Filomat 37:27 (2023), 9259–9271
https://doi.org/10.2298/FIL2327259A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Sequential warped product submanifolds in locally product Riemannian manifolds

Najwa Mohammed AL-Asmari[a,b], Siraj Uddin[c], Monia Fouad Naghi[a]

[a]*Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia*
[b]*Department of Mathematics, College of Science, King Khalid University, Muhayil Asir, Saudi Arabia*
[c]*Department of Mathematics, Faculty of Natural Sciences, Jamia Millia Islamia, New Delhi-110025, India*

Abstract. In this paper, we present sequential warped product submanifolds of locally product Riemannian manifolds and show that there exists a class of non-trivial sequential warped product submanifolds of a locally product Riemannian manifold of the form $(M_\theta \times_{\sigma_1} M_\perp) \times_{\sigma_2} M_T$ by giving non-trivial examples. Also, we prove some useful results for such warped products and establish Chen's inequality which represents a relationship between the squared norm of the second fundamental form for the warping functions σ_1 and σ_2 . Further, some applications of our main result are given.

1. Introduction

In the 1960s, Bishop and O'Neill [7] introduced singly or ordinary warped product manifolds in order to create Riemannian manifolds with negative sectional curvature. It is well-known that warped product manifolds play an important role in differential geometry as well as in physics. Several classes of warped product submanifolds have appeared in the last twenty years. Also, warped product submanifolds have been studied for the different structures on manifolds, Chen's books are useful resources for a detailed study of warped product manifolds and warped product submanifolds [11, 12].

As a generalization of Riemannian products and warped products, Nolker in [20] defined and studied multiply warped products. For more details, we refer to (see [13], [16], [36]). Bi-warped products are special classes of multiply warped products and studied in almost Hermitian manifolds as well as almost contact manifolds (for instance, see; [1], [31], [32], [34], [35], CSA22).

In [27], Shenawy introduced a new concept of warped products as sequential warped products such that the base or fiber or both in sequential warped product are warped product itself (see [15]). Recently, Sahin in [26] studied sequential warped product submanifolds having the factors as holomorphic submanifolds, totally real submanifolds and pointwise slant submanifolds of Kaehler manifolds.

In this article, we study sequential warped product submanifolds which are defined by harmonizing an invariant submanifold M_T, an anti-invariant submanifold $M_\perp$ and a proper slant submanifold M_θ of locally

2020 *Mathematics Subject Classification*. Primary 53C15, 53C40, 53C42, 53B25; Secondary 53C21 53C25, 53C50
Keywords. slant submanifolds; mixed geodesic; skew semi-invariant submanifolds; bi-warped products; sequential warped products; locally product Riemannian manifolds
Received: 25 February 2023; Accepted: 26 April 2023
Communicated by Mića S. Stanković
The project was funded by the Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah, under grant no. G: 443-247-1440. The authors, therefore, acknowledge with thanks DSR for technical and financial support.
Email addresses: nalasmri@kku.edu.sa (Najwa Mohammed AL-Asmari), siraj.ch@gmail.com (Siraj Uddin), mnaghi@kau.edu.sa (Monia Fouad Naghi)

product Riemannian manifolds. We provide several useful lemmas for the proof of our main theorem. Finally, we establish B.-Y. Chen's inequality for the squared norm of second fundamental form of such submanifolds. The equality case is also considered.

The paper is organized as follows: Section 2, in this section, we recall preliminaries and definitions needed further study. In Section 3, we study sequential warped product submanifolds. In this section, we give non-trivial examples of sequential warped product submanifolds of the form $(M_\theta \times_{\sigma_1} M_\perp) \times_{\sigma_2} M_T$ such that M_θ is a slant submanifold and $M_\perp$ is an anti-invariant submanifold, M_T is an invariant submanifold. Some useful results are deriven in section 3 which we need to prove the main result. In Section 4, the main result of this paper is proved. The main result is leading to several applications given in the last section.

2. Preliminaries

In this section, we give preliminaries and definitions needed for this paper. In fact, in subsection 2.1, we recall the definition of the sequential warped product manifolds. In subsection 2.2, we give the basic background for submanifolds of Riemannian manifolds, definition of locally product Riemannian manifolds and some classes of submanifolds.

2.1. Sequentail warped product manifolds

The formal definition of sequential warped product manifolds is given as follows:

Definition 2.1. *[15] Let M_i be three pseudo-Riemannian manifolds with metrics g_i, for $i = 1, 2, 3$. Let $\sigma_1 : M_1 \to (0, \infty)$ and $\sigma_2 : M_1 \times M_2 \to (0, \infty)$ be two smooth positive functions on M_1 and $M_1 \times M_2$, respectively. Then the sequential warped product manifold, denoted by $(M_1 \times_{\sigma_1} M_2) \times_{\sigma_2} M_3$, is the triple product manifold $M = (M_1 \times M_2) \times M_3$ furnished with the metric tensor*

$$g = (g_1 \oplus {\sigma_1}^2 g_2) \oplus {\sigma_2}^2 g_3,$$

the functions σ_1 and σ_2 are called warping functions.

In particular, if warping functions σ_1 and σ_2 are constant, then M is a Riemannian product manifold; and if σ_2 is constant, then M is an ordinary warped product; if σ_1 is constant, then M is a generic warped product submanifold of order 1 (e.g. [29]), if σ_2 is defined only on M_1, then M is called a bi-warped product manifold $M_1 \times_{\sigma_1} M_2 \times_{\sigma_2} M_3$ (e.g., [1], [33]) with two fibres which is a special case of multiply warped products.

The following proposition gives the basic formula for the Levi-Civita connection of the sequential warped products that will be used during this study.

Proposition 2.2. *[15] Let $M = (M_1 \times_{\sigma_1} M_2) \times_{\sigma_2} M_3$ be a sequential warped product manifolds with metric $\tilde{g} = (g_1 \oplus {\sigma_1}^2 g_2) \oplus {\sigma_2}^2 g_3$. Then, we have*

(1) $\nabla_{X_1} X_2 = \nabla_{X_2} X_1 = X_1(\ln \sigma_1) X_2$
(2) $\nabla_{X_1} X_3 = \nabla_{X_3} X_1 = X_1(\ln \sigma_2) X_3$
(3) $\nabla_{X_2} X_3 = \nabla_{X_3} X_2 = X_2(\ln \sigma_2) X_3$,

for each $X_i \in \Gamma(TM_i)$, for $i = 1,\ 2,\ 3$.

2.2. Submanifolds of locally product Riemannian manifolds

Let $\tilde{M}$ be an m-dimensional Riemannian manifold with a tensor field F of type $(1, 1)$ such that $F^2 = I$ ($F \neq \pm I$) where I denotes the identity transformation. Then we say that $\tilde{M}$ is an almost product manifold with almost product structure F [37]. If an almost product manifold $\tilde{M}$ admits a Riemannian metric g such that

$$g(FX, FY) = g(X, Y), \qquad g(FX, Y) = g(X, FY), \tag{1}$$

for any vector fields X and Y on $\tilde{M}$, then $\tilde{M}$ is called an almost product Riemannian manifold. Let $\tilde{\nabla}$ denotes the Levi Cevita connection on $\tilde{M}$ with respect to g. If $(\tilde{\nabla}_X F)Y = 0$, for all $X, Y \in \Gamma(T\tilde{M})$, where $\Gamma(T\tilde{M})$

denotes the Lie algebra of vector fields in $\tilde{M}$, then $(\tilde{M}, g)$ is called a locally product Riemannian manifold with Riemannian metric g [6]. Let M be a submanifold of a locally product Riemannian manifold $\tilde{M}$ with induced Riemannian metric g and if ∇ and $\nabla^{\perp}$ are the induced Riemannian connections on the tangent bundle TM and the normal bundle $T^{\perp}M$ of M, respectively. Then, the formulas of Gauss and Weingarten are respectively given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2}$$

