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Characterization of sequential warped product gradient Ricci-Bourguignon soliton

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Abstract. In this paper, we study characterization of sequential warped product gradient Ricci-Bourguignon soliton. We derive applications of some vector fields like torse-forming vector field, torqued vector field, conformal vector field on Ricci-Bourguignon soliton. We show that for torse-forming vector field, a Ricci-Bourguignon soliton becomes an almost quasi-Einstein manifold. Next, we obtain the inheritance properties of the Einstein-like sequential warped product gradient Ricci-Bourguignon almost soliton of class type $\mathbb{P}, \mathbb{A}, \mathbb{B}$. We prove that, when the manifold is complete, the potential function depends only on M_1 and M_3 must be an Einstein manifold. We also present for a gradient Ricci-Bourguignon soliton sequential warped product, the warping functions are constants under some certain conditions.

1. Introduction

The concept of warped products was introduced by O'Neill and Bishop to build Riemannian manifolds with negative sectional curvature [4]. The study of warped products provides some important insights in differential geometry as well as in the field of physics, since warped product space-time models are used to obtain exact solutions to Einstein's equation [1, 2, 17]. Let (B, g_B) and (F, g_F) be two Riemannian manifolds with dim $B = m > 0$, dim$F = k > 0$ and $f : B \longrightarrow (0, \infty)$, $f \in C^\infty(B)$. Consider the product manifold $B \times F$ with its projections $\pi : B \times F \to B$ and $\sigma : B \times F \to F$. The warped product $B \times_f F$ is the manifold $B \times F$ with the Riemannian structure such that $\|X\|^2 = \|\pi^*(X)\|^2 + f^2(\pi(p))\|\sigma^*(X)\|^2$, for any vector field X on M. Thus we have $g_M = g_B + f^2 g_F$ holds on M. Here B is called the base of M and F the fiber. The function f is called the warping function of the warped product [17].
In 2015, S. Shenawy introduced a new class of warped product manifolds, namely sequential warped products where the base factor of the warped product is itself a new warped product manifold [20]. Sequential warped products can be considered as a generalization of singly warped products. Let (M_i, g_j) $i = 1, 2, 3$ be three Riemannian manifolds. Let $f : M_1 \to (0, \infty)$ and $h : M_1 \times M_2 \to (0, \infty)$ be two smooth positive

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functions on M_1 and $M_1 \times M_2$ respectively. Then the sequential warped product manifold, denoted by $(M_1 \times_f M_2) \times_h M_3$, is the triple product manifold $(M_1 \times M_2) \times M_3$, with the metric tensor $\bar{g} = (g_1 \oplus f^2 g_2) \oplus h^2 g_3$. The functions f and h are called warping functions.
Let (M_i, g_i), $i = 1, 2$ be two n_i-dimensional Riemannian manifolds. Let $\bar{f} : M_1 \times M_2 \to (0, \infty)$ and $f : M_1 \to (0, \infty)$ be two smooth positive functions. Then $(n_1 + n_2 + 1)$-dimensional product manifold $I \times_h (M_1 \times_f M_2)$, with the metric tensor $\bar{g} = -h^2 dt^2 \oplus (g_1 \oplus f^2 g_2)$ is a standard static space-time, where I is an open, connected subinterval of $\mathbb{R}$, and dt^2 is the Euclidean metric tensor on I. Also $(n_1 + n_2 + 1)$-dimensional product manifold $I_h \times (M_1 \times_f M_2)$, with the metric tensor $\bar{g} = -dt^2 \oplus h^2(g_1 \oplus f^2 g_2)$ is a generalized Robertson-Walker space-time, where I is an open, connected subinterval of $\mathbb{R}$, $h : I \to (0, \infty)$ and $f : M_1 \to (0, \infty)$ are smooth functions, and dt^2 is the Euclidean metric tensor on I.

