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Remark on the dilation of truncated Toeplitz operators

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Abstract. An operator $S^u_{\varphi,\psi}$ on L^2 is called the *dilation of a truncated Toeplitz operator* if for two symbols $\varphi, \psi \in L^\infty$ and an inner function u,

$$S^u_{\varphi,\psi} f = \varphi P_u f + \psi Q_u f$$

holds for $f \in L^2$ where P_u is the orthogonal projection of L^2 onto $\mathcal{K}^2_u$ and $Q_u = I - P_u$. In this paper, we study the squares of the dilation of truncated Toeplitz operators and the relation among its component operators. In particular, we provide characterizations for the square of the dilation of truncated Toeplitz operators $S^u_{\varphi,\psi}$ to be an isometry and a self-adjoint operator, respectively. As applications of the results, we find the cases where $(S^u_{\varphi,\psi})^2$ is self-adjoint (resp., isometric) but $S^u_{\varphi,\psi}$ is not self-adjoint (resp., isometric).

1. Introduction

Let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on a separable complex Hilbert space $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *self-adjoint* if $T^* = T$, *unitary* if $T^*T = TT^* = I$, and *isometric* if $T^*T = I$, respectively, where T^* denotes the adjoint of T.

Let $\mathcal{H}$ be a subspace of a Hilbert space $\mathcal{K}$ and let P be the orthogonal projection from $\mathcal{K}$ onto $\mathcal{H}$. Then R is called a (weak) *dilation* of T to $\mathcal{K}$ if $T = PRP$, i.e., $Tf = PRf$ for each $f \in \mathcal{H}$ (see [1] or [9]). In this case, the operator T is called the *compression* of R to $\mathcal{H}$. Since $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^\perp$, it follows that R is a dilation of T if and only if the matrix representation of R has the following form

$$\begin{pmatrix} T & X \\ Y & Z \end{pmatrix}.$$

The concept of a dilation is related to model theory which means the representation of some class as pieces of operators in a smaller, better-understood, class.

Using the concept of singular integral operators, the authors in [13] introduced the dilation of truncated Toeplitz operators on L^2. Moreover, the authors in ([10], [11], and [13]) have studied normality and

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hyponormality of the dilation of a truncated Toeplitz operator on L^2. In 2018, Gu and Kang [7] gave a complete characterization of self-adjoint, isometric, coisometric and normal truncated singular integral operators.

Let $T \in \mathcal{L}(\mathcal{H})$. Then we consider the following question. When does T have a square root?, i.e., does there exist an operator $A \in \mathcal{L}(\mathcal{H})$ such that $T = A^2$? Any operator does not have a square root, in general. In 1970, Halmos ([9]) proved that the unilateral shift given by $Sf = zf$ on the Hilbert Hardy space does not have a square root. In 1971, H. Radjavi and P. Rosenthal ([18]) studied roots of normal operators. In 1999, M. K. Kim and E. Ko ([14]) studied square roots of hyponormal operators. In 2003, E. Ko ([15]) studied scalar extension of square roots of semi-hyponormal operators. Recently, J. Mashreghi, M. Ptak, and W. Ross ([12]) have studied the square roots of the unilateral shift, Toeplitz operators, truncated Toeplitz operators, the Hilbert matrix, certain compressed shifts, and the Volterra integral operator, respectively.

In this paper, we concentrate on the following questions; *When operators have special properties, i.e., isomerty, unitary, etc, we study their square roots. In particular, when such square roots become the dilation of truncated Toeplitz operators, we consider the connections of such dilation and their symbol functions.*

2. Preliminaries

Let L^2 be the Lebesgue (Hilbert) space on the unit circle and let L^∞ be the Banach space of all functions in L^2 essentially bounded on $\partial\mathbb{D}$. The Hilbert Hardy space, denoted by H^2, consists of all analytic functions f on $\mathbb{D}$ having square-summable Taylor coefficients at 0. Let P denote the orthogonal projection of L^2 onto H^2. Then $Q = I - P$ is the orthogonal projection of L^2 onto $(H^2)^\perp := L^2 \ominus H^2 = L^2 \cap (H^2)^\perp$. For any $\varphi \in L^\infty$, let M_φ denote the multiplication operator on L^2 such that $M_\varphi f = \varphi f$ for $f \in L^2$. For any $\varphi \in L^\infty$, the *Toeplitz operator* $T_\varphi : H^2 \to H^2$ is defined by the formula

$$T_\varphi f = P(\varphi f), \ f \in H^2$$

where P is the orthogonal projection of L^2 onto H^2. It is known that T_φ is bounded if and only if $\varphi \in L^\infty$, and in this case, $\|T_\varphi\| = \|\varphi\|_\infty$.

In 2007, Sarason [16] initiated the study of truncated Toeplitz operators which are the compressions of Toeplitz operators. A function $u \in H^2$ is called *inner* if $|u| = 1$ a.e. The *model space* is given by $\mathcal{K}_u^2 := H^2 \ominus uH^2$ for a nonconstant inner function u. For any $\varphi \in L^\infty$ and an inner function u, the *truncated Toeplitz operator* $A_\varphi^u : \mathcal{K}_u^2 \to \mathcal{K}_u^2$ is defined by the formula

$$A_\varphi^u f = P_u(\varphi f) \text{ for } f \in \mathcal{K}_u^2$$

where P_u is the orthogonal projection of L^2 onto $\mathcal{K}_u^2$. Various aspects of this operator were studied in [3]-[8], [16], and [17]. For any $\varphi \in L^\infty$ and an inner function u, let $\widetilde{A_\varphi^u}$ denote the operator on $(\mathcal{K}_u^2)^\perp := L^2 \ominus \mathcal{K}_u^2$ such that

$$\widetilde{A_\varphi^u} f = Q_u(\varphi f) \text{ for } f \in (\mathcal{K}_u^2)^\perp$$

where Q_u is the orthogonal projection of L^2 onto $(\mathcal{K}_u^2)^\perp$. Recently, $\widetilde{A_\varphi^u}$ is called a *dual truncated Toeplitz operator* (see [2] and [5]). Let Γ_φ^u be the *truncated Hankel operator* of $\mathcal{K}_u^2$ to $(\mathcal{K}_u^2)^\perp$ such that

$$\Gamma_\varphi^u f = Q_u(\varphi f) \text{ for } f \in \mathcal{K}_u^2.$$

Let $\widetilde{\Gamma_\varphi^u}$ be the operator of $(\mathcal{K}_u^2)^\perp$ to $\mathcal{K}_u^2$ such that

$$\widetilde{\Gamma_\varphi^u} f = P_u(\varphi f) \text{ for } f \in (\mathcal{K}_u^2)^\perp.$$

It is obvious that $(A_\varphi^u)^* = A_{\overline{\varphi}}^u$ and $(\widetilde{A_\varphi^u})^* = \widetilde{A_{\overline{\varphi}}^u}$. So we can consider the following dilation of a truncated Toeplitz operator A_φ^u on $\mathcal{K}_u^2 \oplus (\mathcal{K}_u^2)^\perp = L^2$ (see [13]).

Definition 2.1. *An operator $S^u_{\varphi,\psi}$ on L^2 is called the dilation of a truncated Toeplitz operator if for two symbols $\varphi, \psi \in L^\infty$ and an inner function u,*

$$S^u_{\varphi,\psi} f = \varphi P_u f + \psi Q_u f$$

holds for $f \in L^2$ where P_u is the orthogonal projection of L^2 onto $\mathcal{K}_u^2$ and $Q_u = I - P_u$. In particular, if $\varphi = \psi$, then $S^u_{\varphi,\varphi} = M_\varphi$ is a multiplication operator on L^2. Moreover, if $\varphi = \psi$ is inner, then $|\varphi| = 1$ a.e. Thus $S^u_{\varphi,\varphi} = M_\varphi$ is a unitary operator on L^2.

Notice that an operator $S^u_{\varphi,\psi} \in \mathcal{L}(L^2)$ has the following block matrix representation:

$$S^u_{\varphi,\psi} = \begin{pmatrix} A^u_\varphi & \widetilde{\Gamma^u_\psi} \\ \Gamma^u_\varphi & \widetilde{A^u_\psi} \end{pmatrix} \tag{1}$$

on $\mathcal{K}_u^2 \oplus (\mathcal{K}_u^2)^\perp = L^2$ where $A^u_\varphi, \widetilde{\Gamma^u_\psi}, \Gamma^u_\varphi$, and $\widetilde{A^u_\psi}$ are defined as before.

We outline the paper as follows. In Section 2 we study the squares of the dilation of truncated Toeplitz operators. In particular, we give necessary and sufficient conditions for the dilation of truncated Toeplitz operators to be the square roots of an isometry and a self-adjoint operator, respectively. As applications for such operators, we find the cases where $(S^u_{\varphi,\psi})^2$ is self-adjoint (resp., isometric) but $S^u_{\varphi,\psi}$ is not self-adjoint (resp., isometric).