$$\tilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{3}$$

for each $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$, where h and A_N are the second fundamental form and the shape operator (corresponding to the normal vector field N) respectively for the immersion of M into $\tilde{M}$. Furthermore, they are related by

$$g(h(X, Y), N) = g(A_N X, Y). \tag{4}$$

Now, for any vector field $X \in \Gamma(TM)$, we put

$$FX = TX + \omega X, \tag{5}$$

where TX and ωX are the tangential and the normal components of FX, respectively. Similarly, for any vector field $N \in \Gamma(T^{\perp}M)$, we write

$$FN = tN + nN, \tag{6}$$

where tN is the tangential component and nN is the normal component of FN. We choose a local field of orthonormal frame $\{e_1, \dots e_n, e_{n+1}, \dots e_m\}$ in $\tilde{M}$ such that $\{e_1, \dots e_n\}$ is an orthonormal basis of the tangent bundle TM and $\{e_{n+1} \dots e_m\}$ is an orthonormal basis of the normal bundle $T^{\perp}M$, we have the squared norm of the second fundamental form h is defined by

$$||h||^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)), \text{ and } \ h^r_{ij} = g\left(h(e_i, e_j), e_r\right), \tag{7}$$

where $i, j = 1, \dots, n$, $r \in \{n+1, \cdots, m\}$, for any $p \in M$. Furthermore, for a differentiable function σ on M, we know that $||\nabla\sigma||^2 = \sum_{i=1}^{n}(e_i(\sigma))^2$, where $\nabla\sigma$ is the gradient of σ which is defined as $g(\nabla\sigma, X) = X(\sigma)$, for any X tangent to M.

A submanifold M of a locally product Riemannian manifold $\tilde{M}$ is said to be *totally umbilical* submanifold if $h(X, Y) = g(X, Y)H$, for any $X, Y \in \Gamma(TM)$, where H is the mean curvature vector field of M is given by $H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i)$. A submanifold M is said to be *totally geodesic* if $h(X, Y) = 0$ for $X, Y \in \Gamma(TM)$. Let $\mathfrak{D}^1$ and $\mathfrak{D}^2$ be any two distributions on M. Then we say that M is $\mathfrak{D}^1$ *totally geodesic*, if $h(X_1, Y_1) = 0$ for all $X_1, Y_1 \in \Gamma(\mathfrak{D}^1)$ and we say that $\mathfrak{D}^1 \oplus \mathfrak{D}^2$-*mixed geodesic* if $h(X_1, X_2) = 0$ for $X_1 \in \Gamma(\mathfrak{D}^1)$ and $X_2 \in \Gamma(\mathfrak{D}^2)$.

By the analogy with submanifolds in a Kaehler manifold, different classes of submanifolds in a locally product Riemannian manifold were considered.

1. A submanifold M of a locally product Riemannian manifold $\tilde{M}$ is said to be *slant* (for instance, see [9] and [23]), if for each non-zero vector X tangent to M, the angle $\theta(X)$ between FX and T_pM is a constant, i.e., it does not depend on the choice of $p \in M$ and $X \in T_pM$.
2. A submanifold M of a locally product Riemannian manifold $\tilde{M}$ is called *semi-slant* (see, [18] and [21]), if it has two orthogonal distributions $\mathfrak{D}^T$ and $\mathfrak{D}^{\theta}$, such that $\mathfrak{D}^T : p \longrightarrow \mathfrak{D}^T_p \subset T_pM$ is an invariant with respect to F i.e., for any $X \in \Gamma(\mathfrak{D}^T)$, we have $FX \in \Gamma(\mathfrak{D}^T)$, and the complementary distribution $\mathfrak{D}^{\theta}$ is slant, i.e., $\theta(X)$ is the angle between FX and $\mathfrak{D}^{\theta}_p$ is constant for any $X \in \mathfrak{D}^{\theta}_p$ and $p \in M$.
3. A submanifold M of a locally product Riemannian manifold $\tilde{M}$ is called *a semi-invariant submanifold* (see [6], [19]) [22]) of $\tilde{M}$ if it has two orthogonal distributions $\mathfrak{D}^T$ and $\mathfrak{D}^{\perp}$ such that $\mathfrak{D}^T$ is invariant distribution with respect to F and the orthogonal complementary distribution $\mathfrak{D}^{\perp}$ is anti-invariant with respect to F, i.e., for any $X \in \Gamma(\mathfrak{D}^{\perp})$, we have $FX \in \Gamma(T^{\perp}M)$.

4. A submanifold M of a locally product Riemannian manifold $\tilde{M}$ is said be *pseudo-slant (or Hemi-slant)* (see [8] and [28]), if it has two orthogonal distributions $\mathfrak{D}^{\perp}$ and $\mathfrak{D}^{\theta}$ such that $\mathfrak{D}^{\perp}$ is anti-invariant and $\mathfrak{D}^{\theta}$ is slant.
Finally, we recall the definition of skew semi-invariant (CR-slant) submanifolds from [17], as follows

Definition 2.3. *[17] Let M be a submanifold of a l.p.R. manifold $\tilde{M}$. Then M is said to be a generic submanifold if there exists an integer k and functions λ_i, $i \in \{1, \dots, k\}$ defined on M with values in $(0, 1)$ such that*

(1) Each $\lambda_i^2(p)$, $1 \leq i \leq k$ is a distinct eigenvalue of T^2 with

$$T_pM = \mathfrak{D}_p \oplus \mathfrak{D}_p^{\perp} \oplus \mathfrak{D}_p^{\lambda_1} \oplus \cdots \oplus \mathfrak{D}_p^{\lambda_k},$$

for any $p \in M$.

(2) The dimensions of $\mathfrak{D}_p$, $\mathfrak{D}_p^{\perp}$ and $\mathfrak{D}_p^{\lambda_i}$, $1 \leq i \leq k$ are independent for any $p \in M$.

Moreover, if each λ_i is constant on M, then we say that M is a skew semi-invariant submanifold of $\tilde{M}$.

Definition 2.4. *A submanifold M of a locally product Riemannian manifold $\tilde{M}$ is said to be a skew semi-invariant submanifold of order 1, if there exist orthogonal distributions $\mathfrak{D}^T$, $\mathfrak{D}^{\perp}$ and $\mathfrak{D}^{\theta}$ on M such that*

(i) $TM = \mathfrak{D}^T \oplus \mathfrak{D}^{\perp} \oplus \mathfrak{D}^{\theta}$.

(ii) The distribution $\mathfrak{D}^T$ is invariant, i.e. $F(\mathfrak{D}^T) = \mathfrak{D}^T$.

(iii) The distribution $\mathfrak{D}^{\perp}$ is anti-invariant, i.e., $F\mathfrak{D}^{\perp} \subset T^{\perp}M$.

(iv) The distribution $\mathfrak{D}^{\theta}$ is proper slant with slant angle $\theta \neq 0, \frac{\pi}{2}$.

It is easy to see that M is a slant [23] submanifold of a locally product Riemannian manifold $\tilde{M}$ if and only if there exists a constant $\lambda \in [0, 1]$ such that

$$T^2 = \lambda I \tag{8}$$

where I denotes the identity transformation of the tangent bundle TM of the submanifold M. Furthermore, in this case, if θ is the slant angle of M, then $\lambda = \cos^2\theta$.

As a result of (8), we have the following relations

$$g(TX, TY) = (\cos^2\theta)g(X, Y),$$
$$g(\omega X, \omega Y) = (\sin^2\theta)g(X, Y),$$

for any $X, Y \in \Gamma(TM)$.

3. Sequential warped product submanifolds

In this section, we study the sequential warped products of a proper slant submanifold M_θ, an invariant submanifold M_T and an anti-invariant submanifold $M_\perp$ in a locally product Riemannian manifold $\tilde{M}$. There are following possible classes of sequential warped products in locally product Riemannian manifolds.

