t\in\mathbb{R}^d_+} |t^p\partial^q\psi(t)|, \quad j \in \mathbb{N}_0.$$

$G^\alpha_{0,\alpha}$ is the inductive limit of the spaces $G^{\alpha,A}_{0,\alpha,A}$. It is a (DFN) space; $G^\alpha_\alpha(\mathbb{R}^d_+) \subset \mathcal{E}^{\alpha/2}(\mathbb{R}^d_+)$.

Furthermore, for $\alpha > 2$, we define $G^\alpha_{c,\alpha}(\mathbb{R}^d_+)$ as the closure of $\mathcal{D}^{\alpha/2}(\mathbb{R}^d_+)$ in $G^\alpha_\alpha(\mathbb{R}^d_+)$. Clearly, $\mathcal{D}^{\alpha/2}(\mathbb{R}^d_+)$ is a dense subspace of $G^\alpha_{c,\alpha}(\mathbb{R}^d_+)$. This gives embedding $G'^\alpha_{c,\alpha}(\mathbb{R}^d_+) \hookrightarrow \mathcal{D}'^{\alpha/2}(\mathbb{R}^d_+)$. Moreover, $G^\alpha_{c,\alpha}(\mathbb{R}^d_+)$ is continuously embedded in $G^\alpha_\alpha(\mathbb{R}^d_+)$ but not densely embedded. See Example 3.3 below and the comment therein.

If $F \in G'^\alpha_\alpha(\mathbb{R}^d_+)$, then its restriction on $G^\alpha_{c,\alpha}(\mathbb{R}^d_+)$ belongs to $G'^\alpha_{c,\alpha}(\mathbb{R}^d_+)$. Space $G'^\alpha_{c,\alpha}(\mathbb{R}^d_+)$ has a good approximation property:

Proposition 3.1. *Let $\alpha > 2$ and $u \in G'^\alpha_{c,\alpha}(\mathbb{R}^d_+)$. Then there is a sequence $(u_j)_j$ in $G^\alpha_{c,\alpha}(\mathbb{R}^d)$, such that $u_j \to u$ in $G'^\alpha_{c,\alpha}(\mathbb{R}^d_+)$.*

In particular, if u is compactly supported, then $(u_j)_j$ is a sequence in $\mathcal{D}^{\alpha/2}(\mathbb{R}^d_+)$ which converges to u in $\mathcal{D}'^{\alpha/2}(\mathbb{R}^d_+)$.

Proof. By (8), the properties of $\mathcal{S}^{\alpha/2}_{\alpha/2}(\mathbb{R}^d)$ are transferred to $G^\alpha_\alpha(\mathbb{R}^d_+)$ through the mapping v. If $u \in L^1(\mathbb{R}^d_+)$ and $\theta \in G^\alpha_{c,\alpha}(\mathbb{R}^d_+)$, then their dual pairing is given by $\langle u,\theta\rangle_{\mathbb{R}^d_+} = \int_{\mathbb{R}^d_+} u(\rho)\theta(\rho)d\rho_1...d\rho_d$. Let $\kappa \in G^\alpha_\alpha(\mathbb{R}^d_+)$ have the properties:

$$\kappa(r) = 1, r \in B(\mathbf{0},1)\cap\mathbb{R}^d_+, \kappa(r) = 0, r \in CB(\mathbf{0},2),$$

where $CB(\mathbf{0},2)$ is a complement of the ball $B(\mathbf{0},2)$ with center at zero and radius equals 2. Put $\kappa_j(r) = \kappa(r/j), r \in \mathbb{R}^d_+,\ j \in \mathbb{N}$, (cut-off function) . Further on, let $\phi(r) \in G^\alpha_\alpha(\mathbb{R}^d_+)$ so that it is supported by the ball $B(\mathbf{1},1)$ so that its integral over $\mathbb{R}^d_+$ equals one ($\mathbf{1} = (1,...,1)$). Put $\phi_j = j^d\phi(jr), r \in \mathbb{R}^d_+,\ j \in \mathbb{N}$ (this is a delta sequence). Let $u \in G'^\alpha_\alpha(\mathbb{R}^d_+)$. Define $u_j(r) = \kappa_j(u * \phi_j(r)),\ r \in \mathbb{R}^d_+,\ j \in \mathbb{N}$. This is a sequence in $G^\alpha_{c,\alpha}(\mathbb{R}^d_+)$ which converges to u in $G'^\alpha_{c,\alpha}(\mathbb{R}^d_+)$ since $\langle u_j(r),\theta(r)\rangle_{\mathbb{R}^d_+} = \langle u(r),\Phi_j(r)\rangle_{\mathbb{R}^d_+},\ j \in \mathbb{N}$, where