3. Main Results

In this section, we study the square of the dilation of a truncated Toeplitz operator $S^u_{\varphi,\psi} = \begin{pmatrix} A^u_\varphi & \widetilde{\Gamma^u_\psi} \\ \Gamma^u_\varphi & \widetilde{A^u_\psi} \end{pmatrix}$ on L^2 and the relation among the components of $S^u_{\varphi,\psi}$. In 2019 the authors in [10] provided a characterization of the self-adjointness of the dilation of truncated Toeplitz operators. We first study the self-adjointness of the square of $S^u_{\varphi,\psi}$ where $\varphi, \psi \in L^\infty$.

Lemma 3.1. *([10]) Let $\varphi \in L^\infty$ and let u be a nonconstant inner function. Then the following statements hold.*
(i) $\Gamma^u_\varphi = 0$ *if and only if* $\varphi \in \mathbb{C}$.
(ii) $\widetilde{\Gamma^u_\varphi} = 0$ *if and only if* $\varphi \in \mathbb{C}$.
(iii) $\widetilde{A^u_\varphi} = 0$ *if and only if* $\varphi = 0$.

Lemma 3.2. *([10, Theorem 4.2]) Let $\varphi, \psi \in L^\infty$ and let u be a nonconstant inner function. Then $S^u_{\varphi,\psi}$ is self-adjoint if and only if for some $a \in \mathbb{C}$, $\varphi - \overline{\varphi} = d \in \mathbb{R}$, $\psi = \overline{\psi}$, $\varphi - \psi \in \mathbb{C}$, and $d = au(0) + \overline{a}\overline{u(0)}$ hold.*

In the following theorem, when the square of the dilation of a truncated Toeplitz operator $S^u_{\varphi,\psi}$ is self-adjoint, we consider the relation among its component operators.

Theorem 3.3. *Let $\varphi, \psi \in L^\infty$ and let u be a nonconstant inner function. If $\nu = \varphi - \psi$, then $(S^u_{\varphi,\psi})^2$ is self-adjoint if and only if the following equations hold.*
(i) $\widetilde{\Gamma^u_\nu}\Gamma^u_\varphi - (\widetilde{\Gamma^u_\nu}\Gamma^u_\varphi)^* = A^u_{\varphi^2 - \overline{\varphi}^2}$,
(ii) $\Gamma^u_\nu\widetilde{\Gamma^u_\psi} - (\Gamma^u_\nu\widetilde{\Gamma^u_\psi})^* = -\widetilde{A^u_{\psi^2 - \overline{\psi}^2}}$, *and*
(iii) $A^u_\nu\widetilde{\Gamma^u_\psi} - (\Gamma^u_\nu A^u_\varphi)^* = -\widetilde{\Gamma^u_{\psi^2 - \overline{\varphi}\overline{\psi}}}$.

Proof. Since $S^u_{\varphi,\psi} = \begin{pmatrix} A^u_\varphi & \widetilde{\Gamma^u_\psi} \\ \Gamma^u_\varphi & \widetilde{A^u_\psi} \end{pmatrix}$ by (1) and so $(S^u_{\varphi,\psi})^* = \begin{pmatrix} A^u_{\overline{\varphi}} & \widetilde{\Gamma^u_{\overline{\varphi}}} \\ \Gamma^u_{\overline{\psi}} & \widetilde{A^u_{\overline{\psi}}} \end{pmatrix}$, we get that

$$(S^u_{\varphi,\psi})^2 = \begin{pmatrix} (A^u_\varphi)^2 + \widetilde{\Gamma^u_\psi}\Gamma^u_\varphi & A^u_\varphi\widetilde{\Gamma^u_\psi} + \widetilde{\Gamma^u_\psi}\widetilde{A^u_\psi} \\ \Gamma^u_\varphi A^u_\varphi + \widetilde{A^u_\psi}\Gamma^u_\varphi & \Gamma^u_\varphi\widetilde{\Gamma^u_\psi} + (\widetilde{A^u_\psi})^2 \end{pmatrix} \tag{2}$$

and

$$((S^u_{\varphi,\psi})^2)^* = \begin{pmatrix} (A^u_{\overline{\varphi}})^2 + \widetilde{\Gamma^u_{\overline{\varphi}}}\Gamma^u_{\overline{\psi}} & A^u_{\overline{\varphi}}\widetilde{\Gamma^u_{\overline{\varphi}}} + \widetilde{\Gamma^u_{\overline{\varphi}}}\widetilde{A^u_{\overline{\psi}}} \\ \Gamma^u_{\overline{\psi}} A^u_{\overline{\varphi}} + \widetilde{A^u_{\overline{\psi}}}\Gamma^u_{\overline{\psi}} & \Gamma^u_{\overline{\psi}}\widetilde{\Gamma^u_{\overline{\varphi}}} + (\widetilde{A^u_{\overline{\psi}}})^2 \end{pmatrix}. \tag{3}$$

If $(S^u_{\varphi,\psi})^2$ is self-adjoint, then we get from (2) and (3) that for any $f \in \mathcal{K}^2_u$,

$$\begin{aligned} 0 &= (A^u_\varphi)^2 f + \widetilde{\Gamma^u_\psi}\Gamma^u_\varphi f - (A^u_{\overline{\varphi}})^2 f - \widetilde{\Gamma^u_{\overline{\varphi}}}\Gamma^u_{\overline{\psi}} f \\ &= P_u[\varphi P_u(\varphi f) + \psi Q_u(\varphi f) - \overline{\varphi} P_u(\overline{\varphi} f) - \overline{\varphi} Q_u(\overline{\psi} f)] \\ &= P_u[\varphi(I - Q_u)(\varphi f) + \psi Q_u(\varphi f) - \overline{\varphi}(I - Q_u)(\overline{\varphi} f) - \overline{\varphi} Q_u(\overline{\psi} f)] \\ &= P_u[\varphi^2 f - (\varphi - \psi) Q_u(\varphi f) - \overline{\varphi}^2 f + \overline{\varphi} Q_u(\overline{\varphi} f) - \overline{\varphi} Q_u(\overline{\psi} f)] \\ &= P_u[(\varphi^2 - \overline{\varphi}^2) f - (\varphi - \psi) Q_u(\varphi f) + \overline{\varphi} Q_u((\overline{\varphi} - \overline{\psi}) f)] \\ &= A^u_{\varphi^2 - \overline{\varphi}^2} f - \widetilde{\Gamma^u_{\varphi - \psi}}\Gamma^u_\varphi f + \widetilde{\Gamma^u_{\overline{\varphi}}}\Gamma^u_{\overline{\varphi} - \overline{\psi}} f. \end{aligned}$$

Set $\nu = \varphi - \psi$. Then

$$\begin{aligned} 0 &= A^u_{\varphi^2 - \overline{\varphi}^2} - \widetilde{\Gamma^u_{\varphi - \psi}}\Gamma^u_\varphi + \widetilde{\Gamma^u_{\overline{\varphi}}}\Gamma^u_{\overline{\varphi} - \overline{\psi}} \\ &= A^u_{\varphi^2 - \overline{\varphi}^2} - \widetilde{\Gamma^u_\nu}\Gamma^u_\varphi + \widetilde{\Gamma^u_{\overline{\varphi}}}\Gamma^u_{\overline{\nu}} \\ &= A^u_{\varphi^2 - \overline{\varphi}^2} - \widetilde{\Gamma^u_\nu}\Gamma^u_\varphi + (\widetilde{\Gamma^u_\nu}\Gamma^u_\varphi)^*. \end{aligned}$$

Hence

$$\widetilde{\Gamma^u_\nu}\Gamma^u_\varphi - (\widetilde{\Gamma^u_\nu}\Gamma^u_\varphi)^* = A^u_{\varphi^2 - \overline{\varphi}^2}.$$

Similarly, we get that

$$\Gamma^u_\nu\widetilde{\Gamma^u_\psi} - (\Gamma^u_\nu\widetilde{\Gamma^u_\psi})^* = -\widetilde{A^u_{\psi^2 - \overline{\psi}^2}}.$$