1. $(M_T \times_{\sigma_1} M_\perp) \times_{\sigma_2} M_\theta$,
2. $(M_\theta \times_{\sigma_1} M_T) \times_{\sigma_2} M_\perp$,
3. $(M_T \times_{\sigma_1} M_\theta) \times_{\sigma_2} M_\perp$,
4. $(M_\perp \times_{\sigma_1} M_T) \times_{\sigma_2} M_\theta$,
5. $(M_\perp \times_{\sigma_1} M_\theta) \times_{\sigma_2} M_T$,
6. $(M_\theta \times_{\sigma_1} M_\perp) \times_{\sigma_2} M_T$.

First, we verify whether the above classes of sequential warped product submanifolds do exist or not in a locally product Riemannian manifold $\tilde{M}$. For this, we have the following results.

The following is an immediate result of Theorem 3.1 in [3] and Theorem 3.1 in [24].

Corollary 3.1. *There do not exist any proper sequential warped product submanifolds* $(M_T \times_{\sigma_1} M_\perp) \times_{\sigma_2} M_\theta$ *and* $(M_\theta \times_{\sigma_1} M_T) \times_{\sigma_2} M_\perp$ *in a locally product Riemannian manifold* $\tilde{M}$ *such that* M_θ *is a proper slant submanifold,* $M_\perp$ *is an anti-invariant submanifold and* M_T *is an invariant submanifold of* $\tilde{M}$.

Also from Theorem 3.3 in [4] and Theorem 3.1 in [25], we deduce the following result.

Corollary 3.2. *There do not exist any proper sequential warped product submanifolds of the types* $(M_T \times_{\sigma_1} M_\theta) \times_{\sigma_2} M_\perp$ *and* $(M_\perp \times_{\sigma_1} M_T) \times_{\sigma_2} M_\theta$ *in a locally product Riemannian manifold* $\tilde{M}$.

Furthermore, from Theorem 3.4 in [4], we obtain the following non-existence result.

Corollary 3.3. *There do not exist any proper sequential warped product submanifolds of the form* $(M_\perp \times_{\sigma_1} M_\theta) \times_{\sigma_2} M_T$ *in a locally product Riemannian manifold* $\tilde{M}$.

From the above results, we find that there do not exist any proper sequential warped products upto the case 1-5 and hence, the remaining class of sequential warped products is the sixth case is of the form $(M_\theta \times_{\sigma_1} M_\perp) \times_{\sigma_2} M_T$ and we see that such warped products exist in locally product Riemannian manifolds.

In the following examples we find the existence of proper sequential warped products of the form $(M_\theta \times_{\sigma_1} M_\perp) \times_{\sigma_2} M_T$.

Example 3.4. Consider the 12-Euclidean space $\mathbb{R}^{12} = \mathbb{R}^7 \times \mathbb{R}^3 \times \mathbb{R}^2$ with the cartesian coordinates $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, y_1, y_2, y_3, z_1, z_2)$ and the almost product structure given by

$$F\Big(\frac{\partial}{\partial x_i}\Big) = \frac{\partial}{\partial x_i}, \;\; F\Big(\frac{\partial}{\partial y_j}\Big) = -\frac{\partial}{\partial y_j}, \;\; F\Big(\frac{\partial}{\partial z_k}\Big) = \frac{\partial}{\partial z_k}, \;\; 1 \le i \le 7, \;\; 1 \le j \le 3, \;\; 1 \le k \le 2,$$

If a submanifold M of $\mathbb{R}^{12}$ is defined by the immersion

$$\phi(u, v, r) = (u\cos v, u\sin v, u, u\cos r, u\sin r, v\cos r, v\sin r, u\sin v, u\cos v, v, u - r, u + r)$$

with $u, v \neq 0$, then its tangent space TM is spanned by the vectors X, Y and Z, where

$$\begin{aligned}
X &= \cos v\frac{\partial}{\partial x_1} + \sin v\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + \cos r\frac{\partial}{\partial x_4} + \sin r\frac{\partial}{\partial x_5} + \sin v\frac{\partial}{\partial y_1} + \cos v\frac{\partial}{\partial y_2} + \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2},\\
Y &= -u\sin v\frac{\partial}{\partial x_1} + u\cos v\frac{\partial}{\partial x_2} + \cos r\frac{\partial}{\partial x_6} + \sin r\frac{\partial}{\partial x_7} + u\cos v\frac{\partial}{\partial y_1} - u\sin v\frac{\partial}{\partial y_2} + \frac{\partial}{\partial y_3},\\
Z &= -u\sin r\frac{\partial}{\partial x_4} + u\cos r\frac{\partial}{\partial x_5} - v\sin r\frac{\partial}{\partial x_6} + v\cos r\frac{\partial}{\partial x_7} - \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2}.
\end{aligned}$$

On the other hand, if $T^\perp M = F\mathfrak{D}^\perp \oplus \omega\mathfrak{D}^\theta \oplus \mu$ is the normal bundle of M, then $T^\perp M$ is spanned by the normal vector fields N_1, N_2, N_3, N_4, N_5, N_5, N_6, N_7, N_8 and N_9 such that

$$\begin{aligned}
&N_1 = FY, \;\; N_2 = \cos v\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_3} + \sin v\frac{\partial}{\partial y_1}, \; N_3 = -\frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_2},\\
&N_4 = (\cos r - u\sin r)\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4} + u\sin r\frac{\partial}{\partial z_1}, \;\; N_5 = -(\cos r + uu\sin r)\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4} + u\sin r\frac{\partial}{\partial z_2},\\
&N_6 = \cos v\frac{\partial}{\partial x_1} + \sin v\frac{\partial}{\partial x_2} - 2\frac{\partial}{\partial x_3} + \cos r\frac{\partial}{\partial x_4} + \sin r\frac{\partial}{\partial x_5}, \;\; N_7 = -\frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_2},\\
&N_8 = \sin v\frac{\partial}{\partial x_1} - \cos v\frac{\partial}{\partial x_2} + u\cos r\frac{\partial}{\partial x_6} + u\sin r\frac{\partial}{\partial x_7}, \;\;\; N_9 = \sin v\frac{\partial}{\partial x_1} - \cos v\frac{\partial}{\partial x_2} + u\frac{\partial}{\partial y_3}.
\end{aligned}$$

It is clear that FY is perpendicular to TM, thus $\mathfrak{D}^\perp = \mathrm{Span}\{Y\}$ is an anti-invariant distribution and $\mathfrak{D}^T = \mathrm{Span}\{Z\}$ is an invariant distribution. Moreover, $\mathfrak{D}^\theta = \mathrm{Span}\{X\}$ is a slant distribution with slant angle $\theta = \cos^{-1}(\frac{2}{3})$. Clearly, M is a CR-slant submanifold. Furthermore, each distribution is integrable. Let $M_\perp$,

M_θ and M_T be integral manifolds of $\mathfrak{D}^\perp$, $\mathfrak{D}^\theta$ and $\mathfrak{D}^T$, respectively. Then the metric tensor g of M is given by

$$ds^2 = 6du^2 + 2(u^2+1)dv^2 + (u^2+v^2+2)dr^2$$
$$g = g_{M_\theta} + {\sigma_1}^2 g_{M_\perp} + {\sigma_2}^2 g_{M_T},$$

where $\sigma_1 = \sqrt{2(u^2+1)}$ and $\sigma_2 = \sqrt{u^2+v^2+2}$ are warping functions defined on M_θ and $M_\theta \times M_\perp$, respectively. Thus M is a sequential warped product submanifold of the form $(M_\theta \times_{\sigma_1} M_\perp) \times_{\sigma_2} M_T$.

