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Duals of generalized Orlicz Hilbert sequence spaces and matrix transformations

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Abstract. In this paper, we define the sequence spaces $h_c\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)$, $h_0\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)$ and $h_\infty\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)$ resulting from the infinite Hilbert matrix and the Musielak-Orlicz function. We give some topological properties and inclusion relations of these newly created spaces. We also identified $\alpha-, \beta-$ and $\gamma-$duals of the spaces. Finally, we tried to characterize some matrix transformations between these spaces.

1. Introduction and Preliminaries:

Hilbert defined the Hilbert matrix in 1894. The Hilbert matrix played both several branches of mathematics and computational sciences.The $n \times n$ matrix $H = \left(h_{i,j}\right) = \frac{1}{i+j-1}$ $(i, j \in \mathbb{N})$ is a Hilbert matrix [1]. We consider the infinite Hilbert matrix H as follows:

$$H = \begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 & ... \\ 1/2 & 1/3 & 1/4 & ... & . \\ 1/3 & 1/4 & ... & . & . \\ 1/4 & ... & . & . & . \\ . & . & . & . & \\ . & . & . & & \\ . & . & & & \end{pmatrix}$$

and it can be showed in integral form as follows:

$$H = (h_{i,j}) = \int_0^1 x^{i+j-2} dx$$

The inverse of Hilbert matrix H^{-1} is defined by

$$H^{-1} = \left(h_{i,j}^{-1}\right) = (-1)^{i+j} (i + j - 1) \begin{pmatrix} n + i - 1 \\ n - j \end{pmatrix} \begin{pmatrix} n + j - 1 \\ n - i \end{pmatrix} \begin{pmatrix} i + j - 1 \\ i - 1 \end{pmatrix}^2$$

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for all $i, j \in \mathbb{N}$.

Let us we denote the space of all real or complex sequence with w. We write the sequence spaces of all convergent, null and bounded sequences by c, c_0 and l_∞, respectively. Also we will denote the space of all bounded, convergent and absolutely convergent series with bs, cs, and l_1 respectively. Let $A = (a_{nk})$ be an infinite matrix of real or complex numbers and X, Y be subsets of w. We write $A_n(x) = \sum_k a_{nk}x_k$ and $Ax = A_n(x)$ for $n, k \in \mathbb{N}$. For a sequence space X, the matrix domain of an infinite matrix A is defined by

$$X_A = \{x = (x_k) \in w : Ax \in X\}$$

which is also a sequence space. We denote with (X, Y) the class of all matrices A such that $A : X \to Y$.

A matrix $A = (a_{nk})$ is called a triangle if $a_{nk} = 0$ for $k > n$ and $a_{nn} \neq 0$ for all $n \in \mathbb{N}$. For the triangle matrices A,B and a sequence x, $A(Bx) = (AB)x$ holds. We remark that the triangle matrix A uniquely has an inverse matrix $A^{-1} = B$ and the matrix B is also triangle.

Let X be a normed sequence space. A sequence (b_n) in X is called a Schauder basis for X if for every $x \in X$ there is a unique sequence (α_n) of scalars such that

$$\lim_n \left\| x - \sum_{k=0}^{n} \alpha_k b_k \right\| = 0.$$

A B-space is a complete normed space. A topological sequence space in which all coordinate functionals $\pi_k, \pi_k(x) = x_k$, are continuous is called a K-space. A BK-space is defined as a K-space which is also a B-space, that is, a BK-space is a Banach space with continuous coordinates. For example, the space $l_p (1 \leq p < \infty)$ is BK-space with $\|x\|_p = \left(\sum_{k=0}^{\infty} |x_k|^p\right)^{\frac{1}{p}}$ and c, c_0 and l_∞ are BK-space with $\|x\|_\infty = \sup_k |x_k|$.

Kızmaz [2] was firstly introduced the concept of the difference operator in the sequence spaces. Further Et and Çolak [3] generalized the idea of difference sequence spaces of Kızmaz.Besides this topic was studied by many authors ([4], [5]) . Now, the difference matrix $\Delta = (\delta_{nk})$ defined by

$$\delta_{nk} = \begin{cases} (-1)^{n-k}, & (n-1 \leq k \leq n) \\ 0, & (0 < n-1 \text{ or } n > k). \end{cases}$$

The difference operator order m is defined $\Delta^{(m)} : w \to w, \left(\Delta^{(1)}x\right)_k = (x_k - x_{k-1})$ and $\Delta^{(m)}x = \left(\Delta^{(1)}x\right)_k \circ \left(\Delta^{(m-1)}x\right)_k$ for $m \geq 2$.

The triangle matrix $\Delta^{(m)} = \left(\delta_{nk}^{(m)}\right)$ defined by

$$\delta_{nk}^{(m)} = \begin{cases} (-1)^{n-k} \begin{pmatrix} m \\ n-k \end{pmatrix}, & (\max\{0, n-m\} \leq k \leq n) \\ 0, & (0 \leq k < \max\{0, n-m\} \text{ or } n > k) \end{cases}$$

for all $k, n \in \mathbb{N}$ and for any fixed $m \in \mathbb{N}$.

Let $v = (v_k)$ be any fixed sequence of nonzero complex numbers. We define operators $\Delta^{(m)} : w \to w$ by $m \in \mathbb{N}$, $\Delta_v^{(0)}x_k = v_k x_k$, $\Delta_v x_k = (v_k x_k - v_{k-1}x_{k-1})$, $\Delta_v^{(m)}x_k = \Delta_v^{(m-1)}x_k - \Delta_v^{(m-1)}x_{k-1}$ and so that

$$\Delta_v^{(m)}x_k = \sum_{i=0}^{m} (-1)^i \begin{pmatrix} m \\ i \end{pmatrix} v_{k-i}x_{k-i}$$

Polat [6] and Kirisci and Polat [7] have defined some new sequence spaces using Hilbert matrix. Let h_c, h_0 and h_∞ be convergent Hilbert , null convergent Hilbert and bounded Hilbert sequence spaces, respectively.