$$\Phi_j(r) = \int_{\mathbb{R}^d_+} j^d\phi(j(r-\rho))(\kappa(\rho/j)\theta(\rho))d\rho_1...d\rho_d \text{ converges to } \theta(r) \text{ in } G^\alpha_{c,\alpha}(\mathbb{R}^d_+), j \to \infty.$$

The particular part of lemma is clear. □

We recall the Faà di Bruno formula for the partial derivatives of the composition of functions [7, Corollary 2.10]:

$$\frac{\partial^q}{\partial x^q}\sigma(g(x)) = \sum_{r=0}^{q} \frac{q!}{k_1!...k_q!}\sigma^{(r)}(g(x)) \prod_{i=1}^{q} \Big(\frac{g^{(i)}(x)}{i!}\Big)^{k_i}, \ q \in \mathbb{N}_0 \tag{9}$$

where $r = k_1 + ... + k_q$ and the sum is over all nonnegative integers $k_1, ..., k_q$ for which $k_1 + 2k_2 + ... + qk_q = q$.

Example 3.2. *Let $\phi \in G^5_{0,5}(\mathbb{R}_+)$ so that $\phi(t) = 1, t \in [0, 1]$. Our aim in this example is to show that*

$$f(t) = e^{-1/t}\phi(t) \in G^6_{0,6}(\mathbb{R}_+) \cap G^5_5(\mathbb{R}_+). \tag{10}$$

We use Faá di Bruno formula (9), in the one dimensional case. Since $(\frac{(t^{-1})^{(i)}}{i!})^{k_i} = (-1)^{ik_i}t^{-(i+1)k_i}$, it follows that

$$\prod_{i=1}^{q}(\frac{(t^{-1})^{(i)}}{i!})^{k_i} = (-1)^q t^{-q-r}.$$

With this, we have

$$(e^{-1/t})^{(q)} = q! \sum_{r=0}^{q} \frac{e^{-1/t}}{k_1!...k_q!}(-1)^q t^{-q-r}.$$

By Stirling's formula, and the fact that $e^{-1/t}t^{-q-r}$ takes its maximum at $t = 1/(q + r)$, we have

$$\max_{t\in[0,1]} \frac{e^{-1/t}}{t^{q+r}} = e^{-q-r}(q + r)^{q+r} \sim \frac{e^{-q-r}(q + r)!e^{q+r}}{\sqrt{2\pi(q + r)}} \le \frac{2^{q+r}q!r!}{\sqrt{2\pi(q + r)}},$$

we have,

$$(e^{-1/t})^{(q)} \le q!^3 \sum_{r=0}^{q} \frac{2^{q+r}}{k_1!...k_q! \sqrt{2\pi(q + r)}} \le c^q q!^3, t \in [0, 1].$$

By the similar calculation

$$|t^{q/2}(e^{-1/t})^{(q)}| \le q! \sum_{r=0}^{q} \frac{(q/2 + r)^{q/2+r}e^{-q/2-r}}{k_1!...k_q! \sqrt{2\pi(q/2 + r)}} \le q! \sum_{r=1}^{q} \frac{(q/2 + 2q/2)!2^{q/2+r}}{k_1!...k_q! \sqrt{2\pi(q/2 + r)}} \le c_2^q q!^{5/2},$$

where we have used that $q! \le 2^q(q/2)!^2 \le 2^q q!$. Now, one can find that for $\alpha = 6$ there exists $h > 0$ such that

$$\sup_{t\in[0,1]} \frac{|\langle t\rangle^{(p+q)/2}(e^{-1/t})^{(q)}|}{h^{p+q}p!^3q!^3} < \infty. \tag{11}$$

and that for $\alpha = 5$, there exists $h > 0$ such that

$$\sup_{t\in[0,1]} \frac{|t^{(p+q)/2}(e^{-1/t})^{(q)}|}{h^{p+q}p!^{5/2}q!^{5/2}} < \infty. \tag{12}$$

