Filomat 37:27 (2023), 9243–9258
https://doi.org/10.2298/FIL2327243G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On the growth of the modulus of the derivative of algebraic polynomials in bounded and unbounded domains with cusps

Cevahir Doğanay Gün[a]

[a]*Department of Mathematics and Science Education, Nizip Faculty of Education, Gaziantep University, Gaziantep, Turkey*

Abstract. In this present work, we study the behavior of the derivatives of algebraic polynomials in the bounded and unbounded regions bounded by piecewise smooth curve with interior and exterior cusps.

1. Introduction and main results

Let $\mathbb{C}$ denote the complex plane; $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$; $G \subset \mathbb{C}$, $0 \in G$, be a finite Jordan domain; $\Gamma := \partial G$, $\Omega := \overline{\mathbb{C}} \backslash \overline{G}$. For $R > 0$ and $t \in \mathbb{C}$, let us set: $\Delta(t, R) := \{w : |w - t| > R\}$ (with respect to $\overline{\mathbb{C}}$). Let $w = \Phi(z)$, $\Phi : \Omega \to \Delta(0, 1)$, be the univalent conformal mapping with normalization: $\Phi(\infty) = \infty$, $\lim_{z\to\infty} \frac{\Phi(z)}{z} > 0$. For any $R > 1$, we denote by $\Gamma_R := \{z : |\Phi(z)| = R\}$, exterior level curve for the Γ and let $G_R := int\Gamma_R$, $\Omega_R := ext\Gamma_R$.

Denote by $\wp_n$ the class of algebraic polynomials $P_n(z)$, $\deg P_n \leq n$, $n \in \mathbb{N}$.

For the Jordan domain G and a given weight function $h(z)$, we introduce:

$$\begin{aligned} \|P_n\|_{A_p(h,G)} &: = \left(\iint_G h(z) \left| P_n(z) \right|^p d\sigma_z \right)^{1/p}, \ 0 < p < \infty, \\ \|P_n\|_{A_\infty(1,G)} &: = \max_{z \in \overline{G}} |P_n(z)|, \ p = \infty, \end{aligned} \tag{1}$$

where σ is the two-dimensional Lebesgue measure, and when Γ is rectifiable:

$$\begin{aligned} \|P_n\|_p &: = \|P_n\|_{\mathcal{L}_p(h,\Gamma)} := \left(\int_\Gamma h(z) \left| P_n(z) \right|^p |dz| \right)^{1/p}, \ 0 < p < \infty, \\ \|P_n\|_\infty &: = \|P_n\|_{\mathcal{L}_\infty(1,\Gamma)} := \max_{z \in \Gamma} |P_n(z)|; \ \mathcal{L}_p(1, \Gamma) =: \mathcal{L}_p(\Gamma). \end{aligned} \tag{2}$$

In the theory of approximation of functions by algebraic polynomials, one of the important auxiliary results is the so-called Bernstein-Walsh lemma [41], which states that for any $P_n \in \wp_n$

$$|P_n(z)| \leq |\Phi(z)|^n \|P_n\|_{C(\overline{G})}, \ \ z \in \Omega \tag{3}$$

2020 *Mathematics Subject Classification*. Primary 30A10; Secondary 30C10, 41A17
Keywords. Algebraic polynomials, conformal mapping, smooth curve
Received: 18 March 2023; Revised: 10 May 2023; Accepted: 15 May 2023
Communicated by Ljubiša D.R. Kočinac
Email address: cevahirdoganaygun@gmail.com (Cevahir Doğanay Gün)

is true. Hence, for the points $z \in \overline{G}_R$ from (3), we have:

$$\|P_n\|_{C(\overline{G}_R)} \leq R^n \|P_n\|_{C(\overline{G})}. \tag{4}$$

This implies that the uniform norm ($C-$norm) of the polynomial P_n, over the domain G increases by at most a constant when the domain G is extended to the domain $G_{1+(1/n)}$.

Further, in 1937, in [31] it was proved that:

$$\|P_n\|_{\mathcal{L}_p(\Gamma_R)} \leq R^{n+\frac{1}{p}} \|P_n\|_{\mathcal{L}_p(\Gamma)}, \ \ p > 0. \tag{5}$$

This shows that if the $C-$norm of the polynomial P_n is replaced by the "$\mathcal{L}_p(\Gamma)-$norm" (2), then this fact remains valid.

Let us give a weight analogue of (5). For this, we define the weight function $h(z) \neq 1$ to be used throughout the work as follows.

Let $\left\{z_j\right\}_{j=1}^m \in \Gamma$ be a fixed system of distinct points. The generalized Jacobi weight function $h(z)$ is defined as follows:

$$h(z) = h_0(z) \prod_{j=1}^m \left|z - z_j\right|^{\gamma_j}, \quad z \in G_{R_0}, \ R_0 > 1, \tag{6}$$

where $\gamma_j > -1$, for all $j = 1, 2, ..., m$, and h_0 is uniformly separated from zero in G_{R_0}, i.e., there exists a constant $c_1(G) > 0$ such that for all $z \in G_{R_0}$, $h_0(z) \geq c_1(G) > 0$.

The weighted analog of the inequality (5) has been proved in [11, Lemma 2.4] for weight function $h(z)$ as defined in (6) as follows:

$$\|P_n\|_{\mathcal{L}_p(h,\Gamma_R)} \leq R^{n+\frac{1+\gamma^*}{p}} \|P_n\|_{\mathcal{L}_p(h,\Gamma)}, \ \gamma^* = \max\left\{0; \gamma_j : 1 \leq j \leq \ m\right\}. \tag{7}$$

The analog of the estimates (3) and (7) for the "$A_p(h, G)-$norm" for the domains with quasiconformal boundary (see: Definition 2.1) and the same weight function (6) with $\gamma_j > -2, \ j = 1, 2, ..., m$, was given in [2] as follows:

$$\|P_n\|_{A_p(h,G_R)} \leq c_2 \, {R^*}^{n+\frac{1}{p}} \|P_n\|_{A_p(h,G)}, \ p > 0,$$

where $R^* := 1 + c_3(R - 1)$ and $c_3 > 0$, $c_2 := c_2(G, p, c_3) > 0$, are constants independent from n and R. Further, for arbitrary Jordan domain G, any $P_n \in \wp_n, \ R_1 = 1 + \frac{1}{n}$, in [6, Theorem1.1] it was obtained that

$$\|P_n\|_{A_p(G_R)} \leq c R^{n+\frac{2}{p}} \|P_n\|_{A_p(G_{R_1})}, \ p > 0,$$

is true for arbitrary $R > R_1 = 1 + \frac{1}{n}$, where $c = \left(\frac{2}{e^p - 1}\right)^{\frac{1}{p}} \left[1 + O(\frac{1}{n})\right], \ n \to \infty$, is asymptotically exact constant.

In [39] N. Stylianopoulos showed that if the curve Γ is rectifiable and quasiconformal, then there exists a constant $C = C(\Gamma) > 0$ depending only on Γ such that

$$|P_n(z)| \leq C \frac{\sqrt{n}}{d(z, \Gamma)} \|P_n\|_{A_2(G)} |\Phi(z)|^{n+1}, \ \ z \in \Omega,$$

where $d(z, \Gamma) := \inf\{|\zeta - z| : \ \zeta \in \Gamma\}$, holds for every $P_n \in \wp_n$. In this paper, firstly we study the following type estimation for the derivatives $|P_n'(z)|$:

In this work, our goal is to study the behavior of the $\left|P_n'(z)\right|$ on the whole complex plane. To do this, we first find estimates for $\left|P_n'(z)\right|$ in an unbounded domain Ω as the following type:

$$\left|P_n'(z)\right| \leq \eta_n \|P_n\|_p, \ z \in \Omega; \tag{8}$$

and, on a bounded domain $\overline{G}$ as the following type:

$$\|P_n^{'}\|_\infty \leq \mu_n \|P_n\|_p, \tag{9}$$

where $\eta_n := \eta_n(\Gamma, h, p, z) \to \infty$ and $\mu_n := \mu_n(\Gamma, h, p) \to \infty$ as $n \to \infty$, are constants depending on the properties of the Γ and h.

Estimates analogously to (8) and (9) for arbitrary $P_n \in \wp_n$, a different weight function h and for some norms were obtained in [7, 17, 18, 20, 21, 30, 33, 36, 39] for an unbounded domain, and in [2, 5, 16, 19, 24–28, 30, 32–35, 37, 40] (also reference therein and others) for a bounded domain.

The objects of study in this work will be piecewise smooth regions with interior and exterior zero angles. For this, we give some definitions and notations that will be used later in the text.

Let S be a rectifiable Jordan curve or an arc and $z = z(s)$, $s \in [0, |S|]$, $|S| := mes\, S$ (linear measure of S), denote the natural representation of S.

Definition 1.1. We say that a Jordan curve or an arc $S \in C_\theta$, if S has a continuous tangent $\theta(z) := \theta(z(s))$ at every point $z(s)$.

Throughout this work we denote by $c, c_0, c_1, c_2, ...$ are positive and $\varepsilon_0, \varepsilon_1, \varepsilon_2, ...$ are sufficiently small positive constants (in general, different in different relations), which depend on G in general. Let for any $j = 1, 2, ...$ and sufficiently small $\varepsilon_1 > 0$, $f_j : [0, \varepsilon_1] \to \mathbb{R}$ denoted the twice differentiable functions, such that $f_j(0) = 0$, $f_j^{(k)}(x) > 0$, $x > 0$ and $k = 0, 1, 2$. For any $k \geq 0$ and $m > k$, notations $j = \overline{k, m}$ denotes $j = k, k+1, ..., m$.

Definition 1.2. We say that a Jordan domain $G \in C_\theta(\lambda_i; f_j)$, $0 < \lambda_i \leq 2$, $i = \overline{1, m_1}$, $f_j = f_j(x)$, $j = \overline{m_1 + 1, m}$, if $\Gamma = \partial G$ consists of a union of finite number of C_θ-arcs $\left\{\Gamma_j\right\}_{j=0}^m$, connecting at the points $\left\{z_j\right\}_{j=0}^m \in \Gamma$ such that Γ is locally smooth at $z_0 \in \Gamma \backslash \left\{z_j\right\}_{j=1}^m$ and:

a) for every $z_i \in \Gamma$, $i = \overline{1, m_1}$, $m_1 \leq m$, the domain G has exterior (with respect to $\overline{G}$) angles $\lambda_i \pi$, $0 < \lambda_i \leq 2$, at the corner z_i;

b) for every $z_j \in \Gamma$, $j = \overline{m_1 + 1, m}$, in the local co-ordinate system (x, y) with origin at z_j the following conditions are satisfied

b_1) $\left\{z = x + iy : |z| < \varepsilon_1,\ c_1 f_j(x) \leq y \leq c_2 f_j(x),\ 0 \leq x \leq \varepsilon_1\right\} \subset \overline{\Omega}$,

b_2) $\left\{z = x + iy : |z| < \varepsilon_1,\ \left|y\right| \geq \varepsilon_2 x,\ 0 \leq x \leq \varepsilon_1\right\} \subset \overline{G}$,

for some constants $-\infty < c_1 < c_2 < +\infty$, $0 < \varepsilon_i < 1$, $i = 1, 2$.

