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A g_n-inverse of multivalued operators

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Abstract. For an everywhere defined closed linear relation in a Banach space the concept of g_n-invertibility is introduced and studied. It is shown that many of the results of S.R. Caradus and other authors for operators remain valid in the context of multivalued linear operators. In particular, we gather some results and characterizations of g_n-invertibility and semi-Fredholm linear relations. Some stability results under perturbations by compact relations are also given for this concept. Part of the results proved in this paper improve and generalize some results known for pseudo-generalized invertible operators [Filomat 36:8 (2022), 2551–2572].

1. Introduction and notations

Let X, Y and Z be three Banach spaces. A multivalued linear operator $T : \mathsf{X} \longrightarrow \mathsf{Y}$ or simply a linear relation is a mapping from a subspace $\mathsf{D}(T) \subset \mathsf{X}$, called the domain of T, into the collection of nonempty subsets of Y such that $T(\lambda x + \mu y) = \lambda T(x) + \mu T(y)$ for all nonzero scalars λ, μ and $x, y \in \mathsf{D}(T)$. We denote by $\mathcal{L}_{\mathcal{R}}(\mathsf{X}, \mathsf{Y})$ the class of all linear relations from X to Y and $\mathcal{L}_{\mathcal{RD}}(\mathsf{X}, \mathsf{Y})$ the class of all everywhere defined linear relations from X to Y. If T maps the points of its domain to singletons, then T is said to be a single valued linear operator or simply an operator. The graph $\mathsf{G}(T)$ of $T \in \mathcal{L}_{\mathcal{R}}(\mathsf{X}, \mathsf{Y})$ is

$$\mathsf{G}(T) := \{(x, y) \in \mathsf{X} \times \mathsf{Y} : x \in \mathsf{D}(T), \ y \in Tx\}.$$

The inverse of T is the linear relation T^{-1} given by $\mathsf{G}(T^{-1}) := \{(y, x) \in \mathsf{Y} \times \mathsf{X} : (x, y) \in \mathsf{G}(T)\}$. The subspaces $\mathsf{N}(T) := T^{-1}(0)$ and $\mathsf{R}(T) := T(\mathsf{D}(T))$ are called respectively the null space and the range space of T. We say that T is surjective if $\mathsf{R}(T) = \mathsf{Y}$ and T is injective if $\mathsf{N}(T) = \{0\}$. If $\mathsf{X} = \mathsf{Y}$, then we write $\mathcal{L}_{\mathcal{R}}(\mathsf{X}, \mathsf{X}) = \mathcal{L}_{\mathcal{R}}(\mathsf{X})$ and similarly $\mathcal{L}_{\mathcal{RD}}(\mathsf{X}, \mathsf{X}) = \mathcal{L}_{\mathcal{RD}}(\mathsf{X})$ for short. For $S, T \in \mathcal{L}_{\mathcal{R}}(\mathsf{X}, \mathsf{Y})$, $L \in \mathcal{L}_{\mathcal{R}}(\mathsf{Z}, \mathsf{Y})$ (where Z is a Banach space) and $\lambda \in \mathbb{C}$, the linear relations $T + S$, λS and LT are respectively defined by $\mathsf{G}(T + S) = \{(x, y + z) : (x, y) \in \mathsf{G}(T), \ (x, z) \in \mathsf{G}(S)\}$, $\mathsf{G}(\lambda S) = \{(x, \lambda y) : (x, y) \in \mathsf{G}(S)\}$ and $\mathsf{G}(LT) = \{(x, y) : (x, z) \in \mathsf{G}(T), (z, y) \in \mathsf{G}(L)\}$. Hence, if $V \in \mathcal{L}_{\mathcal{R}}(\mathsf{X})$, the iterate V^n, $n \in \mathbb{N}$ of V is defined as usual with $V^0 = I$ and $V^1 = V$. It is easy to show

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that $(V^{-1})^n = (V^n)^{-1}$, for all $n \in \mathbb{N}$. By the notation $S \subset T$ we mean that $\mathsf{D}(S) \subset \mathsf{D}(T)$ and $Sx \subset Tx$, for all $x \in \mathsf{D}(S)$. From [18] we recall that for $T \in \mathcal{L}_{\mathcal{R}}(\mathsf{X})$, the ascent, $\boldsymbol{a}(T)$, and the descent, $\boldsymbol{d}(T)$, of T are defined by

$$\boldsymbol{a}(T) := \inf\{k \in \mathbb{N} \ : \ \mathsf{N}(T^{k+1}) = \mathsf{N}(T^k)\},$$

$$\boldsymbol{d}(T) := \inf\{k \in \mathbb{N} \ : \ \mathsf{R}(T^{k+1}) = \mathsf{R}(T^k)\},$$

where the infimum over the empty set is taken to be infinite. It is obvious to see that $T^n(0) \subset T^{n+1}(0)$, for all $n \in \mathbb{N}$. It is also well known that if $T^n(0) = T^{n+1}(0)$, for some non-negative integer n, then $T^k(0) = T^n(0)$, for every $k \geq n$. We say that $T \in \mathcal{L}_{\mathcal{R}}(\mathsf{X})$ has a trivial singular chain manifold if $\mathsf{R}_c(T) = \{0\}$, where

$$\mathsf{R}_c(T) = \Big[\bigcup_{i=1}^{\infty} \mathsf{N}(T^i)\Big] \cap \Big[\bigcup_{i=1}^{\infty} T^i(0)\Big].$$

Let $\lambda \in \mathbb{C}$, by [18, Lemma 7.1], we know that $\mathsf{R}_c(T) = \{0\}$ if and only if $\mathsf{R}_c(\lambda I - T) = \{0\}$. Hence, $\mathsf{R}_c(T) = \{0\}$ when $\lambda I - T$ is injective for some $\lambda \in \mathbb{C}$.

For a given linear relation $T \in \mathcal{L}_{\mathcal{R}}(\mathsf{X}, \mathsf{Y})$, we denote by Q_T the quotient map from X onto $\mathsf{Y}/\overline{T(0)}$. We shall denote the linear relation $Q_T T$ by QT. It is easy to see that QT is an operator. We define $\|Tx\| = \|QTx\|$, $x \in \mathsf{D}(T)$ and $\|T\| = \|QT\|$. We say that T is continuous if for each open set $\Omega \subset \mathsf{R}(T)$, $T^{-1}(\Omega)$ is an open set in $\mathsf{D}(T)$ equivalently $\|T\| < +\infty$, bounded if it is continuous and everywhere defined, open if its inverse is continuous, and bounded below if it is injective and open. A linear relation T is said to be closed if its graph is a closed subspace of $\mathsf{X} \times \mathsf{Y}$. It is well known (see [6]) that T^{-1} is closed if and only if T is closed if and only if QT is a closed operator and $T(0)$ is a closed subspace. If T is closed, then $\mathsf{N}(T)$ is closed. Recall also that if T is closed and $\mathsf{D}(T) = \mathsf{X}$, then T is bounded. The set of all bounded linear relations from X to Y is denoted by $\mathcal{B}_{\mathcal{R}}(\mathsf{X}, \mathsf{Y})$ and the set of all everywhere defined closed linear relations from X to Y is denoted by $C_{\mathcal{RD}}(\mathsf{X}, \mathsf{Y})$, and as useful we write $\mathcal{B}_{\mathcal{R}}(\mathsf{X}, \mathsf{X}) := \mathcal{B}_{\mathcal{R}}(\mathsf{X})$ and $C_{\mathcal{RD}}(\mathsf{X}, \mathsf{X}) := C_{\mathcal{RD}}(\mathsf{X})$. It is not difficult to show that

$$C_{\mathcal{RD}}(\mathsf{X}, \mathsf{Y}) = \{T \in \mathcal{B}_{\mathcal{R}}(\mathsf{X}, \mathsf{Y}) : T(0) \text{ is closed}\} \subset \mathcal{B}_{\mathcal{R}}(\mathsf{X}, \mathsf{Y}) \subset \mathcal{L}_{\mathcal{RD}}(\mathsf{X}, \mathsf{Y}).$$

