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Some Euclidean Berezin number inequalities of a pair of operators and their applications

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Abstract. In this paper, we present several sharp lower and upper bounds for the Euclidean Berezin number of a pair of bounded linear operators defined on a reproducing kernel Hilbert space. As applications of these bounds we deduce some new results in this field. Further, some applications of the newly obtained inequalities are also provided.

1. Introduction and Preliminaries

Let $\mathcal{B}(\mathcal{H})$ denote the C^*- algebra of all bounded linear operators on a non trivial complex Hilbert space $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$ and associated norm $\|.\|$. Recall that an operator $A \in \mathcal{B}(\mathcal{H})$ is said to be positive if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{H}$. For $A \in \mathcal{B}(\mathcal{H})$, let $W(A)$, $\omega(A)$ and $\|A\|$ denote the numerical range, numerical radius and the usual operator norm of A, respectively. Recall that

$$\begin{aligned} W(A) &= \{\langle Ax, x \rangle : x \in \mathcal{H},\ \|x\| = 1\}, \\ \omega(A) &= \sup\{|\langle Ax, x \rangle| : x \in \mathcal{H},\ \|x\| = 1\}, \end{aligned}$$

and

$$\|A\| = \sup\left\{\left|\langle Ax, y \rangle\right| : x, y \in \mathcal{H},\ \|x\| = \|y\| = 1\right\}.$$

It is well known that $\omega(\cdot)$ defines a norm on $\mathcal{B}(\mathcal{H})$, which is equivalent to the usual operator norm. In fact, for every $A \in \mathcal{B}(\mathcal{H})$, we have [15, 17]

$$\frac{1}{2}\|A\| \leq \omega(A) \leq \|A\|.$$

For some results about the numerical radius inequalities and their applications, we refer to see [2, 3, 11–13].

Let Ω be a nonempty set. A functional Hilbert space $\mathcal{H} = \mathcal{H}(\Omega)$ is a Hilbert space of complex valued functions, which has the property that point evaluations are continuous i.e., for each $\lambda \in \Omega$ the map $f \longmapsto f(\lambda)$ is a continuous linear functional on $\mathcal{H}$. The Riesz representation theorem ensues that for each

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$\lambda \in \Omega$ there exists a unique element $k_\lambda \in \mathcal{H}$ such that $f(\lambda) = \langle f, k_\lambda \rangle$ for all $f \in \mathcal{H}$. The set $\{k_\lambda : \lambda \in \Omega\}$ is called the reproducing kernel of the space $\mathcal{H}$. If $\{e_n\}_{n\geq 0}$ is an orthonormal basis for a functional Hilbert space $\mathcal{H}$, then the reproducing kernel of $\mathcal{H}$ is given by $k_\lambda(z) = \sum_{n=0}^{+\infty} \overline{e_n(\lambda)} e_n(z)$ (see [17]). For $\lambda \in \Omega$, let $\hat{k}_\lambda = \frac{k_\lambda}{\|k_\lambda\|}$ be the normalized reproducing kernel of $\mathcal{H}$.

Let A be a bounded linear operator on $\mathcal{H}$, the Berezin symbol of A, which firstly have been introduced by Berezin [6, 7] is the function $\tilde{A}$ on Ω defined by

$$\tilde{A}(\lambda) := \left\langle A\hat{k}_\lambda, \hat{k}_\lambda \right\rangle.$$

The Berezin set and the Berezin number of the operator A are defined respectively by

$$\mathbf{Ber}(A) := \left\{ \left\langle A\hat{k}_\lambda, \hat{k}_\lambda \right\rangle : \lambda \in \Omega \right\},$$

and

$$\mathbf{ber}(A) := \sup \left\{ \left| \left\langle A\hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right| : \lambda \in \Omega \right\}.$$

Recall that, for $A \in \mathcal{B}(\mathcal{H})$ its Crawford number $c(A)$ is defined by

$$c(A) = \inf \{ |\langle Ax, x \rangle| : x \in \mathcal{H}, \ \|x\| = 1 \}.$$

Similarly, the Crawford-Berezin number of A, denoted by $\tilde{c}(A)$, was defined in [24] as

$$\tilde{c}(A) := \inf \left\{ \left| \left\langle A\hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right| : \lambda \in \Omega \right\}.$$

It is clear that the Berezin symbol $\tilde{A}$ is the bounded function on Ω whose value lies in the numerical range of the operator A and hence for any $A \in \mathcal{B}(\mathcal{H})$,

$$\mathbf{Ber}(A) \subset W(A) \text{ and } \mathbf{ber}(A) \leq \omega(A),$$

Moreover, the Berezin number of an operator A satisfies the following properties:
(i) $\mathbf{ber}(A) \leq \|A\|$.
(ii) $\mathbf{ber}(\alpha A) = |\alpha| \, \mathbf{ber}(A)$ for all $\alpha \in \mathbb{C}$.
(iii) $\mathbf{ber}(A + B) \leq \mathbf{ber}(A) + \mathbf{ber}(B)$ for all $A, B \in \mathcal{B}(\mathcal{H})$.

Notice that, in general, the Berezin number does not define a norm. However, if $\mathcal{H}$ is a reproducing kernel Hilbert space of analytic functions, (for instance on the unit disc $D = \{z \in \mathbb{C} : |z < 1|\}$), then $\mathbf{ber}(.)$ defines a norm on $\mathcal{B}(\mathcal{H}(D))$ (see [19, 20]).

The Berezin symbol has been studied in detail for Toeplitz and Hankel operators on Hardy and Bergman spaces. A nice property of the Berezin symbol is mentioned next. If $\tilde{A}(\lambda) = \tilde{B}(\lambda)$ for all $\lambda \in \Omega$, then $A = B$. Therefore, the Berezin symbol uniquely determines the operator. The Berezin symbol and Berezin number have been studied by many mathematicians over the years, a few of them are [4, 5, 8, 14, 16, 25–28].

Now, for any operator $A \in \mathcal{B}(\mathcal{H})$, the Berezin norm of A denoted as $\|A\|_{ber}$ is defined by

$$\|A\|_{ber} := \sup \left\{ \left| \left\langle A\hat{k}_\lambda, \hat{k}_\mu \right\rangle \right| : \lambda, \mu \in \Omega \right\},$$

where $\hat{k}_\lambda, \hat{k}_\mu$ are normalized reproducing kernels for λ, μ, respectively.