Thus, by (11), $e^{-1/t}\phi(t) \in G^6_{0,6}(\mathbb{R}_+)$. On the other hand, by (12), we have $e^{-1/t}\phi(t) \in G^5_5(\mathbb{R}_+)$. As we noted, we are not able to prove that $f \in G^5_{0,5}(\mathbb{R}_+)$.

Example 3.3. *The space $G^\alpha_{c,\alpha}$, $\alpha > 2$, is not dense in $G^\alpha_{0,\alpha}$, $\alpha > 2$. Let $\phi \in G^\alpha_{0,\alpha}(\mathbb{R}_+)$ so that $\phi(t) = 1, t \in [0, 1]$. There does not exist a sequence $(\psi_n)_n$ in $\mathcal{D}^{\alpha/2}(\mathbb{R})$ such that $\psi_n \to \phi$, $n \to \infty$, in $G^\alpha_{0,\alpha}$ because* $\sup_{t\in[0,1]} |\phi(t) - \psi_n(t)| = 1, n \in \mathbb{N}$.

The proof of the next proposition is based on the multivariate Faà di Bruno formula (9). First, we formulate the one -dimensional case with $g(x) = 2x^2$. Note that i in (9) can take only values 1 and 2. This further, implies $k_1 + k_2 = k$ and $k_1 + 2k_2 = p$, that is, $k_1 = 2k - p, k_2 = p - k$ and $k \geq [p/2]$. This, with $j = p - k$, gives

$$\frac{\partial^p}{\partial x^p}\sigma(2x^2) = \sum_{j=0}^{[\frac{p}{2}]} \frac{p!}{j!(p-2j)!}\sigma^{(p-j)}(2x^2)(4x)^{p-2j}2^j, \tag{13}$$

where $[x]$ is the largest integer $\leq x$. With the help of multi-indices, the identity (13) in the $2d$-dimensional case becomes

$$\begin{aligned} &\frac{\partial^{p,q}_{x,\xi}}{\partial x^p \partial \xi^q}\sigma(2x^2 + 2\xi^2) = \frac{\partial^{p,q}_{x,\xi}}{\partial x^p \partial \xi^q}\sigma(2x_1^2 + 2\xi_1^2, \dots, 2x_d^2 + 2\xi_d^2) \\ = \;& p!q! \sum_{\mathbf{j=0}}^{[\frac{p}{2}]}\sum_{\mathbf{i=0}}^{[\frac{q}{2}]} \frac{\sigma^{(p+q-j-i)}(2x^2+2\xi^2)}{j!i!(p-2j)!(q-2i)!}(4x)^{p-2j}2^j(4\xi)^{q-2i}2^i, \quad x,\xi \in \mathbb{R}^d, \end{aligned} \tag{14}$$

where we use notation

$$\sum_{\mathbf{j=0}}^{[\frac{p}{2}]}\sum_{\mathbf{i=0}}^{[\frac{q}{2}]} = \sum_{j_1=0}^{[\frac{p_1}{2}]}\sum_{i_1=0}^{[\frac{q_1}{2}]} \cdots \sum_{j_d=0}^{[\frac{p_d}{2}]}\sum_{i_d=0}^{[\frac{q_d}{2}]}.$$

Proposition 3.4. *Let $\alpha \geq 2$ and $\psi \in G^{\alpha}_{0,\alpha}(\mathbb{R}^d_+)$ and let $\varphi(x,\xi) = w^*\psi(x,\xi) = \psi(2x^2 + 2\xi^2) = \psi(2x_1^2 + 2\xi_1^2, ..., 2x_d^2 + 2\xi_d^2)$, (cf. (3)). Then $\varphi \in S^{\alpha/2}_{\alpha/2}(\mathbb{R}^{2d})$ and the mapping*

$$w^* : G^{\alpha}_{\alpha}(\mathbb{R}^d_+) \to S^{\alpha/2}_{\alpha/2}(\mathbb{R}^{2d}),\ \psi \mapsto w^*\psi = \psi \circ w = \varphi.$$

is continuous.