An Orlicz function is a function $M:[0,\infty)\to[0,\infty)$ which is continuous, non-decreasing and convex such that $M(0)=0$, $M(x)>0$ for $x>0$ and $M(x)\to\infty$ as $x\to\infty$. Lindenstrauss and Tzafriri [8] used the idea of Orlicz function to construct the following sequence space

$$\ell_M=\left\{x\in w:\sum_{k=1}^{\infty}M\left(\frac{|x_k|}{\rho}\right)<\infty,\ \text{for some } \rho>0\right\}$$

l_M is a Banach space with the norm

$$\|x\|=\inf\left\{\rho>0:\sum_{k=1}^{\infty}M\left(\frac{|x_k|}{\rho}\right)\leq 1\right\}$$

which is called an Orlicz sequence space. A sequence $\mathcal{M}=(M_k)$ of Orlicz functions is called the Musielak-Orlicz function (see [9], [10]). For more details on sequence spaces, see ([11], [12], [13]) and the references there in.

Let$X$ be a linear metric space. A function $p:X\to\mathbb{R}$ is called a paranorm, if

$(P1)$ $p(x)\geq 0$ for $x\in X$,

$(P2)$ $p(-x)=p(x)$ for all $x\in X$,

$(P3)$ $p(x+y)\leq p(x)+p(y)$ for all $x,y\in X$,

$(P4)$ If (λ_n) is a sequence of scalars with $\lambda_n\to\lambda$ as $n\to\infty$ and (x_n) is a sequence of vectors with $p(x_n-x)\to 0$ as $n\to\infty$, then $p(\lambda_n x_n-\lambda x)\to 0$ as $n\to\infty$.

A paranorm p for which $p(x)=0$ implies $x=0$ is called total paranorm and the pair (X,p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see ([9], Theorem 10.4.2, page 183)).

2. Main Results

In this section we define the sequence spaces $h_c\left(\Delta_v^{(m)},\mathcal{M},u,p\right)$, $h_0\left(\Delta_v^{(m)},\mathcal{M},u,p\right)$ and $h_\infty\left(\Delta_v^{(m)},\mathcal{M},u,p\right)$ and give some relations between them. These sequence spaces are linear and BK-spaces. We prove that the new Hilbert sequence spaces $h_c\left(\Delta_v^{(m)},\mathcal{M},u,p\right)$, $h_0\left(\Delta_v^{(m)},\mathcal{M},u,p\right)$ and $h_\infty\left(\Delta_v^{(m)},\mathcal{M},u,p\right)$ are isometrically isomorphic to the space c, c_0 and l_∞ respectively.

Definition 2.1. Let $\mathcal{M}$=(M_k) be a sequence of Orlicz functions, $v=(v_k)$ be any fixed sequence of non-zero complex numbers. Also, let $p=(p_k)$ and $u=(u_k)$ be the bounded sequence and sequence of positive real numbers, respectively and $H=\left(h_{i,j}\right)$ be an infinite Hilbert matrix. In the present paper we have defined the following sequence spaces:

$$h_c\left(\Delta_v^{(m)},\mathcal{M},u,p\right)=\left\{x=(x_k)\in w:\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k-1}\left[M_k\left(\frac{\left|u_k\Delta_v^{(m)}x_k\right|}{\rho}\right)\right]^{p_k}\text{ exists, for some }\rho>0\right\},$$

$$h_0\left(\Delta_v^{(m)},\mathcal{M},u,p\right)=\left\{x=(x_k)\in w:\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k-1}\left[M_k\left(\frac{\left|u_k\Delta_v^{(m)}x_k\right|}{\rho}\right)\right]^{p_k}=0,\text{ for some }\rho>0\right\},$$

$$h_\infty\left(\Delta_v^{(m)},\mathcal{M},u,p\right)=\left\{x=(x_k)\in w:\sup_{n}\sum_{k=1}^{n}\frac{1}{n+k-1}\left[M_k\left(\frac{\left|u_k\Delta_v^{(m)}x_k\right|}{\rho}\right)\right]^{p_k}<\infty,\text{ for some }\rho>0\right\}.$$

If we take $M_k(x)=x$ for all $k\in\mathbb{N}$ and $(v_k)=(1,1,...)$, $M_k(x)=x$ for all $k\in\mathbb{N}$, we obtain that $h_c\left(\Delta_v^{(m)},u,p\right)$, $h_0\left(\Delta_v^{(m)},u,p\right)$, $h_\infty\left(\Delta_v^{(m)},u,p\right)$ and $h_c\left(\Delta^{(m)},u,p\right)$, $h_0\left(\Delta^{(m)},u,p\right)$, $h_\infty\left(\Delta^{(m)},u,p\right)$, respectively. Also if $(u_k)=(1)$ and $(p_k)=(1)$, for all $k\in\mathbb{N}$, we obtain $h_c\left(\Delta_v^{(m)},\mathcal{M}\right)$, $h_0\left(\Delta_v^{(m)},\mathcal{M}\right)$ and $h_\infty\left(\Delta_v^{(m)},\mathcal{M}\right)$.

We define the sequence $y = (y_n)$ which will be frequently used, as the $H\Delta_v^{(m)}$-transform of a sequence as follows:

$$(y_n) = \left(H\Delta_v^{(m)}x\right)^{(\mathcal{M},u,p)} = \sum_{k=1}^{n} \frac{1}{n+k-1}\left[M_k\left(\frac{\left|u_k \sum_{i=k}^{n}(-1)^{i-k}\binom{m}{i-k}x_k v_k\right|}{\rho}\right)\right]^{p_k} \tag{2.1}$$

for each $k, m, n \in \mathbb{N}$.

We will use the following inequality throughout the paper. If $0 < p_k \leq \sup p_k = H$, $D = \max\left(1, 2^{H-1}\right)$, then

$$|a_k + b_k|^{p_k} \leq D\left\{|a_k|^{p_k} + |b_k|^{p_k}\right\} \tag{2.2}$$

for all k and $a_k, b_k \in \mathbb{C}$.

Theorem 2.2. Let $\mathcal{M}$=(M_k) be a sequence of Orlicz functions, $v = (v_k)$ be any fixed sequence of non-zero complex numbers. Also, let $p = (p_k)$ and $u = (u_k)$ be the bounded sequence and sequence of positive real numbers, respectively. Then $h_c\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right), h_0\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)$ and $h_\infty\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)$ are linear spaces over the complex field $\mathbb{C}$.