Example 3.5. Let M be a submanifold of the Euclidean 10-space $\mathbb{R}^{10}$ defined by

$$\psi(u,v,w) = (u\cos v, u\sin v, w\sin v, w\cos v, 2u, u\sin w, u\cos w, -w, u\sin w, u\cos w),$$

with $u,\ v \neq\ 0$ and the product structure is given by

$$F\Big(\frac{\partial}{\partial x_i}\Big) = -\frac{\partial}{\partial x_i},\ \ F\Big(\frac{\partial}{\partial y_j}\Big) = -\frac{\partial}{\partial y_j},\ \ F\Big(\frac{\partial}{\partial z_k}\Big) = \frac{\partial}{\partial z_k},\ \ i = 1,\dots,5.\ \ j = 1,2,3,\ \ k = 1,2.$$

Then the tangent space is spanned by

$$Z_1 = \cos v\frac{\partial}{\partial x_1} + \sin v\frac{\partial}{\partial x_2} + 2\frac{\partial}{\partial x_5} + \sin w\frac{\partial}{\partial y_1} + \cos w\frac{\partial}{\partial y_2} + \sin w\frac{\partial}{\partial z_1} + \cos w\frac{\partial}{\partial z_2},$$
$$Z_2 = -u\sin v\frac{\partial}{\partial x_1} + u\cos v\frac{\partial}{\partial x_2} + w\cos v\frac{\partial}{\partial x_3} - w\sin v\frac{\partial}{\partial x_4},$$
$$Z_3 = \sin\frac{\partial}{\partial x_3} + \cos v\frac{\partial}{\partial x_4} + u\cos w\frac{\partial}{\partial y_1} - u\sin w\frac{\partial}{\partial y_2} - \frac{\partial}{\partial y_3}u\cos w\frac{\partial}{\partial z_1} - u\sin w\frac{\partial}{\partial z_2}.$$

Also, the normal bundle is spanned by

$$W_1 = \cos v\frac{\partial}{\partial x_1} + \sin v\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_5} + \sin w\frac{\partial}{\partial y_1} + \cos w\frac{\partial}{\partial y_2},$$
$$W_2 = -\frac{\partial}{\partial y_1} + \frac{\partial}{\partial z_1},\ \ W_3 = -\frac{1}{2}\frac{\partial}{\partial x_5} + \sin w\frac{\partial}{\partial y_1} + \cos w\frac{\partial}{\partial y_2},$$
$$W_4 = -\sin v\frac{\partial}{\partial x_3} - \cos v\frac{\partial}{\partial x_4} - u\cos w\frac{\partial}{\partial y_1} + u\sin w\frac{\partial}{\partial y_2} + \frac{\partial}{\partial y_3} + u\cos w\frac{\partial}{\partial z_1} - u\sin w\frac{\partial}{\partial z_2},$$
$$W_5 = \sin v\frac{\partial}{\partial x_3} + \cos v\frac{\partial}{\partial x_4} + \frac{\partial}{\partial y_3},\ \ W_6 = -\cos w\frac{\partial}{\partial y_1} + \sin w\frac{\partial}{\partial y_2} - u\frac{\partial}{\partial y_3},$$
$$W_7 = -\cos w\frac{\partial}{\partial y_1} - u\frac{\partial}{\partial y_3} + \sin w\frac{\partial}{\partial z_2}.$$

Clearly, $\mathfrak{D}^\theta = \mathrm{Span}\{Z_1\}$ is slant distribution with slant angle $\theta = \cos^{-1}(\frac{5}{7})$, $\mathfrak{D}^\perp = \mathrm{Span}\{Z_3\}$ is an anti-invariant distribution and $\mathfrak{D}^T = \mathrm{Span}\{Z_2\}$ is an invariant distribution. Obviously, all three distributions are integrable. If the integral manifolds of $\mathfrak{D}^\theta$, $\mathfrak{D}^\perp$ and $\mathfrak{D}^T$ are M_θ, $M_\perp$, and M_T, respectively. Then, the metric g is given by

$$ds^2 = 7du^2 + (1+2u^2)dw^2 + (u^2+w^2)dv^2$$
$$g = g_{M_\theta} + {\sigma_1}^2 g_{M_\perp} + {\sigma_2}^2 g_{M_T}.$$

Thus M is is a sequential warped product submanifold of $\mathbb{R}^{10}$ of the form $(M_\theta \times_{\sigma_1} M_\perp) \times_{\sigma_2} M_T$ with warping functions $\sigma_1 = \sqrt{1+2u^2}$ and $\sigma_2 = \sqrt{u^2+w^2}$.

In the following, we use conveniently the vector fields X_1, Y_1 are in $\mathfrak{D}^\theta$; X_2, Y_2 are vector fields in $\mathfrak{D}^\perp$ and X_3, Y_3 are vector fields in $\mathfrak{D}^T$.

Now, we prove the following preparatory lemmas which are useful to prove the main result.

Lemma 3.6. *Let $M = (M_\theta \times_{\sigma_1} M_\perp) \times_{\sigma_2} M_T$ be a sequential warped product submanifold of a locally product Riemannian manifold $\tilde{M}$. Then we have*

(i) $g(h(X_1, Y_1), FX_2) = -g(h(X_1, X_2), \omega Y_1)$,

(ii) $g(h(X_2, Y_2), \omega X_1) = TX_1(\ln \sigma_1) g(X_2, Y_2) - g(h(X_1, X_2), FY_2)$,

(iii) $g(h(X_2, X_3), FY_2) = -g(h(Y_2, X_3), FX_2) = 0$,

for each $X_1, Y_1 \in \Gamma(\mathfrak{D}^\theta)$, $X_2, Y_2 \in \Gamma(\mathfrak{D}^\perp)$ and $X_3 \in \Gamma(\mathfrak{D}^T)$.

Proof. For any vector fields $X_1, Y_1 \in \Gamma(\mathfrak{D}^\theta)$ and $X_2 \in \Gamma(\mathfrak{D}^\perp)$, we have

$$g(h(X_1, Y_1), FX_2) = g(\tilde{\nabla}_{X_1} Y_1, FX_2) = g(\tilde{\nabla}_{X_1} FY_1, X_2).$$

Using (5), Proposition 2.2 (1) and (3), we get

$$g(h(X_1, Y_1), FX_2) = -X_1(\ln \sigma_2) g(TY_1, X_2) - g(A_{\omega Y_1} X_1, X_2).$$

Using the orthogonality of vector fields of $\mathfrak{D}^\theta$ and $\mathfrak{D}^\perp$, we find

$$g(h(X_1, Y_1), FX_2) = -g(h(X_1, X_2), \omega Y_1),$$

which is (i). For (ii), from (2) and (5), we have

$$g(h(X_2, Y_2), \omega X_1) = g(\tilde{\nabla}_{X_2} Y_2, FX_1) - g(\tilde{\nabla}_{X_2} Y_2, TX_1) = g(\tilde{\nabla}_{X_2} FY_2, X_1) + g(Y_2, \nabla_{X_2} TX_1).$$

Then, (3), (4) and Proposition 2.2 (1) imply that

$$g(h(X_2, Y_2), \omega X_1) = -g(h(X_1, X_2), FY_2) + TX_1(\ln \sigma_1) g(X_2, Y_2)$$

which gives (ii). For the third part, we have

$$g(h(X_2, X_3), FY_2) = g(\tilde{\nabla}_{X_2} X_3, FY_2) = g(\tilde{\nabla}_{X_2} FX_3, Y_2) = X_2(\ln \sigma_2) g(FX_3, Y_2) = 0,$$

which is the second equality of (iii). Furthermore, we find

$$g(h(X_2, X_3), FY_2) = g(\tilde{\nabla}_{X_3} X_2, FY_2) = g(\tilde{\nabla}_{X_3} FX_2, Y_2) = -g(A_{FX_2} X_3, Y_2) = -g(h(Y_2, X_3), FX_2),$$

which proves the first equality of (iii). Hence, the proof is complete. □

Lemma 3.7. *Let $M = (M_\theta \times_{\sigma_1} M_\perp) \times_{\sigma_2} M_T$ be a sequential warped product submanifold of a locally product Riemannian manifold $\tilde{M}$. Then, we have*

(i) $g(h(X_1, X_3), \omega Y_1) = -g(h(Y_1, X_3), \omega X_1) = 0$,

(ii) $g(h(X_1, X_3), FX_2) = -g(h(X_2, X_3), \omega X_1) = 0$,

for any $X_1, Y_1 \in \Gamma(\mathfrak{D}^\theta)$, $X_2 \in \Gamma(\mathfrak{D}^\perp)$ and $X_3 \in \Gamma(\mathfrak{D}^T)$.