Proof. We use the change of variables $t_i = 2x_i^2 + 2\xi_i^2, i = 1, ..., d$ and the inequalities $p!q! \leq (p+q)! \leq 2^{p+q}\, p!\, q!,\ p, q \in \mathbb{N}_0^d$ $(0! = 1)$,

$$|\langle x_1\rangle^{p_1}\langle \xi_1\rangle^{q_1} \cdots \langle x_d\rangle^{p_d}\langle \xi_d\rangle^{q_d}| \leq (1 + 2x_1^2 + 2\xi_1^2)^{p_1/2+q_1/2} \cdots (1 + 2x_d^2 + 2\xi_d^2)^{p_d/2+q_d/2},$$

$$|x^{q_1-2j}\xi^{q_2-2i}| \leq \langle x\rangle^{q_1-2j}\langle \xi\rangle^{q_2-2i} \leq 2^{q_1+q_2-2j-2i}\langle t\rangle^{\frac{q_1+q_2-2j-2i}{2}},\ \ \langle x\rangle^{m_1}\langle \xi\rangle^{m_2} \leq 2^{m_1+m_2}\langle t\rangle^{(m_1+m_2)/2}.$$

It is enough to show that there exist $A > 0$ and $C > 0$ such that for given $h > 0$ (which depends on ψ),

$$\sup_{\substack{p,q\in\mathbb{N}_0^d \\ x,\xi\in\mathbb{R}^d}} \frac{|\partial_x^{q_1}\partial_\xi^{q_2}\varphi(x,\xi)|(1+|x|^2)^{m_1/2}(1+|\xi|^2)^{m_2/2}}{A^{q_1+q_2+m_1+m_2}(q_1+q_2)!^{\alpha/2}(m_1)!^{\alpha/2}(m_2)!^{\alpha/2}} \leq C \sup_{\substack{p,q\in\mathbb{N}_0^d \\ t\in\mathbb{R}^d_+}} \frac{|\langle t\rangle^{(p+q)/2}\partial^q\psi(t)|}{h^{p+q}p!^{\alpha/2}q!^{\alpha/2}}.$$

It should be noticed that in the successive calculation we obtain the expression $(|x|^2 + |\xi|^2)^{-j-i}$ which can not be controlled by other factors. Therefore, the assumption that $\psi \in G^{\alpha}_{0,\alpha}(\mathbb{R}_+)$ is needed.
Let $c^{q_1,q_2}_{A,m_1,m_2} = A^{q_1+q_2+m_1+m_2}(q_1+q_2)!^{\alpha/2}(m_1)!^{\alpha/2}(m_2)!^{\alpha/2}$. Using (14) and the change of variable $t = 2|x|^2 + 2|\xi|^2$ (also written as $t = 2x^2 + 2\xi^2$), we conclude, with the use of suitable constants

$$\|\varphi\|^{\alpha/2}_{\mathcal{S},A,\alpha/2} = \sup_{\substack{q_1,q_2,m_1,m_2\in\mathbb{N}_0^d \\ x,\xi\in\mathbb{R}^d}} \frac{|\partial_x^{q_1}\partial_\xi^{q_2}\varphi(x,\xi)|\langle x\rangle^{m_1}\langle \xi\rangle^{m_2}}{c^{q_1,q_2}_{A,m_1,m_2}}$$

$$\leq \sup_{\substack{q_1,q_2,m_1,m_2\in\mathbb{N}_0^d \\ t\in\mathbb{R}_+^d}} \sum_{j=0}^{[\frac{q_1}{2}]}\sum_{i=0}^{[\frac{q_2}{2}]} \frac{q_1!q_2!|\psi^{(q_1+q_2-i-j)}(t)|}{c_{A,m_1,m_2}^{q_1,q_2} j!i!(q_1-2j)!(q_2-2i)!} \langle t\rangle^{\frac{q_1+q_2-2i-2j+m_1+m_2}{2}} 2^{2(q_1+q_2)-3(i+j)+m_1+m_2}$$