Proof. We shall prove the assertion for $h_\infty\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)$ only and others can be proved similarly. Let $x = (x_k), y = (y_k) \in h_\infty\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\sup_n \sum_{k=1}^{n} \frac{1}{n+k-1}\left[M_k\left(\frac{\left|u_k\Delta_v^{(m)}x_k\right|}{\rho_1}\right)\right]^{p_k} < \infty,$$

$$\sup_n \sum_{k=1}^{n} \frac{1}{n+k-1}\left[M_k\left(\frac{\left|u_k\Delta_v^{(m)}y_k\right|}{\rho_2}\right)\right]^{p_k} < \infty,$$

for some $\rho_1, \rho_2 > 0$. Let $\rho_3 = \max\left(2|\alpha|\rho_1, 2\left|\beta\right|\rho_2\right)$. Since $\mathcal{M} = (M_k)$ is a non-decreasing and convex, using (2.2) inequality, we have

$$\begin{aligned}
&\sup_n \textstyle\sum_{k=1}^{n} \frac{1}{n+k-1}\left[M_k\left(\frac{\left|u_k\Delta_v^{(m)}(\alpha x_k + \beta y_k)\right|}{\rho_3}\right)\right]^{p_k} \\
\leq\ & \sup_n \sum_{k=1}^{n} \frac{1}{n+k-1}\left[M_k\left(\frac{\left|u_k\Delta_v^{(m)}\alpha x_k\right|}{\rho_3} + \frac{\left|u_k\Delta_v^{(m)}\beta y_k\right|}{\rho_3}\right)\right]^{p_k} \\
\leq\ & D\sup_n \sum_{k=1}^{n} \frac{1}{n+k-1}\left[M_k\left(\frac{\left|u_k\Delta_v^{(m)}x_k\right|}{\rho_1}\right)\right]^{p_k} \\
& +D\sup_n \sum_{k=1}^{n} \frac{1}{n+k-1}\left[M_k\left(\frac{\left|u_k\Delta_v^{(m)}y_k\right|}{\rho_2}\right)\right]^{p_k} \\
<\ & \infty.
\end{aligned}$$

Thus, it becomes $\alpha x + \beta y \in h_\infty\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)$. This proves that $h_\infty\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)$ is linear space. Similarly, it can be proved that $h_c\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)$ and $h_0\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)$ are linear spaces.

Theorem 2.3. Let $\mathcal{M}$=(M_k) be a sequence of Orlicz functions, $v = (v_k)$ be any fixed sequence of non-zero complex numbers. Also, let $p = (p_k)$ and $u = (u_k)$ be the bounded sequence and sequence of positive real numbers, respectively. Then $h_\infty\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)$ is a paranorm space with the following paranorm,

$$g(x)=\inf\left\{(\rho)^{\frac{p_k}{G}}:\left(\sup_n\sum_{k=1}^{n}\frac{1}{n+k-1}\left[M_k\left(\frac{\left|u_k\Delta_v^{(m)}x_k\right|}{\rho}\right)\right]^{p_k}\right)^{\frac{1}{G}}\leq 1,\ \text{for some }\rho>0\right\},$$

where $0<p_k\leq\sup p_k=H$ and $G=\max(1,H)$.

Proof. (i) Clearly $g(x)\geq 0$ for $x=(x_k)\in h_\infty\left(\Delta_v^{(m)},\mathcal{M},u,p\right)$. Since $M_k(0)=0$, we get $g(\theta)=0$.

(ii) $g(-x)=g(x)$ is trivial.

(iii) Let x, $y\in h_\infty\left(\Delta_v^{(m)},\mathcal{M},u,p\right)$. Then there exist pozitive numbers ρ_1, ρ_2 such that

$$\sup_n\sum_{k=1}^{n}\frac{1}{n+k-1}\left[M_k\left(\frac{\left|u_k\Delta_v^{(m)}x_k\right|}{\rho_1}\right)\right]^{p_k}\leq 1$$

$$\sup_n\sum_{k=1}^{n}\frac{1}{n+k-1}\left[M_k\left(\frac{\left|u_k\Delta_v^{(m)}y_k\right|}{\rho_2}\right)\right]^{p_k}\leq 1$$

Let $\rho=\rho_1+\rho_2$. Then by using Minkowski's inequality, we have

$$\begin{aligned}
&\sup_n\sum_{k=1}^{n}\frac{1}{n+k-1}\left[M_k\left(\frac{\left|u_k\Delta_v^{(m)}(x_k+y_k)\right|}{\rho}\right)\right]^{p_k}\\
=\ &\sup_n\sum_{k=1}^{n}\frac{1}{n+k-1}\left[M_k\left(\frac{\left|u_k\Delta_v^{(m)}(x_k+y_k)\right|}{\rho_1+\rho_2}\right)\right]^{p_k}\\
\leq\ &\frac{\rho_1}{\rho_1+\rho_2}\sup_n\sum_{k=1}^{n}\frac{1}{n+k-1}\left[M_k\left(\frac{\left|u_k\Delta_v^{(m)}x_k\right|}{\rho_1}\right)\right]^{p_k}\\
&+\frac{\rho_2}{\rho_1+\rho_2}\sup_n\sum_{k=1}^{n}\frac{1}{n+k-1}\left[M_k\left(\frac{\left|u_k\Delta_v^{(m)}y_k\right|}{\rho_2}\right)\right]^{p_k}\\
\leq\ &1
\end{aligned}$$

and thus,

$$\begin{aligned}
g(x+y)&=\inf\left\{(\rho)^{\frac{p_k}{G}}:\left(\sup_n\sum_{k=1}^{n}\frac{1}{n+k-1}\left[M_k\left(\frac{\left|u_k\Delta_v^{(m)}(x_k+y_k)\right|}{\rho}\right)\right]^{p_k}\right)^{\frac{1}{G}}\leq 1,\ \text{for some }\rho>0\right\}\\
&\leq\inf\left\{(\rho_1)^{\frac{p_k}{G}}:\left(\sup_n\sum_{k=1}^{n}\frac{1}{n+k-1}\left[M_k\left(\frac{\left|u_k\Delta_v^{(m)}(x_k+y_k)\right|}{\rho_1}\right)\right]^{p_k}\right)^{\frac{1}{G}}\leq 1,\ \text{for some }\rho>0\right\}\\
&\quad+\inf\left\{(\rho_2)^{\frac{p_k}{G}}:\left(\sup_n\sum_{k=1}^{n}\frac{1}{n+k-1}\left[M_k\left(\frac{\left|u_k\Delta_v^{(m)}(x_k+y_k)\right|}{\rho_2}\right)\right]^{p_k}\right)^{\frac{1}{G}}\leq 1,\ \text{for some }\rho>0\right\}.
\end{aligned}$$

Therefore, $g(x+y)\leq g(x)+g(y)$.