Proof. From (2), (5) and Proposition 2.2, we find

$$\begin{aligned} g(h(X_1, X_3), \omega Y_1) &= g(\tilde{\nabla}_{X_1} FX_3, Y_1) - g(\tilde{\nabla}_{X_1} X_3, TY_1) \\ &= X_1(\ln \sigma_2) g(FX_3, Y_1) - X_1(\ln \sigma_2) g(X_3, TY_1) = 0, \end{aligned}$$

which is the second equality of (i). Similarly, we have

$$\begin{aligned} g(h(X_1, X_3), \omega Y_1) &= g(\tilde{\nabla}_{X_3} TX_1, Y_1) + g(\tilde{\nabla}_{X_3} \omega X_1, Y_1) - g(\tilde{\nabla}_{X_3} X_1, TY_1) \\ &= TX_1(\ln \sigma_2) g(X_3, Y_1) - g(h(Y_1, X_3), \omega X_1) - X_1(\ln \sigma_2) g(X_3, TY_1), \end{aligned}$$

By orthogonality of two distribution, we get from the above relation the first equality of (i). In a similar fashion, we find

$$g(h(X_1, X_3), FX_2) = g(\tilde{\nabla}_{X_1} FX_3, X_2) = X_1(\ln \sigma_2) g(FX_3, X_2) = 0,$$

which is the second equality of (ii). On the other hand,

$$g(h(X_1, X_3), FX_2) = g(\tilde{\nabla}_{X_3} TX_1, X_2) + g(\tilde{\nabla}_{X_3} \omega X_1, X_2) = TX_1(\ln \sigma_2) g(X_2, X_3) - g(A_{\omega X_1} X_3, X_2).$$

Hence, by the orthogonality of vector fields and from (4), we get the first equality of (ii). Hence, the proof is complete. □

Lemma 3.8. *Let $M = (M_\theta \times_{\sigma_1} M_\perp) \times_{\sigma_2} M_T$ be a sequential warped product submanifold of a locally product Riemannian manifold $\tilde{M}$. Then we have*

(i) $g(h(X_3, Y_3), FX_2) = -X_2(\ln \sigma_2) g(X_3, FY3)$,

(ii) $g(h(X_3, Y_3), \omega X_1) = -X_1(\ln \sigma_2) g(X_3, FY_3) + TX_1(\ln \sigma_2) g(X_3, Y_3)$,

for $X_1 \in \Gamma(\mathfrak{D}^\theta)$, $X_2 \in \Gamma(\mathfrak{D}^\perp)$ and $X_3, Y_3 \in \Gamma(\mathfrak{D}^T)$.

Proof. The first part simply implies from (2) and Proposition 2.2. For (ii), again using (2) and (5), we have

$$g(h(X_3, Y_3), \omega X_1) = g(\tilde{\nabla}_{X_3} Y_3 - \nabla_{X_3} Y_3, FX_1 - TX_1) = -g(FY_3, \tilde{\nabla}_{X_3} X_1) - g(\tilde{\nabla}_{X_3} Y_3, TX_1).$$

Then, from Proposition 2.2, we derive

$$g(h(X_3, Y_3), \omega X_1) = -X_1(\ln \sigma_2) g(X_3, FY_3) + TX_1(\ln \sigma_2) g(X_3, Y_3)$$

which is (ii) and hence the lemma is proved completely. □

Proposition 3.9. *There is no proper sequential warped product submanifold in a locally product Riemannian manifold $\tilde{M}$ of the form $(M_\theta \times_{\sigma_1} M_\perp) \times_{\sigma_2} M_T$, where M_θ is slant submanifold and $M_\perp$ is anti-invariant submanifold, M_T is invariant submanifold of $\tilde{M}$ if and only if $h(\mathfrak{D}^T, \mathfrak{D}^T) \perp F\mathfrak{D}^\perp$.*

Proof. follows from Lemma 3.8 (i). □

Proposition 3.10. *There does not exist any proper sequential warped product submanifold in a locally product Riemannian manifold $\tilde{M}$ of the form $(M_\theta \times_{\sigma_1} M_\perp) \times_{\sigma_2} M_T$ if and only if $h(\mathfrak{D}^T, \mathfrak{D}^T) \perp \omega\mathfrak{D}^\theta$.*

Proof. Suppose that $h(\mathfrak{D}^T, \mathfrak{D}^T) \perp \omega\mathfrak{D}^\theta$, which means that

$$g(h(X_3, Y_3), \omega X_1) = 0$$

for any $X_1 \in \Gamma(\mathfrak{D}^\theta)$ and $X_3, Y_3 \in \Gamma(\mathfrak{D}^T)$, using that fact in Lemma 3.8 (ii), we find that

$$TX_1(\ln \sigma_2)\, g(X_3, Y_3) = X_1(\ln \sigma_2)\, g(X_3, FY_3). \tag{9}$$

Interchanging X_1 by TX_1 and Y_3 by FY_3 in (9) and using (8) and (1), we obtain

$$\cos^2 \theta X_1(\ln \sigma_2)\, g(X_3, FY_3) = TX_1(\ln \sigma_2)\, g(X_3, Y_3). \tag{10}$$

Then, from (9) and(10), we derive

$$\sin^2 \theta X_1(\ln \sigma_2)\, g(X_3, FY_3) = 0,$$

Since M_θ is proper slant and g is Riemannian metric we obtain $X_1(\ln \sigma_2) = 0$, hence σ_2 is constant, the converse follows directly. □

From Proposition 3.9, Proposition 3.10, and Lemma 3.8, we have the following result.

Theorem 3.11. *For a non-trivial proper sequential warped product of the form $(M_\theta \times_{\sigma_1} M_\perp) \times_{\sigma_2} M_T$ in a locally product Riemannian manifold $\tilde{M}$: the components of second fundamental form $h(\mathfrak{D}^T, \mathfrak{D}^T)$ neither orthogonal to both $F\mathfrak{D}^\perp$ and $\omega\mathfrak{D}^\theta$ nor M is $\mathfrak{D}^T$ totally geodesic.*

4. B.-Y. Chen's inequality for Sequential warped products

In this section, we prove the Chen's first inequality for the squared norm of the second fundamental form in terms of the warping functions of a sequential warped product submanifold $(M_\theta \times_{\sigma_1} M_\perp) \times_{\sigma_2} M_T$ in a locally product Riemannian manifold.

For this, let $M = (M_\theta \times_{\sigma_1} M_\perp) \times_{\sigma_2} M_T$ be an n-dimensional sequential warped product submanifold of an m-dimensional locally product Riemannian manifold $\tilde{M}$ with $\dim(M_\theta) = q$, $dim(M_T) = p$ and $\dim(M_\perp) = r$, and we denote the tangent bundles of $M_\perp$, M_T and M_θ by $\mathfrak{D}^\perp$, $\mathfrak{D}^T$ and $\mathfrak{D}^\theta$, respectively. The tangent bundles TM and the normal bundle $T^\perp M$ are decomposed by

$$TM = \mathfrak{D}^\perp \oplus \mathfrak{D}^T \oplus \mathfrak{D}^\theta, \;\; T^\perp M = F\mathfrak{D}^\perp \oplus \omega\mathfrak{D}^\theta \oplus \mu$$