Ricci solitons, which are important geometric partial differential equation highlighted in many fields of theoretical research and practical applications, are a natural generalization of the Einstein manifolds. In 1979, the idea of the Ricci-Bourguignon flow (or RB flow) as a generalization of Ricci flow was developed by Jean-Pierre Bourguignon [5] using some unpublished work of Lichnerowicz and a paper of Aubin [3]. The Ricci-Bourguignon flow is an evolution equation for metrics on a Riemannian manifold given by

$$\frac{\partial}{\partial t} g(t) = -2(Ric - R\rho g), \tag{1}$$

where $\rho \in \mathbb{R}$ is a constant, Ric is the Ricci curvature and R is the scalar curvature of the Riemannian metric g. It should be observed that the right hand side of the evolution equation (1) is of special interest for special values of ρ in particular [10].
1. $\rho = \frac{1}{2}$, the Einstein tensor $Ric - \frac{R}{2} g$ (Einstein soliton).
2. $\rho = \frac{1}{n}$, the traceless Ricci tensor $Ric - \frac{R}{2} g$.
3. $\rho = \frac{1}{2(n-1)}$, the Schouten tensor $Ric - \frac{R}{2} g$ (Schouten soliton).
4. $\rho = 0$, the Einstein tensor Ric (Ricci soliton).

In [10], S. Dwivedi introduced the notions of Ricci-Bourguignon and Ricci-Bourguignon almost solitons, which generalize the Ricci and almost Ricci solitons, respectively. In the paper, the author explained integral formulas for compact gradient Ricci-Bourguignon solitons and compact gradient Ricci-Bourguignon almost solitons. Using the integral formula he observed that a compact gradient Ricci-Bourguignon almost soliton is isometric to an Euclidean sphere under a certain condition.
A Riemannian manifold (M, g) is called a Ricci-Bourguignon soliton (or RB soliton) if there exists a smooth vector field V satisfying the following equation

$$Ric + \frac{1}{2} \pounds_V g = (\mu + R\rho) g, \tag{2}$$

for some real constant μ and the Lie derivative $\pounds_V g$. Ricci-Bourguignon soliton appears as a self-similar solution to Ricci-Bourguignon flow and often arises as a limit of dilation of singularities in the Ricci-Bourguignon flow [6]. The Ricci-Bourguignon soliton is said to be shrinking, steady or expanding if μ is positive, zero or negative, respectively.

If the vector field V is the gradient of a smooth function f, then (M, g) is called a gradient Ricci-Bourguignon soliton and equation (2) becomes

$$\nabla\nabla f + Ric = (\mu + R\rho) g. \tag{3}$$

A Riemannian manifold (M, g) is called a Ricci-Bourguignon almost soliton (or RB almost soliton) if there exists a smooth vector field V and a soliton function $\mu : M \to \mathbb{R}$ such that the Ricci tensor Ric satisfies the following equation

$$Ric + \frac{1}{2} \pounds_V g = (\mu + R\rho) g. \tag{4}$$

The Ricci-Bourguignon almost soliton is called shrinking, steady or expanding if μ is positive, zero or negative, respectively. A Ricci-Bourguignon almost soliton is said to be a gradient Ricci-Bourguignon almost soliton if $V = \nabla f$ for some smooth function f on M.

Motivated by the above studies, the first aim of this paper is to give a classification for Ricci-Bourguignon solitons with torse-forming vector field, torqued vector field, conformal vector field. Secondly, we obtain some geometric properties of the Einstein-like sequential warped product gradient Ricci-Bourguignon almost soliton of class type $\mathbb{P}, \mathbb{A}, \mathbb{B}$. Finally We also characterize the warping functions of sequential warped product with gradient Ricci-Bourguignon soliton.

2. Some vector fields on Ricci-Bourguignon soliton

In this section we investigate applications of some vector fields like torse-forming vector field, torqued vector field, conformal vector field on Ricci-Bourguignon soliton. We begin by defining certain terms and developing certain propositions that will lay the groundwork for the exposition that follows.

Definition 2.1 : A vector field τ on a Riemannian or pseudo Riemannian manifold M is called torse-forming if for any vectors $X \in \chi(M)$ it satisfies [16, 21]

$$\nabla_X \tau = \phi X + \psi(X)\tau, \tag{5}$$

where ϕ is a function, ψ is a 1-form, ∇ is Levi-Civita connection on M.
The vector field τ is called concircular [22] if the 1-form ψ vanishes identically in the above equation.