Finally, we prove that the scalar multiplication is continuous. Let λ be any complex number. By definition,

$$\begin{aligned} g(\lambda x) &= \inf\left\{(\rho)^{\frac{p_k}{G}} : \left(\sup_n \sum_{k=1}^{n} \frac{1}{n+k-1}\left[M_k\left(\frac{\left|u_k\Delta_v^{(m)}(\lambda x_k)\right|}{\rho}\right)\right]^{p_k}\right)^{\frac{1}{G}} \leq 1, \text{ for some } \rho > 0\right\} \\ &= \inf\left\{(|\lambda|\, t)^{\frac{p_k}{G}} : \left(\sup_n \sum_{k=1}^{n} \frac{1}{n+k-1}\left[M_k\left(\frac{\left|u_k\Delta_v^{(m)}(x_k)\right|}{t}\right)\right]^{p_k}\right)^{\frac{1}{G}} \leq 1, \text{ for some } \rho > 0\right\} \end{aligned}$$

where $t = \dfrac{\rho}{|\lambda|} > 0$. Since $|\lambda|^{p_k} \leq \max(1, |\lambda|^{\sup p_k})$, we have

$$g(\lambda x) \leq \max(1, |\lambda|^{\sup p_k}) . \inf\left\{(t)^{\frac{p_k}{G}} : \left(\sup_n \sum_{k=1}^{n} \frac{1}{n+k-1}\left[M_k\left(\frac{\left|u_k\Delta_v^{(m)}(x_k)\right|}{t}\right)\right]^{p_k}\right)^{\frac{1}{G}} \leq 1, \text{ for some } \rho > 0\right\}.$$

So the fact that scalar multiplication is continuous is due to the above inequality. This completes the proof of the theorem.

Theorem 2.4. Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions, $u = (u_k)$ be a sequence of positive real numbers and $v = (v_k)$ be any fixed sequence of non-zero complex numbers. If for each k, $0 \leq p_k \leq q_k < \infty$, $p = (p_k)$ and $q = (q_k)$ are bounded sequences of positive real numbers, then $h_0\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right) \subseteq h_0\left(\Delta_v^{(m)}, \mathcal{M}, u, q\right)$.

Proof. Let $x \in h_0\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)$.Then

$$\sum_{k=1}^{n} \frac{1}{n+k-1}\left[M_k\left(\frac{\left|u_k\Delta_v^{(m)}(x_k)\right|}{\rho}\right)\right]^{p_k} \to 0 \text{ as } n \to \infty$$

This means that

$$\frac{1}{n+k-1}\left[M_k\left(\frac{\left|u_k\Delta_v^{(m)}(x_k)\right|}{\rho}\right)\right]^{p_k} < 1$$

for large enough k values. Since M_k is increasing and $p_k \leq q_k$, we have as $n \to \infty$

$$\sum_{k=1}^{n} \frac{1}{n+k-1}\left[M_k\left(\frac{\left|u_k\Delta_v^{(m)}(x_k)\right|}{\rho}\right)\right]^{q_k} \leq \sum_{k=1}^{n} \frac{1}{n+k-1}\left[M_k\left(\frac{\left|u_k\Delta_v^{(m)}(x_k)\right|}{\rho}\right)\right]^{p_k} \to 0.$$

Thus $x \in h_0\left(\Delta^{(m)}, \mathcal{M}, u, q\right)$. This completes the proof.

Theorem 2.5. Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions, $v = (v_k)$ be any fixed sequence of non-zero complex numbers and $\varphi = \lim\limits_{t\to\infty} \dfrac{M_k(t)}{t} > 0$. Then $h_0\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right) \subseteq h_0\left(\Delta_v^{(m)}, u, p\right)$.

Proof. Let $\varphi > 0$ to prove that $h_0\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right) \subseteq h_0\left(\Delta_v^{(m)}, u, p\right)$. From the definition of φ, $M_k(t) \geq \varphi(t)$, for all $t > 0$. Since $\varphi > 0$, we have $t \leq \frac{1}{\varphi} M_k(t)$ for all $t > 0$.

Let $x = (x_k) \in h_0\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)$. Thus, we have

$$\sum_{k=1}^{n} \frac{1}{n+k-1}\left[\frac{\left|u_k\Delta_v^{(m)}(x_k)\right|}{\rho}\right]^{p_k} \leq \sum_{k=1}^{n} \frac{1}{n+k-1}\left[M_k\left(\frac{\left|u_k\Delta_v^{(m)}(x_k)\right|}{\rho}\right)\right]^{p_k}$$

which means that $x = (x_k) \in h_0\left(\Delta_v^{(m)}, u, p\right)$. This completes the proof.

Theorem 2.6. Let $\mathcal{M}' = \left(M_k'\right)$ and $\mathcal{M}'' = \left(M_k''\right)$ be sequences of Orlicz functions and $v = (v_k)$ be any fixed sequence of non-zero complex numbers, then

$$h_0\left(\Delta_v^{(m)}, \mathcal{M}', u, p\right) \cap h_0\left(\Delta_v^{(m)}, \mathcal{M}'', u, p\right) \subseteq h_0\left(\Delta_v^{(m)}, \left(\mathcal{M}' + \mathcal{M}''\right), u, p\right).$$

Proof. Let $x = (x_k) \in h_0\left(\Delta_v^{(m)}, \mathcal{M}', u, p\right) \cap h_0\left(\Delta_v^{(m)}, \mathcal{M}'', u, p\right)$. Therefore,

$$\sum_{k=1}^{n} \frac{1}{n+k-1}\left[M_k'\left(\frac{\left|u_k \Delta^{(m)}(x_k)\right|}{\rho}\right)\right]^{p_k} \text{ as } n \to \infty$$

$$\sum_{k=1}^{n} \frac{1}{n+k-1}\left[M_k''\left(\frac{\left|u_k \Delta_v^{(m)}(x_k)\right|}{\rho}\right)\right]^{p_k} \text{ as } n \to \infty.$$

Then, we have

$$\begin{aligned}
&\textstyle\sum_{k=1}^{n} \frac{1}{n+k-1}\left[\left(M_k' + M_k''\right)\left(\frac{\left|u_k \Delta_v^{(m)}(x_k)\right|}{\rho}\right)\right]^{p_k} \\
&\leq K\left\{\sum_{k=1}^{n} \frac{1}{n+k-1}\left[M_k'\left(\frac{\left|u_k \Delta_v^{(m)}(x_k)\right|}{\rho}\right)\right]^{p_k}\right\} \\
&\quad + K\left\{\sum_{k=1}^{n} \frac{1}{n+k-1}\left[M_k''\left(\frac{\left|u_k \Delta_v^{(m)}(x_k)\right|}{\rho}\right)\right]^{p_k}\right\} \\
&\to 0 \text{ as } n \to \infty.
\end{aligned}$$

Thus,

$$\sum_{k=1}^{n} \frac{1}{n+k-1}\left[\left(M_k' + M_k''\right)\left(\frac{\left|u_k \Delta_v^{(m)}(x_k)\right|}{\rho}\right)\right]^{p_k} \to 0 \text{ as } n \to \infty.$$

Therefore, $x = (x_k) \in h_0\left(\Delta_v^{(m)}, \left(\mathcal{M}' + \mathcal{M}''\right), u, p\right)$ and this completes the proof.