For $A, B \in \mathcal{B}(\mathcal{H})$ it is clear from the definition of the Berezin norm that the following properties hold:
(i) $\|\lambda A\|_{ber} = |\lambda| \, \|A\|_{ber}$ for all $\lambda \in \mathbb{C}$,
(ii) $\|A + B\|_{ber} \leq \|A\|_{ber} + \|B\|_{ber}$,

(iii) $\|A\|_{ber} = \|A^*\|_{ber}$,
(iv) $\mathbf{ber}(A) \leq \|A\|_{ber}$.
Recall that the Euclidean operator radius of $A, B \in \mathcal{B}(\mathcal{H})$ defined as

$$\omega_e(A,B) = \sup\left\{\sqrt{|\langle Ax,x\rangle|^2 + |\langle Bx,x\rangle|^2} : x \in \mathcal{H},\ \|x\| = 1\right\}.$$

For more details of Euclidean operator radius and related inequalities we refer to [10, 18, 22, 23].

Similarly, the Euclidean Berezin number of a pair of operators $\mathbf{T} = (T_1, T_2) \in \mathcal{B}(\mathcal{H})^2$ denoted by $\mathbf{ber}(\mathbf{T})$ or $\mathbf{ber}(T_1, T_2)$, is defined as

$$\mathbf{ber}(\mathbf{T}) := \sup\left\{\sqrt{\left|\left\langle T_1\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 + \left|\left\langle T_2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2} : \lambda \in \Omega\right\}.$$

In this paper, we prove several lower and upper bounds for the Euclidean Berezin number of a pair of bounded linear operators defined on a reproducing kernel Hilbert space.

2. Main results

Our first result can be stated as follows.

Theorem 2.1. *Let $\mathbf{T} = (T_1, T_2) \in \mathcal{B}(\mathcal{H})^2$ be a 2-tuple of operators. Then*

$$\frac{1}{2}\max\left\{\mathbf{ber}^2(T_1+T_2) + \tilde{c}^2(T_1-T_2), \mathbf{ber}^2(T_1-T_2) + \tilde{c}^2(T_1+T_2)\right\} \leq \mathbf{ber}^2(\mathbf{T}).$$

Proof. Let $\hat{k}_\lambda$ be the normalized reproducing kernel of $\mathcal{H}$. Then, we have

$$\begin{aligned} & \left|\left\langle T_1\hat{k}_\lambda, \hat{k}_\lambda\right\rangle + \left\langle T_2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 + \left|\left\langle T_1\hat{k}_\lambda, \hat{k}_\lambda\right\rangle - \left\langle T_2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 \\ = \ & 2\left(\left|\left\langle T_1\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 + \left|\left\langle T_2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2\right). \end{aligned}$$

Therefore,

$$\begin{aligned} \left|\left\langle (T_1+T_2)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 + \left|\left\langle (T_1-T_2)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 &= 2\left(\left|\left\langle T_1\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 + \left|\left\langle T_2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2\right) \\ &\leq 2\mathbf{ber}^2(\mathbf{T}). \end{aligned}$$

This implies that

$$\begin{aligned} \left|\left\langle (T_1+T_2)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 &\leq 2\mathbf{ber}^2(\mathbf{T}) - \left|\left\langle (T_1-T_2)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 \\ &\leq 2\mathbf{ber}^2(\mathbf{T}) - \tilde{c}^2(T_1-T_2). \end{aligned}$$

Taking supremum over $\lambda \in \Omega$, we get

$$\mathbf{ber}^2(T_1+T_2) \leq 2\mathbf{ber}^2(\mathbf{T}) - \tilde{c}^2(T_1-T_2).$$

So,

$$\mathbf{ber}^2(T_1+T_2) + \tilde{c}^2(T_1-T_2) \leq 2\mathbf{ber}^2(\mathbf{T}). \tag{1}$$

Replacing T_2 by $-T_2$ in (1) we obtain

$$\mathbf{ber}^2(T_1-T_2) + \tilde{c}^2(T_1+T_2) \leq 2\mathbf{ber}^2(\mathbf{T}). \tag{2}$$

Combining the inequalities (1) and (2), we get

$$\frac{1}{2}\max\left\{\mathbf{ber}^2(T_1+T_2)+\tilde{c}^2(T_1-T_2),\mathbf{ber}^2(T_1-T_2)+\tilde{c}^2(T_1+T_2)\right\}\leq\mathbf{ber}^2(\mathbf{T}),$$

as desired. □

Lemma 2.2. *[8] Let $A\in\mathcal{B}(\mathcal{H})$ be a positive operator. Then*

$$\|A\|_{ber}=\mathbf{ber}(A).$$

The following corollary is an immediate consequence of Theorem 2.1 and Lemma 2.2.

Corollary 2.3. *If $\mathbf{T}=(T_1,T_2)\in\mathcal{B}(\mathcal{H})^2$ is a 2-tuple of operators, such that T_1 and T_2 are positive operators, then*

$$\frac{1}{2}\max\left\{\|T_1+T_2\|_{ber}^2+\tilde{c}^2(T_1-T_2),\mathbf{ber}^2(T_1-T_2)+\tilde{c}^2(T_1+T_2)\right\}\leq\mathbf{ber}^2(\mathbf{T}).$$

Next lower bound for Euclidean Berezin number of a pair of operators reads as follows.

Theorem 2.4. *Let $\mathbf{T}=(T_1,T_2)\in\mathcal{B}(\mathcal{H})^2$ be a 2-tuple of operators. Then*

$$\max\left\{\mathbf{ber}^2(T_1)+\tilde{c}^2(T_2),\mathbf{ber}^2(T_2)+\tilde{c}^2(T_1)\right\}\leq\mathbf{ber}^2(\mathbf{T}).$$

Proof. Let $\hat{k}_\lambda$ be the normalized reproducing kernel of $\mathcal{H}$. Then, we have

$$\left|\left\langle(T_1+T_2)\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right|^2+\left|\left\langle(T_1-T_2)\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right|^2=2\left(\left|\left\langle T_1\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right|^2+\left|\left\langle T_2\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right|^2\right).$$

This implies that

$$\mathbf{ber}^2(T_1+T_2,T_1-T_2)=2\mathbf{ber}^2(\mathbf{T}). \tag{3}$$

Replacing T_1 by T_1+T_2 and T_2 by T_1-T_2 in Theorem 2.1, we get

$$2\max\left\{\mathbf{ber}^2(T_1)+\tilde{c}^2(T_2),\mathbf{ber}^2(T_2)+\tilde{c}^2(T_1)\right\}\leq\mathbf{ber}^2(T_1+T_2,T_1-T_2).$$

Now, by (3) we deduce the desired result. This completes the proof. □

As a consequence of Theorem 2.4, we have the following result.