Definition 2.2: On a Riemannian or pseudo-Riemannian manifold a nowhere zero vector field τ is called a torqued vector field if it satisfies [7]

$$\nabla_X \tau = \varphi X + \alpha(X)\tau, \qquad \alpha(\tau) = 0. \tag{6}$$

The function φ is called the torqued function and the 1-form α is called the torqued form of τ.

Definition 2.3 : A vector field τ on a (pseudo) Riemannian manifold M with metric g called a conformal vector field [9] if

$$\pounds_\tau g = 2\phi g, \tag{7}$$

for a smooth function $\phi \in C^\infty(M)$. In particular, τ is called conformal killing if $\phi = 0$.

Definition 2.4 : A pseudo Riemannian manifold (M, g) is called an almost quasi-Einstein manifold if

$$S = ag + b(\beta \otimes \gamma + \gamma \otimes \beta), \tag{8}$$

where a, b are functions and β, γ are 1-forms.

Proposition 2.5 : If the smooth vector field of a Ricci-Bourguignon soliton is a torse-forming τ, then (M, g) is an almost quasi-Einstein manifold.

Proof: Let (M, g, ξ, λ) be a Ricci-Bourguignon soliton where the smooth vector field ξ is a torse-forming τ. From the definition of Lie-derivative we have

$$(\pounds_\tau g)(X, Y) = g(\nabla_X \tau, Y) + g(X, \nabla_Y \tau). \tag{9}$$

Using (6) in (9) we obtain

$$\begin{aligned}(\pounds_\tau g)(X, Y) &= g(\phi X + \alpha(X)\tau, Y) + g(X, \phi Y + \alpha(Y)\tau) \\ &= g(\phi X, Y) + \alpha(X)g(\tau, Y) + g(\phi Y, X) + \alpha(Y)g(X, \tau) \\ &= 2\phi g(X, Y) + \alpha(X)g(\tau, Y) + \alpha(Y)g(\tau, X),\end{aligned} \tag{10}$$

for any vector fields X, Y tangent to M.
Combining (2) and (10) we have

$$2(\mu + \rho R)g(X, Y) - S(X, Y) = 2\phi g(X, Y) + \alpha(X)g(\tau, Y) + \alpha(Y)g(\tau, X),$$

which gives

$$S(X, Y) = [(\mu + \rho R) - \phi]g(X, Y) - \frac{1}{2}\alpha(X)g(\tau, Y) - \frac{1}{2}\alpha(Y)g(\tau, X). \tag{11}$$

If we denote the dual 1-form of τ by γ, then from (11) we get

$$S = [(\mu + \rho R) - \phi]]g - \frac{1}{2}[\alpha \otimes \gamma + \gamma \otimes \alpha]. \tag{12}$$

Hence (M, g) is an almost quasi-Einstein manifold.

Note 2.6 : If the smooth vector field of a Ricci-Bourguignon soliton is a torqued vector field τ, then (M, g) is an almost quasi-Einstein manifold.

Corollary 2.7 : If the smooth vector field of a Ricci-Bourguignon soliton is a concircular vector field τ, then (M, g) becomes an Einstein manifold.

Corollary 2.8 : If the smooth vector field of a Ricci-Bourguignon soliton is a conformal vector field τ, then (M, g) becomes an Einstein manifold.

3. Einstein-Like Sequential warped product and gradient Ricci-Bourguignon soliton

In [8] and [12], many authors have explored the geometric properties of sequential warped product manifolds which are the natural generalizations of singly warped products. The existence of Einstein sequential warped product manifolds has been studied in [18]. In [14] F.Karaca, C. Özg*ü*r have studied on quasi-Einstein sequential warped product manifolds. In this section, we will study the inheritance properties of the Einstein-like sequential warped product manifolds of class type $\mathbb{P}$, $\mathbb{A}$,$\mathbb{B}$ [19] inspired by the above studies of sequential warped product manifolds.

Now we consider the following proposition 3.1 from [20] and proposition 3.2 from [11] , which will be helpful to prove our main results of this section.