Also, it follows from [6, Proposition II.1.7, Corollary II.3.13] that if $T, S \in \mathcal{B}_{\mathcal{R}}(\mathsf{X}, \mathsf{Y})$ and $L \in \mathcal{B}_{\mathcal{R}}(\mathsf{Y}, \mathsf{Z})$, then $S + T \in \mathcal{B}_{\mathcal{R}}(\mathsf{X}, \mathsf{Y})$ and $LT \in \mathcal{B}_{\mathcal{R}}(\mathsf{X}, \mathsf{Z})$. Let $T \in \mathcal{L}_{\mathcal{R}}(\mathsf{X})$ be a closed linear relation. The resolvent set of T is defined by

$$\varrho(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is injective and surjective}\}.$$

We know from [7, Lemma 3.1] that if $\varrho(T) \neq \emptyset$, then for every $n \in \mathbb{N}$, the linear relation T^n is closed. Let M and N be a nonempty subset of X. We define the distance between M and $x \in \mathsf{X}$ by the formula

$$\operatorname{dist}(\mathsf{M}, x) := \inf\{\|x - y\| : y \in \mathsf{M}\}.$$

We shall also write $\operatorname{dist}(x, \mathsf{M})$, for the distance between $\{x\}$ and M. The reduced minimum modulus $\gamma(T)$ of $T \in \mathcal{L}_{\mathcal{R}}(\mathsf{X}, \mathsf{Y})$ (see [6, Definition II.2.1]) is defined by

$$\gamma(T) := \sup\{\lambda : \|Tx\| \geq \lambda \operatorname{dist}(x, \mathsf{N}(T)), \ \text{ for } x \in \mathsf{D}(T)\}.$$

Throughout this paper the symbol $\dotplus$ denotes the direct sum of closed subspaces, i.e., $\mathsf{X}_0 = \mathsf{X}_1 \dotplus \mathsf{X}_2$ if the linear space $\mathsf{X}_0 = \mathsf{X}_1 + \mathsf{X}_2$ is closed and $\mathsf{X}_1 \cap \mathsf{X}_2 = \{0\}$. We shall say that X_1 is complemented in X_0 if there is a closed subspace $\mathsf{X}_2 \subseteq \mathsf{X}_0$ such that $\mathsf{X}_0 = \mathsf{X}_1 \dotplus \mathsf{X}_2$.

Using arguments similar to [1, Lemma 2.5] we can show the following lemma that will be needed in the sequel.

Lemma 1.1. *Let $S, T \in \mathcal{L}_{\mathcal{R}}(\mathsf{X}, \mathsf{Y})$, $L \in \mathcal{L}_{\mathcal{R}}(\mathsf{Z}, \mathsf{X})$ and $U \in \mathcal{L}_{\mathcal{R}}(\mathsf{Y}, \mathsf{Z})$, then*

1) *$(T + S)L \subset TL + SL$ with equality if $L(0) \subset \mathsf{N}(T) \cup \mathsf{N}(S)$.*
2) *$U(T + S) = UT + US$ if U is everywhere defined.*

In this paper, $\mathscr{B}(\mathsf{X},\mathsf{Y})$ is the Banach algebra of all bounded linear operators from X to Y and abbreviate $\mathscr{B}(\mathsf{X},\mathsf{X})$ to $\mathscr{B}(\mathsf{X})$. We say that $T\in\mathcal{L}_{\mathcal{RD}}(\mathsf{X},\mathsf{Y})$ is g_1-invertible if $TST=T$ and $ST(0)=\{0\}$ for some $S\in\mathscr{B}(\mathsf{Y},\mathsf{X})$. In this case, we say that S is a g_1-inverse of T. The set of all g_1-invertible relations from X into Y is denoted by $\mathcal{R}_g^1(\mathsf{X},\mathsf{Y})$. If $T\in\mathcal{R}_g^1(\mathsf{X},\mathsf{Y})$, let

$$\mathcal{G}_1(T)=\Big\{S\in\mathscr{B}(\mathsf{Y},\mathsf{X}):TST=S\text{ and }ST(0)=\{0\}\Big\}.$$

Let $T\in\mathcal{B}_{\mathcal{R}}(\mathsf{X},\mathsf{Y})$ and $S\in\mathcal{G}_1(T)$, and set $S_0=STS$. Then, it is easy to see that

$$S_0\in\mathscr{B}(\mathsf{Y},\mathsf{X}),\ TS_0T=T,\ S_0TS_0=S_0\text{ and }S_0T(0)=\{0\}.$$

S_0 is called a g_1^2-inverse of T. Set

$$\mathcal{G}_1^2(T)=\{S\in\mathscr{B}(\mathsf{Y},\mathsf{X}):S\text{ is a }g_1^2\text{-inverse of }T\}.$$

The class $\mathcal{R}_g^1(\mathsf{X}):=\mathcal{R}_g^1(\mathsf{X},\mathsf{X})$ was introduced and studied by I. Issaoui and M. Mnif in [13].

For $n\in\mathbb{N}\backslash\{0\}$, a bounded linear operator $S\in\mathscr{B}(\mathsf{X})$ is said to be a g_n-inverse of $T\in\mathcal{L}_{\mathcal{RD}}(\mathsf{X})$, if

$$T^nST=T^n\text{ and }ST(0)=\{0\}.$$

In this case we will say that T is g_n-invertible. We denote

$$\mathcal{R}_g^n(\mathsf{X})=\Big\{T\in\mathcal{L}_{\mathcal{RD}}(\mathsf{X}):\exists\,S\in\mathscr{B}(\mathsf{X})\text{ such that }T^nST=T^n\text{ and }ST(0)=\{0\}\Big\}.$$

It is clear that $\mathcal{R}_g^1(\mathsf{X})\subset\mathcal{R}_g^n(\mathsf{X})\subset\mathcal{R}_g^{n+1}(\mathsf{X})$, for all $n\geq1$. If $T\in\mathcal{R}_g^n(\mathsf{X})$, let

$$\mathcal{G}_n(T):=\Big\{S\in\mathscr{B}(\mathsf{X}):T^nST=T^n\text{ and }ST(0)=\{0\}\Big\}.$$

If S is a g_n-inverse (resp. g_1-inverse) of $T\in\mathcal{B}_{\mathcal{R}}(\mathsf{X})$ (resp. $T\in\mathcal{B}_{\mathcal{R}}(\mathsf{X},\mathsf{Y})$), we note that $S':=STS$ is also a g_n-inverse (resp. g_1-inverse) of T. Notice that the first equality $T^nST=T^n$ (resp. $TST=T$) and $T(0)$ is complemented are sufficient conditions for T to have a g_n-inverse (resp. g_1-inverse). Indeed, let P be a linear projection with domain X (resp. Y) and kernel $T(0)$ and $L:=SP$. Since PT is single valued, for all $x\in\mathsf{X}$ and $y\in Tx$, we have $Py=PTx$ and

$$Tx=y+T(0)=Py+(I-P)y+T(0)=PTx+T(0).$$

We denote by $B_T:=PT$, then

$$T=B_T+T(0)\text{ and }\mathsf{R}(B_T)\cap T(0)=\{0\}.$$

If $m=n$ (resp. $m=1$), by Lemma 1.1, we have $LT(0)=\{0\}$ and

$$\begin{aligned}T^m&=T^mS(PT+T(0))\\&=T^mLT+T^mST(0)\\&=T^mLT+T^m(0)\\&=T^mLT.\end{aligned}$$

From [11, 12] recall that a linear relation $T\in\mathcal{L}_{\mathcal{RD}}(\mathsf{X})$ is said to be Drazin invertible of degree $n\in\mathbb{N}$ if there exists $S\in\mathscr{B}(\mathsf{X})$ such that

$$T^{n+1}S=T^n+T^{n+1}(0),\quad STS=S,\quad TS=ST+T(0).$$

Hence, Lemma 1.1 implies that $T^{n+1}ST=T^{n+1}$, and as $ST(0)=STS(0)=S(0)=\{0\}$, we obtain that T is g_{n+1}-invertible. We note also that if $T\in\mathcal{C}_{\mathcal{RD}}(\mathsf{X})$ is Drazin invertible of degree $n\geq1$ such that $\varrho(T)\neq\emptyset$, then by [16, Lemma 3], we have T has a g_n-inverse.