Corollary 2.5. *If $\mathbf{T}=(T_1,T_2)\in\mathcal{B}(\mathcal{H})^2$ is a 2-tuple of operators, such that T_1 and T_2 are positive operators, then*

$$\max\left\{\|T_1\|_{ber}^2+\tilde{c}^2(T_2),\|T_2\|_{ber}^2+\tilde{c}^2(T_1)\right\}\leq\mathbf{ber}^2(\mathbf{T}).$$

The following lemma plays a crucial role in our next proofs, which can be found in [9].

Lemma 2.6. *Let a,b,e be vectors in $\mathcal{H}$ and $\|e\|=1$. Then*

$$|\langle a,e\rangle\langle e,b\rangle|\leq\frac{1}{2}(\|a\|\,\|b\|+|\langle a,b\rangle|).$$

Now, we are in a position to prove the following result.

Theorem 2.7. *Let* $\mathbf{T} = (T_1, T_2) \in \mathcal{B}(\mathcal{H})^2$ *be a 2-tuple of operators. Then*

$$\mathbf{ber}^2(\mathbf{T}) \leq \min\left\{\mathbf{ber}^2(T_1+T_2), \mathbf{ber}^2(T_1-T_2)\right\} + \frac{1}{2}\left\|T_1T_1^* + T_2^*T_2\right\|_{ber} + \mathbf{ber}(T_1T_2).$$

Proof. Let $\hat{k}_\lambda$ be the normalized reproducing kernel of $\mathcal{H}$. Then, we have

$$\begin{aligned}
& \left|\left\langle T_2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 - 2\,\mathrm{Re}\left(\left\langle T_2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle \overline{\left\langle T_1\hat{k}_\lambda, \hat{k}_\lambda\right\rangle}\right) + \left|\left\langle T_1\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 \\
= & \left|\left\langle T_2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle - \left\langle T_1\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 \\
= & \left|\left\langle (T_2-T_1)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 \\
\leq & \ \mathbf{ber}^2(T_2-T_1).
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \left|\left\langle T_2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 + \left|\left\langle T_1\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 \\
\leq & \ \mathbf{ber}^2(T_2-T_1) + 2\,\mathrm{Re}\left(\left\langle T_2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle \overline{\left\langle T_1\hat{k}_\lambda, \hat{k}_\lambda\right\rangle}\right) \\
\leq & \ \mathbf{ber}^2(T_2-T_1) + 2\left|\left\langle T_2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle \left\langle T_1\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right| \\
\leq & \ \mathbf{ber}^2(T_2-T_1) + \left\|T_2\hat{k}_\lambda\right\| \left\|T_1^*\hat{k}_\lambda\right\| + \left|\left\langle T_2\hat{k}_\lambda, T_1^*\hat{k}_\lambda\right\rangle\right| \\
& \text{(by Lemma 2.6)} \\
\leq & \ \mathbf{ber}^2(T_2-T_1) + \frac{1}{2}\left(\left\|T_2\hat{k}_\lambda\right\|^2 + \left\|T_1^*\hat{k}_\lambda\right\|^2\right) + \mathbf{ber}(T_1T_2) \\
= & \ \mathbf{ber}^2(T_2-T_1) + \frac{1}{2}\left\langle\left(T_1T_1^* + T_2^*T_2\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle + \mathbf{ber}(T_1T_2) \\
\leq & \ \mathbf{ber}^2(T_2-T_1) + \frac{1}{2}\left\|T_1T_1^* + T_2^*T_2\right\|_{ber} + \mathbf{ber}(T_1T_2).
\end{aligned}$$

Thus,

$$\begin{aligned}
& \left|\left\langle T_2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 + \left|\left\langle T_1\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 \\
\leq & \ \mathbf{ber}^2(T_2-T_1) + \frac{1}{2}\left\|T_1T_1^* + T_2^*T_2\right\|_{ber} + \mathbf{ber}(T_1T_2).
\end{aligned}$$

Taking supremum over $\lambda \in \Omega$ in the above inequality, we get

$$\mathbf{ber}^2(\mathbf{T}) \leq \mathbf{ber}^2(T_2-T_1) + \frac{1}{2}\left\|T_1T_1^* + T_2^*T_2\right\|_{ber} + \mathbf{ber}(T_1T_2). \tag{4}$$

Now, replacing T_2 by $-T_2$ in (4), we obtain

$$\mathbf{ber}^2(\mathbf{T}) \leq \mathbf{ber}^2(T_2+T_1) + \frac{1}{2}\left\|T_1T_1^* + T_2^*T_2\right\|_{ber} + \mathbf{ber}(T_1T_2). \tag{5}$$

From the inequalities (4) and (5), we have

$$\mathbf{ber}^2(\mathbf{T}) \leq \min\left\{\mathbf{ber}^2(T_1+T_2), \mathbf{ber}^2(T_1-T_2)\right\} + \frac{1}{2}\left\|T_1T_1^* + T_2^*T_2\right\|_{ber} + \mathbf{ber}(T_1T_2),$$

as desired. □

If we take $T_1 = T_2 = T$, then we get the following new inequality.

Corollary 2.8. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$\mathbf{ber}^2(T) \leq \frac{1}{4}\|TT^* + T^*T\|_{ber} + \frac{1}{2}\mathbf{ber}\left(T^2\right).$$

Remark 2.9. *Corollary 2.8 is an improvement of the inequality* $\mathbf{ber}(T) \leq \|T\|$. *Indeed, since* $\mathbf{ber}(X) \leq \|X\|_{ber} \leq \|X\|$ *for every $X \in B(\mathcal{H})$, it can observe that*

$$\begin{aligned} \mathbf{ber}^2(T) &\leq \frac{1}{4}\|TT^* + T^*T\|_{ber} + \frac{1}{2}\mathbf{ber}\left(T^2\right) \\ &\leq \frac{1}{4}\|TT^* + T^*T\| + \frac{1}{2}\left\|T^2\right\| \\ &\leq \frac{1}{4}(\|TT^*\| + \|T^*T\|) + \frac{1}{2}\|T\|^2 \\ &= \|T\|^2. \end{aligned}$$

Thus,

$$\mathbf{ber}(T) \leq \sqrt{\frac{1}{4}\|TT^* + T^*T\|_{ber} + \frac{1}{2}\mathbf{ber}(T^2)} \leq \|T\|.$$

The following lemma will be useful in the proof of the next result.