$$\leq (\frac{C_1}{A})^{q_1+q_2+m_1+m_2} \sup_{\substack{q_1,q_2,m_1,m_2\in\mathbb{N}_0^d \\ t\in\mathbb{R}_+^d}} \sum_{j=0}^{[\frac{q_1}{2}]}\sum_{i=0}^{[\frac{q_2}{2}]} \frac{|\psi^{(q_1+q_2-j-i)}(t)|\langle t\rangle^{\frac{q_1+q_2-j-i}{2}}\langle t\rangle^{\frac{m_1+m_2}{2}}\langle t\rangle^{-\frac{j+i}{2}}}{(q_1+q_2-j-i)!^{\alpha/2}(m_1+m_2)!^{\alpha/2}}$$

$$\frac{(q_1+q_2-j-i)!(i+j)!}{(i+j)!^{\alpha/2}(q_1+q_2-j-i)!} \leq (\frac{C_2(h)}{A})^{q_1+q_2+m_1+m_2} \sup_{\substack{q_1,q_2,m_1,m_2\in\mathbb{N}_0^d \\ t\in\mathbb{R}_+^d}} \sum_{j=0}^{[\frac{q_1}{2}]}\sum_{i=0}^{[\frac{q_2}{2}]} \frac{c_2^{i+j}}{(i+j)^{(\alpha/2-1)}}$$

This implies that for given $h > 0$ there exist $A, C > 0$ such that fpr every $j \in \mathbb{N}_0$,

$$\|\varphi\|^{\alpha/2}_{\mathcal{S},A,\alpha/2} \leq C\|\psi\|^{\alpha}_{G,0,h,j,\alpha} \quad \text{(see Section 2).}$$

This completes the proof of the proposition. □

We note that this proposition also follows from our result related to (8). Here we obtain this result by quite different methods.

4. Weyl pseudo-differential operators with radial symbols and hypoellipticity

Let $\alpha \geq 1$ and $f, g \in \mathcal{S}^{\alpha/2}_{\alpha/2}(\mathbb{R}^d)$. We refer to [6] for the Wigner transform of $f, g \in \mathcal{S}^{\alpha/2}_{\alpha/2}(\mathbb{R}^d)$. It is well known that the bilinear mapping $(f, g) \mapsto W(f, g)$, $\mathcal{S}^{\alpha/2}_{\alpha/2}(\mathbb{R}^d) \times \mathcal{S}^{\alpha/2}_{\alpha/2}(\mathbb{R}^d) \to \mathcal{S}^{\alpha/2}_{\alpha/2}(\mathbb{R}^{2d})$ is continuous.

The Weyl pseudo-differential operator with a symbol $\widetilde{a} \in \mathcal{S}'^{\alpha/2}_{\alpha/2}(\mathbb{R}^{2d})$ defined by (1) is a continuous and linear mapping from $\mathcal{S}^{\alpha/2}_{\alpha/2}(\mathbb{R}^d)$ into $\mathcal{S}'^{\alpha/2}_{\alpha/2}(\mathbb{R}^d)$ (see [20, Theorem 2]). Using the Hermite expansions of elements from $\mathcal{S}^{\alpha/2}_{\alpha/2}(\mathbb{R}^d)$, we obtain

$$W(f, g) = \sum_{(m,k)\in\mathbb{N}_0^{2d}} \langle f, h_m\rangle\langle \overline{g}, h_k\rangle W(h_m, h_k),$$

where the sum converges absolutely in $\mathcal{S}^{\alpha/2}_{\alpha/2}(\mathbb{R}^{2d})$.