Then we choose a local orthonormal frame of TM is $\{e_1, ..., e_n\}$, such that the orthonormal frames of $\mathfrak{D}^\perp$, $\mathfrak{D}^T$ and $\mathfrak{D}^\theta$, respectively are given by

$$\mathfrak{D}^\perp = \mathrm{Span}\{\mathrm{e_1}, \cdots, \mathrm{e_r}\}, \;\; \mathfrak{D}^\mathrm{T} = \mathrm{Span}\;\{\mathrm{e_{r+1}} = \tilde{\mathrm{e}}_1 = \mathrm{F}\tilde{\mathrm{e}}_1, \cdots, \mathrm{e_{r+k}} = \tilde{\mathrm{e}}_\mathrm{k} = \mathrm{F}\tilde{\mathrm{e}}_\mathrm{k}, \mathrm{e_{r+k+1}} = \tilde{\mathrm{e}}_{\mathrm{k+1}} = -\mathrm{F}\tilde{\mathrm{e}}_{\mathrm{k+1}}, \cdots,$$
$$e_{r+p} = \tilde{e}_p = -F\tilde{e}_p\}, \;\; \mathfrak{D}^\theta = \mathrm{Span}\{\mathrm{e_{r+p+1}} = \hat{\mathrm{e}}_1 = \sec\theta\,\mathrm{T}\hat{\mathrm{e}}_1, \cdots, \mathrm{e_n} = \hat{\mathrm{e}}_\mathrm{q} = \sec\theta\mathrm{T}\hat{\mathrm{e}}_\mathrm{q}\}.$$

Then, the orthonormal frame fields of the normal subbundle of $F\mathfrak{D}^\perp$, $\omega\mathfrak{D}^\theta$ and μ, respectively are

$$F\mathfrak{D}^\perp = \mathrm{Span}\{\mathrm{e_{n+1}} = \mathrm{e}_1^* = \mathrm{Fe_1}, \cdots, \mathrm{e_{n+r}} = \mathrm{e_r^*} = \mathrm{Fe_r}\},$$
$$\omega\mathfrak{D}^\theta = \mathrm{Span}\{\mathrm{e_{n+r+1}} = \mathrm{e^*_{r+1}} = \csc\theta\omega\hat{\mathrm{e}}_1, \cdots, \mathrm{e_{n+r+q}} = \mathrm{e^*_{r+q}} = \csc\theta\omega\hat{\mathrm{e}}_\mathrm{q}\}$$
$$\mu = \mathrm{Span}\{\mathrm{e_{r+n+q+1}} = \mathrm{e^*_{r+q+1}}, \cdots, \mathrm{e_m} = \mathrm{e^*_{m-n-r-q}}\}.$$

Now, we are able to establish the main result of this section as follows.

Theorem 4.1. *Let $M = (M_\theta \times_{\sigma_1} M_\perp) \times_{\sigma_2} M_T$ be a $\mathfrak{D}^\theta \oplus \mathfrak{D}^\perp$ mixed geodesic sequential warped product submanifold in a locally product Riemannian manifold $\tilde{M}$ where M_θ, $M_\perp$ and M_T are proper slant, anti-invariant and invariant submanifolds of $\tilde{M}$, respectively. Then, the second fundamental form h satisfies*

$$||h||^2 \geq r\;\cot^2\theta\,\left\|\nabla^\theta \ln\sigma_1\right\|^2 + p\,\left\|\nabla^\perp \ln\sigma_2\right\|^2 + p\csc^2\theta(1-\cos\theta)^2\left\|\nabla^\theta \ln\sigma_2\right\|^2 \tag{11}$$

where $r = \dim(M_\perp)$, $p = \dim(M_T)$ and $\nabla^\theta \ln\sigma_i$ is the gradient of $\ln\sigma_i$ along M_θ , for $i = 1, 2$ and $\nabla^\perp \ln\sigma_2$ is the gradient of $\ln\sigma_2$ along $M_\perp$.

Moreover, the equality in (11) holds identically if and only if

(i) $M_\theta \times_{\sigma_1} M_\perp$ is totally geodesic in $\tilde{M}$ and M_T is totally umbilical in $\tilde{M}$.

(ii) M is also $\mathfrak{D}^\theta \oplus \mathfrak{D}^T$ and $\mathfrak{D}^\perp \oplus \mathfrak{D}^T$ mixed geodesic but not $\mathfrak{D}^T$ totally geodesic in $\tilde{M}$.

Proof. From (7), we have

$$||h||^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)) = \sum_{k=n+1}^{m}\sum_{i,j=1}^{n} g(h(e_i, e_j), e_k)^2.$$

since M is $\mathfrak{D}^\theta \oplus \mathfrak{D}^\perp$ mixed geodesic and using the frame fields of $\mathfrak{D}^T$, $\mathfrak{D}^\perp$, $\mathfrak{D}^\theta$, $F\mathfrak{D}^\perp$, $\omega\mathfrak{D}^\theta$ and μ , we derive

$$\begin{aligned}
\|h\|^2 =& \sum_{k=1}^{r}\sum_{i,j=1}^{q} g(h(\hat{e}_i,\hat{e}_j),Fe_k)^2 + \sum_{i,j,k=1}^{q} g(h(\hat{e}_i,\hat{e}_j),\omega\hat{e}_k)^2\csc^2\theta + \sum_{i,j=1}^{q}\sum_{k=r+n+q+1}^{m} g(h(\hat{e}_i,\hat{e}_j),e_k)^2 \\
&+ \sum_{i,j,k=1}^{r} g(h(e_i,e_j),Fe_k)^2 + \sum_{i,j=1}^{r}\sum_{k=1}^{q} g(h(e_i,e_j),\omega\hat{e}_k)^2\csc^2\theta + \sum_{i,j=1}^{q}\sum_{k=r+n+q+1}^{m} g(h(e_i,e_j),e_k)^2 \\
&+ \sum_{k=1}^{r}\sum_{i,j=1}^{p} g(h(\tilde{e}_i,\tilde{e}_j),Fe_k)^2 + \sum_{i,j=1}^{p}\sum_{k=1}^{q} g(h(\tilde{e}_i,\tilde{e}_j),\omega\hat{e}_k)^2\csc^2\theta + \sum_{i,j=1}^{q}\sum_{k=r+n+q+1}^{m} g(h(\tilde{e}_i,\tilde{e}_j),e_k)^2 \\
&+ 2\sum_{k=1}^{r}\sum_{i=1}^{p}\sum_{j=1}^{q} g(h(\tilde{e}_i,\hat{e}_j),Fe_k)^2 + 2\sum_{i=1}^{p}\sum_{j,k=1}^{q} g(h(\tilde{e}_i,\hat{e}_j),\omega\hat{e}_k)^2\csc^2\theta + 2\sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{k=r+n+q+1}^{m} g(h(\tilde{e}_i,\hat{e}_j),e_k)^2 \\
&+ 2\sum_{j,k=1}^{r}\sum_{i=1}^{p} g(h(\tilde{e}_i,e_j),Fe_k)^2 + 2\sum_{j=1}^{r}\sum_{i=1}^{p}\sum_{k=1}^{q} g(h(\tilde{e}_i,e_j),\omega\hat{e}_k)^2\csc^2\theta + 2\sum_{j=1}^{r}\sum_{i=1}^{p}\sum_{k=r+n+q+1}^{m} g(h(\tilde{e}_i,e_j),e_k)^2.
\end{aligned} \tag{12}$$

By leaving the μ-components terms in the right hand side and using Lemma 3.6, Lemma 3.7 and Lemma 3.8 with the hypothesis of theorem, we obtain

$$\begin{aligned}
\|h\|^2 \geq& \csc^2\theta\sum_{i,j=1}^{r}\sum_{k=1}^{q}\Big[T\hat{e}_k(\ln\sigma_1)g(e_i,e_j)\Big]^2 + \sum_{k=1}^{r}\sum_{i,j=1}^{p}\Big[-e_k(\ln\sigma_2)g(\tilde{e}_i,F\tilde{e}_j)\Big]^2 \\
&+ \csc^2\theta\sum_{i,j=1}^{p}\sum_{k=1}^{q}\Big[-\hat{e}_k(\ln\sigma_2)g(\tilde{e}_i,F\tilde{e}_j) + T\hat{e}_k(\ln\sigma_2)g(\tilde{e}_i,\tilde{e}_j)\Big]^2.
\end{aligned}$$