Lemma 2.10. *Let $\alpha, \beta \in \mathbb{C}$. Then*

$$\left|\alpha - \beta\right|^2 = |\alpha|^2 + \left|\beta\right|^2 - 2\,\mathrm{Re}(\overline{\alpha}\beta).$$

Proof. The proof is trivial. □

We prove now another upper bound for the Euclidean Berezin number of a pair of operators on $\mathcal{B}(\mathcal{H})$.

Theorem 2.11. *Let $\mathbf{T} = (T_1, T_2) \in \mathcal{B}(\mathcal{H})^2$ be a 2-tuple of operators. Then*

$$\mathbf{ber}(\mathbf{T}) \leq \left(\mathbf{ber}^2(T_1 - T_2) + 2\mathbf{ber}(T_1)\,\mathbf{ber}(T_2)\right)^{\frac{1}{2}}.$$

Proof. Let $\hat{k}_\lambda$ be the normalized reproducing kernel of $\mathcal{H}$. Applying Lemma 2.10 with $\alpha := \left\langle T_1\hat{k}_\lambda, \hat{k}_\lambda\right\rangle$ and $\beta := \left\langle T_2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle$, we get

$$\begin{aligned} &\left|\left\langle T_2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 - 2\,\mathrm{Re}\left(\left\langle T_2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\overline{\left\langle T_1\hat{k}_\lambda, \hat{k}_\lambda\right\rangle}\right) + \left|\left\langle T_1\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 \\ &= \left|\left\langle T_2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle - \left\langle T_1\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 \\ &= \left|\left\langle (T_2 - T_1)\,\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 \\ &\leq \mathbf{ber}^2(T_2 - T_1). \end{aligned}$$

Thus,

$$\begin{aligned} \left|\left\langle T_2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 + \left|\left\langle T_1\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 &\leq \mathbf{ber}^2(T_2 - T_1) + 2\,\mathrm{Re}\left(\left\langle T_2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\overline{\left\langle T_1\hat{k}_\lambda, \hat{k}_\lambda\right\rangle}\right) \\ &\leq \mathbf{ber}^2(T_2 - T_1) + 2\left|\left\langle T_2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|\left|\left\langle T_1\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|. \end{aligned}$$

Taking supremum over all $\lambda \in \Omega$, we obtain the desired inequality. □

The following corollary follows from Theorem 2.11.

Corollary 2.12. *If* $\mathbf{T} = (T_1, T_2) \in \mathcal{B}(\mathcal{H})^2$ *is a* 2*-tuple of operators, then*

$$\mathbf{ber}(\mathbf{T}) \leq \left(\|T_1 - T_2\|_{ber}^2 + 2\|T_1\|_{ber}\|T_2\|_{ber} \right)^{\frac{1}{2}}.$$

Proof. The result follows immediately by using the fact $\mathbf{ber}(X) \leq \|X\|_{ber}$ for every $X \in B(\mathcal{H})$ in Theorem 2.11. □

Another upper bound of Euclidean Berezin number of a pair of operators can be stated as follows.

Theorem 2.13. *Let* $\mathbf{T} = (T_1, T_2) \in \mathcal{B}(\mathcal{H})^2$ *be a* 2*-tuple of operators. Then*

$$\mathbf{ber}(\mathbf{T}) \leq \left[2\min\left\{ \mathbf{ber}^2(T_1), \mathbf{ber}^2(T_2) \right\} + \mathbf{ber}(T_1 - T_2)\,\mathbf{ber}(T_1 + T_2) \right]^{\frac{1}{2}}.$$

Proof. By using Theorem 2.11 for $T_1 - T_2$ and $T_1 + T_2$ instead of T_1, T_2 we obtain

$$\mathbf{ber}^2(T_1 - T_2, T_1 + T_2) \leq 4\mathbf{ber}^2(T_2) + 2\mathbf{ber}(T_1 - T_2)\,\mathbf{ber}(T_1 + T_2).$$

Therefore, it follows from (3) that

$$\mathbf{ber}^2(\mathbf{T}) \leq 2\mathbf{ber}^2(T_2) + \mathbf{ber}(T_1 - T_2)\,\mathbf{ber}(T_1 + T_2). \tag{6}$$

Now, in (6) we swap T_2 with T_1 we have

$$\mathbf{ber}^2(\mathbf{T}) \leq 2\mathbf{ber}^2(T_1) + \mathbf{ber}(T_1 - T_2)\,\mathbf{ber}(T_1 + T_2). \tag{7}$$

By (6) and (7) we conclude the result. □

The following lemma is useful in proving our next result.

Lemma 2.14. *[10] If* $y, u, v \in \mathcal{H}$*, then*

$$\left|\langle y, u\rangle\right|^2 + \left|\langle y, v\rangle\right|^2 \leq \left\|y\right\|^2 \left[\max\left\{\|u\|^2, \|v\|^2\right\} + |\langle u, v\rangle|\right].$$

Theorem 2.15. *Let* $\mathbf{T} = (T_1, T_2) \in \mathcal{B}(\mathcal{H})^2$ *be a* 2*-tuple of operators. Then*

$$\mathbf{ber}^2(\mathbf{T}) \leq \max\left(\left\|T_1^* T_1\right\|_{ber}, \left\|T_2^* T_2\right\|_{ber} \right) + \mathbf{ber}\left(T_2^* T_1\right).$$

Proof. Let $\hat{k}_\lambda$ be the normalized reproducing kernel of $\mathcal{H}$. Replacing $y = \hat{k}_\lambda$, $u = T_1\hat{k}_\lambda$ and $v = T_2\hat{k}_\lambda$ in Lemma 2.14, we conclude that

$$\begin{aligned} \left|\left\langle T_1\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 + \left|\left\langle T_2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 &\leq \max\left\{\left\|T_1\hat{k}_\lambda\right\|^2, \left\|T_2\hat{k}_\lambda\right\|^2\right\} + \left|\left\langle T_1\hat{k}_\lambda, T_2\hat{k}_\lambda\right\rangle\right| \\ &= \max\left\{\left\langle T_1^* T_1\hat{k}_\lambda, \hat{k}_\lambda\right\rangle, \left\langle T_2^* T_2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right\} + \left|\left\langle T_2^* T_1\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|. \end{aligned}$$