Example 1.2. *Let $T \in \mathcal{L}_{\mathcal{RD}}(\mathsf{X})$ be such that $T^{n+1} = T^n$, for some $n \in \mathbb{N}\backslash\{0\}$.*

1) *If $\|T\| < +\infty$ and $T^n(0)$ is complemented, then $T \in \mathcal{R}_g^n(\mathsf{X})$. Indeed, let $S := PT^n$, where P is a linear projection with domain X and kernel $T^n(0)$. For $x \in \mathsf{X}$, we note that $T^n(I-P)x \subset T^{2n}(0) = T^n(0)$, and so*

$$T^n x = T^n Px + T^n(I-P)x = T^n Px, \quad \forall\, x \in \mathsf{X}.$$

It is easy to see that $S \in \mathscr{B}(\mathsf{X})$, $ST(0) = \{0\}$ and

$$T^n = T^{2n+1} = T^n ST,$$

i.e., S is a g_n-inverse of T.

2) *If T is closed and $\varrho(T) \neq \emptyset$, then $T \in \mathcal{R}_g^n(\mathsf{X})$. Indeed, since T has finite ascent and descent, then [11, Theorem 2.10] shows that $T \in \mathcal{R}_g^n(\mathsf{X})$.*

Example 1.3. *Let H be a Hilbert space and $T \in C_{\mathcal{RD}}(\mathsf{H})$ be a partial isometry (see [10, Definition 3.1]). We denote by T^* the adjoint of T and P the orthogonal projection onto the orthogonal subspace of $T(0)$ in H. From [10, Theorem 3.3], we have $T^\star := T^*P$ is a g_n-inverse of T, for all $n \geq 1$.*

The concept of the g_1-inverse of bounded linear operators or of linear relations in a Banach space, and even on a Banach algebra, were introduced and investigated by several authors, for instance, [3–5, 13–15, 19, 20] among others. Furthermore, if $T \in \mathcal{R}_g^n(\mathsf{X})$ is an operator, then we find the case of n-left pseudo-generalized invertible operator introduced by Lahmar and the second author in [17]. The motivation of this paper is to explore new additive properties of the g_1-inverse for linear relations in Banach spaces. In addition to some results of these last articles are extended to the case of g_n-invertible linear relations.

The paper is organized as follows. In the next section some basic properties and many results related to the concepts of g_1-invertible for bounded operators on a Banach space are extended to the case of closed linear relations everywhere defined. In particular, we give a sufficient condition for an everywhere defined linear relation to have a g_1-inverse. In the third section, we study some results and characterizations of g_n-invertible and Fredholm linear relations. Also, we are concerned with the stability of the g_n-invertible linear relations, under perturbations by compact relations. Part of the results proved in this paper improve and generalize some results known for g_n-invertible bounded operator [4, 14, 17, 19].

2. Some basic properties

We start this section with the following remark.

Remark 2.1. *Let $T \in \mathcal{R}_g^n(\mathsf{X})$, for some $n \in \mathbb{N}\backslash\{0\}$ and S be a g_n-inverse of T such that $\|ST\| < 1$, then $\|T^n\| = 0$. Indeed, if $S = 0$, the result is obvious. Now, if $0 < \|ST\| < 1$ and as $ST(0) = \{0\}$, we have $I - ST$ is invertible. It follows that $\mathsf{R}(T^n) = \mathsf{R}[T^n(I-ST)] = \mathsf{R}(T^n - T^nST) = T^n(0)$. Hence, if $x \in \mathsf{X}$ and $y \in T^n(x) \subset T^n(0)$, then $T^n(x) = y + T^n(0) = T^n(0)$.*

The next lemma is used to prove Theorem 2.3.

Lemma 2.2. *Let $T, S \in C_{\mathcal{RD}}(\mathsf{X}, \mathsf{Y})$ be such that $S(0) = T(0)$. If $\mathsf{N}(S)$ is complemented and $\mathsf{R}(T) \subset \mathsf{R}(S)$, then there exists $C \in \mathscr{B}(\mathsf{X})$ such that $T = SC$. Moreover if $T^2(0) = T(0)$, then $CT(0) = \{0\}$.*

Proof. Let $\mathsf{M} \subset \mathsf{X}$ be such that $\mathsf{X} = \mathsf{M} \dotplus \mathsf{N}(S)$. Let $x \in \mathsf{X}$ and $y \in Tx$, and as $y \in \mathsf{R}(S)$, then there exists $z \in \mathsf{M}$ such that $y \in S(z)$. Set $C(x) = z$, then

$$Tx = y + T(0) = y + S(0) = S(z) = SCx$$

and it remains only to prove that C is bounded. Since C is defined on all of X, to do this it suffices to show that C has a closed graph.

If $\{(x_n, z_n)\}_{n\in\mathbb{N}}$ is a sequence of elements each in the graph of C so that $(x_n, z_n) \xrightarrow[n\to+\infty]{} (x, z)$, then $\lim\limits_{n\to+\infty} QTx_n = QTx$ and $\lim\limits_{n\to+\infty} QSz_n = QSz$. Thus, $Tx = Sz$ and further, because M is closed, it follows that $z \in \mathsf{M}$ so that $Cx = z$. Hence C has been shown to be bounded. Now, if $T^2(0) = T(0)$ and $x \in T(0)$, then $T(x) \subset T^2(0) = S(0)$, this implies that $C(x) = 0$. Consequently, $CT(0) = \{0\}$, when $T^2(0) = T(0)$. This completes the proof of the lemma. □

Theorem 2.3. *Let $T \in C_{\mathcal{RD}}(\mathsf{X})$ be such that $\boldsymbol{d}(T) < +\infty$ and $T^m(0) = T^{m+1}(0)$, for some $m \in \mathbb{N}$. If $\mathsf{N}(T^n)$ is complemented and $T^n(0)$ is closed, for some $n \geq \max\{m, \boldsymbol{d}(T)\} + 1$, then $T \in \mathcal{R}_g^1$.*

Proof. First of all, we see that $T^{n-1}(0) = T^n(0)$ is closed, and so $T^{n-1}, T^n \in C_{\mathcal{RD}}(\mathsf{X})$. Since $\mathsf{R}(T^{n-1}) \subset \mathsf{R}(T^n)$ and $T^{2(n-1)}(0) = T^{n-1}(0)$, it follows from Lemma 2.2 that there exists $C \in \mathscr{B}(\mathsf{X})$ such that $T^{n-1} = T^nC$ and $CT(0) = \{0\}$. This leads to $T^n = T^nCT$. □

For bounded operators in Banach spaces, Proposition 2.4 was proved in [4, Theorem 1].

Proposition 2.4. *Let $T \in C_{\mathcal{RD}}(\mathsf{X}, \mathsf{Y})$.*

1) *If $\mathsf{R}(T)$ and $\mathsf{N}(T)$ are both complemented, then T has a g_1-inverse. In particular, every Fredholm relation has a g_1-inverse.*
2) *If T has a g_1-inverse S such that $\mathsf{R}_c(TS) = \{0\}$, then $\mathsf{R}(T)$ and $\mathsf{N}(T)$ are both complemented.*
3) *If T is g_1-invertible such that $T(0)$ is complemented, then $\mathsf{R}(T)$ and $\mathsf{N}(T)$ are both complemented.*

Proof. 1) $\mathsf{Y} = \mathsf{R}(T) \dotplus \mathsf{M}$, $\mathsf{X} = \mathsf{N}(T) \dotplus \mathsf{N}$ and P (resp. Q) is a linear projection with domain Y (resp. X), range $\mathsf{R}(T)$ (resp. N) and kernel M (resp. $\mathsf{N}(T)$). We note that if $y \in T(x_1) \cap T(x_2)$, then $x_1 - x_2 \in \mathsf{N}(T)$, and so $Q(x_1) = Q(x_2)$. Hence, we consider the following operator linear

$$\begin{array}{rccl} S_1 : & \mathsf{R}(T) & \longrightarrow & \mathsf{X} \\ & y & \longmapsto & QT^{-1}y = Q(x), \text{ with } y \in T(x). \end{array}$$

Put $S := S_1P$, from [6, Theorem II.2.5], we get $\|S\| \leq \frac{\|Q\|\,\|P\|}{\gamma(T)}$. As $ST(0) = S_1T(0) = \{0\}$ and $TSTx = TS_1Tx = TQ(x) = Tx$, for all $x \in \mathsf{X}$, we conclude that S is a g_1-inverse of T.