Finally, it suffices to show that (iii) holds. For any $g \in (\mathcal{K}^2_u)^\perp$, we get from (2) and (3) that

$$\begin{aligned} 0 &= A^u_\varphi\widetilde{\Gamma^u_\psi} g + \widetilde{\Gamma^u_\psi}\widetilde{A^u_\psi} g - A^u_{\overline{\varphi}}\widetilde{\Gamma^u_{\overline{\varphi}}} g - \widetilde{\Gamma^u_{\overline{\varphi}}}\widetilde{A^u_{\overline{\psi}}} g \\ &= P_u[\varphi P_u(\psi g) + \psi Q_u(\psi g) - \overline{\varphi} P_u(\overline{\varphi} g) - \overline{\varphi} Q_u(\overline{\psi} g)] \\ &= P_u[(\nu + \psi) P_u(\psi g) + \psi Q_u(\psi g) - \overline{\varphi} P_u(\overline{\varphi} g) - \overline{\varphi} Q_u(\overline{\psi} g)] \\ &= P_u[\nu P_u(\psi g) + \psi\{P_u(\psi g) + Q_u(\psi g)\} - \overline{\varphi} P_u(\overline{\varphi} g) - \overline{\varphi} Q_u(\overline{\psi} g)] \\ &= P_u[\nu P_u(\psi g) + \psi^2 g - \overline{\varphi} P_u((\overline{\nu} + \overline{\psi}) g) - \overline{\varphi} Q_u(\overline{\psi} g)] \\ &= P_u[\nu P_u(\psi g) + \psi^2 g - \overline{\varphi} P_u(\overline{\nu} g) - \overline{\varphi}\overline{\psi} g] \\ &= A^u_\nu\widetilde{\Gamma^u_\psi} g + \widetilde{\Gamma^u_{\psi^2 - \overline{\varphi}\overline{\psi}}} g - (\Gamma^u_\nu A^u_\varphi)^* g. \end{aligned}$$

Hence we have

$$A^u_\nu\widetilde{\Gamma^u_\psi} - (\Gamma^u_\nu A^u_\varphi)^* = -\widetilde{\Gamma^u_{\psi^2 - \overline{\varphi}\overline{\psi}}}.$$

The converse statement holds by a similar method. So we complete the proof. □

As some applications of Theorem 3.3, we obtain the following corollaries.

Corollary 3.4. *Let $\varphi, \psi \in L^\infty$, let $\nu = \varphi - \psi$, and let u be a nonconstant inner function. Assume that $(S^u_{\varphi,\psi})^2$ is self-adjoint. Then the following statements hold.*
(i) $\widetilde{\Gamma^u_\nu}\Gamma^u_\varphi$ and $\Gamma^u_\nu\widetilde{\Gamma^u_\psi}$ are self-adjoint if and only if $\psi^2 = \overline{\psi}^2$ and $\varphi^2 - \overline{\varphi}^2 \in uH^2 + \overline{uH^2}$.
(ii) If $\nu = \varphi - \psi \in \mathbb{C}$, then $\nu\widetilde{\Gamma^u_\psi} = (\nu\Gamma^u_\psi)^*$ holds.

Proof. (i) If $(S^u_{\varphi,\psi})^2$ is self-adjoint, then (i), (ii), and (iii) in Theorem 3.3 hold. If $\varphi^2 - \overline{\varphi}^2 \in uH^2 + \overline{uH^2}$ and $\psi^2 = \overline{\psi}^2$, it follows from [10] and [16] that $A^u_{\varphi^2-\overline{\varphi}^2} = 0$ and $\widetilde{A^u_{\psi^2-\overline{\psi}^2}} = 0$. Hence $\widetilde{\Gamma^u_\nu}\Gamma^u_\varphi$ and $\Gamma^u_\nu\widetilde{\Gamma^u_\psi}$ are self-adjoint from Theorem 3.3.

Conversely, if $\widetilde{\Gamma^u_\nu}\Gamma^u_\varphi$ and $\Gamma^u_\nu\widetilde{\Gamma^u_\psi}$ are self-adjoint, then $A^u_{\varphi^2-\overline{\varphi}^2} = 0$ and $\widetilde{A^u_{\psi^2-\overline{\psi}^2}} = 0$ from Theorem 3.3. Hence $\varphi^2 - \overline{\varphi}^2 \in uH^2 + \overline{uH^2}$ and $\psi^2 = \overline{\psi}^2$ from [10] and [16].

(ii) If $\nu = \varphi - \psi \in \mathbb{C}$, then $\Gamma^u_\nu = 0 = \widetilde{\Gamma^u_\nu}$ by Lemma 3.1. Since $(S^u_{\varphi,\psi})^2$ is self-adjoint, it follows from Theorem 3.3 that $A^u_{\varphi^2-\overline{\varphi}^2} = 0$, $\widetilde{A^u_{\psi^2-\overline{\psi}^2}} = 0$, and $A^u_\nu\widetilde{\Gamma^u_\psi} = -\widetilde{\Gamma^u_{\psi^2-\overline{\varphi}\overline{\psi}}}$. Then $\varphi^2 - \overline{\varphi}^2 \in uH^2 + \overline{uH^2}$ by [16] and $\psi^2 = \overline{\psi}^2$ by [10]. Hence $0 = A^u_\nu\widetilde{\Gamma^u_\psi} + \widetilde{\Gamma^u_{\overline{\psi}(\overline{\psi}-\overline{\varphi})}} = A^u_\nu\widetilde{\Gamma^u_\psi} + \widetilde{\Gamma^u_{\overline{\psi}(-\overline{\nu})}}$. For any $g \in (\mathcal{K}^2_u)^\perp$, we have

$$\begin{aligned} 0 &= (A^u_\nu\widetilde{\Gamma^u_\psi} + \widetilde{\Gamma^u_{\overline{\psi}(-\overline{\nu})}})g \\ &= P_u[\nu P_u(\psi g) - \overline{\psi\nu g}] \\ &= \nu P_u(\psi g) - \overline{\nu} P_u(\overline{\psi} g) \\ &= (\nu\widetilde{\Gamma^u_\psi} - \overline{\nu}\widetilde{\Gamma^u_{\overline{\psi}}})g. \end{aligned}$$

Hence $\nu\widetilde{\Gamma^u_\psi} - \overline{\nu}\widetilde{\Gamma^u_{\overline{\psi}}} = 0$. So, $\nu\widetilde{\Gamma^u_\psi} = (\nu\Gamma^u_\psi)^*$ holds. □

Corollary 3.5. *Let $\varphi, \psi \in L^\infty$. If $\varphi^2 - \overline{\varphi}^2 \in uH^2 + \overline{uH^2}$, $\psi = -\overline{\psi}$, and $\varphi - \psi = id \in \mathbb{C}$ for $d \in \mathbb{R}$, then $(S^u_{\varphi,\psi})^2$ is self-adjoint, but $S^u_{\varphi,\psi}$ is not self-adjoint.*

Proof. If $\psi = -\overline{\psi}$, then $\psi^2 = \overline{\psi}^2$ and so $\widetilde{A^u_{\psi^2-\overline{\psi}^2}} = 0$. Since $\varphi^2 - \overline{\varphi}^2 \in uH^2 + \overline{uH^2}$, $A^u_{\varphi^2-\overline{\varphi}^2} = 0$ by [16]. If $\varphi - \psi = id \in \mathbb{C}$, then $\Gamma^u_{\varphi-\psi} = 0$ and $\widetilde{\Gamma^u_{\varphi-\psi}} = 0$ by Lemma 3.1. Thus (i) and (ii) of Theorem 3.3 hold. On the other hand, since $\psi = -\overline{\psi}$ and $\Gamma^u_{\varphi-\psi} = 0$, we get that for $\nu = \varphi - \psi$,

$$\begin{aligned} A^u_\nu\widetilde{\Gamma^u_\psi} - (\Gamma^u_\nu A^u_\varphi)^* + \widetilde{\Gamma^u_{\psi^2-\overline{\varphi}\overline{\psi}}} &= A^u_{id}\widetilde{\Gamma^u_\psi} + \widetilde{\Gamma^u_{\psi^2-\overline{\varphi}\overline{\psi}}} \\ &= A^u_{id}\widetilde{\Gamma^u_\psi} + \widetilde{\Gamma^u_{(\overline{\psi}-\overline{\varphi})\overline{\psi}}} \\ &= A^u_{id}\widetilde{\Gamma^u_\psi} + \widetilde{\Gamma^u_{(id)(-\psi)}}. \end{aligned}$$

For $g \in (\mathcal{K}^2_u)^\perp$, we have

$$(A^u_\nu\widetilde{\Gamma^u_\psi} - (\Gamma^u_\nu A^u_\varphi)^* + \widetilde{\Gamma^u_{\psi^2-\overline{\varphi}\overline{\psi}}})g = (A^u_{id}\widetilde{\Gamma^u_\psi} + \widetilde{\Gamma^u_{id(-\psi)}})g = (id - id)P_u(\psi g) = 0.$$

Hence $(S^u_{\varphi,\psi})^2$ is self-adjoint from Theorem 3.3. But, since $\psi \neq \overline{\psi}$, $S^u_{\varphi,\psi}$ is not self-adjoint from Lemma 3.2. □

As some applications, for given φ, we want to find conditions of ψ for $(S^u_{\varphi,\psi})^2$ to be self-adjoint.