Therefore,

$$\left|\left\langle T_1\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 + \left|\left\langle T_2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 \leq \max\left\{\left\langle T_1^* T_1\hat{k}_\lambda, \hat{k}_\lambda\right\rangle, \left\langle T_2^* T_2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right\} + \left|\left\langle T_2^* T_1\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|. \tag{8}$$

Taking supremum over all $\lambda \in \Omega$ in (8), we deduce that

$$\mathbf{ber}^2(\mathbf{T}) \leq \max\left(\mathbf{ber}\left(T_1^*T_1\right), \mathbf{ber}\left(T_2^*T_2\right)\right) + \mathbf{ber}\left(T_2^*T_1\right).$$

Using the fact $\mathbf{ber}(X) \leq \|X\|_{ber}$ for every $X \in B(\mathcal{H})$, we get

$$\mathbf{ber}^2(\mathbf{T}) \leq \max\left(\left\|T_1^*T_1\right\|_{ber}, \left\|T_2^*T_2\right\|_{ber}\right) + \mathbf{ber}\left(T_2^*T_1\right).$$

This completes the proof. □

Corollary 2.16. *Let $T \in \mathcal{B}(\mathcal{H})$, then we have*

$$\mathbf{ber}^2(T) \leq \frac{1}{2}\left(\max\left(\|T^*T\|_{ber}, \|TT^*\|_{ber}\right) + \mathbf{ber}\left(T^2\right)\right).$$

Proof. In view of Theorem 2.15 we choose that $T_1 = T$ and $T_2 = T^*$, then we can see that

$$\mathbf{ber}^2(T, T^*) \leq \max\left(\|T^*T\|_{ber}, \|TT^*\|_{ber}\right) + \mathbf{ber}\left(T^2\right).$$

Let $\hat{k}_\lambda$ be the normalized reproducing kernel of $\mathcal{H}$. Then, we have

$$\begin{aligned}
\mathbf{ber}^2(T, T^*) &= \sup_{\lambda\in\Omega}\left(\left|\left\langle T\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 + \left|\left\langle T^*\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2\right) \\
&= \sup_{\lambda\in\Omega}\left(\left|\left\langle T\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 + \left|\left\langle \hat{k}_\lambda, T\hat{k}_\lambda\right\rangle\right|^2\right) \\
&= 2\sup_{\lambda\in\Omega}\left|\left\langle T\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 = 2\mathbf{ber}^2(T).
\end{aligned}$$

Therefore, we infer that

$$\mathbf{ber}^2(T) \leq \frac{1}{2}\left(\max\left(\|T^*T\|_{ber}, \|TT^*\|_{ber}\right) + \mathbf{ber}\left(T^2\right)\right).$$

Hence, we get the desired inequality. □

Remark 2.17. *From Corollary 2.16, we obviously have*

$$\begin{aligned}
\mathbf{ber}^2(T) &\leq \frac{1}{2}\left(\max\left(\|T^*T\|_{ber}, \|TT^*\|_{ber}\right) + \mathbf{ber}\left(T^2\right)\right) \\
&\leq \frac{1}{2}\left(\max\left(\|T^*T\|, \|TT^*\|\right) + \left\|T^2\right\|\right) \\
&= \frac{1}{2}\left(\|T^*T\| + \left\|T^2\right\|\right) \\
&= \frac{1}{2}\left(\|T\|^2 + \left\|T^2\right\|\right) \\
&\leq \|T\|^2.
\end{aligned}$$

Then, Corollary 2.16 is another improvement of inequality $\mathbf{ber}(T) \leq \|T\|$.

Corollary 2.18. *If $T \in \mathcal{B}(\mathcal{H})$ is a selfadjoint operator, then*

$$\mathbf{ber}^2(T) \leq \mathbf{ber}\left(T^2\right).$$

Proof. From Corollary 2.16 and Lemma 2.2, it follows that

$$\begin{aligned}
\mathbf{ber}^2(T) &\leq \frac{1}{2}\left(\max\left(\|T^*T\|_{ber}, \|TT^*\|_{ber}\right) + \mathbf{ber}\left(T^2\right)\right) \\
&= \frac{1}{2}\left(\max\left(\mathbf{ber}(T^*T), \mathbf{ber}(TT^*)\right) + \mathbf{ber}\left(T^2\right)\right) \\
&= \frac{1}{2}\left(\mathbf{ber}\left(T^2\right) + \mathbf{ber}\left(T^2\right)\right) \\
(\text{since } T^* &= T) \\
&= \mathbf{ber}\left(T^2\right).
\end{aligned}$$

Thus,

$$\mathbf{ber}^2(T) \leq \mathbf{ber}\left(T^2\right).$$

□

Next, we prove the following inequality.

Theorem 2.19. *Let $\mathbf{T} = (T_1, T_2) \in \mathcal{B}(\mathcal{H})^2$ be a 2-tuple of operators. Then*

$$\mathbf{ber}^2(\mathbf{T}) \leq \frac{1}{2}\left(\left\|T_1^*T_1 + T_2^*T_2\right\|_{ber} + \left\|T_1^*T_1 - T_2^*T_2\right\|_{ber}\right) + \mathbf{ber}\left(T_2^*T_1\right). \tag{9}$$

Proof. Let $\hat{k}_\lambda$ be the normalized reproducing kernel of $\mathcal{H}$. By using (8) and since $\max\{\alpha, \beta\} = \frac{1}{2}\left(\alpha + \beta + \left|\alpha - \beta\right|\right)$ for any $\alpha, \beta \in \mathbb{R}^+$, we can observe that

$$\begin{aligned}
&\left|\left\langle T_1\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 + \left|\left\langle T_2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 \\
\leq\ & \frac{1}{2}\left(\left\|T_1\hat{k}_\lambda\right\|^2 + \left\|T_2\hat{k}_\lambda\right\|^2 + \left|\left\|T_1\hat{k}_\lambda\right\|^2 - \left\|T_2\hat{k}_\lambda\right\|^2\right|\right) + \left|\left\langle T_1\hat{k}_\lambda, T_2\hat{k}_\lambda\right\rangle\right|.
\end{aligned} \tag{10}$$