2) Since

$$\mathsf{N}(T) \subset \mathsf{N}(ST) \subseteq \mathsf{N}(TST) = \mathsf{N}(T)$$

and $(ST)^2 = ST$, then $\mathsf{X} = \mathsf{R}(ST) \dotplus \mathsf{N}(T)$. Now, let $S' = STS$, it is clear that $S'TS' = S'$, $TS'T = T$ and $TS' = TS$. This gives that $(TS')^2 = (TS')$ and $\mathsf{Y} = \mathsf{R}(TS') \dotplus \mathsf{N}(TS')$ according to [18, Theorem 5.8]. On the other hand, since

$$\mathsf{R}(T) = \mathsf{R}(TS'T) \subset \mathsf{R}(TS') \subset \mathsf{R}(T),$$

$$\mathsf{N}(S') \subset \mathsf{N}(TS') \subset \mathsf{N}(S'TS') = \mathsf{N}(S'),$$

it follows that $\mathsf{Y} = \mathsf{R}(T) \dotplus \mathsf{N}(S')$.

3) follows from [13, Lemma 2.3]. This completes the proof. □

The following lemma is important for future use.

Lemma 2.5. *Let $n \in \mathbb{N}\backslash\{0\}$ and $T \in \mathcal{L}_{\mathcal{RD}}(\mathsf{X})$ be such that $\|T\| < +\infty$. Then*

$$T^n \in \mathcal{R}_g^1(\mathsf{X}) \Longrightarrow T \in \mathcal{R}_g^n(\mathsf{X}).$$

Proof. Let $L \in \mathscr{B}(\mathsf{X})$ be such that $T^nLT^n = T^n$ and $LT^n(0) = \{0\}$. Let $S := LT^{n-1}$, we have $\|S\| < +\infty$, $S(0) \subset ST(0) = LT^n(0) = \{0\}$ and $T^nST = T^nLT^n = T^n$. □

Corollary 2.6. *Let $T \in C_{\mathcal{RD}}(\mathsf{X})$ be such that $T^n(0)$ is closed, for some $n \in \mathbb{N}\backslash\{0\}$. If $\mathsf{R}(T^n)$ and $\mathsf{N}(T^n)$ are both complemented, then T has a g_n-inverse.*

Proof. First, we note that $T^n \in C_{\mathcal{RD}}(\mathsf{X})$, because $T^n \in \mathcal{B}_{\mathcal{R}}(\mathsf{X})$ and $T^n(0)$ is closed. Therefore Proposition 2.4 and Lemma 2.5 imply that $T \in \mathcal{R}_g^n(\mathsf{X})$. □

Recall that if $S, T \in \mathscr{B}(\mathsf{X})$ and $n \in \mathbb{N}\backslash\{0\}$, then $I - ST \in \mathcal{R}_g^n(\mathsf{X})$ if and only if $I - TS \in \mathcal{R}_g^n(\mathsf{X})$ (see [17, Proposition 2.10] and [4, Section 6, P. 26]). The objective of the following proposition is to prove that this result remains valid if $T \in C_{\mathcal{RD}}(\mathsf{X}, \mathsf{Y})$ and $S \in \mathscr{B}(\mathsf{Y}, \mathsf{X})$ such that $ST(0) = \{0\}$ and $T(0)$ is complemented.

Proposition 2.7. *Let $T \in C_{\mathcal{RD}}(\mathsf{X}, \mathsf{Y})$ and $S \in \mathscr{B}(\mathsf{Y}, \mathsf{X})$ be such that $ST(0) = \{0\}$. For all $n \in \mathbb{N}\backslash\{0\}$, we have*

$$I - TS \in \mathcal{R}_g^n(\mathsf{Y}) \Longrightarrow I - ST \in \mathcal{R}_g^n(\mathsf{X}),$$

$$\left\{\begin{array}{c} T(0) \textit{ is complemented} \\ \textit{and} \\ I - ST \in \mathcal{R}_g^n(\mathsf{X}) \end{array}\right. \Longrightarrow I - TS \in \mathcal{R}_g^n(\mathsf{Y}).$$

Proof. First, it is shown that

$$(*) \qquad (I - TS)^n(0) = T(0), \quad \forall\, n \in \mathbb{N}\backslash\{0\}.$$

Assume that $(I - TS)^n(0) = T(0)$. It will be shown that $(I - TS)^{n+1}(0) = T(0)$, and then, the equality will follow by induction. Since

$$T(0) = (I - TS)(0) \subset (I - TS)^{n+1}(0) \subset (I - TS)T(0) \subset T(0) - TST(0) = T(0),$$

we obtain $(I - TS)^{n+1}(0) = T(0)$, which implies $(*)$. Now we will show that

$$(**) \qquad (I - TS)^n = \sum_{k=0}^{n} C_n^k(-1)^k(TS)^k, \quad \forall\, n \in \mathbb{N}\backslash\{0\}.$$

We proceed by induction, clearly $(I - TS)^j = \sum\limits_{k=0}^{j} C_j^k(-1)^k(TS)^k$, for $j = 1$. Suppose we have shown its validity for $1 \leq j \leq n$. Then, we can complete the proof of $(**)$ by showing

$$(I - TS)^{n+1} = \sum_{k=0}^{n+1} C_{n+1}^k(-1)^k(TS)^k.$$

Let $V := I + T(0)$, it is clear that $\mathsf{N}(V) = T(0)$, and as $(I - TS)^n(0) \subset \mathsf{N}(V)$, by Lemma 1.1, we obtain

$$\begin{aligned}
(I - TS)^{n+1} &= (V - TS)(I - TS)^n = V(I - TS)^n - TS(I - TS)^n \\
&= (I - TS)^n + T(0) - TS \sum_{k=0}^{n} C_n^k(-1)^k(TS)^k \\
&= \sum_{k=0}^{n} C_n^k(-1)^k(TS)^k + \sum_{k=0}^{n} C_n^k(-1)^{k+1}(TS)^{k+1} \\
&= \sum_{k=0}^{n} C_n^k(-1)^k(TS)^k + \sum_{k=1}^{n+1} C_n^{k-1}(-1)^k(TS)^k \\
&= I + \sum_{k=1}^{n} (C_n^k + C_n^{k-1})(-1)^k(TS)^k + (-1)^{n+1}(TS)^{n+1} \\
&= I + \sum_{k=1}^{n} C_{n+1}^k(-1)^k(TS)^k + (-1)^{n+1}(TS)^{n+1} \\
&= \sum_{k=0}^{n+1} C_{n+1}^k(-1)^k(TS)^k.
\end{aligned}$$

• Let W be a g_n-inverse of $(I - TS)$ and $A := (I - TS)^n - (I - TS)^n W(I - TS)$. Since $WT(0) = W(I - TS)(0) = \{0\}$, then it follows from $(*)$, $(**)$ and Lemma 1.1 that

$$A = \underbrace{\sum_{k=0}^{n} C_n^k(-1)^k (TS)^k}_{F_1} - \underbrace{\sum_{k=0}^{n} C_n^k(-1)^k (TS)^k W}_{F_2} + \underbrace{\sum_{k=0}^{n} C_n^k(-1)^k (TS)^k WTS}_{F_3} = T(0).$$

However, if we set $B = (I - ST)^n(I + SWT)(I - ST)$, then

$$B = (I - ST)^n + \Big(\sum_{k=0}^{n} C_n^k(-1)^k (ST)^k SW\Big)T - \sum_{k=0}^{n} C_n^k(-1)^k (ST)^{k+1} - \Big(\sum_{k=0}^{n} C_n^k(-1)^k (ST)^k SWTS\Big)T.$$

On the other hand, we have

$$\Big(\sum_{k=0}^{n} C_n^k(-1)^k (ST)^k SW\Big)T = \Big(\sum_{k=0}^{n} C_n^k(-1)^k S(ST)^k W\Big)T = SF_2T,$$

$$\begin{aligned}
\sum_{k=0}^{n} C_n^k(-1)^k (ST)^{k+1} &= \sum_{k=0}^{n} C_n^k(-1)^k S(TS)^k T \\
&= S\Big(\sum_{k=0}^{n} C_n^k(-1)^k (TS)^k T\Big) \\
&= SF_1T, \text{ because } T(0) \subset \mathsf{N}[(TS)^k], \ \forall\, 1 \le k \le n,
\end{aligned}$$

and

$$\Big(\sum_{k=0}^{n} C_n^k(-1)^k (ST)^k SWTS\Big)T = \Big(\sum_{k=0}^{n} C_n^k(-1)^k S(TS)^k WTS\Big)T = SF_3T.$$

This implies that

$$B = (I - ST)^n + SF_2T - SF_1T - SF_3T = (I - ST)^n - SAT.$$

Therefore, since $SAT = ST(0) = \{0\}$, we infer that $I + SWT$ is a g_n-inverse of $(I - ST)$.