Thus, Definition1.2 show that each domain $G \in C_\theta(\lambda_i; f_j)$ may have exterior non zero angles with opening $\lambda_i \pi$, $0 < \lambda_i < 2$, interior zero angles for $\lambda_i = 2$ at the points $z_i \in \Gamma$, $i = \overline{1, m_1}$, and exterior zero angles at the points $z_j \in \Gamma$, $j = \overline{m_1 + 1, m}$, at which the boundary arcs are touching with $f_j(x)$−speed. If $m = 0$, then the domain G does not have such angles, and in this case we will write: $G \in C_\theta$; if $m_1 = m \geq 1$, then G has only $\lambda_i \pi$, $0 < \lambda_i \leq 2$, $i = \overline{1, m_1}$, exterior angles (when $\lambda_i = 2$ −interior zero angles) and in this case we will write: $G \in C_\theta(\lambda_i; 0)$; if $m_1 = 0$ and $m \geq 1$, then G has only exterior zero angles and in this case we will write: $G \in C_\theta(1; f_j)$.

We assume that the points $\left\{z_j\right\}_{j=1}^m \in \Gamma$ defined in (6) and Definition 1.2 are the same. Also we assume that these points on the curve $\Gamma = \partial G$ are located in the positive direction such that, G has $\lambda_j \pi$, $0 < \lambda_j \leq 2$, $j = \overline{1, m_1}$, exterior angles (when $\lambda_j = 2-$ interior zero angles) at the points $\left\{z_j\right\}_{j=1}^{m_1}$, $m_1 \leq m$, and has exterior zero angles on the points $\left\{z_j\right\}_{j=m_1+1}^m$ and $w_j := \Phi(z_j)$.

Before stating our main results, we introduce some notation. For clarity of results, we will consider $m_1 = 1$, $m = 2$, i.e. the piesewise smooth curve Γ has two singular points $z_1 \in \Gamma$ and $z_2 \in \Gamma$. The reasoning for the cases $m_1 > 1$, $m > 2$ is carried out in exactly the same way. For $0 < \delta_j < \delta_0 := \frac{1}{4} \min \left\{\left|z_i - z_j\right| : i, j = 1, 2, ..., \Gamma,\ i \neq j\right\}$, let $\Omega(z_j,\ \delta_j) := \Omega \cap \left\{z : \left|z - z_j\right| \leq \delta_j\right\}$; $\delta := \min\limits_{1 \leq j \leq \Gamma} \delta_j$. For $\Gamma = \partial G$, we put:

$U_\infty(\Gamma,\delta) := \bigcup\limits_{\zeta\in\Gamma} U(\zeta,\ \delta)$ - infinite open cover of the curve Γ; $U_N(\Gamma,\delta) := \bigcup\limits_{j=1}^{N} U_j(\Gamma,\delta) \subset U_\infty(\Gamma,\delta)$ - finite open cover of the curve Γ. For any $t \geq 1$ we put: $\Omega_t(\delta) := \Omega(\Gamma_t,\delta) := \Omega_t \cap U_N(\Gamma_t,\delta)$, $\widehat{\Omega}_t := \Omega_t \setminus \Omega_t(\delta)$.

Now we proceed to the formulation of new results. Throughout this work we will assume that $p > 1$ and we put:

$$\widetilde{\lambda}_k := \begin{cases} \max\{1,\lambda_k\}+\varepsilon, & 0<\lambda_k<2, \\ 2, & \lambda_k = 2, \end{cases} \quad k=1,2.\ ; \quad \widetilde{\lambda} := \max\left\{\widetilde{\lambda}_1,\widetilde{\lambda}_2\right\}; \quad \widetilde{\gamma} := \max\{\gamma_1,\gamma_2\}, \quad \gamma_k^* := \max\{0,\gamma_k\},$$

$$\widetilde{\gamma}^* := \max\left\{\gamma_1^*, v\gamma_2^*\right\}; \quad \alpha_* := \min\{\alpha_1,\alpha_2\}; \quad \alpha^* := \max\{\alpha_1,\alpha_2\}.$$

Theorem 1.3. *Assume that $G \in C_\theta(\lambda_1; cx^{1+\alpha_2})$ for some $0 < \lambda_1 \leq 2$, $\alpha_2 > 0$ and $h(z)$ is defined by (6) for $j = 2$. Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$,*

$$\left|P_n'(z)\right| \leq c_1 \frac{\left|\Phi^{2(n+1)}(z)\right|}{d(z,\Gamma)} \|P_n\|_p \begin{cases} D_n^{(1)} + D_n^{(2)}, & z \in \Omega_R(\delta), \\ E_n, & z \in \widehat{\Omega}_R(\delta), \end{cases} \tag{10}$$

is true, where $c_1 = c_1(\Gamma,p) > 0$ is a constant independent of z and n, c is defined as in Definition 1.2;

$$D_n^{(1)} := \begin{cases} n^{\frac{(\gamma_1+1)}{p}\widetilde{\lambda}_1}, & \gamma_1 \geq \frac{\gamma_2+1}{(1+\alpha_2)\widetilde{\lambda}_1} - 1, \gamma_2 \geq (1+\alpha_2)\widetilde{\lambda}_1 - 1, \\ n^{\frac{(\gamma_2+1)}{p(1+\alpha_2)}+\varepsilon}, & 0<\gamma_1 < \frac{\gamma_2+1}{(1+\alpha_2)\widetilde{\lambda}_1} - 1,\ \gamma_2 \geq (1+\alpha_2)\widetilde{\lambda}_1 - 1, \\ n^{\frac{1}{p}\widetilde{\lambda}_1}, & -1<\gamma_1<0,\ -1<\gamma_2<(1+\alpha_2)\widetilde{\lambda}_1 - 1; \end{cases}$$

$$D_n^{(2)} := \begin{cases} n^{\left(\frac{\gamma_1+1}{p}\right)\widetilde{\lambda}_1}, & \gamma_1 > p-1+\frac{\gamma_2+1-p}{(1+\alpha_2)\widetilde{\lambda}_1}, \gamma_2 \geq p-1, \\ n^{\widetilde{\lambda}_1+\left(\frac{\gamma_2+1}{p}-1\right)\frac{1}{1+\alpha_2}+\varepsilon}, & p-1<\gamma_1 \leq p-1+\frac{\gamma_2+1-p}{(1+\alpha_2)\widetilde{\lambda}_1}, \gamma_2 > p-1, \\ (\ln n)^{1-\frac{1}{p}}, & \gamma_1,\gamma_2 = p-1, \\ 1, & -1<\gamma_1,\gamma_2<p-1; \end{cases}$$

$$E_n := \begin{cases} n^{\left(\frac{\gamma_1+1}{p}-1\right)\widetilde{\lambda}_1} + n^{\left(\frac{\gamma_2+1}{p}-1\right)\frac{1}{1+\alpha_2}+\varepsilon}, & \gamma_1,\gamma_2 > p-1, \\ (\ln n)^{1-\frac{1}{p}}, & \begin{cases} \text{if } \gamma_1 = p-1,\ -1<\gamma_2 \leq p-1, \\ \text{or } -1<\gamma_1 \leq p-1,\ \gamma_2 = p-1, \end{cases} \\ 1, & -1<\gamma_1,\gamma_2<p-1. \end{cases}$$

Let us consider separate cases when the domain G has only one type of singular points on the boundary Γ: an exterior non-zero (interior zero) angle or an exterior zero angle. In these cases, from Theorem 1.3 we get the following:

Corollary 1.4. *Assume that $G \in C_\theta(\lambda_1;\lambda_2)$, for some $0 < \lambda_1, \lambda_2 \leq 2$ and $h(z)$ is defined by (6) for $j = 2$. Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$,*

$$\left|P_n'(z)\right| \leq c_2 \frac{\left|\Phi^{2(n+1)}(z)\right|}{d(z,\Gamma)} \|P_n\|_p \begin{cases} D_{n,1}^{(1)} + D_{n,2}^{(1)}, & z \in \Omega_R(\delta), \\ E_{n,1} & z \in \widehat{\Omega}_R(\delta), \end{cases} \tag{11}$$

is true, where $c_2 = c_2(\Gamma,p) > 0$ is a constant independent of z and n;

$$D_{n,1}^{(1)} := n^{\frac{(\widetilde{\gamma}^*+1)}{p}\widetilde{\lambda}}; \quad D_{n,2}^{(1)} := \begin{cases} n^{\frac{\widetilde{\gamma}+1}{p}\widetilde{\lambda}}, & \gamma_1,\gamma_2 > p-1, \\ n^{\widetilde{\lambda}}(\ln n)^{1-\frac{1}{p}}, & \begin{cases} \text{if } \gamma_1 = p-1,\ -1<\gamma_2 \leq p-1, \\ \text{or } -1<\gamma_1 \leq p-1,\ \gamma_2 = p-1, \end{cases} \\ n^{\widetilde{\lambda}}, & -1<\gamma_1,\gamma_2<p-1; \end{cases}$$

$$E_{n,1} := \begin{cases} n^{\left(\frac{\widetilde{\gamma}+1}{p}-1\right)\widetilde{\lambda}}, & \gamma_1,\gamma_2 > p-1, \\ (\ln n)^{1-\frac{1}{p}}, & \begin{cases} \text{if } \gamma_1 = p-1,\ -1<\gamma_2 \leq p-1, \\ \text{or } -1<\gamma_1 \leq p-1,\ \gamma_2 = p-1, \end{cases} \\ 1, & -1<\gamma_1,\gamma_2<p-1. \end{cases}$$

Corollary 1.5. *Assume that $G \in C_\theta(cx^{1+\alpha_1}; cx^{1+\alpha_2})$, for some $\alpha_1, \alpha_2 > 0$ and $h(z)$ is defined by (6) for $j = 2$. Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$,*

$$\left|P_n'(z)\right| \le c_3 \frac{\left|\Phi^{2(n+1)}(z)\right|}{d(z,\Gamma)} \|P_n\|_p \begin{cases} D_{n,1}^{(2)} + D_{n,2}^{(2)}, & z \in \Omega_R(\delta), \\ E_{n,2} & z \in \widehat{\Omega}_R(\delta), \end{cases} \tag{12}$$

is true, where $c_3 = c_3(\Gamma, p) > 0$ is a constant independent of z and n, c is defined as in Definition 1.2;

$$D_{n,1}^{(2)} := n^{\frac{(\widetilde{\gamma}_2^*+1)}{p(1+\alpha_*)}+\varepsilon}, \quad D_{n,2}^{(2)} := \begin{cases} n^{\frac{\widetilde{\gamma}+1}{p(1+\alpha_*)}+\varepsilon}, & \gamma_1, \gamma_2 > p-1, \\ n^{\frac{1}{1+\alpha_*}+\varepsilon} (\ln n)^{1-\frac{1}{p}}, & \text{if } \gamma_1 = p-1,\ -1 < \gamma_2 \le p-1, \\ & \text{or } -1 < \gamma_1 \le p-1,\ \gamma_2 = p-1, \\ n^{\frac{1}{1+\alpha_*}+\varepsilon}, & -1 < \gamma_1, \gamma_2 < p-1; \end{cases}$$

$$E_{n,2} := \begin{cases} n^{\left(\frac{\widetilde{\gamma}+1}{p}-1\right)\frac{1}{1+\alpha_*}+\varepsilon}, & \gamma_1, \gamma_2 > p-1, \\ (\ln n)^{1-\frac{1}{p}}, & \left\{ \begin{array}{l} \text{if } \gamma_1 = p-1,\ -1 < \gamma_2 \le p-1, \\ \text{or } -1 < \gamma_1 \le p-1,\ \gamma_2 = p-1, \end{array} \right. \\ 1, & -1 < \gamma_1, \gamma_2 < p-1. \end{cases}$$

Now we give a theorem that gives an estimate for $\left|P_n'(z)\right|$ on $\overline{G}$.