Since

$$\left\|T_1\hat{k}_\lambda\right\|^2 = \left\langle T_1\hat{k}_\lambda, T_1\hat{k}_\lambda\right\rangle = \left\langle \hat{k}_\lambda, T_1^*T_1\hat{k}_\lambda\right\rangle,$$

and

$$\left\|T_2\hat{k}_\lambda\right\|^2 = \left\langle \hat{k}_\lambda, T_2^*T_2\hat{k}_\lambda\right\rangle.$$

Then (10) can be written as

$$\begin{aligned}
&\left|\left\langle T_1\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 + \left|\left\langle T_2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 \\
\leq\ & \frac{1}{2}\left(\left\langle \hat{k}_\lambda, \left(T_1^*T_1 + T_2^*T_2\right)\hat{k}_\lambda\right\rangle + \left|\left\langle \hat{k}_\lambda, \left(T_1^*T_1 - T_2^*T_2\right)\hat{k}_\lambda\right\rangle\right|\right) + \left|\left\langle T_2^*T_1\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|.
\end{aligned}$$

By taking supremum over $\lambda \in \Omega$, we get

$$\begin{aligned}
\mathbf{ber}^2(\mathbf{T}) &\leq \frac{1}{2}\left(\mathbf{ber}\left(T_1^*T_1 + T_2^*T_2\right) + \mathbf{ber}\left(T_1^*T_1 - T_2^*T_2\right)\right) + \mathbf{ber}\left(T_2^*T_1\right) \\
&\leq \frac{1}{2}\left(\left\|T_1^*T_1 + T_2^*T_2\right\|_{ber} + \left\|T_1^*T_1 - T_2^*T_2\right\|_{ber}\right) + \mathbf{ber}\left(T_2^*T_1\right).
\end{aligned}$$

Hence, we obtain the desired inequality. □

The following corollary follows from Theorem 2.19.

Corollary 2.20. *Let* $\mathbf{T} = (T_1, T_2) \in \mathcal{B}(\mathcal{H})^2$ *be a* 2*-tuple of operators. Then*

$$\mathbf{ber}^2(\mathbf{T}) \leq \frac{1}{2}\left(\left\|T_1^*T_1 + T_2^*T_2\right\|_{ber} + \left\|T_1^*T_2 + T_2^*T_1\right\|_{ber} + \mathbf{ber}\left[\left(T_1^* - T_2^*\right)(T_1 + T_2)\right]\right).$$

Proof. By the inequality (9) for $T_1 + T_2, T_1 - T_2$ instead of T_1, T_2 and perform the required calculations, then we get

$$\mathbf{ber}^2(T_1 + T_2, T_1 - T_2) \leq \left\|T_1^*T_1 + T_2^*T_2\right\|_{ber} + \left\|T_1^*T_2 + T_2^*T_1\right\|_{ber} + \mathbf{ber}\left(\left(T_1^* - T_2^*\right)(T_1 + T_2)\right).$$

Furthermore, by using the inequality (3) we get

$$2\mathbf{ber}^2(\mathbf{T}) \leq \left\|T_1^*T_1 + T_2^*T_2\right\|_{ber} + \left\|T_1^*T_2 + T_2^*T_1\right\|_{ber} + \mathbf{ber}\left(\left(T_1^* - T_2^*\right)(T_1 + T_2)\right).$$

Thus,

$$\mathbf{ber}^2(\mathbf{T}) \leq \frac{1}{2}\left(\left\|T_1^*T_1 + T_2^*T_2\right\|_{ber} + \left\|T_1^*T_2 + T_2^*T_1\right\|_{ber} + \mathbf{ber}\left(\left(T_1^* - T_2^*\right)(T_1 + T_2)\right)\right),$$

as required. □

The next theorem provides an upper and lower bound of Euclidean Berezin number of a pair of operators.

Theorem 2.21. *Let* $\mathbf{T} = (T_1, T_2) \in \mathcal{B}(\mathcal{H})^2$ *be a* 2*-tuple of operators. Then*

$$\frac{\sqrt{2}}{2}\max\left\{\mathbf{ber}(T_1 + T_2), \mathbf{ber}(T_1 - T_2)\right\} \leq \mathbf{ber}(\mathbf{T}) \leq \frac{\sqrt{2}}{2}\sqrt{\mathbf{ber}^2(T_1 + T_2) + \mathbf{ber}^2(T_1 - T_2)}.$$

Proof. Let $\hat{k}_\lambda$ be the normalized reproducing kernel of $\mathcal{H}$. Then, we have

$$\begin{aligned}
\left|\left\langle T_1\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 + \left|\left\langle T_2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 &\geq \frac{1}{2}\left(\left|\left\langle T_1\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right| + \left|\left\langle T_2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|\right)^2 \\
&\geq \frac{1}{2}\left(\left|\left\langle T_1\hat{k}_\lambda, \hat{k}_\lambda\right\rangle \pm \left\langle T_2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|\right)^2 \\
&= \frac{1}{2}\left(\left|\left\langle (T_1 \pm T_2)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|\right)^2.
\end{aligned}$$

Therefore,

$$\sqrt{\left|\left\langle T_1\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 + \left|\left\langle T_2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2} \geq \frac{\sqrt{2}}{2}\left|\left\langle (T_1 \pm T_2)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|.$$

Taking supremum over $\lambda \in \Omega$, we get

$$\mathbf{ber}(\mathbf{T}) \geq \frac{\sqrt{2}}{2}\mathbf{ber}(T_1 \pm T_2).$$

This implies that

$$\mathbf{ber}(\mathbf{T}) \geq \frac{\sqrt{2}}{2}\max\left\{\mathbf{ber}(T_1 + T_2), \mathbf{ber}(T_1 - T_2)\right\}.$$

This proves the first inequality.