Proposition 3.1 : Let $\bar{M} = (M_1 \times_f M_2) \times_h M_3$ be a sequential warped product with metric $g = (g_1 \oplus f^2 g_2) \oplus h^2 g_3$ and also let $X_i, Y_i, Z_i \in \chi(M_i)$. Then
(i) $\bar{Ric}(X_1, Y_1) = Ric^1(X_1, Y_1) - \frac{n_2}{f} H_1^f(X_1, Y_1) - \frac{n_3}{h} H^h(X_1, Y_1)$,
(ii)) $\bar{Ric}(X_2, Y_2) = Ric^2(X_2, Y_2) - f^2 g_2(X_2, Y_2) f^* - \frac{n_3}{h} H^h(X_2, Y_2)$,
(iii) $\bar{Ric}(X_3, Y_3) = Ric^3(X_3, Y_3) - h^2 g_3(X_3, Y_3) h^*$,
(iv) $\bar{Ric}(X_i, Y_j) = 0, i \neq j$,
where $n_1 = dimM_1$, $n_2 = dimM_2$,$n_3 = dimM_3$,$f^* = \frac{\Delta^1 f}{f} + (n_2 - 1)\frac{|\nabla^1 f|^2}{f^2}$ and $h^* = \frac{\Delta h}{h} + (n_1 + n_2 - 1)\frac{|\nabla h|^2}{h^2}$.

Proposition 3.2 : Let $M = (M_1 \times_f M_2) \times_h M_3$ be a sequential warped product endowed with the metric $g = (g_1 \oplus f^2 g_2) \oplus h^2 g_3$. If (M, g) is a gradient Ricci-Bourguignon soliton, then the potential function ψ depends on (M_1, g_1).

Class $\mathbb{A}$: A sequential warped product is called Einstein-like sequential warped product of class $\mathbb{A}$ if the Ricci tensor satisfies

$$(\nabla_X Ric)(Y, Z) + (\nabla_Y Ric)(X, Z) + (\nabla_Z Ric)(X, Y) = 0,$$

for any vector fields $X, Y, Z \in \chi(M)$. The above equation is equivalent to

$$(\nabla_X Ric)(X, X) = 0, \tag{13}$$

for any vector field $X \in \chi(M)$ and and the Ricci tensor is also called cyclic parallel.

Class $\mathbb{B}$: A sequential warped product is called Einstein-like sequential warped product of class $\mathbb{B}$ if the Ricci tensor satisfies

$$(\nabla_X Ric)(Y, Z) - (\nabla_Y Ric)(X, Z) = 0,$$

for any vector fields $X, Y, Z \in \chi(M)$. The Ricci tensor is called Codazzi tensor.

Class $\mathbb{P}$: A sequential warped product is called Einstein-like sequential warped product of class $\mathbb{P}$ if the Ricci tensor satisfies

$$(\nabla_X Ric)(Y, Z) = 0,$$

for any vector fields $X, Y, Z \in \chi(M)$.Then M is called Ricci symmetric.

Theorem 3.3 : In a sequential warped product manifold $M = (M_1 \times_f M_2) \times_h M_3$ where M is of class type $\mathbb{A}$. Then (a) (M_1, g_1) is an Einstein-like manifold of class $\mathbb{A}$ iff $n_2 h(\nabla^1_{X_1} H^f)(X_1, X_1) + n_3 f(\nabla^1_{X_1} H^h)(X_1, X_1) = h n_2 X_1(\ln f) H^f(X_1, X_1) + n_3 f X_1(\ln h) H^h(X_1, X_1)$.
(b) (M_2, g_2) is an Einstein-like manifold of class $\mathbb{A}$ iff $(\nabla^2_{X_2} H^h)(X_2, X_2) = X_2(\ln h) H^h(X_2, X_2)$.

Proof. Let $M = (M_1 \times_f M_2) \times_h M_3$ be a sequential warped product of class $\mathbb{A}$. So

$$0 = (\nabla_X Ric)(X, X). \tag{14}$$

Taking $X = X_1$ and using proposition 3.1 (i) we obtain

$$\begin{aligned} 0 &= X_1[Ric^1(X_1, X_1) - \frac{n_2}{f} H^f(X_1, X_1) - \frac{n_3}{h} H^h(X_1, X_1) \\ &- 2[Ric^1(\nabla^1_{X_1} X_1, X_1) + \frac{2n_2}{f} H^f(\nabla^1_{X_1} X_1, X_1) + \frac{2n_3}{h} H^h(\nabla^1_{X_1} X_1, X_1). \end{aligned}$$