We introduce $\mathcal{A}^{\alpha}_{\alpha}(\mathbb{R}^d_+)$ as the space of all locally integrable functions $a(r), r \in \mathbb{R}^d_+$, with the properties

$$a(r) \in G'^{\alpha}_{\alpha}(\mathbb{R}^d_+) \text{ and } \widetilde{a}(x,\xi) = a(2x_1^2 + 2\xi_1^2, ..., 2x_d^2 + 2\xi_d^2) \in S'^{\alpha/2}_{\alpha/2}(\mathbb{R}^{2d}).$$

(Here we use the fact that the composition of a locally integrable function and a smooth function is locally integrable.)

We considered in [16] the symbols $a_w(x,\xi) := a(2x_1^2 + 2\xi_1^2, \ldots, 2x_d^2 + 2\xi_d^2), a \in \mathcal{R}^{\{\alpha\};p}(\mathbb{R}^d_+)$ (see introduction). We obtained the characterisation of a in terms of the growths of their Laguerre coefficients for which the Weyl quantisation of a_w extends to a well-defined and continuous operator $a_w^W : S'^{\alpha/2}_{\alpha/2}(\mathbb{R}^d) \to S'^{\alpha/2}_{\alpha/2}(\mathbb{R}^d)$.

Example 4.1. *Let $\alpha \geq 2$ and $s = \alpha/2$. With the following example, in the case when $d = 1$, we show that $\mathcal{R}^{\{s\};p}(\mathbb{R}_+)$ is a proper subset of $\mathcal{A}^{\alpha}_{\alpha}(\mathbb{R}_+)$. Let $f(x) = e^x, x \in [n, n + e^{-n}), n \in \mathbb{N}, n > N_0$ and $f(x) = 0$ in the rest of $\mathbb{R}$. Let $\varphi \in G^{\alpha}_{\alpha}(\mathbb{R}^d_+)$. We use the fact that $\varphi(x) \leq Ce^{h|x|^{2/\alpha}}, x > N_0$, for suitable $C, h, N_0 > 0$. With this, we have*

$$\Big|\int_{\mathbb{R}_+} f(x)\varphi(x)dx\Big| \leq C\sum_{n>N_0}^{\infty} \frac{1}{e^{hn^{2/\alpha}}} < \infty,$$

so that $f \notin \mathcal{R}^{\{\alpha/2\};p}(\mathbb{R}^d_+)$ since $\int_{\mathbb{R}^d_+} e^{px^{2/\alpha}-hpx^{2/\alpha}}dx = \infty$ for $h = 1$. Thus, $f \in \mathcal{A}^{\alpha}_{\alpha}(\mathbb{R}^d_+) \setminus \mathcal{R}^{\{\alpha/2\};p}(\mathbb{R}^d_+)$.

Because of that, the next proposition has a sense although its proof is the same as in [16]; so, the proof is skipped.

Proposition 4.2. *Let $a(r) \in \mathcal{A}^{\alpha}_{\alpha}(\mathbb{R}^d_+)$.*

(i) The Weyl pseudo-differential operator defined as

$$(\widetilde{a}^W(x,D)\varphi)(\psi) = \langle \widetilde{a}^W(x,D)\varphi, \psi\rangle = \langle a(2x^2+2\xi^2), W(\varphi,\psi)(x,\xi)\rangle, \qquad \varphi,\psi \in \mathcal{S}^{\alpha/2}_{\alpha/2}(\mathbb{R}^d)$$

is a continuous bilinear mapping $\mathcal{S}^{\alpha/2}_{\alpha/2}(\mathbb{R}^d)\times\mathcal{S}^{\alpha/2}_{\alpha/2}(\mathbb{R}^d)\to\mathbb{C}$. It is given by $(\varphi,\psi)\mapsto\sum_{k\in\mathbb{N}^d_0}\varphi_k\psi_k a_k$, where $\varphi(x)=\sum_{k\in\mathbb{N}^d_0}\langle\varphi,h_k\rangle h_k(x)$, $\psi(x)=\sum_{k\in\mathbb{N}^d_0}\langle\psi,h_k\rangle h_k(x)$ and $a(\rho)=\sum_{k\in\mathbb{N}^d_0}a_k\ell_k(\rho)$, $a_k=(2\pi)^{d/2}(-1)^{|k|}2^{-d}\langle a,\ell_k\rangle$.