Theorem 1.6. *Assume that $G \in C_\theta(\lambda_1; cx^{1+\alpha_2})$, for some $0 < \lambda_1 \le 2$, $\alpha_2 > 0$ and $h(z)$ is defined by (6) for $j = 2$. Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$,*

$$\left\|P_n'\right\|_\infty \le c_4 \|P_n\|_p \left[F_{n,1} + F_{n,2}\right], \tag{13}$$

is true, where $c_4 = c_4(\Gamma, p) > 0$ is a constant independent of z and n, c is defined as in Definition 1.2;

$$F_{n,1} ;= n^{\left(\frac{(\gamma_1^*+1)}{p}+1\right)\widetilde{\lambda}_1}; \; F_{n,2} := \begin{cases} n^{\frac{\gamma_2^*+1}{p(1+\alpha_2)}+\frac{\alpha_2}{p(1+\alpha_2)}+\widetilde{\lambda}_1+\varepsilon}, & 1 < p < 2 + \frac{\gamma_2^*}{1+\alpha_2}, \\ n^{\widetilde{\lambda}_1+1-\frac{1}{p}} (\ln n)^{1-\frac{1}{p}}, & p = 2 + \frac{\gamma_2^*}{1+\alpha_2}, \\ n^{\widetilde{\lambda}_1+1-\frac{1}{p}}, & p > 2 + \frac{\gamma_2^*}{1+\alpha_2}. \end{cases}$$

Accordingly, as in Theorem 1.3, from here we will write out the consequences related to individual cases.

Corollary 1.7. *Assume that $G \in C_\theta(\lambda_1; \lambda_2)$, for some $0 < \lambda_1, \lambda_2 \le 2$ and $h(z)$ is defined by (6) for $j = 2$. Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$,*

$$\left\|P_n'\right\|_\infty \le c_5 n^{\left(\frac{(\widetilde{\gamma}^*+1)}{p}+1\right)\widetilde{\lambda}} \|P_n\|_p, \tag{14}$$

is true, where $c_5 = c_5(\Gamma, p) > 0$ is a constant independent of z and n.

Corollary 1.8. *Assume that $G \in C_\theta(cx^{1+\alpha_1}; cx^{1+\alpha_2})$, for some $\alpha_1, \alpha_2 > 0$ and $h(z)$ is defined by (6) for $j = 2$. Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$,*

$$\left\|P_n'\right\|_\infty \le c_6 \|P_n\|_p \begin{cases} n^{\frac{\widetilde{\gamma}^*+1}{p(1+\alpha_*)}+\frac{\alpha^*}{p(1+\alpha_*)}+\varepsilon}, & 1 < p < 2 + \frac{\widetilde{\gamma}^*}{1+\alpha_2}, \\ n^{2-\frac{1}{p}} (\ln n)^{1-\frac{1}{p}}, & p = 2 + \frac{\gamma_1^*}{1+\alpha_2} = 2 + \frac{\gamma_2^*}{1+\alpha_2}, \\ n^{2-\frac{1}{p}}, & p > 2 + \frac{\widetilde{\gamma}^*}{1+\alpha_2}, \end{cases} \tag{15}$$

is true, where $c_6 = c_6(\Gamma, p) > 0$ is a constant independent of z and n, c is defined as in Definition 1.2.

Remark 1.9. The estimates (13) and, consequently, (14), (15) are sharp.

Therefore, combining Theorems 1.3, 1.6, we obtain an estimation of the growth of $\left|P_n'(z)\right|$ on the whole complex plane:

Theorem 1.10. *Assume that $G \in C_\theta(\lambda_1; cx^{1+\alpha_2})$, for some $0 < \lambda_1 \leq 2$, $\alpha_2 > 0$ and $h(z)$ is defined as in (6). Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$,*

$$\left|P_n'(z)\right| \preceq c_7 \|P_n\|_p \begin{cases} \frac{|\Phi^{2(n+1)}(z)|}{d(z,\Gamma)} \begin{cases} D_n^{(1)} + D_n^{(2)}, & z \in \Omega_R(\delta), \\ E_n, & z \in \widehat{\Omega}_R(\delta), \end{cases} \\ F_{n,1} + F_{n,2}, \; z \in \overline{G}_R, \end{cases}$$

is true, where $c_7 = c_7(G,p) > 0$ is the constant independent of z and n; c is defined as in Definition 1.2; $D_n^{(k)}$, E_n; $F_{n,k}$, $k = 1,2$, are defined as in (10) and (13), respectively.

Also combining the Corollaries 1.4-1.7 and 1.5-1.8, we can give estimates separately for the cases $G \in C_\theta(\lambda_1; \lambda_2)$, $0 < \lambda_1, \lambda_2 \leq 2$, and $G \in C_\theta(cx^{1+\alpha_1}; cx^{1+\alpha_2})$, $\alpha_1, \alpha_2 > 0$, in the whole plane.

2. Some auxiliary results

For the nonnegative functions $a > 0$ and $b > 0$, we shall use the notations "$a \preceq b$" (order inequality), if $a \leq cb$ and "$a \asymp b$" are equivalent to $c_1 a \leq b \leq c_2 a$ for some constants c, c_1, c_2 (independent of a and b) respectively.

Definition 2.1. ([38]) The Jordan curve (or arc) Γ is called $K-$quasiconformal ($K \geq 1$), if there is a $K-$quasiconformal mapping f of the domain $D \supset \Gamma$ such that $f(\Gamma)$ is a circle (or line segment). Let $F(\Gamma)$ denote the set of all sense preserving plane homeomorphisms f of the domain $D \supset \Gamma$ such that $f(\Gamma)$ is a line segment (or circle) and let

$$K_\Gamma := \inf\{K(f) : f \in F(\Gamma)\},$$

where $K(f)$ is the maximal dilatation of a such mapping f. Then Γ is a quasiconformal curve, if $K_\Gamma < \infty$, and Γ is a $K-$quasiconformal curve, if $K_\Gamma \leq K$.

According to [29], there exists quasiconformal curves which are not rectifiable. Also, according to the "three-point" criterion [22, p.100], every piecewise smooth curve (without cusps) is is quasiconformal.

On the other hand, from [38], we have:

Corollary 2.2. *If $S \in C_\theta$, then S is $(1+\varepsilon)$-quasiconformal for arbitrary small $\varepsilon > 0$.*

Corollary 2.3. *If S is an analytic curve or an arc, then S is $1-$quasiconformal.*

Lemma 2.4. ([1]) *Let Γ be a $K-$quasiconformal curve, $z_1 \in \Gamma$, $z_2, z_3 \in \Omega \cap \{z : |z - z_1| \leq d(z_1, \Gamma_{r_0})\}$; $w_j = \Phi(z_j)$, ($z_2, z_3 \in G \cap \{z : |z - z_1| \leq d(z_1, \Gamma_{R_0})\}$; $w_j = \varphi(z_j)$), $j = 1,2,3$. Then,*

a) The statements $|z_1 - z_2| \preceq |z_1 - z_3|$ and $|w_1 - w_2| \preceq |w_1 - w_3|$ are equivalent.
So $|z_1 - z_2| \asymp |z_1 - z_3|$ and $|w_1 - w_2| \asymp |w_1 - w_3|$ also are equivalent;

b) If $|z_1 - z_2| \preceq |z_1 - z_3|$, then

$$\left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{K^{-2}} \preceq \left|\frac{z_1 - z_3}{z_1 - z_2}\right| \preceq \left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{K^2},$$

where $0 < r_0 < 1$, $R_0 := r_0^{-1}$ are constants, depending on G.

Corollary 2.5. *Under the assumptions of Lemma 2.4, if* $z_3 \in \Gamma_{R_0}$*, then*

$$|w_1 - w_2|^{K^2} \le |z_1 - z_2| \le |w_1 - w_2|^{K^{-2}}.$$

Corollary 2.6. *If* $\Gamma \in C_\theta$*, then*

$$|w_1 - w_2|^{1+\varepsilon} \le |z_1 - z_2| \le |w_1 - w_2|^{1-\varepsilon},$$

for all $\varepsilon > 0$.

Recall that for $0 < \delta_j < \delta_0 := \frac{1}{4}\min\left\{\left|z_i - z_j\right| : i, j = 1, 2, ..., m,\ i \neq j\right\}$, we put $\Omega(z_j, \delta_j) := \Omega \cap \left\{z : \left|z - z_j\right| \le \delta_j\right\}$; $\delta := \min\limits_{1\le j\le m} \delta_j$, $\Omega(\delta) := \bigcup\limits_{j=1}^{m} \Omega(z_j,\ \delta)$, $\widehat{\Omega} := \Omega \setminus \Omega(\delta)$. Additionally, let $\Delta_j := \Phi(\Omega(z_j,\ \delta))$, $\Delta(\delta) := \bigcup\limits_{j=1}^{m} \Phi(\Omega(z_j,\ \delta))$, $\widehat{\Delta}(\delta) := \Delta \backslash \Delta(\delta)$. Let $w_j := \Phi(z_j)$ and for $\varphi_j := \arg w_j$, $j = 1, 2, ..., m$, we put $\Delta_j^{'} := \left\{t = Re^{i\theta} : R > 1,\ \frac{\varphi_{j-1}+\varphi_j}{2} \le \theta < \frac{\varphi_j+\varphi_{j+1}}{2}\right\}$, where $\varphi_0 \equiv \varphi_m$, $\varphi_1 \equiv \varphi_{m+1}$; $\Omega_j := \Psi(\Delta_j^{'})$, $\Gamma^j := \Gamma \cap \overline{\Omega}_j$, $i = 1, 2, ..., m$. Clearly, $\Omega = \bigcup\limits_{j=1}^{m} \Omega_j$. $\Gamma_R^j := \Gamma_R \cap \overline{\Omega}^j$. $F^i := \Phi(\Gamma^i) = \overline{\Delta}_i^{'} \cap \{\tau : |\tau| = 1\}$, $F_R^i := \Phi(\Gamma_R^i) = \overline{\Delta}_i^{'} \cap \{\tau : |\tau| = R\}$, $i = \overline{1, m}$.