• Let W be a g_n-inverse of $(I - ST)$ and $A := (I - ST)^n - (I - ST)^n W(I - ST)$. It is clear that

$$A = \underbrace{\sum_{k=0}^{n} C_n^k(-1)^k (ST)^k}_{F_1} - \underbrace{\sum_{k=0}^{n} C_n^k(-1)^k (ST)^k W}_{F_2} + \underbrace{\sum_{k=0}^{n} C_n^k(-1)^k (ST)^k WST}_{F_3} = 0.$$

Let $B_T := PT$, where P is a linear projection with domain Y and kernel $T(0)$, then B_T is a bounded operator. We consider $B = (I - TS)^n(I + B_TWS)(I - TS)$, from $(**)$, we get

$$\begin{aligned}
B = (I - TS)^n + \overbrace{\sum_{k=0}^{n} C_n^k(-1)^k (TS)^k B_T WS}^{G_1} - \overbrace{\Big(\sum_{k=0}^{n} C_n^k(-1)^k (TS)^k\Big) TS}^{G_2} \\
- \overbrace{\sum_{k=0}^{n} C_n^k(-1)^k (TS)^k B_T WSTS}^{G_3}.
\end{aligned}$$

Since $(TS)^k B_T = (TS)^k B_T + T(0) = (TS)^k(B_T + T(0)) = (TS)^k T$ and $T(0) \subset \mathsf{N}[(TS)^k]$, for all $k \ge 1$, we deduce that

$$G_1 = T\Big(\sum_{k=0}^{n} C_n^k(-1)^k (ST)^k W\Big)S = TF_2S, \quad G_3 = T\Big(\sum_{k=0}^{n} C_n^k(-1)^k (ST)^k WST\Big)S = TF_3S,$$

$$G_2 = \Big(\sum_{k=0}^{n} C_n^k(-1)^k(TS)^kT\Big)S = T\Big(\sum_{k=0}^{n} C_n^k(-1)^k(ST)^k\Big)S = TF_1S.$$

Consequently,

$$B = (I - TS)^n + T(F_2 - F_3 - F_1)S = (I - TS)^n - TAS = (I - TS)^n + T(0) = (I - TS)^n,$$

and so

$$(* * *) \qquad (I - TS)^n(I + B_TWS)(I - TS) = (I - TS)^n.$$

Since $(I - TS)^n(0) = T(0)$ and $(I - TS)^n(I - P) \subset (I - TS)^nT(0) = T(0)$, we obtain

$$(I - TS)^n = (I - TS)^nP + (I - TS)^n(I - P) + T(0) = (I - TS)^nP + T(0) = (I - TS)^nP.$$

Therefore from $(* * *)$, $L := P(I + B_TWS) \in \mathscr{B}(\mathsf{Y})$ is a g_n-inverse of $(I - TS)$, since $L(I - TS)(0) \subset P(T(0) - B_TWST(0)) = \{0\}$. This completes the proof. □

The nullity and the defect of a linear relation $T \in \mathcal{L}_{\mathcal{R}}(\mathsf{X}, \mathsf{Y})$ are defined by $\alpha(T) = \dim \mathsf{N}(T)$ and $\beta(T) = \dim \mathsf{Y}/\mathsf{R}(T)$, respectively. Let $T \in \mathcal{L}_{\mathcal{R}}(\mathsf{X}, \mathsf{Y})$ be a closed relation. Recall that T is said to be upper semi-Fredholm if T has closed range and $\alpha(T) < +\infty$, and T is said to be lower semi-Fredholm if $\beta(T) < +\infty$. If T is upper or lower semi-Fredholm we say that T is semi-Fredholm, and we denote by $\Phi_\pm$ the class of all semi-Fredholm relations. For $T \in \Phi_\pm$ we define the index of T by

$$\operatorname{ind}(T) = \alpha(T) - \beta(T).$$

A closed linear relation $T \in \mathcal{L}_{\mathcal{R}}(\mathsf{X}, \mathsf{Y})$ is Fredholm if $\max\{\alpha(T), \beta(T)\} < +\infty$. We denote by Φ (resp. Φ_+, Φ_-) the class of all Fredholm (resp. upper semi-Fredholm, lower semi-Fredholm) relations. In the sequel, we denote by $\mathcal{R}_g^1 := \mathcal{R}_g^1(\mathsf{X}, \mathsf{Y})$ or $\mathcal{R}_g^1(\mathsf{X}, \mathsf{X})$ and $\mathcal{R}_g^n := \mathcal{R}_g^n(\mathsf{X})$, for all $n \geq 2$.

Corollary 2.8. *Let $T \in \mathcal{R}_g^n$, for some $n \in \mathbb{N}\backslash\{0\}$ be such that $\alpha(T) < +\infty$. If S is a g_n-inverse of T, then*

1) *$I - ST \in \mathcal{R}_g^1$, $T \in \Phi_+$, $S \in \Phi_-$, ST is a Drazin invertible,*
2) *TS is a Drazin invertible, whenever $T(0) = \{0\}$,*
3) *$I - TS \in \mathcal{R}_g^1$, whenever $T(0)$ is complemented.*

Proof. We see that $T^n(I - ST) = T^n(0)$, this implies that $\mathsf{R}(I - ST) \subset \mathsf{N}(T^n)$. Therefore [18, Lemma 5.4] shows that $\dim \mathsf{R}(I - ST) < +\infty$, and so [14, Theorem 6.3.4] implies that $I - ST \in \mathcal{R}_g^1$. Now, by using [6, Theorem V.8.5], we obtain $ST \in \Phi$, and so $\beta(S) < +\infty$ and [6, Proposition V.2.16] shows that $T \in \Phi_+$. Now, Proposition 2.7 gives that $I - TS \in \Omega_1^\ell$, when $T(0)$ is complemented.

Since $\boldsymbol{a}(ST) = \boldsymbol{a}(I - (I - ST)) < +\infty$ and $\boldsymbol{d}(ST) = \boldsymbol{d}(I - (I - ST)) < +\infty$, then ST is a Drazin invertible. Hence, if T is an operator, [2, Theorem 1.124] proves that TS is also Drazin invertible. This completes the proof. □

The following lemma extends [4, Section 6, P. 25] and [17, Lemma 3.1] to the case of bounded linear relations.

Lemma 2.9. *Let $T \in \mathcal{B}_{\mathcal{R}}(\mathsf{X}, \mathsf{Y})$ and $n \in \mathbb{N}\backslash\{0\}$, then*

$$T \in \mathcal{R}_g^m \Longleftrightarrow \begin{cases} \exists\, S \in \mathscr{B}(\mathsf{Y}, \mathsf{X}) \text{ such that } ST(0) = \{0\}, \\ \qquad \text{and} \\ T^mST - T^m \in \mathcal{R}_g^1, \end{cases}$$

where $m = n$ if $\mathsf{X} = \mathsf{Y}$ and $m = 1$ when $\mathsf{X} \neq \mathsf{Y}$.