This implies

$$
\begin{aligned}
0 &= (\nabla^1_{X_1} Ric^1)(X_1, X_1) + \frac{n_2(X_1 f)}{f^2} H^f(X_1, X_1) + \frac{n_3(X_1 h)}{h^2} H^h(X_1, X_1) \\
&- \frac{n_2}{f}(\nabla^1_{X_1} H^f)(X_1, X_1) - \frac{n_3}{h}(\nabla^1_{X_1} H^h)(X_1, X_1).
\end{aligned} \tag{15}
$$

Then we derive

$$
\frac{n_2(X_1 f)}{f^2} H^f(X_1, X_1) + \frac{n_3(X_1 h)}{h^2} H^h(X_1, X_1) = \frac{n_2}{f}(\nabla^1_{X_1} H^f)(X_1, X_1) + \frac{n_3}{h}(\nabla^1_{X_1} H^h)(X_1, X_1).
$$

Hence we get the result.

For part (b), taking $X = X_2$ in the equation (14) and using proposition 3.1 (ii) we obtain

$$
\begin{aligned}
0 &= X_2[Ric^2(X_2, X_2) - f^* g_2(X_2, X_2) - \frac{n_3}{h} H^h(X_2, X_2) \\
&- 2[Ric^2(\nabla^2_{X_2} X_2, X_2) - f^* g_2(\nabla^2_{X_2} X_2, X_2) - \frac{n_3}{h} H^h(\nabla^2_{X_2} X_2, X_2),
\end{aligned} \tag{16}
$$

where $f^* = \frac{\Delta^1 f}{f} + (n_2 - 1)\frac{|\nabla^1 f|^2}{f^2}$.
Thus we get

$$
\begin{aligned}
0 &= (\nabla^2_{X_2} Ric^2)(X_2, X_2) + \frac{n_3(X_2 h)}{h^2} H^h(X_2, X_2) \\
&- \frac{n_3}{h}(\nabla^2_{X_2} H^h)(X_2, X_2).
\end{aligned} \tag{17}
$$

Hence we get the result.

Proposition 3.4 : Let $M = (M_1 \times_f M_2) \times_h M_3$ be a sequential warped product gradient Ricci-Bourguignon almost soliton with potential function ψ. If (M_i, g_i), $i = 1, 2$ is of class type $\mathbb{A}$. Then $(\nabla^i_{X_i} H^\psi)(X_i, X_i) = (X_i\mu) g_i(X_i, X_i)$.
Proof : Let $M = (M_1 \times_f M_2) \times_h M_3$ be a sequential warped product gradient Ricci-Bourguignon almost soliton of class $\mathbb{A}$. Then we obtain

$$
0 = (\nabla^i_{X_i} Ric^i)(X_i, X_i).
$$

Hence we have

$$
\begin{aligned}
0 &= X_1[(\mu + \rho R) g_i(X_i, X_i) - (H^\psi)(X_i, X_i)] \\
&- 2[(\mu + \rho R) g_i(\nabla^i_{X_i} X_i, X_i) - (H^\psi)(\nabla^i_{X_i} X_i, X_i)].
\end{aligned}
$$

Thus we derive

$$
0 = (\nabla^i_{X_i} H^\psi)(X_i, X_i) - (X_i \mu) g_i(X_i, X_i).
$$

This completes the proof.

Theorem 3.5 : In a sequential warped product manifold $M = (M_1 \times_f M_2) \times_h M_3$ where M is of class type $\mathbb{B}$. Then
(a) (M_1, g_1) is an Einstein-like manifold of class $\mathbb{A}$ iff

$$
\begin{aligned}
n_2 h[(\nabla^1_{X_1} H^f)(Y_1, Z_1) &- (\nabla^1_{Y_1} H^f)(X_1, Z_1)] + n_3 f[(\nabla^1_{X_1} H^h)(Y_1, Z_1) - (\nabla^1_{Y_1} H^h)(X_1, Z_1)] \\
&= h n_2[X_1(\ln f) H^f(Y_1, Z_1) \\
&- Y_1(\ln f) H^f(X_1, Z_1)] + f n_3[X_1(\ln h) H^h(Y_1, Z_1) - Y_1(\ln h) H^h(X_1, Z_1)].
\end{aligned}
$$