Then, from the frame fields of $\mathfrak{D}^\theta$ we have $\hat{e}_k = \sec\theta T\hat{e}_k$, for $k = 1,\cdots,q$, then $T\hat{e}_k = \cos\theta\hat{e}_k$ the above expression takes the form

$$\begin{aligned}
\|h\|^2 \geq& \csc^2\theta\sum_{i,j=1}^{r}\sum_{k=1}^{q}\Big[T\hat{e}_k(\ln\sigma_1)g(e_i,e_j)\Big]^2 + \sum_{k=1}^{r}\sum_{i,j=1}^{p}\Big[-e_k(\ln\sigma_2)g(\tilde{e}_i,F\tilde{e}_j)\Big]^2 \\
&+ \csc^2\theta\sum_{i,j=1}^{p}\sum_{k=1}^{q}\Big[-\hat{e}_k(\ln\sigma_2)g(\tilde{e}_i,F\tilde{e}_j) + T\hat{e}_k(\ln\sigma_2)g(\tilde{e}_i,\tilde{e}_j)\Big]^2
\end{aligned}$$

and using gradient definition, we arrive at

$$\|h\|^2 \geq r\cot^2\theta\|\nabla^\theta\ln\sigma_1\|^2 + p\|\nabla^\perp\ln\sigma_2\|^2 + p\csc^2\theta\,\|\nabla^\theta\ln\sigma_2\|^2(1-\cos\theta)^2.$$

which is inequality (11). For the equality case, from the leaving μ-components, we get

$$h(TM,TM)\perp\mu. \tag{13}$$

From Lemma 3.6 (i) and leaving the second term in (11), we get

$$h(\mathfrak{D}^\theta,\mathfrak{D}^\theta)\perp F\mathfrak{D}^\perp,\;\; h(\mathfrak{D}^\theta,\mathfrak{D}^\theta)\perp\omega\mathfrak{D}^\theta. \tag{14}$$

Then from (13) and (14), we obtain

$$h(\mathfrak{D}^\theta,\mathfrak{D}^\theta) = \{0\}. \tag{15}$$

Also, from Lemma 3.7 (i) and (iii), we obtain

$$h(\mathfrak{D}^{\theta}, \mathfrak{D}^{T}) \perp F\mathfrak{D}^{\perp}, \quad h(\mathfrak{D}^{\theta}, \mathfrak{D}^{T}) \perp \omega\mathfrak{D}^{\theta}. \tag{16}$$

Thus, from (13) and (16), we have

$$h(\mathfrak{D}^{\theta}, \mathfrak{D}^{T}) = \{0\}. \tag{17}$$

From Lemma 3.6 (iii) and Lemma 3.7 (ii), we derive

$$h(\mathfrak{D}^{\perp}, \mathfrak{D}^{T}) = \{0\}. \tag{18}$$

Since M is $\mathfrak{D}^{\theta} \oplus \mathfrak{D}^{\perp}$ mixed geodesic, we get

$$h(\mathfrak{D}^{\theta}, \mathfrak{D}^{\perp}) = \{0\}. \tag{19}$$

Also, from Theorem 3.11, we have

$$h(\mathfrak{D}^{T}, \mathfrak{D}^{T}) \neq \{0\}. \tag{20}$$

From ([7],[10]), we have $M_\theta \times_{\sigma_1} M_\perp$ is totally geodesic in M, then M_θ is also totally geodesic in M [7], using this fact with (15), (17) and (19), we conclude that $M_\theta \times_{\sigma_1} M_\perp$ is totally geodesic in $\tilde{M}$. Also, since M_T is totally umbilical in M with Theorem 3.11 and (13), (17), (18) we observe that M_T is a totally umbilical in $\tilde{M}$, which is (i). On the other hand, all conditions together with (17) and (18) imply that M is $\mathfrak{D}^{\theta} \oplus \mathfrak{D}^{T}$ and $\mathfrak{D}^{\perp} \oplus \mathfrak{D}^{T}$ mixed geodesic and M is $\mathfrak{D}^{\theta}$ totally geodesic but from (20) M is not $\mathfrak{D}^{T}$, which is (ii). The converse of equality follows directly from the assumptions. Hence, the proof of the theorem is complete. □

5. Some Applications of the main result

W have the following consequences of Theorem 4.1:

If σ_2 is constant on $M_\perp$, then M becomes bi-warped product, which has been discussed in [1]. In this case Theorem 4.1 provide the following result:

Theorem 5.1. *Let $M = M_\theta \times_{\sigma_1} M_\perp \times_{\sigma_2} M_T$ be a bi-warped product submanifold of a locally product Riemannian manifold $\tilde{M}$ such that M is $\mathfrak{D}^{\perp} \oplus \mathfrak{D}^{\theta}$ -mixed geodesic. Then*

(i) The second fundamental form h and the warping functions σ_1 and σ_2 of M satisfy

$$\|h\|^2 \geq r\cot^2\theta\,\|\nabla^{\theta}(\ln\sigma_1)\|^2 + p\csc^2\theta\,(1-\cos\theta)^2\,\|\nabla^{\theta}(\ln\sigma_2)\|^2 \tag{21}$$

where $r = \dim M_\perp$, $p = \dim M_T$ and $\nabla^{\theta}(\ln\sigma_i)$ is gradient of $\ln\sigma_i$, $i = 1, 2$.

(ii) If equality sign in (i) holds identically, then M_θ is a totally geodesic submanifold of $\tilde{M}$ and $M_\perp$ and M_T are totally umbilical submanifolds of $\tilde{M}$ with their mean curvature vectors $-\nabla^{\theta}(\ln\sigma_1)$ and $-\nabla^{\theta}(\ln\sigma_2)$, respectively; M is also a $\mathfrak{D} \oplus \mathfrak{D}^{\perp}$-mixed geodesic submanifold of $\tilde{M}$.

Therefore, Theorem 4.1 extended to Theorem 6 in [1] which is Theorem 5.1.

If $M_\theta = \{0\}$, then $M = M_\perp \times_{\sigma_2} M_T$ is a warped product semi-invariant submanifold and studied in [24] and [5]. Hence, Theorem 4.1 implies that

Theorem 5.2. *Let $M = M_\perp \times_{\sigma_2} M_T$ be a semi-invariant warped product submanifold of a locally product Riemannian manifold $\tilde{M}$. Then, the squared norm of the second fundamental form of M satisfies*

$$\|h\|^2 \geq p\left\|\nabla^{\perp}(\ln\sigma_2)\right\|^2, \tag{22}$$

where $\nabla^{\perp}(\ln\sigma_2)$ is the gradient of $\ln\sigma_2$ and p is the dimension of M_T. If the equality sign holds identically, then, $M_\perp$ is totally geodesic submanifold of $\tilde{M}$ and M is mixed geodesic. Moreover, M is never a minimal submanifold of $\tilde{M}$.

Hence, Theorem 5.2 is a special case of Theorem 4.1 which is Theorem 4.2 in [24] and Theorem 4.1 in [5].

If $\dim M_\perp = 0$, then M changes to a warped product semi-slant submanifold of the form $M = M_\theta \times_{\sigma_2} M_T$, studied in [4], [25] and [2]. In this case Theorem 4.1 has the following form:

Theorem 5.3. *Let $M = M_\theta \times_{\sigma_2} M_T$ be a proper warped product semi-slant submanifold of a locally product Riemannian manifold $\tilde{M}$, then the squared norm of the second fundamental form of the warped product immersion satisfies*

$$\|h\|^2 \geq p(\csc\theta - \cot\theta)^2 \|\nabla^\theta \ln\sigma_2\|^2,$$

where $p = \dim M_T$ and $\nabla^\theta \ln\sigma_2$ is gradient of the function $\ln\sigma_2$ along M_θ. If the equality sign in (i) holds identically, then M_θ is totally geodesic in $\tilde{M}$ and M_T is a totally umbilical submanifold of $\tilde{M}$. Furthermore, $M_\theta \times_{\sigma_2} M_T$ is a mixed geodesic submanifold of $\tilde{M}$.