For the second inequality, we have

$$\left|\left\langle T_1\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right|^2+\left|\left\langle T_2\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right|^2\leq \mathbf{ber}^2\left(\mathbf{T}\right).$$

Now, we apply the parallelogram identity for complex numbers, we obtain

$$\begin{aligned}\left|\left\langle T_1\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right|^2+\left|\left\langle T_2\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right|^2 &= \frac{1}{2}\left(\left|\left\langle T_1\hat{k}_\lambda,\hat{k}_\lambda\right\rangle+\left\langle T_2\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right|^2+\left|\left\langle T_1\hat{k}_\lambda,\hat{k}_\lambda\right\rangle-\left\langle T_2\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right|^2\right)\\ &= \frac{1}{2}\left(\left|\left\langle \left(T_1+T_2\right)\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right|^2+\left|\left\langle \left(T_1-T_2\right)\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right|^2\right)\\ &\leq \frac{1}{2}\left(\mathbf{ber}^2\left(T_1+T_2\right)+\mathbf{ber}^2\left(T_1-T_2\right)\right).\end{aligned}$$

Thus,

$$\sqrt{\left|\left\langle T_1\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right|^2+\left|\left\langle T_2\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right|^2}\leq\frac{\sqrt{2}}{2}\sqrt{\mathbf{ber}^2\left(T_1+T_2\right)+\mathbf{ber}^2\left(T_1-T_2\right)}.$$

Taking supremum over $\lambda\in\Omega$, we get

$$\mathbf{ber}\left(\mathbf{T}\right)\leq\frac{\sqrt{2}}{2}\sqrt{\mathbf{ber}^2\left(T_1+T_2\right)+\mathbf{ber}^2\left(T_1-T_2\right)}.$$

Hence, the proof of this theorem is complete. □

As a consequence of Theorem 2.21, we have the following two corollaries.

Corollary 2.22. *Let $\mathbf{T}=(T_1,T_2)\in\mathcal{B}(\mathcal{H})^2$ be a 2-tuple of operators. If T_1 and T_2 are positive operators, then*

$$\frac{\sqrt{2}}{2}\max\left\{\left\|T_1+T_2\right\|_{ber},\mathbf{ber}\left(T_1-T_2\right)\right\}\leq\mathbf{ber}\left(\mathbf{T}\right)\leq\frac{\sqrt{2}}{2}\sqrt{\left\|T_1+T_2\right\|_{ber}^2+\left\|T_1-T_2\right\|_{ber}^2}.$$

Corollary 2.23. *Let $T\in\mathcal{B}(\mathcal{H})$ and $\alpha,\beta\in\mathbb{C}$. Then*

$$\begin{aligned}\frac{\sqrt{2}}{2}\max\left\{\mathbf{ber}\left(\alpha T+\beta T^*\right),\mathbf{ber}\left(\alpha T-\beta T^*\right)\right\} &\leq \sqrt{|\alpha|^2+\left|\beta\right|^2}\,\mathbf{ber}\left(T\right)\\ &\leq \frac{\sqrt{2}}{2}\sqrt{\mathbf{ber}^2\left(\alpha T+\beta T^*\right)+\mathbf{ber}^2\left(\alpha T-\beta T^*\right)}.\end{aligned}$$

Proof. Choosing $T_1=\alpha T$ and $T_2=\beta T^*$ in Theorem 2.21, we get the desired result. □

Next, we need the following lemma.

Lemma 2.24. *[1] Let $a_1,a_2,...,a_n$ be finite positive sequence of real numbers and $p\geq 2$. Then*

$$\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^p\leq\frac{1}{n}\sum_{k=1}^{n}a_k^p-\frac{1}{n}\sum_{k=1}^{n}\left|a_k-\frac{1}{n}\sum_{j=1}^{n}a_j\right|^p.$$

We now prove the following theorem.

Theorem 2.25. *Let $\mathbf{T}=(T_1,T_2)\in\mathcal{B}(\mathcal{H})^2$ be a 2-tuple of operators. Then*

$$\mathbf{ber}^{2r}\left(\mathbf{T}\right)\leq\frac{1}{2}\left(\mathbf{ber}^{2r}\left(T_1+T_2\right)+\mathbf{ber}^{2r}\left(T_1-T_2\right)\right)-2^r\inf_{\lambda\in\Omega}\left|\mathrm{Re}\left\langle T_1\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\overline{\left\langle T_2\hat{k}_\lambda,\hat{k}_\lambda\right\rangle}\right|,$$

for all $r \geq 1$.

Proof. Let $\hat{k}_\lambda$ be the normalized reproducing kernel of $\mathcal{H}$. Then, we have

$$\begin{aligned}
&\left(\left|\left\langle T_1\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 + \left|\left\langle T_2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2\right)^r \\
= &\left(\frac{1}{2}\left|\left\langle T_1\hat{k}_\lambda, \hat{k}_\lambda\right\rangle + \left\langle T_2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 + \frac{1}{2}\left|\left\langle T_1\hat{k}_\lambda, \hat{k}_\lambda\right\rangle - \left\langle T_2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2\right)^r \\
= &\left(\frac{1}{2}\left|\left\langle (T_1 + T_2)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 + \frac{1}{2}\left|\left\langle (T_1 - T_2)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2\right)^r \\
\leq &\frac{1}{2}\left|\left\langle (T_1 + T_2)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^{2r} + \frac{1}{2}\left|\left\langle (T_1 - T_2)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^{2r} \\
&-\frac{1}{2}\left|\left|\frac{1}{2}\left\langle (T_1 + T_2)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 - \frac{1}{2}\left|\left\langle (T_1 - T_2)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2\right|^r \\
&-\frac{1}{2}\left|\left|\frac{1}{2}\left\langle (T_1 - T_2)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 - \frac{1}{2}\left|\left\langle (T_1 + T_2)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2\right|^r \\
&\text{(by Lemma 2.24)} \\
= &\frac{1}{2}\left|\left\langle (T_1 + T_2)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^{2r} + \frac{1}{2}\left|\left\langle (T_1 - T_2)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^{2r} \\
&-\frac{1}{2^r}\left|\left|\left\langle (T_1 - T_2)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 - \left|\left\langle (T_1 + T_2)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2\right|^r \\
= &\frac{1}{2}\left|\left\langle (T_1 + T_2)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^{2r} + \frac{1}{2}\left|\left\langle (T_1 - T_2)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^{2r} \\
&-\frac{2^{2r}}{2^r}\left|\mathrm{Re}\left\langle T_1\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\overline{\left\langle T_2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle}\right| \\
\leq &\frac{1}{2}\left(\mathbf{ber}^{2r}(T_1 + T_2) + \mathbf{ber}^{2r}(T_1 - T_2)\right) - 2^r \inf_{\lambda\in\Omega}\left|\mathrm{Re}\left\langle T_1\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\overline{\left\langle T_2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle}\right|.
\end{aligned}$$

Therefore, we infer that

$$\begin{aligned}
\left(\left|\left\langle T_1\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 + \left|\left\langle T_2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2\right)^r \leq &\frac{1}{2}\left(\mathbf{ber}^{2r}(T_1 + T_2) + \mathbf{ber}^{2r}(T_1 - T_2)\right) \\
&-2^r \inf_{\lambda\in\Omega}\left|\mathrm{Re}\left\langle T_1\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\overline{\left\langle T_2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle}\right|.
\end{aligned}$$

Taking supremum over $\lambda \in \Omega$ in the above inequality, we get the desired inequality. □

Next, we need the following lemma.