(b) (M_2, g_2) is an Einstein-like manifold of class $\mathbb{B}$ iff

$$n_2[(\nabla^2_{X_2} H^h)(Y_2, Z_2) - (\nabla^2_{Y_2} H^h)(X_2, Z_2)] = n_3[X_2(\ln h)H^h(Y_2, Z_2) - Y_2(\ln h)H^h(X_2, Z_2)].$$

Proof. Let $M = (M_1 \times_f M_2) \times_h M_3$ be a sequential warped product of class . Let us take the deviation tensor $B(X, Y, Z)$ as follows

$$B(X, Y)Z = (\nabla_X Ric)(Y, Z) - (\nabla_Y Ric)(X, Z). \tag{18}$$

We consider $X = X_1$, $Y = Y_1$, $Z = Z_1$ and then we have

$$B(X_1, Y_1)Z_1 = (\nabla_{X_1} Ric)(Y_1, Z_1) - (\nabla_{Y_1} Ric)(X_1, Z_1). \tag{19}$$

Firstly we derive $(\nabla_{X_1} Ric)(Y_1, Z_1)$ as

$$(\nabla_{X_1} Ric)(Y_1, Z_1) = X_1 Ric(Y_1, Z_1) - Ric(\nabla_{X_1} Y_1, Z_1) - Ric(\nabla_{X_1} Z_1, Y_1). \tag{20}$$

Using proposition 3.1 (i) we obtain

$$\begin{aligned}(\nabla_{X_1} Ric)(Y_1, Z_1) &= (\nabla^1_{X_1} Ric^1)(Y_1, Z_1) + \frac{n_2(X_1 f)}{f^2} H^f(Y_1, Z_1) \\ &+ \frac{n_3(X_1 h)}{h^2} H^h(Y_1, Z_1) - \frac{n_2}{f}(\nabla^1_{X_1} H^f)(Y_1, Z_1) \\ &- \frac{n_3}{h}(\nabla^1_{X_1} H^h)(Y_1, Z_1).\end{aligned} \tag{21}$$

By exchanging X_1 and Y_1 in the last equation (21) we get the deviation tensor. For Einstein-like manifolds of class $\mathbb{B}$, the deviation tensor vanishes from which we get the result (a).

For part (b), taking $X = X_2$, $Y = Y_2$ and $Z = Z_2$ in the equation (18) and using proposition 3.1 (ii) we obtain the result.

Proposition 3.6 : Let $M = (M_1 \times_f M_2) \times_h M_3$ be a sequential warped product gradient Ricci-Bourguignon almost soliton with potential function ψ. If (M_i, g_i), $i = 1, 2$ is of class type $\mathbb{B}$. Then $(\nabla^i_{Y_i} H^\psi)(X_i, Z_i) - (\nabla^i_{X_i} H^\psi)(Y_i, Z_i) = (Y_i\mu)g_i(X_i, Z_i) - (X_i\mu)g_i(Y_i, Z_i)$.
Proof : Let $M = (M_1 \times_f M_2) \times_h M_3$ be a sequential warped product gradient Ricci-Bourguignon almost soliton of class $\mathbb{B}$. Now we obtain

$$\begin{aligned}(\nabla^i_{X_i} Ric^i)(Y_i, Z_i) &= X_i[(\mu + \rho R)g_i(Y_i, X_i) - H^\psi(Y_i, Z_i)] \\ &- [(\mu + \rho R)g_i(\nabla^i_{X_i} Y_i, Z_i) - H^\psi(\nabla^i_{X_i} Y_i, Z_i)] \\ &- [(\mu + \rho R)g_i(\nabla^i_{X_i} Z_i, Y_i) - H^\psi(\nabla^i_{X_i} Z_i, Y_i)].\end{aligned} \tag{22}$$

After simplification we get

$$(\nabla^i_{X_i} Ric^i)(Y_i, Z_i) = (X_i\mu)g_i(Y_i, Z_i) - (\nabla^i_{X_i} H^\psi)(Y_i, Z_i) \tag{23}$$

Similarly we have

$$(\nabla^i_{Y_i} Ric^i)(X_i, Z_i) = (Y_i\mu)g_i(X_i, Z_i) - (\nabla^i_{Y_i} H^\psi)(X_i, Z_i) \tag{24}$$

By these two equations (23) and (24) we obtain the result.