Corollary 3.6. *Let $u(z) = z$ and $\varphi(z) = a_1 z + a_{-1}\overline{z}$ for nonzero $a_1, a_{-1} \in \mathbb{C}$. If $(S^u_{\varphi,\psi})^2$ is self-adjoint where $\psi(z) = \sum_{j=-\infty}^{\infty} b_j z^j$, then $a_1 b_{-1} + a_{-1} b_1$ is real.*

Proof. Assume that $(S^u_{\varphi,\psi})^2$ is self-adjoint. If $u(z) = z$, then $\mathcal{K}_u^2 = \vee\{1\}$. Form Theorem 3.3, we get that

$$\widetilde{\Gamma}^u_\nu \Gamma^u_\varphi 1 - (\widetilde{\Gamma}^u_\nu \Gamma^u_\varphi)^* 1 = A^u_{\varphi^2 - \overline{\varphi}^2} 1. \tag{4}$$

Now, for $\nu = \varphi - \psi$,

$$\begin{aligned}\widetilde{\Gamma}^u_\nu \Gamma^u_\varphi 1 &= \widetilde{\Gamma}^u_\nu (Q_u(a_1 z + a_{-1}\overline{z})) \\ &= \widetilde{\Gamma}^u_\nu (a_1 z + a_{-1}\overline{z}) \\ &= P_u[(a_1 z + a_{-1}\overline{z} - \psi)(a_1 z + a_{-1}\overline{z})] \\ &= 2a_1 a_{-1} - P_u[(a_1 z + a_{-1}\overline{z})\psi],\end{aligned}$$

$$\begin{aligned}(\widetilde{\Gamma}^u_\nu \Gamma^u_\varphi)^* 1 = \widetilde{\Gamma^u_{\overline{\varphi}}} \Gamma^u_{\overline{\nu}} 1 &= P_u[(\overline{a_1 z} + \overline{a_{-1}} z)(\overline{a_1 z} + \overline{a_{-1}} z - Q_u(\overline{\psi}))] \\ &= 2\overline{a_1 a_{-1}} - P_u[(\overline{a_1 z} + \overline{a_{-1}} z) Q_u(\overline{\psi})],\end{aligned}$$

and $A^u_{\varphi^2 - \overline{\varphi}^2} 1 = 2(a_1 a_{-1} - \overline{a_1 a_{-1}})$ holds. Thus (4) implies

$$P_u[(\overline{a_1 z} + \overline{a_{-1}} z) Q_u(\overline{\psi}) - (a_1 z + a_{-1}\overline{z})\psi] = 0. \tag{5}$$

Set $\psi(z) = \sum_{j=-\infty}^{\infty} b_j z^j$. Then $Q_u(\overline{\psi}) = \sum_{j=1}^{\infty} \overline{b_j} \overline{z}^j + \sum_{j=-\infty}^{-1} \overline{b_j} \overline{z}^j$. Therefore, (5) becomes

$$\begin{aligned} 0 &= P_u[(\overline{a_1 z} + \overline{a_{-1}} z)(\sum_{j=1}^{\infty} \overline{b_j} \overline{z}^j + \sum_{j=-\infty}^{-1} \overline{b_j} \overline{z}^j) - (a_1 z + a_{-1}\overline{z}) \sum_{j=-\infty}^{\infty} b_j z^j] \\ &= \overline{a_{-1} b_1} + \overline{a_1 b_{-1}} - a_1 b_{-1} - a_{-1} b_1. \end{aligned} \tag{6}$$

Hence $a_1 b_{-1} + a_{-1} b_1$ is real. □

Example 3.7. *Let $u(z) = z^n$, $\varphi(z) = iz + \frac{1}{2}\overline{z}$, and $\psi(z) = 2iz + \overline{z}$. Since $a_{-1} b_1 + a_1 b_{-1} = 2i$ is not real, $(S^u_{\varphi,\psi})^2$ is not self-adjoint from Corollary 3.6.*

Corollary 3.8. *Let $u(z) = z^2$ and $\varphi(z) = a_1 z + a_{-1}\overline{z}$ for nonzero $a_1, a_{-1} \in \mathbb{C}$.. If $(S^u_{\varphi,\psi})^2$ is self-adjoint where $\psi(z) = \sum_{j=-\infty}^{\infty} b_j z^j$, then $a_1(a_{-1} + b_{-1})$ is real and $\overline{a_{-1} b_2} = a_1 b_{-2}$.*

Proof. If $u(z) = z^2$, then $\mathcal{K}_u^2 = \vee\{1, z\}$. Thus, for $\nu = \varphi - \psi$,

$$\begin{aligned}\widetilde{\Gamma}^u_\nu \Gamma^u_\varphi z &= \widetilde{\Gamma}^u_\nu (Q_u(a_1 z^2 + a_{-1})) = \widetilde{\Gamma}^u_\nu (a_1 z^2) \\ &= P_u[a_1 z^3 + a_{-1} a z - a_1 z^2 \psi] \\ &= a_{-1} a_1 z - P_u[a_1 z^2 \psi],\end{aligned} \tag{7}$$

$$\begin{aligned}(\widetilde{\Gamma}^u_\nu \Gamma^u_\varphi)^* z &= \widetilde{\Gamma^u_{\overline{\varphi}}} \Gamma^u_{\overline{\nu}} z \\ &= P_u[(\overline{a_1 z} + \overline{a_{-1}} z)(\overline{a_{-1}} z^2 - Q_u(\overline{\psi} z))] \\ &= P_u[\overline{a_1 a_{-1}} z + \overline{a_{-1}} z^3 - (\overline{a_1 z} + \overline{a_{-1}} z) Q_u(\overline{\psi} z)] \\ &= \overline{a_1 a_{-1}} z - P_u[(\overline{a_1 z} + \overline{a_{-1}} z) Q_u(\overline{\psi} z)],\end{aligned} \tag{8}$$

and

$$A^u_{\varphi^2 - \overline{\varphi}^2} z = P_u[(\varphi^2 - \overline{\varphi}^2) z]$$

$$\begin{aligned} &= P_u[(a_1^2-\overline{a_{-1}}^2)z^3+(a_{-1}^2-\overline{a_1}^2)\overline{z}+2(a_1a_{-1}-\overline{a_1a_{-1}})z] \\ &= 2(a_1a_{-1}-\overline{a_1a_{-1}})z \end{aligned} \tag{9}$$

holds. From (7), (8), (9), we obtain that

$$a_{-1}a_1z-P_u[a_1z^2\psi]-\overline{a_1a_{-1}}z+P_u[(\overline{a_1}z+\overline{a_{-1}}z)Q_u(\overline{\psi}z)]=2(a_1a_{-1}-\overline{a_1a_{-1}})z. \tag{10}$$

Thus

$$P_u[(\overline{a_1z}+\overline{a_{-1}}z)Q_u(\overline{\psi}z)-a_1z^2\psi]=(a_1a_{-1}-\overline{a_1a_{-1}})z.$$

Set $\psi(z)=\sum_{j=-\infty}^{\infty}b_jz^j$. Then (10) implies that

$$\begin{aligned} 0 &= P_u[(\overline{a_1z}+\overline{a_{-1}}z)Q_u(\sum_{j=-\infty}^{\infty}\overline{b_j}z^{1-j})-a_1\sum_{j=-\infty}^{\infty}b_jz^{j+2}]-(a_1a_{-1}-\overline{a_1a_{-1}})z \\ &= P_u[(\overline{a_1z}+\overline{a_{-1}}z)(\sum_{j=-\infty}^{-1}\overline{b_j}z^{1-j}+\sum_{j=2}^{\infty}\overline{b_j}z^{1-j})-a_1\sum_{j=-\infty}^{\infty}b_jz^{j+2}]-(a_1a_{-1}-\overline{a_1a_{-1}})z \\ &= P_u[\overline{a_1}\sum_{j=-\infty}^{-1}\overline{b_j}z^{-j}+\overline{a_1}\sum_{j=2}^{\infty}\overline{b_j}z^{-j}+\overline{a_{-1}}\sum_{j=-\infty}^{-1}\overline{b_j}z^{2-j}+\overline{a_{-1}}\sum_{j=2}^{\infty}\overline{b_j}z^{2-j}-a_1\sum_{j=-\infty}^{\infty}b_jz^{j+2}] \\ &\quad -(a_1a_{-1}-\overline{a_1a_{-1}})z \\ &= P_u[\overline{a_1}(\overline{b_{-1}}z+\overline{b_{-2}}z^2+\cdots)+\overline{a_1}(\overline{b_2}\overline{z}^2+\overline{b_3}\overline{z}^3+\cdots) \\ &\quad +\overline{a_{-1}}(\overline{b_{-1}}z^3+\overline{b_{-2}}z^4+\overline{b_{-3}}z^5+\cdots)+\overline{a_{-1}}(\overline{b_2}+\overline{b_3z}+\overline{b_4z^2}+\cdots)-a_1(b_{-2}+b_{-1}z)] \\ &\quad -(a_1a_{-1}-\overline{a_1a_{-1}})z \\ &= \overline{a_1b_{-1}}z+\overline{a_{-1}b_2}-a_1(b_{-2}+b_{-1}z)-(a_1a_{-1}-\overline{a_1a_{-1}})z \\ &= (\overline{a_1b_{-1}}-a_1b_{-1}-a_1a_{-1}+\overline{a_1a_{-1}})z+\overline{a_{-1}b_2}-a_1b_{-2}. \end{aligned}$$