Proof. If $T \in \mathcal{R}_g^m$ and S is a g_m-inverse of T, then $T^mST - T^m = T^m(0) \in \mathcal{R}_g^1$ and $ST(0) = \{0\}$. Conversely, since $T^mST - T^m \in \mathcal{R}_g^1$, then there exists a bounded operator L such that

$$T^mST - T^m = (T^mST - T^m)L(T^mST - T^m)$$

and

$$LT^m(0) = L(T^mST - T^m)(0) = \{0\}.$$

Since $T(0) \subset \mathsf{N}(S) \subset \mathsf{N}(T^mS)$, it follows from Lemma 1.1 that

$$T^m = T^mST - (T^mST - T^m)L(T^mST - T^m) = T^m(S - (ST - I)L(T^mS - T^{n-1}))T,$$

and as $(S - (ST - I)L(T^mS - T^{m-1}))(0) = -(ST - I)LT^m(0) = \{0\}$, we deduce that $T \in \mathcal{R}_g^m$. This completes the proof of Lemma 2.9. □

In the sequel, we will need the following three lemmas.

Lemma 2.10. *Let $T \in \mathcal{L}_{\mathcal{RD}}(\mathsf{X}, \mathsf{Y})$ be such that $T(0)$ is closed.*

1) *If $QT \in \mathcal{R}_g^1$, then $T \in \mathcal{R}_g^1$.*
2) *If $T(0)$ is complemented, then*

$$T \in \mathcal{R}_g^1 \Longleftrightarrow B_T \in \mathcal{R}_g^1 \Longleftrightarrow QT \in \mathcal{R}_g^1.$$

Proof. 1) Let $S : \mathsf{Y}/T(0) \longrightarrow \mathsf{Y}$ be a bounded operator such that $QTSQT = QT$. Therefore

$$Q_TT = Q_T(T(SQ_T)T) \Longrightarrow T(SQ_T)T - T \subset T(0),$$

this shows that $T = T + T(SQ_T)T - T = T(SQ)T$, and as $SQ_TT(0) = \{0\}$, we deduce that SQ_T is a g_1-inverse of T.

2) From the proof of [13, Lemma 2.1], we have

$$T \in \mathcal{R}_g^1 \Longleftrightarrow B_T \in \mathcal{R}_g^1.$$

Assume now that $T \in \mathcal{R}_g^1$ and let S be a g_1-inverse of T. We consider the linear application

$$\begin{array}{rccc} \pi : & \mathsf{Y}/T(0) & \longrightarrow & \mathsf{Y} \\ & \overline{x} & \longmapsto & Px, \end{array}$$

where P is a linear projection with domain Y and kernel $T(0)$. We note that

$$QTS\pi QT = QTSPT = QTS(PT + T(0)) = Q_T(TST) = QT,$$

and therefore $QT \in \mathcal{R}_g^1$. The proof is completed. □

A linear relation $T \in \mathcal{L}_{\mathcal{RD}}(\mathsf{X}, \mathsf{Y})$ is called compact if QT is a compact operator (see [6, Definition V.1.1]). It is clear that if $T \in \mathcal{C}_{\mathcal{RD}}(\mathsf{X}, \mathsf{Y})$ such that $\dim \mathsf{R}(T)/T(0) < +\infty$, then T is compact. Notice that an everywhere defined linear relation, that is compact, is necessary bounded, see [6, Corollary V.2.3].

Lemma 2.11. *Let $T, S \in \mathcal{L}_{\mathcal{RD}}(\mathsf{X}, \mathsf{Y})$ be compacts, $U \in \mathcal{B}_{\mathcal{R}}(\mathsf{Y}, \mathsf{Z})$ and $V \in \mathcal{L}_{\mathcal{RD}}(\mathsf{Z}, \mathsf{X})$ be a linear relation with a bounded selection, then $T + S$, UT and TV are compacts.*

Proof. From [6, Proposition IV.2.12, Theorem V.2.2, Proposition V.2.10], we have $T + S$ and UT are compacts. Let A be a bounded selection of V. Then $V = A + V - V$, and by Lemma 1.1, we get $TV = TA + (TV - TV)$. On one hand, since A is a bounded operator, it follows from [6, Proposition V.2.12] that TA is compact, and as $TV - TV$ is compact, we obtain $TV = TA + (TV - TV)$ is compact. This completes the proof. □

The following lemma follows immediately from [6, Theorem V.5.12, Corollary V.711].

Lemma 2.12. *Let $T \in \mathcal{L}_{\mathcal{RD}}(\mathsf{X})$ be compact.*

1) $\lambda I - T \in \Phi_-$, *for all* $\lambda \in \mathbb{C}\backslash\{0\}$.
2) *If* $\dim T(0) < +\infty$, *then* $\lambda I - T \in \Phi_+$, *for all* $\lambda \in \mathbb{C}\backslash\{0\}$.

We know from [14, Theorem 6.3.4, Theorem 6.8.5] that every finite rank operator is g_1-invertible, and a compact operator is g_1-invertible if and only if it is of finite rank. The following theorem extends this result to linear relations.

Theorem 2.13. *Let $T \in C_{\mathcal{RD}}(\mathsf{X})$ and $n \in \mathbb{N}\backslash\{0\}$.*

1) *If* $\dim \mathsf{R}(T^n)/T^n(0) < +\infty$ *and $T^n(0)$ is closed, then T has a g_n-inverse.*
2) *Suppose that T is compact. If T has a g_n-inverse, then* $\dim \mathsf{R}(T^n)/T^n(0) < +\infty$.

Proof. 1) First, we note from Lemma 2.10 and [14, Theorem 6.3.4] that $T^n \in \mathcal{R}_g^1$. Therefore by the proof of Corollary 2.6, we obtain $T \in \mathcal{R}_g^n$.

2) Let S be a g_n-inverse of T. As T is a compact relation, then by Lemma 2.11, ST is a bounded compact operator, and hence $I - ST$ is Fredholm. Now, since $\mathsf{R}(I - ST) \subset \mathsf{N}(T^n)$, we have

$$\dim \mathsf{R}(T^n)/T^n(0) = \dim \mathsf{X}/\mathsf{N}(T^n) \leq \beta(I - ST) < +\infty.$$

This completes the proof. □

Corollary 2.14. *Let $n \in \mathbb{N}\backslash\{0\}$ and $T \in C_{\mathcal{RD}}(\mathsf{X})$ be such that* $\dim \mathsf{R}(T^n) < +\infty$, *then T has a g_n-inverse.*

Proof. Since $\dim T^n(0) \leq \dim \mathsf{R}(T^n) < +\infty$ and $\|T^n\| < +\infty$, then $T^n \in C_{\mathcal{RD}}(\mathsf{X})$. Therefore by Theorem 2.13, we have $T \in \mathcal{R}_g^n$. □

Theorem 2.15. *If $T \in C_{\mathcal{RD}}(\mathsf{X})$ has a g_n-inverse compact, for some $n \in \mathbb{N}\backslash\{0\}$, we have* $\dim \mathsf{R}(T^n)/T^n(0) < +\infty$.

Proof. Let $S \in \mathscr{B}(\mathsf{X})$ be compact such that $T^nST = T^n$ and $ST(0) = \{0\}$. Since ST is a compact operator according to Lemma 2.11, from the proof of Theorem 2.13, we deduce that $\dim \mathsf{R}(T^n)/T^n(0) < +\infty$. □

By a similar argument to the one in the proof of Theorems 2.13 and 2.15, we obtain :

Theorem 2.16. *Let $T \in C_{\mathcal{RD}}(\mathsf{X}, \mathsf{Y})$.*

1) *If* $\dim \mathsf{R}(T)/T(0) < +\infty$, *then T has a g_1-inverse.*
2) *Let S be a g_1-inverse of T. If T or S is compact, then* $\dim \mathsf{R}(T)/T(0) < +\infty$.

3. Semi-Fredholm and g_n-invertible linear relations

This section focuses on some properties of g_n-invertible and semi-Fredholm operators from X to Y. These properties are an extension of similar properties obtained in [4] for the class of bounded operators acting on a Banach space.

We start this section with the following lemma.