Theorem 2.7. Let $\mathcal{M}' = \left(M_k'\right)$ and $\mathcal{M}'' = \left(M_k''\right)$ be sequences of Orlicz functions and $v = (v_k)$ be any fixed sequence of non-zero complex numbers, then

$$h_0\left(\Delta_v^{(m)}, \mathcal{M}', u, p\right) \subseteq h_0\left(\Delta_v^{(m)}, \left(\mathcal{M}' \circ \mathcal{M}''\right), u, p\right)$$

Proof. Let $x = (x_k) \in h_0\left(\Delta_v^{(m)}, \mathcal{M}', u, p\right)$. Then we have

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k-1}\left[M_k'\left(\frac{\left|u_k \Delta_v^{(m)}(x_k)\right|}{\rho}\right)\right]^{p_k} = 0.$$

Let $\varepsilon > 0$ and choose $\delta > 0$ with $0 < \delta < 1$ such that $M_k(t) < \varepsilon$, for $0 \leq t \leq \delta$.

Write $y_k = \frac{1}{n+k-1}\left[M_k'\left(\frac{\left|u_k\Delta_v^{(m)}(x_k)\right|}{\rho}\right)\right]$ and consider

$$\sum_{k=1}^{n}\left[M_k(y_k)\right]^{p_k} = \sum_{1}\left[M_k(y_k)\right]^{p_k} + \sum_{2}\left[M_k(y_k)\right]^{p_k}$$

where the first summation is over $y_k \le \delta$ and the second summation is over $y_k > \delta$. Since M_k is continuos, we have

$$\sum_{1}\left[M_k(y_k)\right]^{p_k} < \varepsilon^H \tag{2.3}$$

and for $y_k > \delta$, we use the fact that

$$y_k < \frac{y_k}{\delta} \le 1 + \frac{y_k}{\delta}.$$

From the definition, we have for $y_k > \delta$

$$M_k(y_k) < 2M_k(1)\frac{y_k}{\delta}.$$

Hence,

$$\sum_{1}\left[M_k(y_k)\right]^{p_k} \le \max\left(1, \left(2M_k(1)\,\delta^{-1}\right)^H\right)\sum_{1}\left[y_k\right]^{p_k} \tag{2.4}$$

From the equation (2.3) and (2.4), we have

$$h_0\left(\Delta_v^{(m)}, \mathcal{M}', u, p\right) \subseteq h_0\left(\Delta_v^{(m)}, \left(\mathcal{M}' \circ \mathcal{M}''\right), u, p\right).$$

Theorem 2.8. Hilbert sequence spaces $h_c\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)$, $h_0\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)$ and $h_\infty\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)$ are isometrically isomorphic to the space c, c_0 and l_∞ respectively, that is, $h_c\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right) \cong c$, $h_0\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right) \cong c_0$ and $h_\infty\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right) \cong l_\infty$.

Proof. We'll just do the proof for $h_0\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right) \cong c_0$ for the others it can be done similarly. To demonstrate the theorem, we must show that there is linear bijection between the space $h_0\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)$ and c_0. For this, we consider the transformation T defined by the notation (2.1), from $h_0\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)$ to c_0 by $x \to y = Tx$. The linearity of T is obvious. Moreover, when $Tx = \theta$ it is trivial that $x = \theta = (0, 0, 0...)$ and hence T is injective. Next, let $y = (y_n) \in c_0$ and the sequence $x = (x_n)$ is defined as follows:

$$x_n = v_n^{-1}\sum_{k=1}^{n}\left[\sum_{k=1}^{n}\binom{m+n-i-1}{i-k}h_{ik}^{-1}\right]y_k$$

where h_{ik}^{-1} is defined by (2.1). Then,

$$\begin{aligned}
\lim_{n\to\infty}\left(H\Delta_v^{(m)}x\right)_n^{\mathcal{M},u,p} &= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k-1}\left[M_k\left(\frac{\left|u_k\Delta_v^{(m)}(x_k)\right|}{\rho}\right)\right]^{p_k}\\
&= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k-1}\left[M_k\left(\frac{\left|u_k\sum_{i=0}^{m}(-1)^i\binom{m}{i}x_{k-i}v_{k-i}\right|}{\rho}\right)\right]^{p_k}\\
&= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k-1}\left[M_k\left(\frac{\left|u_k\sum_{i=k}^{n}(-1)^{i-k}\binom{m}{i-k}x_kv_k\right|}{\rho}\right)\right]^{p_k}\\
&= \lim_{n\to\infty}y_n=0.
\end{aligned}$$

Thus, $x \in h_0\left(\Delta_v^{(m)},\mathcal{M},u,p\right)$. As a result, it is clear that T is surjective. Since it is linear bijection, $h_0\left(\Delta_v^{(m)},\mathcal{M},u,p\right)$ and c_0 are linear isomorphic. This completes the proof.