Lemma 2.7. ([9]) *Let* $G \in C_\theta(\lambda_1, ..., \lambda_m)$, $0 < \lambda_j < 2$, $j = 1, 2, ..., m$,. *Then*
i) for any $w \in \Delta_j$, $|w - w_j|^{\lambda_j+\varepsilon} \prec |\Psi(w) - \Psi(w_j)| \prec |w - w_j|^{\lambda_j-\varepsilon}$, $|w - w_j|^{\lambda_j-1+\varepsilon} \prec |\Psi'(w)| \prec |w - w_j|^{\lambda_j-1-\varepsilon}$,

ii) for any $w \in \overline{\Delta}\backslash\Delta_j$, $(|w| - 1)^{1+\varepsilon} \prec d(\Psi(w), \Gamma)| \prec (|w| - 1)^{1-\varepsilon}$, $(|w| - 1)^{\varepsilon} \prec |\Psi'(w)| \prec (|w| - 1)^{-\varepsilon}$.

Recall that, $\left\{z_j\right\}_{j=1}^{m}$ be a fixed system of the points on Γ and the weight function $h(z)$ is defined as in(6).

Lemma 2.8. ([9]) *Let* Γ *be a rectifiable Jordan curve;* $h(z)$ *as defined in (6). Then, for arbitrary* $P_n(z) \in \wp_n$*, any* $R > 1$ *and* $n \in \mathbb{N}$*, we have*

$$\|P_n\|_{\mathcal{L}_p(h,\Gamma_R)} \le R^{n+\frac{1+\gamma_*}{p}} \|P_n\|_{\mathcal{L}_p(h,\Gamma)},\ p > 0, \tag{16}$$

where $\gamma^* = \max\{\gamma_k :\ k \le\ m\}$.

3. Proofs

Proof. [Proof of Theorem 1.3] Suppose that $G \in C_\theta(\lambda_1; f_2)$, for some $0 < \lambda_1 \le 2$, $f_2(x) = cx^{1+\alpha_2}$, $\alpha_2 > 0$ and $h(z)$ is defined as in (6). Let us put:

$$T_n(z) := \frac{P_n(z)}{\Phi^{n+1}(z)},\ z \in \Omega. \tag{17}$$

Hence, we obtain:

$$P_n'(z) = \Phi^{n+1}(z)\left[T_n'(z) - P_n(z)\left(\frac{1}{\Phi^{n+1}(z)}\right)'\right],\ z \in \Omega. \tag{18}$$

By Cauchy integral representation for the unbounded domain Ω, for $T_n(z)$ and $\left(\frac{1}{\Phi^{n+1}(z)}\right)'$, respectively, we have:

$$T_n'(z) = -\frac{1}{2\pi i}\int\limits_{\Gamma} \frac{P_n(\zeta)}{\Phi^{n+1}(\zeta)}\frac{d\zeta}{(\zeta - z)^2},\ \left(\frac{1}{\Phi^{n+1}(z)}\right)' = -\frac{1}{2\pi i}\int\limits_{\Gamma} \frac{1}{\Phi^{n+1}(\zeta)}\frac{d\zeta}{(\zeta - z)^2},\ z \in \Omega_R.$$

Putting them in (18), we find:

$$\left|P_n'(z)\right| \leq \frac{\left|\Phi^{n+1}(z)\right|}{2\pi}\left[\int\limits_{\Gamma}\left|\frac{P_n(\zeta)}{\Phi^{n+1}(\zeta)}\right|\frac{|d\zeta|}{|\zeta-z|^2}+|P_n(z)|\int\limits_{\Gamma}\frac{|d\zeta|}{\left|\Phi^{n+1}(\zeta)\right||\zeta-z|^2}\right]$$

$$\begin{aligned} &\leq \left|\Phi^{n+1}(z)\right|\left[\int\limits_{\Gamma}\frac{|P_n(\zeta)||d\zeta|}{|\zeta-z|^2}+|P_n(z)|\int\limits_{\Gamma}\frac{|d\zeta|}{|\zeta-z|^2}\right] \qquad (19)\\ &\leq \left|\Phi^{n+1}(z)\right|\left[\frac{1}{d(z,\Gamma)}\int\limits_{\Gamma}\frac{|P_n(\zeta)||d\zeta|}{|\zeta-z|}+|P_n(z)|\int\limits_{\Gamma}\frac{|d\zeta|}{|\zeta-z|^2}\right], \end{aligned}$$

since $|\Phi(\zeta)|=1$, for $\zeta\in\Gamma$.

Denoted the last integrals by

$$A_n(z):=\int\limits_{\Gamma}\frac{|P_n(\zeta)||d\zeta|}{|\zeta-z|};\quad B_n(z):=\int\limits_{\Gamma}\frac{|d\zeta|}{|\zeta-z|^2}, \qquad (20)$$

and start evaluating them separately.

Multiplying the numerator and denominator of the first integral by $h^{\frac{1}{p}}(\zeta)$ and applying Hölder inequality, we obtain:

$$A_n(z)\leq \|P_n\|_p\left(\int\limits_{\Gamma}\frac{|d\zeta|}{h^{\frac{q}{p}}(\zeta)|\zeta-z|^q}\right)^{\frac{1}{q}},\ \frac{1}{p}+\frac{1}{q}=1. \qquad (21)$$

Denote by $J_n(z)$ the last integral, we get:

$$\begin{aligned} [J_n(z)]^q &:= \int\limits_{\Gamma}\frac{|d\zeta|}{h^{q-1}(\zeta)|\zeta-z|^q} \qquad (22)\\ &\leq \int\limits_{\Gamma^1}\frac{|d\zeta|}{|\zeta-z_1|^{(q-1)\gamma_1}|\zeta-z|^q}+\int\limits_{\Gamma^2}\frac{|d\zeta|}{|\zeta-z_2|^{(q-1)\gamma_2}|\zeta-z|^q}\\ &=: J_{n,1}^1(z)+J_{n,2}^2(z). \end{aligned}$$

To make the following calculations easier, we put: $z_1=-1$, $z_2=1$; $(-1,1)\subset G$ and let local coordinate axis in Definition 1.2 be parallel to natural axis OX and OY in the coordinate system XOY; $\Gamma=\Gamma^+\cup\Gamma^-$, where $\Gamma^+:=\{z\in\Gamma:\mathrm{Im}z\geq 0\}$, $\Gamma^-:=\{z\in\Gamma:\mathrm{Im}z<0\}$; $w^\pm:=\left\{w=e^{i\theta}:\theta=\frac{\varphi_1\pm\varphi_2}{2}\right\}$, $z^\pm\in\Psi(w^\pm)$ and let $\Gamma_i^\pm(z_i,z^\pm)$– denoted the arcs, connected the points z_i which $z^\pm$, respectively; $\left|\Gamma_i^\pm\right|:=mes\,\Gamma_i^\pm(z_i,z^\pm)$, $i=1,2$. Let $z_0\in\Gamma^+$ be taken as an arbitrary point (or $z_0\in\Gamma^-$ subject to the chosen direction). Then, from (21) and (22), we have:

$$A_n(z)\leq \|P_n\|_p\left\{\left[J_{n,1}^1(z)\right]^{\frac{1}{q}}+\left[J_{n,2}^2(z)\right]^{\frac{1}{q}}\right\}. \qquad (23)$$

Let us introduce some notations: $R=1+\frac{1}{n}$; $d_{i,R}:=d(z_i,\Gamma_R)$; $\mathcal{F}_1^{1,\pm}:=\left\{\zeta\in\Gamma^1:\ |\zeta-z_1|<c_1d_{1,R}\right\}$, $\mathcal{F}_2^{1,\pm}:=\left\{\zeta\in\Gamma^1:\ c_1d_{1,R}\leq|\zeta-z_1|\leq\left|\Gamma_1^\pm\right|\right\}$, $\mathcal{F}_1^{2,\pm}:=\left\{\zeta\in\Gamma^2:\ |\zeta-z_2|<c_2d_{2,R}\right\}$, $\mathcal{F}_2^{2,\pm}:=\left\{\zeta\in\Gamma^2:\ c_2d_{2,R}\leq|\zeta-z_2|\leq\left|\Gamma_2^\pm\right|\right\}$;

$$Y_{n,k}^{i,\pm}(z):=\int\limits_{\mathcal{F}_k^{i,\pm}}\frac{|d\zeta|}{|\zeta-z_i|^{\gamma_i(q-1)}|\zeta-z|^q}\ ;\ i,k=1,2.$$

Then, under the these notations, (23) can be written as:

$$A_n(z) \quad \leq \quad \|P_n\|_p \sum_{i=1}^{2} \left[Y_{n,1}^{i,\pm}(z) + Y_{n,2}^{i,\pm}(z) \right]^{\frac{1}{q}}, \ i = 1, 2. \tag{24}$$

Let us start to estimate the integrals $Y_{n,k}^{i,\pm}$ for each $i, k = 1, 2$.

1. Let $z \in \Omega_R(\delta)$. Denote by

$$\begin{aligned} \left(\mathcal{F}_k^{i,\pm}\right)_1 \quad &:= \quad \left\{\zeta \in \mathcal{F}_k^{i,\pm} : \ |\zeta - z_i| < \ |\zeta - z|\right\}, \ \left(\mathcal{F}_k^{i,\pm}\right)_2 := \ \mathcal{F}_k^{i,\pm} \ \setminus \left(\mathcal{F}_k^{i,\pm}\right)_1; \\ \left[Y_{n,k}^{i,\pm}(z)\right]_1 \quad &:= \quad \int\limits_{\left(\mathcal{F}_k^{i,\pm}\right)_1} \frac{|d\zeta|}{|\zeta - z_i|^{\gamma_i(q-1)+q}}; \ \left[Y_{n,k}^{i,\pm}(z)\right]_2 := \int\limits_{\left(\mathcal{F}_k^{i,\pm}\right)_2} \frac{|d\zeta|}{|\zeta - z|^{\gamma_i(q-1)+q}}. \end{aligned}$$