Lemma 3.1. *Let $T \in \mathcal{L}_{\mathcal{RD}}(\mathsf{X})$ and M be a subspace of X. If* $\dim \mathsf{M} < +\infty$, *then*

$$\dim T(\mathsf{M})/T(0) < +\infty.$$

Proof. Let A be a selection of T, then $T(\mathsf{M}) = A(\mathsf{M}) + T(0)$. Therefore

$$\dim T(\mathsf{M})/T(0) = \dim(A(\mathsf{M}) + T(0))/T(0) \leq \dim A(\mathsf{M}) < +\infty.$$

What needed to be demonstrated. □

Theorem 3.2. *Let $T \in C_{\mathcal{RD}}(\mathsf{X}, \mathsf{Y})$ and $S \in C_{\mathcal{RD}}(\mathsf{Y}, \mathsf{Z})$ be such that $ST \in \Phi$. Then*

1) *T has a g_1-inverse,*
2) *S has a g_1-inverse whenever $T(0) = \{0\}$,*
3) *S has a g_1-inverse whenever T has a continuous selection and $ST(0) = \{0\}$.*

Proof. 1) First, we have $ST \in C_{\mathcal{RD}}(\mathsf{X}, \mathsf{Z})$, because $ST(0)$ is closed and $\|ST\| < +\infty$. Since $STUST = ST$ and $UST(0) = \{0\}$, for some $U \in \mathscr{B}(\mathsf{Z}, \mathsf{X})$, then $ST(UST - I) = ST(0)$, and so $\dim \mathsf{R}(UST - I) \leq \dim \mathsf{N}(ST) < +\infty$. Then it follows by Lemma 3.1 that $\dim \mathsf{R}(TUST - T)/T(0) < +\infty$. Let $L := TUST - T$, we have $L(0) = T(0)$ is closed and $\|QL\| < +\infty$, and hence $L \in C_{\mathcal{RD}}(\mathsf{X})$. Now, Theorem 2.16 implies that $TUST - T \in \mathcal{R}_g^1$. Let $S' := US$, since $\|S'\| < +\infty$, $S'(0) \subset S'T(0) = UST(0) = \{0\}$, we infer by Lemma 2.9 that $T \in \mathcal{R}_g^1$.

2) and 3) Since $(STU - I)ST = ST(0)$, then $\mathsf{R}(ST) \subset \mathsf{N}(STU - I)$, and so

$$\dim \mathsf{R}(STU - I)/ST(0) = \dim \mathsf{Z}/\mathsf{N}(STU - I) \leq \beta(ST) < +\infty.$$

We note that

$$STU(0) \subset STUS(0) \subset STUST(0) = ST(0) = STU(0),$$

this gives that $S(0) \subset \mathsf{N}(STU)$ and $STUS(0) = ST(0)$. Therefore

$$\dim \mathsf{R}[Q(STUS - S)] = \dim \mathsf{R}[(STU - I)S]/ST(0) \leq \dim \mathsf{R}(STU - I)/ST(0) < +\infty,$$

and $STUS - S \in C_{\mathcal{RD}}(\mathsf{Y}, \mathsf{Z})$, because $(STUS - S)(0) = ST(0)$ is closed and $\|STUS - S\| < +\infty$, and hence Theorem 2.16 implies that $STUS - S \in \mathcal{R}_g^1$. Now, we can deduce from Lemma 2.9 that $S \in \mathcal{R}_g^1$, when T is an operator. Finally, if T has a continuous selection A and $ST(0) = \{0\}$, then $SAUS - S = STUS - S \in \mathcal{R}_g^1$ and $S \in \mathcal{R}_g^1$. This completes the proof. $\square$

Corollary 3.3. *Let $T \in C_{\mathcal{RD}}(\mathsf{X})$ and $S \in C_{\mathcal{RD}}(\mathsf{X}, \mathsf{Y})$ be such that $ST \in \Phi$, then T has a g_n-inverse, for all $n \geq 1$.*

For $T \in \mathcal{L}_{\mathcal{R}}(\mathsf{X})$, the g-essential ascent, $\widetilde{\boldsymbol{a}}_e(T)$, is defined by (see [8, 9])

$$\widetilde{\boldsymbol{a}}_e(T) = \inf\{n \in \mathbb{N} : \dim \mathsf{R}(T^n) \cap \mathsf{N}(T) < +\infty\},$$

the infimum over the empty set is taken to be $+\infty$.

The following result goes back to [17, Theorem 5.3] for bounded operators. We will improve it for closed linear relations everywhere defined.

Theorem 3.4. *If $T \in \mathcal{R}_g^n$, for some $n \in \mathbb{N}\backslash\{0\}$ is such that $T^2(0) = T(0)$ and $\widetilde{\boldsymbol{a}}_e(T) < +\infty$, then $T \in \mathcal{R}_g^m$, where $m = \max\{1, \widetilde{\boldsymbol{a}}_e(T)\}$.*

Proof. We note that the result is obvious when $n \leq \widetilde{\boldsymbol{a}}_e(T)$, and hence we suppose that $n > \widetilde{\boldsymbol{a}}_e(T)$. Let S be a bounded operator such that $T^nST = T^n$ and $ST(0) = \{0\}$. If $\alpha(T) < +\infty$, then $UT - I$ is finite dimensional. This implies that $UT = I + (UT - I)$ is Fredholm, and so by Theorem 3.2, $T \in \mathcal{R}_g^1$. Now, we suppose that $1 \leq \widetilde{\boldsymbol{a}}_e(T)$, then there exists $k \in \{1, 2, \cdots, n-1\}$ such that $\widetilde{\boldsymbol{a}}_e(T) = n - k$. Since $T^k(T^{n-k}ST - T^{n-k}) = T^n(0) = T^k(0)$, we obtain $\mathsf{R}(T^{n-k}ST - T^{n-k}) \subset \mathsf{R}(T^{n-k}) \cap \mathsf{N}(T^k) = \mathsf{N}[(T_{|\mathsf{R}(T^{n-k})})^k]$. This gives that

$$\dim \mathsf{R}(T^{n-k}ST - T^{n-k}) \leq k \dim \mathsf{N}[T_{|\mathsf{R}(T^{n-k})}] = k \dim \mathsf{R}(T^{n-k}) \cap \mathsf{N}(T) < +\infty.$$

On the other hand, since $(T^{n-k}ST - T^{n-k})(0) = T(0)$ is closed and $\|T^{n-k}ST - T^{n-k}\| < +\infty$, then $T^{n-k}ST - T^{n-k} \in C_{\mathcal{RD}}(\mathsf{X})$. Hence, Theorem 2.13 proves that $T^{n-k}ST - T^{n-k} \in \mathcal{R}_g^1$. Then it follows from Lemma 2.9 that $T \in \mathcal{R}_g^{n-k}$. This completes the proof. $\square$

Corollary 3.5. *If $T \in \mathcal{R}_g^n$, for some $n \in \mathbb{N}\backslash\{0\}$ is such that $T^2(0) = T(0)$ and $\dim \mathsf{R}(T) \cap \mathsf{N}(T) < +\infty$, then $T \in \mathcal{R}_g^1$.*

Theorem 3.6. *Let $n \in \mathbb{N}\backslash\{0\}$ and $T, S \in C_{\mathcal{RD}}(\mathsf{X})$ have g_n-inverses be such that* $\max\{\alpha(T), \alpha(S)\} < +\infty$. *Then ST has a g_n-inverse.*

Proof. There exist U and V such that $S^nUS = S^n$, $T^nVT = T^n$ and $US(0) = VT(0) = \{0\}$. Hence $S^n(US - I) = S^n(0)$ and $T^n(VT - I) = T^n(0)$ and therefore $US - I$ and $VT - I$ are finite dimensional. Since $T(0) \subset \mathsf{N}(V)$, we obtain

$$\begin{aligned} VUST &= I + VT - I + VUST - VT \\ &= I + VT - I + V(US - I)T, \end{aligned}$$

and as $\dim \mathsf{R}(VT - I + V(US - I)T) < +\infty$, it follows from [6, Theorem V.8.5] that $VUST \in \Phi$. By Theorem 3.2, $ST \in \mathcal{R}_g^n$, which proves the theorem. □

Remark 3.7. Let $T \in C_{\mathcal{RD}}(\mathsf{X}, \mathsf{Y})$ and $S \in C_{\mathcal{RD}}(\mathsf{Y}, \mathsf{Z})$ have g_1-inverses. If $\max\{\alpha(T), \alpha(S)\} < +\infty$, by the proof of Theorem 3.6, we see that ST has a g_1-inverse.