Remark 2.9. It is well known that the spaces c, c_0 and l_∞ are BK-spaces. Let us considering the fact that $\Delta_v^{(m)}$ is a triangle, we can say that the Hilbert sequence spaces $h_c\left(\Delta_v^{(m)},\mathcal{M},u,p\right)$, $h_0\left(\Delta_v^{(m)},\mathcal{M},u,p\right)$ and $h_\infty\left(\Delta_v^{(m)},\mathcal{M},u,p\right)$ are BK-spaces with the norm defined by

$$\begin{aligned}
\|x\|_{\Delta}^{\mathcal{M},u,p} &= \left\|H\Delta_v^{(m)}x\right\|_\infty^{\mathcal{M},u,p}\\
&=\sup_n\left|\sum_{k=1}^{n}\frac{1}{n+k-1}\left[M_k\left(\frac{\left|u_k\sum_{i=0}^{m}(-1)^i\binom{m}{i}x_{k-i}v_{k-i}\right|}{\rho}\right)\right]^{p_k}\right| \qquad (2.5)
\end{aligned}$$

Corollary 2.10. Define the space $d^{(k)} = \left(d_n^{(k)}\left(\Delta_v^{(m)},\mathcal{M},u,p\right)\right)_{n\in\mathbb{N}}$

$$d_n^{(k)}\left(\Delta_v^{(m)},\mathcal{M},u,p\right)=\begin{cases}\sum_{k=1}^{n}\left[M_k\left(\frac{\left|u_k\sum_{i=k}^{n}\binom{m+n-i-1}{n-i}h_{ik}^{-1}v_k\right|}{\rho}\right)\right]^{p_k}, & n\geq k\\ 0, & n<k\end{cases}$$

for every fixed $k\in\mathbb{N}$. The following statements hold:

(i) The sequence $d_n^{(k)}\left(\Delta_v^{(m)},\mathcal{M},u,p\right)$ is a basis for the space $h_0\left(\Delta_v^{(m)},\mathcal{M},u,p\right)$ and every $x\in h_0\left(\Delta_v^{(m)},\mathcal{M},u,p\right)$ has a unique representation of the form

$$x=\sum_k\left(H\Delta_v^{(m)}x\right)_k^{\mathcal{M},u,p}d^{(k)}$$

(ii) The set $\left\{t,d^{(1)},d^{(2)},...\right\}$ is a basis or the space $h_c\left(\Delta_v^{(m)},\mathcal{M},u,p\right)$ and every $x\in h_c\left(\Delta_v^{(m)},\mathcal{M},u,p\right)$ has a unique representation of the form

$$x=st+\sum_k\left[\left(H\Delta_v^{(m)}x\right)_k^{\mathcal{M},u,p}-s\right]d^{(k)}$$

where $t = t_n\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right) = \sum_{k=1}^{n}\left[M_k\left(\frac{\left|u_k \sum_{i=k}^{n}\binom{m+n-i-1}{n-i} h_{ik}^{-1} v_k\right|}{\rho}\right)\right]^{p_k}$ for all $k \in \mathbb{N}$ and $s = \lim_{k\to\infty}\left(H\Delta_v^{(m)}x\right)_k^{\mathcal{M},u,p}$.

Corollary 2.11. The Hilbert sequence spaces $h_0\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)$ and $h_c\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)$ are separable.

3. Characterizations of Matrix Transformation and $\alpha-$, $\beta-$ and $\gamma-$duals

Let $A = (a_{nk})$ be an infinite matrix of complex numbers, X and Y be subsets of the sequence space w. Let $x = (x_k)$ and $y = (y_k)$ be two sequences. Thus, we can write $xy = (x_k y_k)$, $x^{-1} * Y = \{a \in w : ax \in Y\}$ and $M(X, Y) = \cap_{x\in X^{-1}} * Y = \{a \in w : ax \in Y,$ for all $x \in X\}$ for the multiplier space of X and Y. In the special cases of $Y = \{l_1, cs, bs\}$, we write $x^{\alpha} = x^{-1} * l_1$, $x^{\beta} = x^{-1} * cs$, $x^{\gamma} = x^{-1} * bs$ and $X^{\alpha} = M(X, l_1)$, $X^{\beta} = M(X, cs)$, $X^{\gamma} = M(X, bs)$ for the $\alpha-$dual, $\beta-$dual, $\gamma-$dual of X. By $A_n = (a_{nk})$ we denote the sequence in the $n^{th}-$row of A and write $A_n(x) = \sum_{k=1}^{\infty} a_{nk}x_k$ $\forall n \in \mathbb{N}$ and $A(x) = (A_n(x))$, provided $A_n \in x^{\beta}$ for all n.

We shall begin with the lemmas due to Stieglitz ve Tietz [15] which will be used in the computation of the $\beta-$ and $\gamma-$duals of the Hilbert sequence spaces.

Lemma 3.1. [16] Let X, Y be any two sequence spaces. $A \in (X : Y_T)$ if and only if $TA \in (X : Y)$, where A is an infinite matrix and T is a triangle matrix.

Lemma 3.2. (i) Let $A_n = (a_{nk})$ be an infinite matrix. Then $A \in (c_0 : l_\infty)$ if and only if

$$\sup_n \sum_k |a_{nk}| < \infty \tag{3.1}$$

(ii) $A \in (c_0 : c)$ if and only if (3.1) holds with

$$\lim_n a_{nk} \text{ exists for all } k. \tag{3.2}$$

(iii) $A \in (c_0 : bs)$ if and only if

$$\sup_n \sum_k \left|\sum_{n=0}^{m} a_{nk}\right| < \infty. \tag{3.3}$$

(iv) $A \in (c_0 : cs)$ if and only if (3.3) holds with

$$\sum_k a_{nk} \text{ convergent for all } k. \tag{3.4}$$

Lemma 3.3. (i) Let $A_n = (a_{nk})$ be an infinite matrix. Then $A \in (c : c)$ if and only if (3.1) and (3.2) hold with

$$\lim_{n\to\infty} \sum_k a_{nk} \text{ exists.}$$

(ii) $A \in (l_\infty : c)$ if and only if (3.2) holds with

$$\lim_n \sum_k |a_{nk}| = \sum_k \left|\lim_{n\to\infty} a_{nk}\right|. \tag{3.5}$$

Lemma 3.4.(i) Let $A_n = (a_{nk})$ be an infinite matrix. Then $A \in (c : cs)$ if and only if (3.3), (3,4) hold and

$$\sum_n \sum_k a_{nk} \text{ convergent.} \tag{3.6}$$

(ii) $A \in (l_\infty : cs)$ if and only if (3.2) holds and

$$\lim_m \sum_k \left| \sum_{n=m}^{\infty} a_{nk} \right| = 0. \tag{3.7}$$

Lemma 3.5. [17] Let $U = (u_{nk})$ be an infinite matrix of complex numbers for all $n, k \in \mathbb{N}$. Let $B^U = (b_{nk})$ be defined via a sequence $a = (a_k) \in w$ and inverse of the triangle matrix $U = (u_{nk})$ by

$$b_{nk} = \sum_{j=k}^{n} a_j u_{jk}$$

for all $n, k \in \mathbb{N}$.Then,

$$X_U^\alpha = \left\{ a = (a_k) \in w : B^U \in (X : l_1) \right\},$$
$$X_U^\beta = \left\{ a = (a_k) \in w : B^U \in (X : c) \right\}$$
$$X_U^\gamma = \left\{ a = (a_k) \in w : B^U \in (X : l_\infty) \right\}.$$