Lemma 2.26. *[21] Let $T \in B(\mathcal{H})$ and let f and g be non-negative continuous functions on $[0, +\infty)$ such that $f(t)g(t) = t$ for all $t \in [0, +\infty)$. Then*

$$\left|\langle Tx, y\rangle\right| \leq \left\|f(|T|)x\right\| \left\|g(|T^*|)y\right\|,$$

for all $x, y \in \mathcal{H}$.
Finally, we present the following result.

Theorem 2.27. *Let $T_1, T_2 \in \mathcal{B}(\mathcal{H})$ and let f, g be two non-negative continuous functions on $[0, +\infty)$ satisfying $f(t)g(t) = t$ for all $t \in [0, +\infty)$. Then*

$$\frac{1}{2}\|T_1 + T_2\|_{ber}^2 \leq \mathbf{ber}\left(f^2(|T_1|), f^2(|T_2|)\right)\mathbf{ber}\left(g^2\left(\left|T_1^*\right|\right), g^2\left(\left|T_2^*\right|\right)\right).$$

Proof. Let $\hat{k}_\lambda$ and $\hat{k}_\mu$ be two normalized reproducing kernels of $\mathcal{H}$. Then, we have

$$\begin{aligned}
&\left|\left\langle (T_1+T_2)\hat{k}_\lambda,\hat{k}_\mu\right\rangle\right|^2\\
=\ &\left|\left\langle T_1\hat{k}_\lambda,\hat{k}_\mu\right\rangle+\left\langle T_2\hat{k}_\lambda,\hat{k}_\mu\right\rangle\right|^2\\
\leq\ &\left(\left|\left\langle T_1\hat{k}_\lambda,\hat{k}_\mu\right\rangle\right|+\left|\left\langle T_2\hat{k}_\lambda,\hat{k}_\mu\right\rangle\right|\right)^2\\
\leq\ &2\left(\left|\left\langle T_1\hat{k}_\lambda,\hat{k}_\mu\right\rangle\right|^2+\left|\left\langle T_2\hat{k}_\lambda,\hat{k}_\mu\right\rangle\right|^2\right)\\
\leq\ &2\left(\left\|f(|T_1|)\hat{k}_\lambda\right\|^2\left\|g\left(\left|T_1^*\right|\right)\hat{k}_\mu\right\|^2+\left\|f(|T_2|)\hat{k}_\lambda\right\|^2\left\|g\left(\left|T_2^*\right|\right)\hat{k}_\mu\right\|^2\right)\\
&\text{(Lemma 2.26)}\\
=\ &2\left(\left\langle f^2(|T_1|)\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\left\langle g^2\left(\left|T_1^*\right|\right)\hat{k}_\mu,\hat{k}_\mu\right\rangle+\left\langle f^2(|T_2|)\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\left\langle g^2\left(\left|T_2^*\right|\right)\hat{k}_\mu,\hat{k}_\mu\right\rangle\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\left|\left\langle (T_1+T_2)\hat{k}_\lambda,\hat{k}_\mu\right\rangle\right|^2\\
\leq\ &2\left(\left\langle f^2(|T_1|)\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\left\langle g^2\left(\left|T_1^*\right|\right)\hat{k}_\mu,\hat{k}_\mu\right\rangle+\left\langle f^2(|T_2|)\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\left\langle g^2\left(\left|T_2^*\right|\right)\hat{k}_\mu,\hat{k}_\mu\right\rangle\right).
\end{aligned}$$

Now, on utilizing the elementary inequality: $ab+cd\leq\sqrt{a^2+c^2}\sqrt{b^2+d^2}$ for $a,b,c,d\in\mathbb{R}$, we get

$$\begin{aligned}
&\left|\left\langle (T_1+T_2)\hat{k}_\lambda,\hat{k}_\mu\right\rangle\right|^2\\
\leq\ &2\left(\left\langle f^2(|T_1|)\hat{k}_\lambda,\hat{k}_\lambda\right\rangle^2+\left\langle f^2(|T_2|)\hat{k}_\lambda,\hat{k}_\lambda\right\rangle^2\right)^{\frac{1}{2}}\left(\left\langle g^2\left(\left|T_1^*\right|\right)\hat{k}_\mu,\hat{k}_\mu\right\rangle^2+\left\langle g^2\left(\left|T_2^*\right|\right)\hat{k}_\mu,\hat{k}_\mu\right\rangle^2\right)^{\frac{1}{2}}\\
\leq\ &2\mathbf{ber}\left(f^2(|T_1|),f^2(|T_2|)\right)\mathbf{ber}\left(g^2\left(\left|T_1^*\right|\right),g^2\left(\left|T_2^*\right|\right)\right).
\end{aligned}$$

Taking supremum over $\lambda,\mu\in\Omega$, we get

$$\frac{1}{2}\|T_1+T_2\|_{ber}^2\leq\mathbf{ber}\left(f^2(|T_1|),f^2(|T_2|)\right)\mathbf{ber}\left(g^2\left(\left|T_1^*\right|\right),g^2\left(\left|T_2^*\right|\right)\right),$$

as desired. □

If we take $f(t)=g(t)=\sqrt{t}$ in the above theorem, then we get the following corollary.

Corollary 2.28. *If $T_1,T_2\in\mathcal{B}(\mathcal{H})$, then*

$$\frac{1}{2}\|T_1+T_2\|_{ber}^2\leq\mathbf{ber}(|T_1|,|T_2|)\,\mathbf{ber}\left(\left|T_1^*\right|,\left|T_2^*\right|\right).$$

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