Theorem 3.8. *Let $T \in C_{\mathcal{RD}}(\mathsf{X}, \mathsf{Y})$ and $S \in C_{\mathcal{RD}}(\mathsf{Y}, \mathsf{Z})$ have g_1-inverses be such that* $\dim ST(0) < +\infty$. *If* $\max\{\beta(T), \beta(S)\} < +\infty$, *then ST has a g_1-inverse.*

Proof. There exist U and V such that $SVS = S$, $TUT = T$ and $VS(0) = UT(0) = \{0\}$. Since $\mathsf{R}(T) \subset \mathsf{N}(TU - I)$, then

$$\dim \mathsf{R}(TU - I)/T(0) = \dim \mathsf{Y}/\mathsf{N}(TU - I) \le \beta(T) < +\infty.$$

It follows that $TU - I$ is compact. In the similar way, we obtain that $SV - I$ is compact. Now, by Lemma 2.11, we get $K := (SV - I) + S(TU - I)V$ is compact, and as $\dim K(0) = \dim ST(0) < +\infty$, Lemma 2.12 implies that $STUV = I + K$ is Fredholm. Finally, by Theorem 3.2, we deduce that $ST \in \mathcal{R}_g^1$. This completes the proof. □

Theorem 3.9. *Let $T \in C_{\mathcal{RD}}(\mathsf{X}, \mathsf{Y})$ and $S \in C_{\mathcal{RD}}(\mathsf{Y}, \mathsf{Z})$ be such that $ST \in \mathcal{R}_g^1$. If $\alpha(ST) < +\infty$ or $\alpha(S) < +\infty$ and $ST(0) = S(0)$, then $T \in \mathcal{R}_g^1$. Similarly, if $T(0) = \{0\}$ and $\min\{\beta(ST), \beta(T)\} < +\infty$, then $S \in \mathcal{R}_g^1$.*

Proof. For some U, $STUST = ST$ such that $UST(0) = \{0\}$. Hence $ST(UST - I) = S(TUST - T) = ST(0)$, and so $\dim \mathsf{R}(UST - I) < +\infty$ or $\dim \mathsf{R}(TUST - T) < +\infty$, when $\alpha(ST) < +\infty$ or $\alpha(S) < +\infty$ and $ST(0) = S(0)$. In either case, it follows that $\dim \mathsf{R}(TUST - T)/T(0) < +\infty$ according to Lemma 3.1. Therefore Lemma 2.9 and Theorem 2.16 imply that $T \in \mathcal{R}_g^1$. Now, we suppose that $T(0) = \{0\}$ and $\min\{\beta(ST), \beta(T)\} < +\infty$. A similar argument can be used to show that $\dim \mathsf{R}(STUS - S)/S(0) \le \dim \mathsf{R}(STU - I)/S(0) < +\infty$. Since $TUS(0) = TUST(0) = \{0\}$, by Lemma 2.9 and Theorem 2.16, we infer that $T \in \mathcal{R}_g^1$. This completes the proof. □

Theorem 3.10. *If $T \in \mathcal{R}_g^n \cap \Phi_+$, for some $n \in \mathbb{N}\backslash\{0\}$, then there exists a positive real number r such that if $\|T - S\| < r$ and $S(0) \subset T(0)$, then $S \in \mathcal{R}_g^1 \cap \Phi_+$ and in fact, $\alpha(S) \le \alpha(T)$.*

Proof. For some U, $T^nUT = T^n$ and $UT(0) = \{0\}$. If $\alpha(T) < +\infty$, then $T^n(UT - I) = T^n(0)$ implies that $UT - I$ is finite dimensional. Suppose we write $A = S - T$, with S is an everywhere defined closed linear relations such that $S(0) \subset T(0)$. Thus

$$US = U(S + T(0)) = U(A + T) = (I + UA) + (UT - I).$$

Choose $r < \|U\|^{-1}$ so that $I + UA$ is invertible. Then US must be Fredholm and by Theorem 3.2, S has a g_1-inverse. Finally, we can prove the inequality $\alpha(S) \le \alpha(T)$ similarly as in the proof of [4, Theorem 4, page 27]. □

Remark 3.11. From the proof of [4, Theorem 4, page 27], with Lemma 2.9 and Theorem 3.2, we can prove that [4, Theorem 4, page 27] remains valid when T is a bounded operator from X to Y.

Theorem 3.12. *If $T \in \mathcal{R}_g^1 \cap \Phi_-$ be such that $T(0)$ is complemented, then there exists a positive real number r such that if $\|T - S\| < r$ and $S(0) \subset T(0)$, then $S \in \mathcal{R}_g^1 \cap \Phi_-$ and in fact, $\beta(S) \le \beta(T)$.*

Proof. Since $T(0)$ is complemented, by Lemma 2.10, we have $QT \in \mathcal{R}_g^1 \cap \Phi_-$. Then it follows from [4, Theorem 4, page 27] and Remark 3.11 that there exists $r > 0$ such that if $\|QT - L\| < r$, then $L \in \mathcal{R}_g^1 \cap \Phi_-$ and $\beta(L) \leq \beta(QT)$. Let $S \in C_{\mathcal{RD}}(\mathsf{X}, \mathsf{Y})$ be such that $S(0) = T(0)$ and $\|T - S\| < r$. Since $\|QT - QS\| = \|T - S\| < r$, then $QS \in \mathcal{R}_g^1 \cap \Phi_-$ and $\beta(S) = \beta(QS) \leq \beta(QT) = \beta(T)$. Now, by Lemma 2.10, we have $S \in \mathcal{R}_g^1$, and the proof is completed. □

Theorem 3.13. *Let $T \in \mathcal{R}_g^n$, for some $n \in \mathbb{N}\backslash\{0\}$, and V be a compact relation.*

1) *Assume that $V(0)$ is finite dimensional or $V(0) \subset T(0)$. If $\alpha(T) < +\infty$, then $T + V \in \mathcal{R}_g^1 \cap \Phi_+$.*
2) *Assume that $T(0)$ is finite dimensional and $V(0) \subset T(0)$. If $\beta(T) < +\infty$, then $T + V \in \mathcal{R}_g^1 \cap \Phi_-$.*

Proof. 1) For some U, $T^nUT = T^n$ and $UT(0) = \{0\}$. Assume $\alpha(T) < +\infty$, then by [18, Lemma 5.4], we have

$$\dim \mathsf{R}(UT - I) \leq \alpha(T^n) < n\,\alpha(T) < +\infty.$$

So Lemma 2.11 implies that $S := UV + (UT - I)$ is compact. If $V(0)$ is finite dimensional (resp. $V(0) \subset T(0)$), then $S(0) = UV(0)$ (resp. $S(0) = \{0\}$) is finite dimensional. Therefore by Lemma 2.12, we obtain that $U(T + V) = I + S$ is Fredholm. Now, Theorem 3.2 and [6, Proposition V.2.16] imply that $T + V \in \Omega_1^\ell \cap \Phi_+$.

2) Let P be a linear projection with domain Y (or X, if $n > 1$) and kernel $T(0)$ and let $B_T := PT$. We have $\beta(B_T) < +\infty$ and PV is a compact operator according to Lemma 2.11. Then it follows from Lemma 2.10 and [4, Theorem 5, page 27] that $B_{T+V} = B_T + PV \in \mathcal{R}_g^1$ and $\beta(B_{T+V}) = \beta(B_T + PV) < +\infty$. This shows that $T + V \in \mathcal{R}_g^1$ and $\beta(T + V) < +\infty$. The proof is therefore completed. □

Corollary 3.14. *Let $T \in \mathcal{R}_g^n$ and S be a g_n-inverse of T. If $\alpha(T) < +\infty$, then $TST \in \mathcal{R}_g^1 \cap \Phi_+$.*

Proof. Since $\alpha(T) < +\infty$, then $\dim \mathsf{R}(I - ST) < +\infty$. Therefore Lemma 3.1 implies that $\dim \mathsf{R}(T - TST)/T(0) < +\infty$. Thus $V := T - TST$ is compact, and as $V(0) \subset T(0)$, by Theorem 3.13 it follows that $TST = T - V \in \mathcal{R}_g^1 \cap \Phi_+$. □

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