Hence $a_1(a_{-1}+b_{-1})=\overline{a_1(a_{-1}+b_{-1})}$ and $\overline{a_{-1}b_2}=a_1b_{-2}$ hold. □

Proposition 3.9. *Let $\varphi,\psi\in L^\infty$ and let u be a nonconstant inner function. Assume that $(S^u_{\varphi,\psi})^2$ is self-adjoint. If $\varphi\in\mathbb{C}$, then*

$$(\widetilde{A^u_{\overline{\psi}}})^2=\psi\widetilde{A^u_{\overline{\psi}}}+\varphi\widetilde{\Gamma^u_{\overline{\psi}}}$$

where $\varphi\in\mathbb{R}$ or $\varphi\in i\mathbb{R}$.

Proof. If $\varphi\in\mathbb{C}$, then the equations of Theorem 3.3 become

$$\begin{cases} A^u_{\varphi^2-\overline{\varphi}^2}=0, \\ -\Gamma^u_\psi\widetilde{\Gamma^u_\psi}+\Gamma^u_{\overline{\psi}}\widetilde{\Gamma^u_{\overline{\psi}}}=-\widetilde{A^u_{\psi^2-\overline{\psi}^2}}, \\ A^u_{\varphi-\psi}\widetilde{\Gamma^u_\psi}+A^u_{\overline{\varphi}}\widetilde{\Gamma^u_{\overline{\psi}}}=-\widetilde{\Gamma^u_{\psi^2-\overline{\varphi}\overline{\psi}}}. \end{cases} \tag{11}$$

Since $\varphi\in\mathbb{C}$ and $\varphi^2-\overline{\varphi}^2\in uH^2+\overline{uH^2}$ by [16], it follows that $\varphi=r$ or $\varphi=ir$ where $r\in\mathbb{R}$. (Indeed, if $\varphi=a+bi$, then $\varphi^2-\overline{\varphi}^2=4abi\in uH^2+\overline{uH^2}$, which means that $a=0$ or $b=0$. Hence $\varphi=r$ or $\varphi=ir$ where $r\in\mathbb{R}$.) If $\varphi=r$, then the second and third equations of (11) give that for all $g\in(\mathcal{K}_u^2)^\perp$,

$$Q_u[\psi P_u(\psi g)]-Q_u(\overline{\psi}P_u(\overline{\psi}g))-Q_u(\psi^2g)+Q_u(\overline{\psi}^2g)=0 \tag{12}$$

and

$$rP_u(\psi g)-P_u[\psi P_u(\psi g)]+rP_u(\overline{\psi}g)+P_u(\psi^2g)-rP_u(\overline{\psi}g)=0. \tag{13}$$

Subtracting (13) from (12), we have

$$\begin{aligned} 0 &= \psi P_u(\psi g) - Q_u(\overline{\psi}P_u(\overline{\psi}g)) - \psi^2 g + Q_u(\overline{\psi}^2 g) - rP_u(\psi g) \\ &= \psi(I-Q_u)(\psi g) - Q_u(\overline{\psi}(I-Q_u)(\overline{\psi}g)) - \psi^2 g + Q_u(\overline{\psi}^2 g) - rP_u(\psi g) \\ &= -\psi Q_u(\psi g) + Q_u(\overline{\psi}Q_u(\overline{\psi}g)) - rP_u(\psi g) \end{aligned}$$

for all $g \in (\mathcal{K}_u^2)^\perp$. Hence $(\widetilde{A^u_{\overline{\psi}}})^2 = \psi\widetilde{A^u_{\overline{\psi}}} + \varphi\widetilde{\Gamma^u_{\psi}}$. Similarly, if $\varphi = ir$, then the second and third equations of (11) give that for all $g \in (\mathcal{K}_u^2)^\perp$,

$$Q_u[\psi P_u(\psi g)] - Q_u(\overline{\psi}P_u(\overline{\psi}g)) - Q_u(\psi^2 g) + Q_u(\overline{\psi}^2 g) = 0 \tag{14}$$

and

$$irP_u(\psi g) - P_u[\psi P_u(\psi g)] - irP_u(\overline{\psi}g) + P_u(\psi^2 g) + irP_u(\overline{\psi}g) = 0. \tag{15}$$

Subtracting (15) from (14), we have

$$\begin{aligned} 0 &= \psi P_u(\psi g) - Q_u(\overline{\psi}P_u(\overline{\psi}g)) - \psi^2 g + Q_u(\overline{\psi}^2 g) - irP_u(\psi g) \\ &= \psi(I-Q_u)(\psi g) - Q_u(\overline{\psi}(I-Q_u)(\overline{\psi}g)) - \psi^2 g + Q_u(\overline{\psi}^2 g) - irP_u(\psi g) \\ &= -\psi Q_u(\psi g) + Q_u(\overline{\psi}Q_u(\overline{\psi}g)) - irP_u(\psi g) \end{aligned}$$

for all $g \in (\mathcal{K}_u^2)^\perp$. Hence $(\widetilde{A^u_{\overline{\psi}}})^2 = \psi\widetilde{A^u_{\overline{\psi}}} + \varphi\widetilde{\Gamma^u_{\psi}}$ where $\varphi \in \mathbb{R}$ or $\varphi \in i\mathbb{R}$. □

Corollary 3.10. *Let $\varphi, \psi \in L^\infty$ and let u be a nonconstant inner function. Assume that $(S^u_{\varphi,\psi})^2$ is self-adjoint. If $\Gamma^u_\varphi = 0$ and $\widetilde{\Gamma^u_\psi} = 0$, then $\varphi \in \mathbb{R}$ or $\varphi \in i\mathbb{R}$ and $\psi \in \mathbb{R}$.*

Proof. Since $\widetilde{\Gamma^u_\psi} = 0$ gives $\psi \in \mathbb{C}$ by [10], it follows from Proposition 3.9 that $\varphi \in \mathbb{R}$ or $\varphi \in i\mathbb{R}$ and $(\widetilde{A^u_{\overline{\psi}}})^2 = \psi\widetilde{A^u_{\overline{\psi}}}$. Since $\psi \in \mathbb{C}$, $(\widetilde{A^u_{\overline{\psi}}})^2 = \psi\widetilde{A^u_{\overline{\psi}}}$ gives that $\overline{\psi}^2 = |\psi|^2$. Thus $\psi = \overline{\psi}$. □

Proposition 3.11. *Let $\varphi, \psi \in L^\infty$ and let u be a nonconstant inner function. Assume that $(S^u_{\varphi,\psi})^2$ is self-adjoint. Then the following statements hold.*
(i) *If $\psi \in \mathbb{C}$, then $(A^u_\varphi)^2$ is self-adjoint and $(A^u_{\overline{\varphi}} + \overline{\psi})\widetilde{\Gamma^u_{\overline{\varphi}}} = 0$ for $\psi \in \mathbb{R}$ or $\psi \in i\mathbb{R}$.*
(ii) *If $\widetilde{A^u_\psi} = 0$, then $(A^u_\varphi)^2$ is self-adjoint and $\varphi A^u_\varphi \mathcal{K}_u^2 \subset \mathcal{K}_u^2$.*

Proof. (i) If $\psi \in \mathbb{C}$, then Theorem 3.3 forces that

$$\begin{cases} \widetilde{\Gamma^u_\varphi}\Gamma^u_\varphi - (\widetilde{\Gamma^u_\varphi}\Gamma^u_\varphi)^* = A^u_{\varphi^2-\overline{\varphi}^2}, \\ \widetilde{A^u_{\psi^2-\overline{\psi}^2}} = 0, \\ (\Gamma^u_\varphi A^u_\varphi)^* + \overline{\psi}\widetilde{\Gamma^u_{\overline{\varphi}}} = 0. \end{cases}$$