The above inequality shows that Theorem 4.1 generalizes the main result Theorem 3.1 of [2].

If $\dim M_T = 0$, then the sequential warped product changes to warped product pseudo-slant submanifold of the form $M = M_\theta \times_{\sigma_1} M_\perp$, which we studied in [30], in this case Theorem 4.1 gives:

Theorem 5.4. *[30] Let $M = M_\theta \times_{\sigma_1} M_\perp$ be a mixed geodesic warped product pseudo-slant submanifold of a locally product Riemannian manifold $\tilde{M}$. Then, we have:*
The squared norm of the second fundamental form h of M satisfies

$$\|h\|^2 \geq r\cot^2\theta \|\nabla^\theta \ln\sigma_1\|^2,$$

where $r = \dim M_\perp$ and $\nabla^\theta \ln\sigma_1$ is the gradient of $\ln\sigma_1$ along M_θ.

As a result, Theorem 5.1 of [30] is a special case of Theorem 4.1.

Acknowledgement. The authors are thankful to the editors and the anonymous referees for their valuable suggestions and constructive comments for improving the quality and the presentation of this paper in the present form.

References

[1] A. AL-Jedani, S. Uddin, A. Alghanemi, I. Mihai, *Bi-warped products and applications in locally product Riemannian manifolds*, Journal of Geometry and Physics. **144** (2019), 358–369.
[2] F. R. Al-Solamy, S. Uddin, *An inequality for warped product submanifolds of a locally product Riemannian manifold*, Hacet. J. Math. Stat. **48** (2019), no. 2, 351–358.
[3] M. Atceken, *Geometry of warped product semi-invariant submanifolds of a locally Riemannian product manifolds*, Serdica Math. J. **35** (2009), 273–289.
[4] M. Atceken, *Warped product semi-slant submanifolds in locally Riemannian product manifolds*, Bull. Austral. Math. Soc. **77** (2008), 177–186.
[5] M. Atceken, *Warped product semi-invariant submanifolds in locally decomposable Riemannian manifolds*, Hacet. J. Math. Stat.**40** (2011), 401–407.
[6] A. Bejancu, *Semi-invariant submanifolds of locally Riemannian product manifolds*, Ann. Univ. Timisoara S. Math. XXII (1984), 3–11.
[7] R.L. Bishop and B. O'Neill, Manifolds of negative curvature, Trans. Amer. Math. Soc. **145** (1969), 1–49.
[8] A. Carriazo, *Bi-slant immersions*, in: Proceedings ICRAMS, (J. C. Misra, S. B. Sinha, Eds.), Narosa Publishing House, (2000), 88-97.
[9] B.-Y. Chen, *Geometry of slant submanifolds*, Katholieke Universiteit Leuven, (1990).
[10] B.-Y. Chen, *Geometry of warped product CR-submanifolds in Kaehler manifolds I*, Monatsh. Math. **133** (2001), 177-195.
[11] B.-Y. Chen, *Pseudo-Riemannian Geometry δ-Invariants and Applications*, World Scientific, Hackensack, NJ, (2011).
[12] B.-Y. Chen, *Differential Geometry of Warped Product Manifolds and Submanifolds*, World Scientific, Hackensack, NJ, (2017).
[13] B.-Y. Chen and F. Dillen, *Optimal inequalities for multiply warped product submanifolds*, Int. Electron. J. Geom. **1** (1) (2018), 1–11.
[14] B.-Y. Chen, S. Uddin, A. Alghanemi, A. Al-Jedani and I. Mihai, *Geometry of bi-warped product submanifolds in Sasakian and cosymplectic manifolds*, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. **116**, 82 (2022). https://doi.org/10.1007/s13398-022-01225-5
[15] U. C. De, S. Shenawy, B. Unal, *Sequential warped products: curvature and conformal vector fields*, Filomat **33** (13) (2019), 4071–4083.
[16] F. Dobarro, B. Unal, *Curvature of multiply warped products*, Journal of Geometry and Physics, , no. 1, **51**(2005), 75–106.
[17] X. Liu , F. M. Shao , *Skew semi-invariant submanifolds of a locally product manifold*. Portugaliae Math. **56** (1999), 319– 327.

[18] H. Li, X. Li, *Semi-slant submanifolds of locally product manifolds*, Georgian Math. J. **12** (2005), 273–282.
[19] K. Matsumoto, *On submanifolds of locally product Riemannian manifolds*, TRU Math. **18** (1982) 145–157.
[20] S. Nolker, *Isometric immersions of warped products*. Differ. Geom. Appl.**6** (1) (1996), 1–30.
[21] N. Papaghiuc, *Semi-slant submanifolds of a Kaehlerian manifold*, Ann. St. Univ. Iasi 9 (1994), 55–61.
[22] B. Sahin, M. Atceken, *Semi-invariant submanifolds of Riemannian product manifolds*, Balkan J. Geom. Appl. **8** (2003) 91–100.
[23] B. Sahin, *Slant submanifolds of an almost product Riemannian manifold*, J. Korean Math. Soc. **43** (2006), 717–732.
[24] B. Sahin, *Warped product Semi-invariant submanifolds of a locally product Riemannian manifold*, Bull. Math. Soc. Sci. Roumanie, **49** (2006), 383–394.
[25] B. Sahin, *Warped product semi-slant submanifolds of a locally product Riemannian manifold*, Studia Sci. Math. Hungar. **46** (2009), 169–184.
[26] B. Sahin , *Sequential warped product submanifolds having holomorphic, totally real and pointwise slant factors*, Period. Math. Hunger., **85** (2022), 128–139. https://doi.org/10.1007/s10998-021-00422-w
[27] S. Shenawy, *A note on sequential warped product manifolds*. arXiv:1506.06056v1 [math.DG] (2015).
[28] H. M. Tastan, F. Ozdemir, *The geometry of hemi-slant submanifolds of a locally product Riemannian manifold*. Turk. J. Math. **39** (2015) , 268-284.
[29] H. M. Tastan, *Warped product skew semi-invariant submanifolds of order* 1 *of a locally product Riemannian manifold*, Turk J Math, **39** (2015), 453–466.
[30] S. Uddin, A. Ali, N. M. Al-Asmari and W. A. Othman, *Warped product pseudo-slant submanifolds in a locally product Riemannian manifold*, Differential Geometry-Dynamical System **18** (2016), 147–158.
[31] S. Uddin, B.-Y. Chen, A. AL-Jedani, A. Alghanemi, *Bi-warped product submanifolds of nearly Kaehler manifolds*, Bulletin of the Malaysian Mathematical Sciences Society (2019), 1-14, doi.org/10.1007/s40840-019-00782.
[32] S. Uddin, A. H. Alkhaldi, *Bi-warped product submanifolds of Kenmotsu manifolds and their applications*, Int. J. Geom. Methods Mod. Phys. **16(1)** (2019). https://doi.org/10.1142/S0219887819500014
[33] S. Uddin, A. Mihai, I. Mihai, A. AL-Jedani, *Geometry of bi-warped product submanifolds of locally product Riemannian manifolds*, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. **114**, 42 (2020). https://doi.org/10.1007/s13398-019-00766-6
[34] S. Uddin, F. R. Al-Solamy, N. M. Al-Houiti, *Generic warped products in locally product Riemannian manifolds*, J. Geom. Phys. **146** (2019), Art. 103515.
[35] S. Uddin, F. Alghamdi, F. R. Al-Solamy, *Geometry of warped product pointwise semi-slant submanifolds of locally product Riemannian manifolds*, J. Geom. Phys. **152** (2020), Art. 103658.
[36] B. Unal, *Multiply warped products*, Journal of Geometry and Physics, **34**(2001), no. 3-4, 287-301.
[37] K. Yano, M. Kon, *Structures on Manifolds*, Series in Pure Mathematics. World Scientific Publishing Co., Singapore (1984)