(ii) It can be continuously extended as a bilinear mapping over $\mathcal{S}^{\alpha/2}_{\alpha/2}(\mathbb{R}^d)\times\mathcal{S}'^{\alpha/2}_{\alpha/2}(\mathbb{R}^d)$, with values in $\mathbb{C}$.

4.1. Hypoellipticity

We continue to assume $\alpha > 2$. We employ the notation $\mathcal{H}^{\alpha}(\mathbb{R}^d_+)$ for a subspace of $G^{\alpha}_{0,\alpha}(\mathbb{R}^d_+)$ consisting of smooth functions $a(r)$, $r\in\mathbb{R}^d_+$, which satisfy the next conditions: There exists $R>0, h>0, C>0, m>0$ such that

$$|a(r)| \geq Ce^{-m|r|^{1/\alpha}}, \ \ |r^{p/2}\partial^p_r a(r)| \leq h^{|p|}|a(r)|p!^{\alpha/2}, \ p\in\mathbb{N}^d_0 \ \ r\in Q^c_R. \tag{15}$$

Proposition 4.3. *Let $\widetilde{a}(x,\xi)=a(2x^2+2\xi^2)$, $x,\xi\in\mathbb{R}^d$, where a function a belongs to $\mathcal{H}^{\alpha}(\mathbb{R}^d_+)$. Then the Weyl pseudo-differential operator $\widetilde{a}^W(x,D)$ is $\Gamma^{\alpha/2,\infty}_{\alpha/2}$-hypoelliptic.*

Proof. We skip the part in which we prove that $\widetilde{a}(x,\xi)\in\Gamma^{\alpha/2,\infty}_{\alpha/2}(\mathbb{R}^{2d})$.

Condition (6) is clear. We will prove (7). Let $(x,\xi)\in B^c_R$. To prove (7), we use our calculation in Proposition 3.4 (without the multiplication with powers of x and ξ). Let $c^{q_1,q_2}_A = A^{q_1+q_2}(q_1+q_2)!^{\alpha/2}$.

$$\sup_{\substack{q_1,q_2\in\mathbb{N}^d_0\\ x,\xi\in\mathbb{R}^d}} \frac{|\partial^{q_1}_x\partial^{q_2}_\xi\widetilde{a}(x,\xi)|}{c^{q_1,q_2}_A} \leq$$

$$\sup_{\substack{q_1,q_2\in\mathbb{N}^d_0\\ t\in\mathbb{R}^d_+}} \sum_{j=0}^{[\frac{q_1}{2}]}\sum_{i=0}^{[\frac{q_2}{2}]} \frac{q_1!q_2!|a^{(q_1+q_2-i-j)}(t)|}{c^{q_1,q_2}_A j!i!(q_1-2j)!(q_2-2i)!}\langle t\rangle^{\frac{q_1+q_2-2i-2j}{2}}2^{2(q_1+q_2)-3(i+j)} \leq$$

$$\Big(\frac{C_1}{A}\Big)^{q_1+q_2}\sup_{\substack{q_1,q_2\in\mathbb{N}^d_0\\ t\in\mathbb{R}^d_+}} \sum_{j=0}^{[\frac{q_1}{2}]}\sum_{i=0}^{[\frac{q_2}{2}]} \frac{(q_1+q_2-j-i)!(i+j)!|a^{(q_1+q_2-j-i)}(t)|\langle t\rangle^{\frac{q_1+q_2-j-i}{2}}\langle t\rangle^{-\frac{j+i}{2}}}{(q_1+q_2-j-i)!^{\alpha/2}(i+j)!^{\alpha/2}(q_1+q_2-j-i)!}$$

$$\leq \Big(\frac{C_2(h)}{A}\Big)^{q_1+q_2}a(r)\sup_{\substack{q_1,q_2,m_1,m_2\in\mathbb{N}^d_0\\ t\in\mathbb{R}^d_+}} \sum_{j=0}^{[\frac{q_1}{2}]}\sum_{i=0}^{[\frac{q_2}{2}]}\frac{c_2^{i+j}}{(i+j)^{(\alpha/2-1)}}$$

Now assumption (15) and the choice of $A>0$ imply the assertion. □

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