Since $\psi \in \mathbb{C}$ and $\widetilde{A^u_{\psi^2-\overline{\psi}^2}} = 0$, it follows that $\psi^2 = \overline{\psi}^2$ by Lemma 3.1. Thus $\psi \in \mathbb{R}$ or $\psi \in i\mathbb{R}$. Hence $\widetilde{\Gamma^u_\varphi}\Gamma^u_\varphi - (\widetilde{\Gamma^u_\varphi}\Gamma^u_\varphi)^* = A^u_{\varphi^2-\overline{\varphi}^2}$ and $(A^u_{\overline{\varphi}} + \overline{\psi})\widetilde{\Gamma^u_{\overline{\varphi}}} = 0$ for $\psi \in \mathbb{R}$ or $\psi \in i\mathbb{R}$. On the other hand, since $\widetilde{\Gamma^u_\varphi}\Gamma^u_\varphi - (\widetilde{\Gamma^u_\varphi}\Gamma^u_\varphi)^* = A^u_{\varphi^2-\overline{\varphi}^2}$, it follows that for all $f \in \mathcal{K}_u^2$,

$$0 = P_u[\varphi Q_u(\varphi f) - \varphi(\varphi f) - \overline{\varphi}Q_u(\overline{\varphi}f) + \overline{\varphi}(\overline{\varphi}f)]$$

$$\begin{aligned} &= P_u[-\varphi P_u(\varphi f) + \overline{\varphi} P_u(\overline{\varphi} f)] \\ &= -(A^u_\varphi)^2 f + (A^u_{\overline{\varphi}})^2 f. \end{aligned} \tag{16}$$

Thus $(A^u_\varphi)^2 = (A^u_{\overline{\varphi}})^2$. Hence $(A^u_\varphi)^2$ is self-adjoint and $(A^u_{\overline{\varphi}} + \overline{\psi})\widetilde{\Gamma^u_{\overline{\varphi}}} = 0$ for $\psi \in \mathbb{R}$ or $\psi \in i\mathbb{R}$.

(ii) If $\widetilde{A^u_\psi} = 0$, then $\psi = 0$ by Lemma 3.1 and so Theorem 3.3 gives that

$$\begin{cases} \widetilde{\Gamma^u_\varphi}\Gamma^u_\varphi - (\widetilde{\Gamma^u_\varphi}\Gamma^u_\varphi)^* = A^u_{\varphi^2 - \overline{\varphi}^2}, \\ (\Gamma^u_\varphi A^u_\varphi)^* = 0. \end{cases}$$

Since (16) means that $(A^u_\varphi)^2 = (A^u_{\overline{\varphi}})^2$, we obtain that $(A^u_\varphi)^2$ is self-adjoint and $\Gamma^u_\varphi A^u_\varphi = 0$. Hence $Q_u[\varphi A^u_\varphi f] = 0$ for any $f \in \mathcal{K}^2_u$. Thus $\varphi A^u_\varphi f \in \mathcal{K}^2_u$ for any $f \in \mathcal{K}^2_u$. Therefore, $\varphi A^u_\varphi \mathcal{K}^2_u \subset \mathcal{K}^2_u$. □

We next consider an isometry of $(S^u_{\varphi,\psi})^2$ where $\varphi, \psi \in L^\infty$. In 2018, Gu and Kang in [7] showed that $S^u_{\varphi,\psi}$ is an isometry if and only if it is unitary. So we omit its proof.

Lemma 3.12. *([7, Theorem 4.4]) Let $\varphi, \psi \in L^\infty$ and let u be a nonconstant inner function. Then the following statements are equivalent.*
(i) *$S^u_{\varphi,\psi}$ is an isometry.*
(ii) *$|\varphi| = 1$, $|\psi| = 1$, and $\overline{\varphi}\psi = c \in \mathbb{C}$, where $|c| = 1$.*
(iii) *$S^u_{\varphi,\psi}$ is unitary.*

From Lemma 3.12, it is easy to show that if $S^u_{\varphi,\psi}$ is an isometry, then $(S^u_{\varphi,\psi})^2$ is also an isometry. However, the converse does not hold in general. So, we next consider $S^u_{\varphi,\psi}$ and the relation among its component operators when $(S^u_{\varphi,\psi})^2$ is an isometry.

Theorem 3.13. *Let $\varphi, \psi \in L^\infty$ and let u be a nonconstant inner function. Then $(S^u_{\varphi,\psi})^2$ is an isometry if and only if*

$$\begin{pmatrix} A^u_{\overline{\varphi}} A^u_{|\varphi|^2} + \widetilde{\Gamma^u_{\overline{\varphi}}}\Gamma^u_{\overline{\psi}\varphi} & A^u_{\overline{\varphi}}\widetilde{\Gamma^u_{\overline{\varphi}\psi}} + \widetilde{\Gamma^u_{\overline{\varphi}}}\widetilde{A^u_{|\psi|^2}} \\ \Gamma^u_{\overline{\psi}} A^u_{|\varphi|^2} + \widetilde{A^u_{\overline{\psi}}}\Gamma^u_{\overline{\psi}\varphi} & \Gamma^u_{\overline{\psi}}\widetilde{\Gamma^u_{\overline{\varphi}\psi}} + \widetilde{A^u_{\overline{\psi}}}\widetilde{A^u_{|\psi|^2}} \end{pmatrix} S^u_{\varphi,\psi} = I.$$

Proof. From (2) and (3), we obtain that

$$((S^u_{\varphi,\psi})^2)^*(S^u_{\varphi,\psi})^2 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{17}$$

where α, β, γ, and δ are to be determined later. Now,

$$\begin{aligned} \alpha &= [(A^u_{\overline{\varphi}})^2 + \widetilde{\Gamma^u_{\overline{\varphi}}}\Gamma^u_{\overline{\psi}}][(A^u_\varphi)^2 + \widetilde{\Gamma^u_\psi}\Gamma^u_\varphi] + [A^u_{\overline{\varphi}}\widetilde{\Gamma^u_{\overline{\varphi}}} + \widetilde{\Gamma^u_{\overline{\varphi}}}\widetilde{A^u_{\overline{\psi}}}][\Gamma^u_\varphi A^u_\varphi + \widetilde{A^u_\psi}\Gamma^u_\varphi] \\ &= (A^u_{\overline{\varphi}})^2(A^u_\varphi)^2 + (A^u_{\overline{\varphi}})^2\widetilde{\Gamma^u_\psi}\Gamma^u_\varphi + \widetilde{\Gamma^u_{\overline{\varphi}}}\Gamma^u_{\overline{\psi}}(A^u_\varphi)^2 + \widetilde{\Gamma^u_{\overline{\varphi}}}\Gamma^u_{\overline{\psi}}\widetilde{\Gamma^u_\psi}\Gamma^u_\varphi \\ &\quad + A^u_{\overline{\varphi}}\widetilde{\Gamma^u_{\overline{\varphi}}}\Gamma^u_\varphi A^u_\varphi + A^u_{\overline{\varphi}}\widetilde{\Gamma^u_{\overline{\varphi}}}\widetilde{A^u_\psi}\Gamma^u_\varphi + \widetilde{\Gamma^u_{\overline{\varphi}}}\widetilde{A^u_{\overline{\psi}}}\Gamma^u_\varphi A^u_\varphi + \widetilde{\Gamma^u_{\overline{\varphi}}}\widetilde{A^u_{\overline{\psi}}}\widetilde{A^u_\psi}\Gamma^u_\varphi. \end{aligned}$$

For any $f \in \mathcal{K}_u^2$, we have

$$\begin{aligned}
\alpha f &= P_u[\overline{\varphi}P_u\{\overline{\varphi}P_u(\varphi P_u(\varphi f))\} + \overline{\varphi}P_u\{\overline{\varphi}P_u(\psi Q_u(\varphi f))\} + \overline{\varphi}Q_u\{\overline{\psi}P_u(\varphi P_u(\varphi f))\} \\
&\quad + \overline{\varphi}Q_u\{\overline{\psi}P_u(\psi Q_u(\varphi f))\} + \overline{\varphi}P_u\{\overline{\varphi}Q_u(\varphi P_u(\varphi f))\} + \overline{\varphi}P_u\{\overline{\varphi}Q_u(\psi Q_u(\varphi f))\} \\
&\quad + \overline{\varphi}Q_u\{\overline{\psi}Q_u(\varphi P_u(\varphi f))\} + \overline{\varphi}Q_u\{\overline{\psi}Q_u(\psi Q_u(\varphi f))\}] \\
&= P_u[\overline{\varphi}P_u\{|\varphi|^2 P_u(\varphi f)\} + \overline{\varphi}P_u\{\overline{\varphi}\psi Q_u(\varphi f)\} \\
&\quad + \overline{\varphi}Q_u\{\overline{\psi}\varphi P_u(\varphi f)\} + \overline{\varphi}Q_u\{|\psi|^2 Q_u(\varphi f)\}] \\
&= [A^u_{\overline{\varphi}}A^u_{|\varphi|^2}A^u_{\varphi} + A^u_{\overline{\varphi}}\widetilde{\Gamma^u_{\overline{\varphi}\psi}}\Gamma^u_{\varphi} + \widetilde{\Gamma^u_{\overline{\varphi}}}\Gamma^u_{\overline{\psi}\varphi}A^u_{\varphi} + \widetilde{\Gamma^u_{\overline{\varphi}}}\widetilde{A^u_{|\psi|^2}}\Gamma^u_{\varphi}]f.
\end{aligned}$$