Theorem 3.6. The $\alpha-, \beta-$ and γ-duals of the Hilbert sequence spaces defined as

$$\begin{aligned}
\left[h_0\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)\right]^\alpha &= \{a = (a_k) \in w : W \in (c_0 : l_1)\}, \\
\left[h_c\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)\right]^\alpha &= \{a = (a_k) \in w : W \in (c : l_1)\}, \\
\left[h_\infty\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)\right]^\alpha &= \{a = (a_k) \in w : W \in (l_\infty : l_1)\}, \\
\left[h_0\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)\right]^\beta &= \{a = (a_k) \in w : W \in (c_0 : c)\}, \\
\left[h_c\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)\right]^\beta &= \{a = (a_k) \in w : W \in (c : c)\}, \\
\left[h_\infty\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)\right]^\beta &= \{a = (a_k) \in w : W \in (l_\infty : c)\}, \\
\left[h_0\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)\right]^\gamma &= \{a = (a_k) \in w : W \in (c_0 : l_\infty)\}, \\
\left[h_c\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)\right]^\gamma &= \{a = (a_k) \in w : W \in (c : l_\infty)\}, \\
\left[h_\infty\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)\right]^\gamma &= \{a = (a_k) \in w : W \in (l_\infty : l_\infty)\}.
\end{aligned}$$

Proof. We shall only compute the $\alpha-, \beta-$ and γ-duals of $h_0\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)$ sequence space. Let h_n^{-1} is defined by (2.1). Let us take any $a = (a_k) \in w$. We define the matrix $W = (w_{nk})$ by

$$w_{nk} = \sum_{k=1}^{n} \left[M_k \left(\frac{u_k \left| \sum_{i=k}^{n} \begin{pmatrix} m+n-i-1 \\ n-i \end{pmatrix} h_{ik}^{-1} a_n v_n^{-1} \right|}{\rho} \right) \right]^{p_k}.$$

Consider the equation

$$\begin{aligned}\sum_{k=1}^{n} a_k x_k &= \sum_{k=1}^{n} \left(a_k v_k^{-1}\right)(v_k x_k)\\ &= \sum_{k=1}^{n}\left[M_k\left(\frac{u_k\left|\sum_{i=1}^{k}\left\{\sum_{j=i}^{k}\begin{pmatrix} m+k-j-1\\ k-j\end{pmatrix} h_{ij}^{-1}\right\} a_k v_k^{-1} y_i\right|}{\rho}\right)\right]^{p_k}\\ &= \sum_{k=1}^{n}\left[M_k\left(\frac{u_k\left|\sum_{i=1}^{k}\left\{\sum_{j=i}^{k}\begin{pmatrix} m+k-j-1\\ k-j\end{pmatrix} h_{ij}^{-1} a_i v_i^{-1}\right\} y_k\right|}{\rho}\right)\right]^{p_k}\\ &= (Wy)_n\,. \end{aligned} \tag{3.8}$$

Using (3.8), we have $ax = (a_k x_k) \in cs$ or bs whenever $x = (x_k) \in h_0\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)$ if and only if $Wy \in l_1, c$ or l_∞ whenever $y = (y_k) \in c_0$. Then, from Lemma 3.1 and Lemma 3.5, we obtain that $a = (a_k) \in \left[h_0\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)\right]^{\alpha}$, $a = (a_k) \in \left[h_0\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)\right]^{\beta}$ or $a = (a_k) \in \left[h_0\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)\right]^{\gamma}$ if and only if $W \in (c_0 : l_1)$, $W \in (c_0 : c)$ or $W \in (c_0 : l_\infty)$, which is required result.

Therefore, the $\alpha-, \beta-$ and γ-duals of Hilbert sequence spaces will be helpful in the characterization of matrix transformations. Let X and Y be arbitrary subsets of w. We will show that the characterization of the classes $(X : Y_T)$ ve $(X_T : Y)$ can be reduced to (X, Y), where T is a triangle. Since if the sequence spaces $h_0\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)$ and c_0 are linearly isomorphic, then the equivalence class $x \in h_0\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right) \Leftrightarrow y \in c_0$ holds. So using Lemma 3.1 and 3.5, we get the desired result.

Theorem 3.7. Let us consider the infinite matrices $A = (a_{nk})$ and $B = (b_{nk})$. These matrices get associated with each other by the relations:

$$b_{nk} = \sum_{k=1}^{n}\left[M_k\left(\frac{u_k\left|\sum_{j=k}^{\infty}\begin{pmatrix} m+n-j-1\\ n-j\end{pmatrix} h_{jk}^{-1} a_{nj} v_k^{-1}\right|}{\rho}\right)\right]^{p_k} \tag{3.9}$$

for all $k, m, n \in \mathbb{N}$. Then the following statements are true:

(i) $A \in \left(h_0\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right) : Y\right)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \left[h_0\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)\right]^{\beta}$ for all $n \in \mathbb{N}$ and $B \in (c_0, Y)$,where Y be any sequence space;

(ii) $A \in \left(h_c\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right) : Y\right)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \left[h_c\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)\right]^{\beta}$ for all $n \in \mathbb{N}$ and $B \in (c, Y)$, where Y be any sequence space;

(iii) $A \in \left(h_\infty\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right) : Y\right)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \left[h_\infty\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)\right]^{\beta}$ for all $n \in \mathbb{N}$ and $B \in (l_\infty, Y)$, where Y be any sequence space.