1.1. Let $\gamma_1, \gamma_2 \geq 0$. We get:

$$\begin{aligned} \left[Y_{n,1}^{1,\pm}(z)\right]_1 \quad &\leq \quad \int\limits_0^{c_1 d_{1,R}} \frac{ds}{s^{\gamma_1(q-1)+q}} \leq d_{1,R}^{-(\gamma_1+1)(q-1)}; \\ \left[Y_{n,1}^{1,\pm}(z)\right]_2 \quad &\leq \quad d_{1,R}^{-[(\gamma_1+1)+q]} \cdot mes\left(\mathcal{F}_1^{1,\pm}\right) \leq d_{1,R}^{-(\gamma_1+1)(q-1)}; \\ Y_{n,1}^{1,\pm}(z) &= \left[Y_{n,1}^{1,\pm}(z)\right]_1 + \left[Y_{n,1}^{1,\pm}(z)\right]_2 \leq d_{1,R}^{-(\gamma_1+1)(q-1)}. \end{aligned} \tag{25}$$

$$\begin{aligned} \left[Y_{n,2}^{1,\pm}(z)\right]_1 \quad &\leq \quad \int\limits_{c_1 d_{1,R}}^{|\Gamma_1^{\pm}|} \frac{ds}{s^{\gamma_1(q-1)+q}} \leq d_{1,R}^{-(\gamma_1+1)(q-1)}; \ \left[Y_{n,2}^{1,\pm}(z)\right]_2 \leq d_{1,R}^{-(\gamma_1+1)(q-1)}; \\ Y_{n,2}^{1,\pm}(z) \quad &= \quad \left[Y_{n,2}^{1,\pm}(z)\right]_1 + \left[Y_{n,2}^{1,\pm}(z)\right]_2 \leq d_{1,R}^{-(\gamma_1+1)(q-1)}. \end{aligned}$$

Analogously, for the $J_{n,2}^2(z)$ in neighborhood of the point z_2, we have:

$$\begin{aligned} \left[Y_{n,1}^{2,\pm}(z)\right]_1 \quad &\leq \quad \int\limits_0^{c_2 d_{2,R}} \frac{ds}{s^{\gamma_2(q-1)+q}} \leq d_{2,R}^{-(\gamma_2+1)(q-1)}; \\ \left[Y_{n,1}^{2,\pm}(z)\right]_2 \quad &\leq \quad \int\limits_{\left(\mathcal{F}_1^{2,\pm}\right)_2} \frac{|d\zeta|}{|\zeta - z|^{\gamma_2(q-1)+q}} \leq d_{2,R}^{-[(\gamma_2+1)+q]} \cdot mes\left(\mathcal{F}_1^{2,\pm}\right) \leq d_{2,R}^{-(\gamma_2+1)(q-1)}; \\ Y_{n,1}^{2,\pm}(z) &= \left[Y_{n,1}^{2,\pm}(z)\right]_1 + \left[Y_{n,1}^{2,\pm}(z)\right]_2 \leq d_{2,R}^{-(\gamma_2+1)(q-1)}. \end{aligned} \tag{26}$$

$$\begin{aligned} \left[Y_{n,2}^{2,\pm}(z)\right]_1 \quad &\leq \quad \int\limits_{\left(\mathcal{F}_2^{2,\pm}\right)_1} \frac{|d\zeta|}{|\zeta - z_2|^{\gamma_2(q-1)+q}} \leq \int\limits_{c_2 d_{2,R}}^{|\Gamma_2^{\pm}|} \frac{ds}{s^{\gamma_2(q-1)+q}} \leq d_{2,R}^{-(\gamma_2+1)(q-1)}; \\ \left[Y_{n,2}^{2,\pm}(z)\right]_2 \quad &\leq \quad \int\limits_{\left(\mathcal{F}_2^{2,\pm}\right)_2} \frac{|d\zeta|}{|\zeta - z|^{\gamma_2(q-1)+q}} \leq \int\limits_{c_2 d_{2,R}}^{|\Gamma_2^{\pm}|} \frac{ds}{s^{\gamma_2(q-1)+q}} \leq d_{2,R}^{-(\gamma_2+1)(q-1)}; \\ Y_{n,2}^{2,\pm}(z) \quad &= \quad \left[Y_{n,2}^{2,\pm}(z)\right]_1 + \left[Y_{n,2}^{2,\pm}(z)\right]_2 \leq d_{2,R}^{-(\gamma_1+1)(q-1)}. \end{aligned}$$

Let $\gamma_1, \gamma_2 < 0$. Then, analogously to the (25) and (26), we obtain:

$$\begin{aligned} Y_{n,1}^{1,\pm}(z) &= \int\limits_{\mathcal{F}_1^{1,\pm}} \frac{|\zeta - z_1|^{-\gamma_1(q-1)}\,|d\zeta|}{|\zeta - z|^q} \le d_{1,R}^{-\gamma_1(q-1)-q} mes\mathcal{F}_1^{1,\pm} \le d_{1,R}^{(-\gamma_1-1)(q-1)}; \qquad (27) \\ Y_{n,2}^{1,\pm}(z) &\le \int\limits_{\mathcal{F}_2^{1,\pm}} \frac{|\zeta - z_1|^{-\gamma_1(q-1)}\,|d\zeta|}{|\zeta - z|^q} \le \int\limits_{c_1 d_{1,R}}^{|\Gamma_1^\pm|} \frac{ds}{s^q} \le d_{1,R}^{-(q-1)}; \end{aligned}$$

and

$$\begin{aligned} Y_{n,1}^{2,\pm}(z) &\le \int\limits_{\mathcal{F}_1^{2,\pm}} \frac{|\zeta - z_2|^{-\gamma_2(q-1)}}{|\zeta - z|^q}\,|d\zeta| \le d_{2,R}^{(-\gamma_2)(q-1)} mes\mathcal{F}_1^{2,\pm} \le d_{2,R}^{(-\gamma_2-1)(q-1)}; \\ Y_{n,2}^{2,\pm}(z) &\le \int\limits_{\mathcal{F}_2^{2,\pm}} \frac{|\zeta - z_2|^{-\gamma_2(q-1)}}{|\zeta - z|^q}\,|d\zeta| \le \int\limits_{c_2 d_{2,R}}^{|\Gamma_2^\pm|} \frac{ds}{s^q} \le d_{2,R}^{-(q-1)}. \end{aligned}$$

Combining (24) - (27), in this case, we get:

$$A_n(z) \le \|P_n\|_p \left[d_{1,R}^{-\frac{(\gamma_1^*+1)}{p}} + d_{2,R}^{-\frac{(\gamma_2^*+1)}{p}} \right], \ \gamma_i^* := \max\{0; \gamma_i, \ i = 1, 2\}. \qquad (28)$$

Let us estimate the $B_n(z)$. By notations from (23), $\Gamma^\pm = \mathcal{F}_1^{1,\pm} \cup \mathcal{F}_2^{1,\pm} \cup \mathcal{F}_1^{2,\pm} \cup \mathcal{F}_2^{2,\pm}$ and, so:

$$B_n(z) = \sum_{i,k=1}^{2} \int\limits_{\mathcal{F}_k^{i,+} \cup \mathcal{F}_k^{i,-}} \frac{|d\zeta|}{|\zeta - z|^2} =: M(\mathcal{F}_k^{i,+}) + M(\mathcal{F}_k^{i,-}). \qquad (29)$$

The integrals $M(\mathcal{F}_k^{i,+})$ and $M(\mathcal{F}_k^{i,-})$, $i, k = 1, 2$, are estimated to be similar, then we will estimate only $M(\mathcal{F}_k^{i,+})$.

$$\begin{aligned} M(\mathcal{F}_1^{1,+}) &= \int\limits_{|z_1 - z|}^{c_1 d_{1,R}} \frac{ds}{s^2} \le \frac{1}{d_{1,R}}; \ M(\mathcal{F}_2^{1,+}) = \int\limits_{d_{1,R}}^{|\Gamma_1^\pm|} \frac{ds}{s^2} \le \frac{1}{d_{1,R}}; \\ M(\mathcal{F}_1^{2,+}) &= \int\limits_{0}^{|z_2 - z_2^+|} \frac{ds}{s^2} \le \int\limits_{0}^{c_2 d_{2,R}} \frac{ds}{s^2} \le \frac{1}{d_{2,R}}; \ M(\mathcal{F}_2^{2,+}) = \int\limits_{|z_2 - z_2^+|}^{|\Gamma_2^\pm|} \frac{ds}{s^2} \le \int\limits_{c_2 d_{2,R}}^{|\Gamma_2^\pm|} \frac{ds}{s^2} \le \frac{1}{d_{2,R}}. \end{aligned}$$

Then, from (29), we have:

$$B_n(z) \le d_{1,R}^{-1} + d_{2,R}^{-1}. \qquad (30)$$

Comparing (19), (20), (28) and (30), we get:

$$\left|P_n'(z)\right| \le \left|\Phi^{n+1}(z)\right| \left[\frac{1}{d(z,\Gamma)} \|P_n\|_p \left(d_{1,R}^{-\frac{(\gamma_1^*+1)}{p}} + d_{2,R}^{-\frac{(\gamma_2^*+1)}{p}} \right) + |P_n(z)| \left(d_{1,R}^{-1} + d_{2,R}^{-1} \right) \right]. \qquad (31)$$

Using [9, Cor.1.3.], for the $|P_n(z)|$ we have the following:

$$|P_n(z)| \le c \frac{B_{n,1}}{d(z,\Gamma)} \|P_n\|_p |\Phi(z)|^{n+1}, \qquad (32)$$

where $c = c(G, p, \gamma_i) > 0$ is a constant independent of n and z, and

$$B_{n,1} := \begin{cases} n^{\frac{(\gamma_1+1-p)}{p}\widetilde{\lambda}_1} + n^{\frac{\gamma_2+1-p}{p(1+\alpha_2)}+\varepsilon}, & \gamma_1, \gamma_2 > p-1, \\ (\ln n)^{1-\frac{1}{p}}, & \gamma_1 = p-1, -1 < \gamma_2 \leq p-1 \\ & \text{or} -1 < \gamma_1 \leq p-1, \gamma_2 = p-1, \\ 1, & -1 < \gamma_1, \gamma_2 < p-1. \end{cases} \tag{33}$$

Then, from (32) and (31), we obtain:

$$\left|P_n'(z)\right| \leq \frac{\left|\Phi^{2(n+1)}(z)\right|}{d(z,\Gamma)} \|P_n\|_p \left[\left(d_{1,R}^{-\frac{(\gamma_1^*+1)}{p}} + d_{2,R}^{-\frac{(\gamma_2^*+1)}{p}}\right) + B_{n,1}\left(d_{1,R}^{-1} + d_{2,R}^{-1}\right)\right]. \tag{34}$$

According to Lemma 2.7, for the $d_{1,R}$, we get:

$$d_{1,R} \geq n^{-\widetilde{\lambda}_1}, \forall \varepsilon > 0. \tag{35}$$

For the estimate $d_{2,R}$, let's set: $z_R \in \Gamma_R$ such that $d_{2,R} = |z_2 - z_R|$, $\zeta^{\pm} \in \Gamma^{\pm}$ such that $d(z_R, \Gamma^2 \cap \Gamma^{\pm}) := d(z_R, \Gamma^+)$; $z_2^{\pm} := \zeta \in \Gamma^2 : |\zeta - z_2| = c_2 d_{2,R}$. Under this notations, from Lemma 2.4, we obtain:

$$d_R^{\pm} := d(z_R, \Gamma^2 \cap \Gamma^{\pm}) \asymp \left|z_R - z_2^{\pm}\right| \asymp d_{2,R}^{1+\alpha_2}. \tag{36}$$