Hence

$$\alpha = A^u_{\overline{\varphi}}A^u_{|\varphi|^2}A^u_{\varphi} + A^u_{\overline{\varphi}}\widetilde{\Gamma^u_{\overline{\varphi}\psi}}\Gamma^u_{\varphi} + \widetilde{\Gamma^u_{\overline{\varphi}}}\Gamma^u_{\overline{\psi}\varphi}A^u_{\varphi} + \widetilde{\Gamma^u_{\overline{\varphi}}}\widetilde{A^u_{|\psi|^2}}\Gamma^u_{\varphi}.$$

On the other hand, we get

$$\begin{aligned}
\beta &= [(A^u_{\overline{\varphi}})^2 + \widetilde{\Gamma^u_{\overline{\varphi}}}\Gamma^u_{\overline{\psi}}][A^u_{\varphi}\widetilde{\Gamma^u_{\psi}} + \widetilde{\Gamma^u_{\psi}}\widetilde{A^u_{\psi}}] + [A^u_{\overline{\varphi}}\widetilde{\Gamma^u_{\overline{\varphi}}} + \widetilde{\Gamma^u_{\overline{\varphi}}}\widetilde{A^u_{\overline{\psi}}}][\Gamma^u_{\varphi}\widetilde{\Gamma^u_{\psi}} + (\widetilde{A^u_{\psi}})^2] \\
&= (A^u_{\overline{\varphi}})^2A^u_{\varphi}\widetilde{\Gamma^u_{\psi}} + (A^u_{\overline{\varphi}})^2\widetilde{\Gamma^u_{\psi}}\widetilde{A^u_{\psi}} + \widetilde{\Gamma^u_{\overline{\varphi}}}\Gamma^u_{\overline{\psi}}A^u_{\varphi}\widetilde{\Gamma^u_{\psi}} + \widetilde{\Gamma^u_{\overline{\varphi}}}\Gamma^u_{\overline{\psi}}\widetilde{\Gamma^u_{\psi}}\widetilde{A^u_{\psi}} \\
&\quad + A^u_{\overline{\varphi}}\widetilde{\Gamma^u_{\overline{\varphi}}}\Gamma^u_{\varphi}\widetilde{\Gamma^u_{\psi}} + A^u_{\overline{\varphi}}\widetilde{\Gamma^u_{\overline{\varphi}}}(\widetilde{A^u_{\psi}})^2 + \widetilde{\Gamma^u_{\overline{\varphi}}}\widetilde{A^u_{\overline{\psi}}}\Gamma^u_{\varphi}\widetilde{\Gamma^u_{\psi}} + \widetilde{\Gamma^u_{\overline{\varphi}}}\widetilde{A^u_{\overline{\psi}}}(\widetilde{A^u_{\psi}})^2.
\end{aligned}$$

For any $g \in (\mathcal{K}_u^2)^{\perp}$, we have

$$\begin{aligned}
\beta g &= P_u[\overline{\varphi}P_u\{\overline{\varphi}P_u(\varphi P_u(\psi g))\} + \overline{\varphi}P_u\{\overline{\varphi}P_u(\psi Q_u(\psi g))\} + \overline{\varphi}Q_u\{\overline{\psi}P_u(\varphi P_u(\psi g))\} \\
&\quad + \overline{\varphi}Q_u\{\overline{\psi}P_u(\psi Q_u(\psi g))\} + \overline{\varphi}P_u\{\overline{\varphi}Q_u(\varphi P_u(\psi g))\} + \overline{\varphi}P_u\{\overline{\varphi}Q_u(\psi Q_u(\psi g))\} \\
&\quad + \overline{\varphi}Q_u\{\overline{\psi}Q_u(\varphi P_u(\psi g))\} + \overline{\varphi}Q_u\{\overline{\psi}Q_u(\psi Q_u(\psi g))\}] \\
&= P_u[\overline{\varphi}P_u\{|\varphi|^2 P_u(\psi g)\} + \overline{\varphi}P_u\{\overline{\varphi}\psi Q_u(\psi g)\} \\
&\quad + \overline{\varphi}Q_u\{\overline{\psi}\varphi P_u(\psi g)\} + \overline{\varphi}Q_u\{|\psi|^2 Q_u(\psi g)\}] \\
&= [A^u_{\overline{\varphi}}A^u_{|\varphi|^2}\widetilde{\Gamma^u_{\psi}} + \widetilde{\Gamma^u_{\overline{\varphi}}}\Gamma^u_{\overline{\psi}\varphi}\widetilde{\Gamma^u_{\psi}} + A^u_{\overline{\varphi}}\widetilde{\Gamma^u_{\overline{\varphi}\psi}}\widetilde{A^u_{\psi}} + \widetilde{\Gamma^u_{\overline{\varphi}}}\widetilde{A^u_{|\psi|^2}}\widetilde{A^u_{\psi}}]g.
\end{aligned}$$

So,

$$\beta = A^u_{\overline{\varphi}}A^u_{|\varphi|^2}\widetilde{\Gamma^u_{\psi}} + \widetilde{\Gamma^u_{\overline{\varphi}}}\Gamma^u_{\overline{\psi}\varphi}\widetilde{\Gamma^u_{\psi}} + A^u_{\overline{\varphi}}\widetilde{\Gamma^u_{\overline{\varphi}\psi}}\widetilde{A^u_{\psi}} + \widetilde{\Gamma^u_{\overline{\varphi}}}\widetilde{A^u_{|\psi|^2}}\widetilde{A^u_{\psi}}.$$

Similarly, we can show that

$$\gamma = \Gamma^u_{\overline{\psi}}A^u_{|\varphi|^2}A^u_{\varphi} + \Gamma^u_{\overline{\psi}}\widetilde{\Gamma^u_{\overline{\varphi}\psi}}\Gamma^u_{\varphi} + \widetilde{A^u_{\overline{\psi}}}\Gamma^u_{\overline{\psi}\varphi}A^u_{\varphi} + \widetilde{A^u_{\overline{\psi}}}\widetilde{A^u_{|\psi|^2}}\Gamma^u_{\varphi}$$

and

$$\delta = \Gamma^u_{\overline{\psi}}A^u_{|\varphi|^2}\widetilde{\Gamma^u_{\psi}} + \Gamma^u_{\overline{\psi}}\widetilde{\Gamma^u_{\overline{\varphi}\psi}}\widetilde{A^u_{\psi}} + \widetilde{A^u_{\overline{\psi}}}\Gamma^u_{\overline{\psi}\varphi}\widetilde{\Gamma^u_{\psi}} + \widetilde{A^u_{\overline{\psi}}}\widetilde{A^u_{|\psi|^2}}\widetilde{A^u_{\psi}}.$$

Hence we can conclude that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} A^u_{\overline{\varphi}}A^u_{|\varphi|^2} + \widetilde{\Gamma^u_{\overline{\varphi}}}\Gamma^u_{\overline{\psi}\varphi} & A^u_{\overline{\varphi}}\widetilde{\Gamma^u_{\overline{\varphi}\psi}} + \widetilde{\Gamma^u_{\overline{\varphi}}}\widetilde{A^u_{|\psi|^2}} \\ \Gamma^u_{\overline{\psi}}A^u_{|\varphi|^2} + \widetilde{A^u_{\overline{\psi}}}\Gamma^u_{\overline{\psi}\varphi} & \Gamma^u_{\overline{\psi}}\widetilde{\Gamma^u_{\overline{\varphi}\psi}} + \widetilde{A^u_{\overline{\psi}}}\widetilde{A^u_{|\psi|^2}} \end{pmatrix} S^u_{\varphi,\psi}. \tag{18}$$

Since $(S^u_{\varphi,\psi})^2$ is an isometry if and only if $\alpha = \delta = I$ and $\beta = \gamma = 0$ by (18), we complete the proof. □

As some applications of Theorem 3.13, we present the following corollaries.