Proof. We suppose that the relation in (3.9) holds between $A = (a_{nk})$ and $B = (b_{nk})$.The spaces $h_0\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)$ and c_0 are linearly isomorphic. Let $A \in \left(h_0\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right) : Y\right)$ and $y = (y_k) \in c_0$.Then $BH\Delta_v^{(m)}$ exists and $(a_{nk}) \in \left[h_0\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)\right]^{\beta}$ for all $k \in \mathbb{N}$, it means that $(b_{nk}) \in c_0$ for all $k, n \in \mathbb{N}$. Hence, By exists for each $y \in c_0$.Thus, if we take $m \to \infty$ in the equality,

$$\sum_{k=1}^{m} a_{nk}x_k = \sum_{k=1}^{m}\left[M_k\left(\frac{\left|u_k\left[\sum_{i=1}^{k}\sum_{j=i}^{k}\begin{pmatrix} m+k-j-1 \\ k-j \end{pmatrix} h_{ij}^{-1}\right] a_{nk}v_k^{-1}\right|}{\rho}\right)\right]^{p_k} = \sum_k b_{nk}y_k$$

for all $m, n \in \mathbb{N}$ which conclude that $B \in (c_0, Y)$. On the contrary, let $(a_{nk})_{k\in\mathbb{N}} \in \left[h_0\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)\right]^{\beta}$ for each $k \in \mathbb{N}$ and $B \in (c_0, Y)$ and $x = (x_k) \in h_0\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)$. Then it is clear that Ax exists. Thus, we attain from the following equality for all $n \in \mathbb{N}$

$$\sum_k b_{nk}y_k = \sum_k a_{nk}x_k$$

as $m \to \infty$ that $Ax = By$ and it is easy to show that $A \in \left(h_0\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right) : Y\right)$. This completes the proof.

Theorem 3.8. Let us assume that components of the infinite matrices $A = (a_{nk})$ and $E = (e_{nk})$ are connected with the following relation

$$e_{nk} = \sum_{k=1}^{n}\sum_{j=k}^{n}\frac{1}{n+j-1}\left[M_k\left(\frac{\left|u_k\sum_{j=k}^{n}(-1)^{j-k}\binom{m}{j-k}a_{jk}v_k^{-1}\right|}{\rho}\right)\right]^{p_k} \tag{3.10}$$

for all $m, n \in \mathbb{N}$ and X be any given sequence space. Then the following statements are true:

(i) $A = (a_{nk}) \in \left(X : h_0\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)\right)$ if and only if $E \in (X : c_0)$;

(ii) $A = (a_{nk}) \in \left(X : h_c\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)\right)$if and only if $E \in (X : c)$;

(iii) $A = (a_{nk}) \in \left(X : h_\infty\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)\right)$ if and only if $E \in (X : l_\infty)$.

Proof. Let us suppose that $z = (z_k) \in X$. Using the relation (3.10), we have

$$\sum_{k=1}^{m} e_{nk}z_k = \sum_{k=1}^{m}\left[\sum_{k=1}^{n}\sum_{j=k}^{n}\frac{1}{n+j-1}\left[M_k\left(\frac{\left|u_k\left[\sum_{j=k}^{n}(-1)^{j-k}\binom{m}{j-k}a_{jk}v_k^{-1}\right]z_k\right|}{\rho}\right)\right]^{p_k}\right] \tag{3.11}$$

for all $m, n \in \mathbb{N}$. Then, for $m \to \infty$ equation (3.11) gives us that $(Ez)_n = \left\{H\Delta_v^{(m)}(Az)\right\}_n$. Thus, we can obtain that $Az \in h_0\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)$ if and only if $Ez \in c_0$. This completes the proof.

Now, we give some conditions:

$$\lim_k a_{nk} = 0 \text{ for all } n, \tag{3.12}$$

$$\lim_{n\to\infty}\sum_k |a_{nk}| = 0, \tag{3.13}$$

$$\lim_{n\to\infty}\sum_k \left|a_{nk} - a_{n,k+1}\right| = 0, \tag{3.14}$$

$$\sup_n \sum_k \left|a_{nk} - a_{n,k+1}\right| < \infty, \tag{3.15}$$

$$\lim_k (a_{nk} - a_{n,k+1}) \text{ exists for all } k, \tag{3.16}$$

$$\lim_{n\to\infty} \sum_k \left|a_{nk} - a_{n,k+1}\right| = \sum_k \left|\lim_{n\to\infty} (a_{nk} - a_{n,k+1})\right|, \tag{3.17}$$

$$\sup_n \left|\lim_{n\to\infty} a_{nk}\right| < \infty, \tag{3.18}$$

Corollary 3.9. Let $A = (a_{nk})$ be an infinite matrix and $X = h_0\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)$, $Y = h_c\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)$ and $Z = h_\infty\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)$. Then, the following statements hold:
(a) $A = (a_{nk}) \in (X, l_\infty)$ if and only if (3.1) holds with b_{nk} instead of a_{nk};
(b) $A = (a_{nk}) \in (X, bs)$ if and only if (3.3) holds with b_{nk} instead of a_{nk};
(c) $A = (a_{nk}) \in (Y, cs)$ if and only if(3.3), (3.4) and (3.6) hold with b_{nk} instead of a_{nk};
(d) $A = (a_{nk}) \in (Z, c)$ if and only if (3.2) and (3.5) hold with b_{nk} instead of a_{nk};
(e) $A = (a_{nk}) \in (Z, cs)$ if and only if (3.7) holds with b_{nk} instead of a_{nk};
(f) $A = (a_{nk}) \in (X, c)$ if and only if (3.1) and (3.2) hold with b_{nk} instead of a_{nk};
(g) $A = (a_{nk}) \in (X, cs)$ if and only if (3.3) and (3.4) holds with b_{nk} instead of a_{nk}.

Corollary 3.10. Let $A = (a_{nk})$ be an infinite matrix and $X = h_0\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)$, $Y = h_c\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)$ and $Z = h_\infty\left(\Delta_v^{(m)}, \mathcal{M}, u, p\right)$. Then, the following statements hold:
(a) $A = (a_{nk}) \in (l_\infty, X)$ if and only if (3.13) holds with e_{nk} instead of a_{nk};
(b) $A = (a_{nk}) \in (bs, X)$ if and only if (3.12) and (3.14) hold with e_{nk} instead of a_{nk};
(c) $A = (a_{nk}) \in (bs, Y)$ if and only if (3.12), (3.16) and (3.17) hold with e_{nk} instead of a_{nk};
(d) $A = (a_{nk}) \in (cs, Y)$ if and only if (3.15) and (3.2) hold with e_{nk} instead of a_{nk};
(e) $A = (a_{nk}) \in (bs, Z)$ if and only if (3.12) and (3.15) hold with e_{nk} instead of a_{nk};
(f) $A = (a_{nk}) \in (cs, Z)$ if and only if (3.15) and (3.18) hold with e_{nk} instead of a_{nk};
(g) $A = (a_{nk}) \in (cs, X)$ if and only if (3.2) and (3.15) hold with e_{nk} instead of a_{nk}.

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