In this case, $d_{2,R} = \left(d_R^{\pm}\right)^{\frac{1}{1+\alpha_2}}$. On the other hand, according to Lemma 2.7 and [23, Corollary 2], we get: $d_R^{\pm} \geq n^{-1-\varepsilon}$. Therefore,

$$d_{2,R} \geq n^{-\frac{1-\varepsilon}{1+\alpha_2}}. \tag{37}$$

From (34)-(37), we get:

$$\left|P_n'(z)\right| \leq \frac{\left|\Phi^{2(n+1)}(z)\right|}{d(z,\Gamma)} \|P_n\|_p \left[\left(n^{\frac{(\gamma_1^*+1)}{p}\widetilde{\lambda}_1} + n^{\frac{(\gamma_2^*+1)}{p(1+\alpha_2)}+\varepsilon}\right) + B_{n,1}\left(n^{\widetilde{\lambda}_1} + n^{\frac{1}{1+\alpha_2}+\varepsilon}\right)\right] \tag{38}$$

$$\leq \frac{\left|\Phi^{2(n+1)}(z)\right|}{d(z,\Gamma)} \|P_n\|_p \left[\begin{cases} n^{\frac{(\gamma_1+1)}{p}\widetilde{\lambda}_1}, & \gamma_1 \geq \frac{\gamma_2+1}{(1+\alpha_2)\widetilde{\lambda}_1} - 1,\ \gamma_2 \geq (1+\alpha_2)\widetilde{\lambda}_1 - 1, \\ n^{\frac{(\gamma_2+1)}{p(1+\alpha_2)}+\varepsilon}, & 0 < \gamma_1 < \frac{\gamma_2+1}{(1+\alpha_2)\widetilde{\lambda}_1} - 1,\ \gamma_2 \geq (1+\alpha_2)\widetilde{\lambda}_1 - 1, \\ n^{\frac{1}{p}\widetilde{\lambda}_1}, & -1 < \gamma_1 < 0,\ -1 < \gamma_2 < (1+\alpha_2)\widetilde{\lambda}_1 - 1, \end{cases} \\ + n^{\widetilde{\lambda}_1+\varepsilon} \cdot B_{n,1} \right]$$

$$\leq \frac{\left|\Phi^{n+1}(z)\right|}{d(z,\Gamma)} \|P_n\|_p \left[\begin{cases} n^{\frac{(\gamma_1+1)}{p}\widetilde{\lambda}_1}, & \gamma_1 \geq \frac{\gamma_2+1}{(1+\alpha_2)\widetilde{\lambda}_1} - 1,\ \gamma_2 \geq (1+\alpha_2)\widetilde{\lambda}_1 - 1, \\ n^{\frac{(\gamma_2+1)}{p(1+\alpha_2)}+\varepsilon}, & 0 < \gamma_1 < \frac{\gamma_2+1}{(1+\alpha_2)\widetilde{\lambda}_1} - 1,\ \gamma_2 \geq (1+\alpha_2)\widetilde{\lambda}_1 - 1, \\ n^{\frac{1}{p}\widetilde{\lambda}_1}, & -1 < \gamma_1 < 0,\ -1 < \gamma_2 < (1+\alpha_2)\widetilde{\lambda}_1 - 1, \end{cases} \\ + \begin{cases} n^{\frac{\gamma_1+1}{p}\widetilde{\lambda}_1} + n^{\frac{\gamma_2+1-p}{p(1+\alpha_2)}+\widetilde{\lambda}_1}, & \gamma_1, \gamma_2 > p-1, \\ (\ln n)^{1-\frac{1}{p}}, & \gamma_1 = p-1, -1 < \gamma_2 \leq p-1 \\ & \text{or} -1 < \gamma_1 \leq p-1, \gamma_2 = p-1, \\ 1, & -1 < \gamma_1, \gamma_2 < p-1, \end{cases} \right]$$

$$\leq \frac{\left|\Phi^{2(n+1)}(z)\right|}{d(z,\Gamma)}\|P_n\|_p \left[\begin{cases} n^{\frac{\gamma_1+1}{p}\widetilde{\lambda}_1}, & \gamma_1 \geq \frac{\gamma_2+1}{(1+\alpha_2)\widetilde{\lambda}_1}-1,\ \gamma_2 \geq (1+\alpha_2)\widetilde{\lambda}_1-1, \\ n^{\frac{(\gamma_2+1)}{p(1+\alpha_2)}+\varepsilon}, & 0<\gamma_1<\frac{\gamma_2+1}{(1+\alpha_2)\widetilde{\lambda}_1}-1,\ \gamma_2 \geq (1+\alpha_2)\widetilde{\lambda}_1-1, \\ n^{\frac{1}{p}\widetilde{\lambda}_1}, & -1<\gamma_1<0,\ -1<\gamma_2<(1+\alpha_2)\widetilde{\lambda}_1-1, \end{cases} + \begin{cases} n^{\left(\frac{\gamma_1+1}{p}\right)\widetilde{\lambda}_1}, & \gamma_1>p-1+\frac{\gamma_2+1-p}{(1+\alpha_2)\widetilde{\lambda}_1}, \gamma_2>p-1, \\ n^{\widetilde{\lambda}_1+\left(\frac{\gamma_2+1}{p}-1\right)\frac{1}{1+\alpha_2}+\varepsilon} & p-1<\gamma_1\leq p-1+\frac{\gamma_2+1-p}{(1+\alpha_2)\widetilde{\lambda}_1}, \gamma_2>p-1, \\ n^{\left(\frac{\gamma_1+1}{p}\right)\widetilde{\lambda}_1}, & \gamma_1>p-1+\frac{\gamma_2+1-p}{(1+\alpha_2)\widetilde{\lambda}_1}, \gamma_2=p-1, \\ (\ln n)^{1-\frac{1}{p}} & -1<\gamma_1\leq p-1+\frac{\gamma_2+1-p}{(1+\alpha_2)\widetilde{\lambda}_1}, \gamma_2=p-1, \\ (\ln n)^{1-\frac{1}{p}} & \gamma_1=p-1, -1<\gamma_2<p-1, \\ 1, & -1<\gamma_1<p-1,\ -1<\gamma_2<p-1, \end{cases} \right]$$

$$\leq \frac{\left|\Phi^{2(n+1)}(z)\right|}{d(z,\Gamma)}\|P_n\|_p \left[\begin{cases} n^{\frac{(\gamma_1+1)}{p}\widetilde{\lambda}_1}, & \gamma_1 \geq \frac{\gamma_2+1}{(1+\alpha_2)\widetilde{\lambda}_1}-1,\ \gamma_2 \geq (1+\alpha_2)\widetilde{\lambda}_1-1, \\ n^{\frac{(\gamma_2+1)}{p(1+\alpha_2)}+\varepsilon}, & 0<\gamma_1<\frac{\gamma_2+1}{(1+\alpha_2)\widetilde{\lambda}_1}-1, \gamma_2 \geq (1+\alpha_2)\widetilde{\lambda}_1-1, \\ n^{\frac{1}{p}\widetilde{\lambda}_1}, & -1<\gamma_1<0,\ -1<\gamma_2<(1+\alpha_2)\widetilde{\lambda}_1-1, \end{cases} + \begin{cases} n^{\left(\frac{\gamma_1+1}{p}\right)\widetilde{\lambda}_1}, & \gamma_1>p-1+\frac{\gamma_2+1-p}{(1+\alpha_2)\widetilde{\lambda}_1}, \gamma_2\geq p-1, \\ n^{\widetilde{\lambda}_1+\left(\frac{\gamma_2+1}{p}-1\right)\frac{1}{1+\alpha_2}+\varepsilon}, & p-1<\gamma_1\leq p-1+\frac{\gamma_2+1-p}{(1+\alpha_2)\widetilde{\lambda}_1}, \gamma_2>p-1, \\ (\ln n)^{1-\frac{1}{p}}, & -1<\gamma_1\leq p-1, -1<\gamma_2\leq p-1, \\ 1, & -1<\gamma_1<p-1,\ -1<\gamma_2<p-1. \end{cases} \right]$$

Therefore, we get:

$$\left|P_n'(z)\right| \leq \frac{\left|\Phi^{2(n+1)}(z)\right|}{d(z,\Gamma)}\|P_n\|_p \left[D_n^{(1)}+D_n^{(2)}\right], \tag{39}$$

where

$$D_n^{(1)} := \begin{cases} n^{\frac{(\gamma_1+1)}{p}\widetilde{\lambda}_1}, & \gamma_1 \geq \frac{\gamma_2+1}{(1+\alpha_2)\widetilde{\lambda}_1}-1,\ \gamma_2 \geq (1+\alpha_2)\widetilde{\lambda}_1-1, \\ n^{\frac{(\gamma_2+1)}{p(1+\alpha_2)}+\varepsilon}, & 0<\gamma_1<\frac{\gamma_2+1}{(1+\alpha_2)\widetilde{\lambda}_1}-1,\ \gamma_2 \geq (1+\alpha_2)\widetilde{\lambda}_1-1, \\ n^{\frac{1}{p}\widetilde{\lambda}_1}, & -1<\gamma_1<0,\ -1<\gamma_2<(1+\alpha_2)\widetilde{\lambda}_1-1; \end{cases}$$

$$D_n^{(2)} := \begin{cases} n^{\left(\frac{\gamma_1+1}{p}\right)\widetilde{\lambda}_1}, & \gamma_1>p-1+\frac{\gamma_2+1-p}{(1+\alpha_2)\widetilde{\lambda}_1}, \gamma_2\geq p-1, \\ n^{\widetilde{\lambda}_1+\left(\frac{\gamma_2+1}{p}-1\right)\frac{1}{1+\alpha_2}+\varepsilon}, & p-1<\gamma_1\leq p-1+\frac{\gamma_2+1-p}{(1+\alpha_2)\widetilde{\lambda}_1}, \\ & \gamma_2>p-1, \\ (\ln n)^{1-\frac{1}{p}}, & \gamma_1,\gamma_2=p-1, \\ 1, & -1<\gamma_1,\gamma_2<p-1, \end{cases}$$

and we complete the proof for the points $z \in \Omega(\delta)$.