Corollary 3.14. *Let $\varphi, \psi \in L^\infty$ and let u be a nonconstant inner function. Then the following statements hold.*
(i) *If $(S^u_{\varphi,\psi})^2$ is an isometry and $\varphi \in \mathbb{C}$, then*

$$\psi\widetilde{A^u_{1+\overline{c}\psi}}(\mathcal{K}_u^2)^\perp \subset (\mathcal{K}_u^2)^\perp \text{ and } \overline{\psi}\Gamma^u_{1+c\overline{\psi}}\mathcal{K}_u^2 \subset \mathcal{K}_u^2$$

where $|c| = 1$.
(ii) *If φ, ψ are inner functions and $(S^u_{\varphi,\psi})^2$ is an isometry, then $S^u_{\varphi,\psi}$ is an isometry if and only if $\Gamma^u_{\varphi\overline{\psi}} = 0$ on $\overline{\text{ran}(S^u_{\varphi,\psi})}$.*

Proof. (i) If $\varphi = c \in \mathbb{C}$, then by Lemma 3.1, $\Gamma^u_\varphi = \Gamma^u_{\overline{\varphi}} = 0$. Then by Theorem 3.13, $(S^u_{\varphi,\psi})^2$ is an isometry if and only if

$$\begin{aligned} I &= \begin{pmatrix} A^u_{\overline{c}}A^u_{|c|^2} & A^u_{\overline{c}}\widetilde{\Gamma^u_{\overline{c}\psi}} \\ \Gamma^u_{\overline{\psi}}A^u_{|c|^2} + \widetilde{A^u_{\overline{\psi}}}\Gamma^u_{c\overline{\psi}} & \Gamma^u_{\overline{\psi}}\widetilde{\Gamma^u_{\overline{c}\psi}} + \widetilde{A^u_{\overline{\psi}}}\widetilde{A^u_{|\psi|^2}} \end{pmatrix}\begin{pmatrix} A^u_c & \widetilde{\Gamma^u_\psi} \\ 0 & \widetilde{A^u_\psi} \end{pmatrix} \\ &= \begin{pmatrix} R & S \\ T & U \end{pmatrix} \end{aligned}$$

where $R = A^u_{\overline{c}}A^u_{|c|^2}A^u_c$, $S = A^u_{\overline{c}}A^u_{|c|^2}\widetilde{\Gamma^u_\psi} + A^u_{\overline{c}}\widetilde{\Gamma^u_{\overline{c}\psi}}\widetilde{A^u_\psi}$, $T = \Gamma^u_{\overline{\psi}}A^u_{|c|^2}A^u_c + \widetilde{A^u_{\overline{\psi}}}\Gamma^u_{c\overline{\psi}}A^u_c$, and $U = \Gamma^u_{\overline{\psi}}A^u_{|c|^2}\widetilde{\Gamma^u_\psi} + \widetilde{A^u_{\overline{\psi}}}\Gamma^u_{c\overline{\psi}}\widetilde{\Gamma^u_\psi} + \Gamma^u_{\overline{\psi}}\widetilde{\Gamma^u_{\overline{c}\psi}}\widetilde{A^u_\psi} + \widetilde{A^u_{\overline{\psi}}}\widetilde{A^u_{|\psi|^2}}\widetilde{A^u_\psi}$. If $(S^u_{\varphi,\psi})^2$ is an isometry, then $R = I$. Hence for any $f \in \mathcal{K}_u^2$ we get that $f = Rf = A^u_{\overline{c}}A^u_{|c|^2}A^u_c f$. So $|c| = 1$. Since $S = 0$, for any $g \in (\mathcal{K}_u^2)^\perp$ we obtain that

$$\begin{aligned} 0 = Sg &= [A^u_{\overline{c}}A^u_{|c|^2}\widetilde{\Gamma^u_\psi} + A^u_{\overline{c}}\widetilde{\Gamma^u_{\overline{c}\psi}}\widetilde{A^u_\psi}]g \\ &= P_u[\overline{c}\psi g + \overline{c}^2\psi Q_u(\psi g)]. \end{aligned}$$

Then $\psi\widetilde{A^u_{1+\overline{c}\psi}}g = \psi g + \overline{c}\psi Q_u(\psi g) \in (\mathcal{K}_u^2)^\perp$. Thus $\psi\widetilde{A^u_{1+\overline{c}\psi}}(\mathcal{K}_u^2)^\perp \subset (\mathcal{K}_u^2)^\perp$ where $|c| = 1$.

Similarly, since $T = 0$, we get that for any $f \in \mathcal{K}_u^2$,

$$0 = [\Gamma^u_{\overline{\psi}}A^u_{|c|^2}A^u_c + \widetilde{A^u_{\overline{\psi}}}\Gamma^u_{c\overline{\psi}}A^u_c]f = Q_u[c\overline{\psi}f + \overline{\psi}Q_u(c^2\overline{\psi}f)].$$

Thus $c\overline{\psi}f + \overline{\psi}Q_u(c^2\overline{\psi}f) \in \mathcal{K}_u^2$ for any $f \in \mathcal{K}_u^2$. Hence $\overline{\psi}\Gamma^u_{1+c\overline{\psi}}\mathcal{K}_u^2 \subset \mathcal{K}_u^2$ where $|c| = 1$.

(ii) From Theorem 3.13, we get that $S^u_{\varphi,\psi}$ is an isometry if and only if

$$\begin{pmatrix} A^u_{\overline{\varphi}}A^u_{|\varphi|^2} + \widetilde{\Gamma^u_{\overline{\varphi}}}\Gamma^u_{\overline{\psi}\varphi} & A^u_{\overline{\varphi}}\widetilde{\Gamma^u_{\overline{\varphi}\psi}} + \widetilde{\Gamma^u_{\overline{\varphi}}}\widetilde{A^u_{|\psi|^2}} \\ \Gamma^u_{\overline{\psi}}A^u_{|\varphi|^2} + \widetilde{A^u_{\overline{\psi}}}\Gamma^u_{\overline{\psi}\varphi} & \Gamma^u_{\overline{\psi}}\widetilde{\Gamma^u_{\overline{\varphi}\psi}} + \widetilde{A^u_{\overline{\psi}}}\widetilde{A^u_{|\psi|^2}} \end{pmatrix} = (S^u_{\varphi,\psi})^*.$$

Hence $S^u_{\varphi,\psi}$ is an isometry if and only if the following equations hold;

$$\begin{cases} A^u_{\overline{\varphi}}(A^u_{|\varphi|^2} - I) + \widetilde{\Gamma^u_{\overline{\varphi}}}\Gamma^u_{\overline{\psi}\varphi} = 0, \\ A^u_{\overline{\varphi}}\widetilde{\Gamma^u_{\overline{\varphi}\psi}} + \widetilde{\Gamma^u_{\overline{\varphi}}}(\widetilde{A^u_{|\psi|^2}} - I) = 0, \\ \Gamma^u_{\overline{\psi}}(A^u_{|\varphi|^2} - I) + \widetilde{A^u_{\overline{\psi}}}\Gamma^u_{\overline{\psi}\varphi} = 0, \text{ and} \\ \Gamma^u_{\overline{\psi}}\widetilde{\Gamma^u_{\overline{\varphi}\psi}} + \widetilde{A^u_{\overline{\psi}}}(\widetilde{A^u_{|\psi|^2}} - I) = 0. \end{cases}$$

Since $A^u_{|\varphi|^2} = \widetilde{A^u_{|\psi|^2}} = I$, it follows that $S^u_{\varphi,\psi}$ is an isometry if and only if the following equations hold;

$$A^u_{\overline{\varphi}}\widetilde{\Gamma^u_{\overline{\varphi}\psi}} = 0, \widetilde{\Gamma^u_{\overline{\varphi}}}\Gamma^u_{\overline{\psi}\varphi} = 0, \Gamma^u_{\overline{\psi}}\widetilde{\Gamma^u_{\overline{\varphi}\psi}} = 0, \text{ and } \widetilde{A^u_{\overline{\psi}}}\Gamma^u_{\overline{\psi}\varphi} = 0.$$

Thus $(S^u_{\varphi,\psi})^*(\widetilde{\Gamma^u_{\overline{\varphi}\psi}} \oplus \Gamma^u_{\overline{\psi}\varphi}) = 0$, i.e., $(\Gamma^u_{\overline{\psi}\varphi} \oplus (\Gamma^u_{\overline{\psi}\varphi})^*)S^u_{\varphi,\psi} = 0$. Therefore, $S^u_{\varphi,\psi}$ is an isometry if and only if $\Gamma^u_{\overline{\psi}\varphi} = 0$ on $\overline{\text{ran}(S^u_{\varphi,\psi})}$. □

Corollary 3.15. *Let $\varphi, \psi \in L^\infty$ be inner functions and let u be a nonconstant inner function. If $(S^u_{\varphi,\psi})^2$ is an isometry and $\varphi\overline{\psi}$ is a nonconstant function on $\overline{\mathrm{ran}(S^u_{\varphi,\psi})}$, then $S^u_{\varphi,\psi}$ is not an isometry.*

Proof. The proof follows from Corollary 3.14 (ii) and Lemma 3.12. □

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