2. Let us now $z \in \widehat{\Omega}_R(\delta)$.

2.1. Let $\gamma_1, \gamma_2 \geq 0$. We get:

$$Y_{n,1}^{1,\pm}(z) \leq \int\limits_0^{c_1 d_{1,R}} \frac{ds}{s^{\gamma_1(q-1)}} \leq \begin{cases} d_{1,R}^{1-\gamma_1(q-1)}, & \gamma_1(q-1)>1, \\ \ln\frac{1}{d_{1,R}}, & \gamma_1(q-1)=1, \\ 1, & \gamma_1(q-1)<1; \end{cases} \quad Y_{n,2}^{1,\pm}(z) \leq \int\limits_{c_1 d_{1,R}}^{|\Gamma_1^{\pm}|} \frac{ds}{s^{\gamma_1(q-1)}} \leq \begin{cases} d_{1,R}^{1-\gamma_1(q-1)}, & \gamma_1(q-1)>1, \\ \ln\frac{1}{d_{1,R}}, & \gamma_1(q-1)=1, \\ 1, & \gamma_1(q-1)<1; \end{cases}$$

$$Y_{n,1}^{1,\pm}(z)+Y_{n,2}^{1,\pm}(z)\leq\begin{cases} d_{1,R}^{1-\gamma_1(q-1)}, & \gamma_1(q-1)>1,\\ \ln\frac{1}{d_{1,R}}, & \gamma_1(q-1)=1,\\ 1, & \gamma_1(q-1)<1; \end{cases} \tag{40}$$

$$Y_{n,1}^{2,\pm}(z) \leq \int\limits_0^{c_2 d_{2,R}} \frac{ds}{s^{\gamma_2(q-1)}} \leq \begin{cases} d_{2,R}^{1-\gamma_2(q-1)}, & \gamma_2(q-1)>1,\\ \ln\frac{1}{d_{2,R}}, & \gamma_2(q-1)=1,\\ 1, & \gamma_2(q-1)<1; \end{cases}$$

$$Y_{n,2}^{2,\pm}(z) \leq \int\limits_{c_2 d_{2,R}}^{|\Gamma_2^{\pm}|} \frac{ds}{s^{\gamma_2(q-1)}} \leq \begin{cases} d_{2,R}^{1-\gamma_2(q-1)}, & \gamma_2(q-1)>1,\\ \ln\frac{1}{d_{2,R}}, & \gamma_2(q-1)=1,\\ 1, & \gamma_2(q-1)<1; \end{cases}$$

$$Y_{n,1}^{2}(z)+Y_{n,2}^{2,\pm}(z)\leq\begin{cases} d_{2,R}^{1-\gamma_2(q-1)}, & \gamma_2(q-1)>1,\\ \ln\frac{1}{d_{2,R}}, & \gamma_2(q-1)=1,\\ 1, & \gamma_2(q-1)<1. \end{cases} \tag{41}$$

Let $\gamma_1,\gamma_2<0$. Then, analogously to the estimates (27), we get:

$$Y_{n,1}^{1,\pm}(z)\leq d_{1,R}^{(-\gamma_1)(q-1)} mes\mathcal{F}_1^1 \leq 1,\ Y_{n,2}^{1,\pm}(z)\leq \left|\Gamma_1^{\pm}\right|^{(-\gamma_1)(q-1)+1}\leq 1;$$

$$Y_{n,1}^{1}(z)+Y_{n,2}^{1}(z)\leq 1; \tag{42}$$

$$Y_{n,1}^{2,\pm}(z)\leq d_{2,R}^{(-\gamma_2)(q-1)} mes\mathcal{F}_1^{2,\pm} \leq 1,\ Y_{n,2}^{2,\pm}(z)\leq \left|\Gamma_2^{\pm}\right|^{(-\gamma_2)(q-1)+1}\leq 1,$$

$$Y_{n,1}^{2,\pm}(z)+Y_{n,2}^{2,\pm}(z)\leq 1. \tag{43}$$

Combining estimates (24) - (43), in this case, we find:

$$A_n(z)\leq \|P_n\|_p \begin{cases} d_{1,R}^{1-\frac{\gamma_1+1}{p}}+d_{2,R}^{1-\frac{\gamma_2+1}{p}}, & \gamma_1,\gamma_2>p-1,\\ \left(\ln\frac{1}{d_{1,R}}\right)^{1-\frac{1}{p}}+\left(\ln\frac{1}{d_{2,R}}\right)^{1-\frac{1}{p}}, & \gamma_1,\gamma_2=p-1,\\ 1, & \gamma_1,\gamma_2<p-1. \end{cases} \tag{44}$$

$$B_n(z)=\int\limits_\Gamma \frac{|d\zeta|}{|\zeta-z|^2}\leq\int\limits_\Gamma |d\zeta|\leq 1. \tag{45}$$

Comparing (19), (20), (29) and (30), we have:

$$\left|P_n'(z)\right|\leq\left|\Phi^{n+1}(z)\right|\left[\frac{\|P_n\|_p}{d(z,\Gamma)}B_{n,1}+|P_n(z)|\right], \tag{46}$$

where $B_{n,1}$ defined as in (33)

According to (33), from (46), we have:

$$\left|P_n'(z)\right|\leq\frac{\left|\Phi^{2(n+1)}(z)\right|}{d(z,\Gamma)}\|P_n\|_p B_{n,1}. \tag{47}$$

Therefore, we complete the proof for the points $z\in\widehat{\Omega}_R(\delta)$, and so, the proof of the Theorem 1.3 is complete. □

Proof. [Proof of Corollaries 1.4 and 1.5]

1) Under the conditions of Corollary 1.4, for the points $z \in \Omega_R(\delta)$, from (34), (33) and (35), we get:

$$\left|P_n'(z)\right| \leq \frac{\left|\Phi^{2(n+1)}(z)\right|}{d(z,\Gamma)} \|P_n\|_p \left[\left(n^{\frac{(\gamma_1^*+1)}{p}\widetilde{\lambda}_1} + n^{\frac{(\gamma_2^*+1)}{p}\widetilde{\lambda}_2}\right) + B_{n,1}\left(n^{\widetilde{\lambda}_1} + n^{\widetilde{\lambda}_2}\right)\right]$$

$$\leq \frac{\left|\Phi^{2(n+1)}(z)\right|}{d(z,\Gamma)} \|P_n\|_p \left[n^{\frac{(\widetilde{\gamma}^*+1)}{p}\widetilde{\lambda}+\varepsilon} + \begin{cases} n^{\frac{\widetilde{\gamma}+1}{p}\widetilde{\lambda}}, & \gamma_1,\gamma_2 > p-1, \\ n^{\widetilde{\lambda}}(\ln n)^{1-\frac{1}{p}}, & \text{if } \gamma_1 = p-1,\ -1 < \gamma_2 \leq p-1, \\ & \text{or } -1 < \gamma_1 \leq p-1,\ \gamma_2 = p-1, \\ n^{\widetilde{\lambda}}, & -1 < \gamma_1,\gamma_2 < p-1, \end{cases} \right]$$

where $\widetilde{\lambda} := \max\left\{\widetilde{\lambda}_1; \widetilde{\lambda}_2\right\}$, $\widetilde{\gamma}^* := \max\left\{\gamma_1^*; \gamma_2^*\right\}$; $\widetilde{\gamma} := \max\{\gamma_1; \gamma_2\}$, and, for the points $z \in \widehat{\Omega}_R(\delta)$, from (47) and (33), we obtain:

$$\left|P_n'(z)\right| \leq \frac{\left|\Phi^{2(n+1)}(z)\right|}{d(z,\Gamma)} \|P_n\|_p \begin{cases} n^{\left(\frac{\widetilde{\gamma}+1}{p}-1\right)\widetilde{\lambda}}, & \gamma_1,\gamma_2 > p-1, \\ (\ln n)^{1-\frac{1}{p}}, & \text{if } \gamma_1 = p-1,\ -1 < \gamma_2 \leq p-1, \\ & \text{or } -1 < \gamma_1 \leq p-1,\ \gamma_2 = p-1, \\ 1, & -1 < \gamma_1,\gamma_2 < p-1. \end{cases}$$

2) Under the conditions of Corollary 1.5, from (34), (33) and (37), we have:

$$\left|P_n'(z)\right| \leq \frac{\left|\Phi^{2(n+1)}(z)\right|}{d(z,\Gamma)} \|P_n\|_p \left[\left(n^{\frac{(\gamma_1^*+1)}{p(1+\alpha_1)}} + n^{\frac{(\gamma_2^*+1)}{p(1+\alpha_2)}+\varepsilon}\right) + B_{n,1}\left(n^{\frac{1}{1+\alpha_1}} + n^{\frac{1}{1+\alpha_2}}\right)\right]$$

$$\leq \frac{\left|\Phi^{2(n+1)}(z)\right|}{d(z,\Gamma)} \|P_n\|_p \left[n^{\frac{(\widetilde{\gamma}_2^*+1)}{p(1+\alpha_*)}+\varepsilon} + \begin{cases} n^{\frac{\widetilde{\gamma}+1}{p(1+\alpha_*)}}, & \gamma_1,\gamma_2 > p-1, \\ n^{\frac{1}{1+\alpha_*}+\varepsilon}(\ln n)^{1-\frac{1}{p}}, & \begin{cases} \text{if } \gamma_1 = p-1,\ -1 < \gamma_2 \leq p-1, \\ \text{or } -1 < \gamma_1 \leq p-1,\ \gamma_2 = p-1, \end{cases} \\ n^{\frac{1}{1+\alpha_*}+\varepsilon}, & -1 < \gamma_1,\gamma_2 < p-1, \end{cases} \right]$$

where $\alpha_* := \min\{\alpha_1; \alpha_2\}$,and, for the points $z \in \widehat{\Omega}_R(\delta)$, from (47) and (33), we obtain:

$$\left|P_n'(z)\right| \leq \frac{\left|\Phi^{2(n+1)}(z)\right|}{d(z,\Gamma)} \|P_n\|_p \begin{cases} n^{\left(\frac{\widetilde{\gamma}+1}{p}-1\right)\frac{1}{1+\alpha_*}+\varepsilon}, & \gamma_1,\gamma_2 > p-1, \\ (\ln n)^{1-\frac{1}{p}}, & \begin{cases} \text{if } \gamma_1 = p-1,\ -1 < \gamma_2 \leq p-1 \\ \text{or } -1 < \gamma_1 \leq p-1,\ \gamma_2 = p-1, \end{cases} \\ 1, & -1 < \gamma_1,\gamma_2 < p-1. \end{cases}$$

□

Proof. [Proof of Theorem 1.6] Assume that $G \in C_\theta(\lambda_1; cx^{1+\alpha_2})$, for some $0 < \lambda_1 \leq 2$, and $\alpha_2 > 0$. Let $z \in \Gamma$ arbitrary fixed point and let us $B(z, d(z,\Gamma_R)) := \{t : |t-z| < d(z,\Gamma_R)\}$. Using the Cauchy integral representation for derivatives $P_n'(z)$ in a bounded domain, we obtain:

$$\left|P_n'(z)\right| = \left|\frac{1}{2\pi i}\int\limits_{\partial B(z,d(z,\Gamma_R))} \frac{P_n(\zeta)\,d\zeta}{(\zeta-z)^2},\right| \leq \frac{1}{2\pi}\int\limits_{\partial B(z,d(z,\Gamma_R))} |P_n(\zeta)|\frac{|d\zeta|}{|\zeta-z|^2} \leq \max_{z\in\overline{G}_R}|P_n(\zeta)| \sup_{z\in\Gamma}\left\{\frac{1}{d(z,\Gamma_R)}\right\}.$$

According [9, Cor.1.3.] and applying (4), we have:

$$\max_{z\in\overline{G}_R}|P_n(\zeta)| \leq \|P_n\|_{C(\overline{G})} \leq \|P_n\|_p \left[n^{\frac{(\gamma_1^*+1)\widetilde{\lambda}_1}{p}} + \begin{cases} n^{\frac{\gamma_2^*+1}{p(1+\alpha_2)}+\frac{\alpha_2}{p(1+\alpha_2)}+\varepsilon}, & 1 < p < 2 + \frac{\gamma_2^*}{1+\alpha_2}, \\ (n\ln n)^{1-\frac{1}{p}}, & p = 2 + \frac{\gamma_2^*}{1+\alpha_2}, \\ n^{1-\frac{1}{p}}, & p > 2 + \frac{\gamma_2^*}{1+\alpha_2}. \end{cases} \right] \quad (48)$$

From (35) and (37), we have:

$$\sup_{z\in\Gamma}\left\{\frac{1}{d(z,\Gamma_R)}\right\} \asymp \sup\left\{\sup_{z\in\Gamma^1}\left\{\frac{1}{d(z,\Gamma_R)}\right\};\sup_{z\in\Gamma^2}\left\{\frac{1}{d(z,\Gamma_R)}\right\}\right\} \preceq n^{\widetilde{\lambda}_1}$$

and, therefore,

$$\left|P_n'(z)\right| \preceq \|P_n\|_p \left[n^{\left(\frac{(\gamma_1^*+1)}{p}+1\right)\widetilde{\lambda}_1} + \begin{cases} n^{\frac{\gamma_2^*+1}{p(1+\alpha_2)}+\frac{\alpha_2}{p(1+\alpha_2)}+\widetilde{\lambda}_1+\varepsilon}, & 1<p<2+\frac{\gamma_2^*}{1+\alpha_2}, \\ n^{\widetilde{\lambda}_1+1-\frac{1}{p}}(\ln n)^{1-\frac{1}{p}}, & p=2+\frac{\gamma_2^*}{1+\alpha_2}, \\ n^{\widetilde{\lambda}_1+1-\frac{1}{p}}, & p>2+\frac{\gamma_2^*}{1+\alpha_2}, \end{cases} \right]$$

and we completed the proof of Theorem 1.6. □

3.0.1. *Proof of Remark*

Proof. The proof of the accuracy of the inequalities (13) - (15) can be divided into two parts: (i) : $\left\|P_n'\right\|_\infty \preceq n\|P_n\|_\infty$; (ii): $\|P_n\|_\infty \preceq \mu_n \|P_n\|_p$. The first inequality- the well known sharp Markov inequality. The sharpness of the inequality (ii) can be verified in the following examples (see, for exp., [17], [18, Rm 2.9]). Let $T_n(z) = 1 + z + ... + z^n$, $h^*(z) = h_0(z)$, $h^{**}(z) = |z-1|^\gamma$, $\gamma > 0$, $\Gamma := \{z : |z| = 1\}$ .Then, for any $n \in \mathbb{N}$, there exist $c_1 = c_1(h^*, p) > 0$, $c_2 = c_2(h^{**}, p) > 0$ such that:

$$a)\ \|T\|_\infty \geq c_1 n^{\frac{1}{p}} \|T\|_{\mathcal{L}_p(h^*,\,\Gamma)}, \quad p > 1;$$

$$b)\ \|T\|_\infty \geq c_2 n^{\frac{\gamma+1}{p}} \|T\|_{\mathcal{L}_p(h^{**},\,\Gamma)}, \quad p > \gamma + 1.$$

□

References

[1] F. G. Abdullayev, V. V. Andrievskii, *Orthogonal polynomials in the domains with quasiconformal boundary*, Izv. Akad. Nauk Az. SSR, Ser. Fiz.-Tekh. Mat. Nauk **4** (1983), 7–11.

[2] F. G. Abdullayev, *On the some properties on orthogonal polynomials over the regions of complex plane 1*, Ukr. Math. J. **52** (2000), 1807–1817.

[3] F. G. Abdullayev, *On the some properties of the orthogonal polynomials over the regions of the complex plane (Part III)*, Ukr. Math. J. **53)** (2001), 1934–1948. (in Russian)

[4] F. G. Abdullayev, U. Deger, *On the orthogonal polynomials with weight having singularities on the boundary of regions in the complex plane*, Bull. Belg. Math. Soc. **16** (2009), 235–250.

[5] F. G. Abdullayev, N. D. Aral, *On Bernstein-Walsh-type lemmas in regions of the complex plane*, Ukr. Math. J. **63** (2011), 337–350.

[6] F. G. Abdullayev, P. Özkartepe, *An analogue of the Bernstein-Walsh lemma in Jordan regions of the complex plane*, J. Inequal. Appl. **2013:570** (2013), 7p.

[7] F. G. Abdullayev, P. Özkartepe, *On the behavior of the algebraic polynomial in unbounded regions with piecewise dini-smooth boundary*, Ukr. Math. J. **66** (2014), 645–665.

[8] F. G. Abdullayev, P. Özkartepe, C. D. Gün, *Uniform and pointwise polynomial inequalities in regions without cusps in the weighted Lebesgue space*, Bull. TICMI. **18** (2014), 146–167.

[9] F. G. Abdullayev, C. D. Gün, P. Özkartepe, *Inequalities for algebraic polynomials in regions with exterior cusps*, J. Nonlin. Func. Anal. **3** (2015), 1–32.

[10] F. G. Abdullayev, P. Özkartepe, *Uniform and pointwise polynomial inequalities in regions with cusps in the weighted Lebesgue space*, Jaen J. Approx. **7** (2015), 231–261.

[11] F.G. Abdullayev, P. Özkartepe, *Uniform and pointwise Bernstein-Walsh-type inequalities on a quasidisk in the complex plane*, Bull. Belg. Math. Soc. **23** (2016), 285–310.

[12] F. G. Abdullayev, N. P. Özkartepe, *Polynomial inequalities in Lavrentiev regions with interior and exterior zero angles in the weighted Lebesgue space*, Publ. Inst. Math. **100(114)** (2016), 209–227.

[13] F. G. Abdullayev, N. P. Özkartepe, *Interference of the weight and boundary contour for algebraic polynomials in the weighted Lebesgue spaces I* , Ukr. Math. J. **68** (2017), 1574–1590.

[14] F. G. Abdullayev, *Polynomial inequalities in regions with corners in the weighted Lebesgue spaces*, Filomat **31(18)** (2017), 5647–5670.

[15] F. G. Abdullayev, M. Imashkyzy, G. Abdullayeva, *Bernstein-Walsh type inequalities in unbounded regions with piecewise asymptotically conformal curve in the weighted Lebesgue space*, J. Math. Sci. **234** (2018), 35–48.

[16] F. G. Abdullayev, T. Tunc, GA. Abdullayev, *Polynomial inequalities in quasidisks on weighted Bergman space*, Ukr. Math. J. **69** (2017), 675–695.

[17] F. G. Abdullayev, N. P. Özkartepe, *The uniform and pointwise estimates for polynomials on the weighted Lebesgue spaces in the general regions of complex plane*, Hacettepe. J. Math. Stat. **48** (2019), 87–101.
[18] F. G. Abdullayev, N. P. Özkartepe, T. Tunç, *Uniform and pointwise estimates for algebraic polynomials in regions with interior and exterior zero angles*, Filomat **33** (2019), 403–413.
[19] F. G. Abdullayev, C. D. Gün, *Bernstein-Nikolskii-type inequalities for algebraic polynomials in Bergman space in regions of complex plane*, Ukr. Math. J. **73** (2021), 513–531.
[20] F. G. Abdullayev, C. D. Gün, *Bernstein-Walsh-type inequalities for derivatives of algebraic polynomials*, B. Korean Math. Soc. **59** (2022), 45–72.
[21] F. G. Abdullayev, *Bernstein-Walsh-type inequalities for derivatives of algebraic polynomials in quasidiscs*, Open Math. **2021** (2022), 1847–1876.
[22] L. Ahlfors, *Lectures on Quasiconformal Mappings*, Princeton, NJ: Van Nostrand, 1966.
[23] V. V. Andrievskii, *On the uniform convergence of the Bieberbach polynomials in the regions with piecewise quasiconformal boundary*, Theory Map. Approx. Func. "Naukovo Dumka", Kyiv (1983), 3–18. (in Russian)
[24] V. V. Andrievskii, *Weighted Polynomial Inequalities in the Complex Plane*, J. Approx. Theory. **164** (2012), 1165–1183.
[25] S. Balcı, M. Imashkyzy, F. G. Abdullayev, *Polynomial inequalities in regions with interior zero angles in the Bergman space*, Ukr. Math. J. **70** (2018), 362–384.
[26] D. Benko, P. Dragnev, V. Totik, *Convexity of harmonic densities*, Rev. Mat. Iberoam. **28** (2012), 1–14.
[27] S. N. Bernstein, *Sur la limitation des derivees des polnomes*, C. R. Acad. Sci. Paris. **190** (1930), 338–341.
[28] S. N. Bernstein, *On the best approximation of continuos functions by polynomials of given degree*. Izd. Akad. Nauk SSSR I. (1952); II; (1954) (O nailuchshem problizhenii nepreryvnykh funktsii posredstrvom mnogochlenov dannoi stepeni), Sobraniye sochinenii. **I(4)** (1912), 11–10.
[29] P. P. Belinskii, *General properties of quasiconformal mappings*, Nauka, Sib. otd., Novosibirsk, 1974 (in Russian).
[30] V. K. Dzjadyk, *Introduction to the theory of uniform approximation of function by polynomials*, Nauka, Moskow, 1977.
[31] E. Hille, G. Szegö, J. D. Tamarkin, *On some generalization of a theorem of A.Markoff*, Duke Math. **3** (1937), 729–739.
[32] D. Jackson, *Certain problems on closest approximations*, B. Am. Math. Soc. **39** (1933), 889–906.
[33] D. I. Mamedhanov, *Inequalities of S.M.Nikol'skii type for polynomials in the complex variable on curves*, Soviet Math. Dokl. **15** (1974), 34–37.
[34] GV. Milovanovic, DS. Mitrinovic, ThM. Rassias, *Topics in polynomials: extremal problems, inequalities, zeros*, Singapore, World Scientific, 1994.
[35] S. M. Nikol'skii, *Approximation of function of several variable and imbeding theorems*, Springer-Verlag, New-York, 1975.
[36] P. Özkartepe, *Uniform and pointwise polynomial estimates in regions with interior and exterior cusps*, Cumhuriyet Sci. J. **39** (2018), 47–65
[37] I. E. Pritsker, *Comparing norms of polynomials in one and several variables*, J. Math. Anal. Appl. **216** (1997), 685–695.
[38] S. Rickman, *Characterization of quasiconformal arcs*, Ann. Acad. Sci. Fenn., Ser. A, Math. **395** (1966), 30 p.
[39] N. Stylianopoulos, *Strong asymptotics for Bergman polynomials over domains with corners and applications*. Constr. Approx. **38** (2013), 59–100.
[40] G. Szegö, A. Zygmund, *On certain mean values of polynomials*, J. Anal. Math. **3** (1953), 225–244.
[41] J. L. Walsh, *Interpolation and approximation by rational functions in the complex domain*, AMS, Rhode Island, 1960.