Corollary 3.7 : In a sequential warped product manifold $M = (M_1 \times_f M_2) \times_h M_3$ where M is of class type $\mathbb{P}$. Then (a) (M_1, g_1) is an Einstein-like manifold of class $\mathbb{A}$ iff

$$\begin{aligned} n_2 h[(\nabla^1_{X_1} H^f)(Y_1, Z_1)] \quad &+ \quad n_3 f[(\nabla^1_{X_1} H^h)(Y_1, Z_1)] \\ &= \quad h n_2[X_1(\ln f) H^f(Y_1, Z_1)] + f n_3[X_1(\ln h) H^h(Y_1, Z_1)]. \end{aligned}$$

(b) (M_2, g_2) is an Einstein-like manifold of class $\mathbb{B}$ iff

$$n_2[(\nabla^2_{X_2} H^h)(Y_2, Z_2)] - n_3[X_2(\ln h) H^h(Y_2, Z_2)] = 0.$$

Corollary 3.8 : Let $M = (M_1 \times_f M_2) \times_h M_3$ be a sequential warped product gradient Ricci-Bourguignon almost soliton with potential function ψ. If (M_i, g_i), $i = 1, 2$ is of class type $\mathbb{P}$. Then Then $(\nabla^i_{X_i} H^\psi)(Y_i, Z_i) - (X_i \mu) g_i(Y_i, Z_i) = 0$.

4. Ricci-Bourguignon soliton on Sequential warped product

In this section we are proving some interesting result to characterize the warping functions of sequential warped product with gradient Ricci-Bourguignon soliton.

Let M be a gradient Ricci-Bourguignon soliton sequential warped product with a potential function ψ as the lift of a smooth function on M_1. Let $\tilde{\phi} = \phi \circ \pi$ be the lift of a smooth function ϕ on M_1. By [13], we get $\psi = \tilde{\phi}$. Now, we have the following proposition.

Now, we state a lemma whose detailed proof is given in [15].

Lemma 4.1 : Let f be a smooth function on a Riemannian manifold M_1, then for any vector X, the divergence of the Hessian tensor H^f satisfies

$$div(H^f)(X) = Ric(\nabla f, X) - \Delta(df)(X), \tag{25}$$

where $\Delta = d\delta + \delta d$ denotes the Laplacian on M_1 acting on differential forms.

Theorem 4.2 : Let $M = (M_1 \times_f M_2) \times_h M_3$ be a complete sequential warped product with the metric with the metric tensor $\bar{g} = (g_1 \oplus f^2 g_2) \oplus h^2 g_3$ where $f : M_1 \to (0, \infty)$ and $h : M_1 \times M_2 \to (0, \infty)$ are two smooth positive functions on M_1 and $M_1 \times M_2$ respectively, . If $(M, \bar{g})$ is a sequential warped product gradient Ricci-Bourguignon soliton, then M_3 is an Einstein manifold.

Proof : Let $M = (M_1 \times_f M_2) \times_h M_3$ be a complete sequential warped product with the potential function $\psi : M_1 \to \mathbb{R}$ which depends only on theM_1 and if $X_3, Y_3 \in \chi(M_3)$, then from Lemma 4.1 we have

$$\bar{Ric}(X_3, Y_3) = Ric^3(X_3, Y_3) - h^2 g_3(X_3, Y_3)\frac{\Delta h}{h} + (n_1 + n_2 - 1)\frac{|\nabla h|^2}{h^2}. \tag{26}$$

From (1) we obtain

$$\bar{Ric}(X_3, Y_3) + H^\psi(X_3, Y_3) = (\mu + R\rho)\bar{g}(X_3, Y_3). \tag{27}$$

Using (26) in (27) we have

$$Ric^3(X_3, Y_3) - h^2 g_3(X_3, Y_3)\frac{\Delta h}{h} + (n_1 + n_2 - 1)\frac{|\nabla h|^2}{h^2} + H^{\psi}(X_3, Y_3) = (\mu + R\rho)h^2 g_3(X_3, Y_3). \tag{28}$$

Also we have

$$\nabla_{X_3}(\nabla_{\bar{g}}\psi) = \frac{(\nabla_{g_{M_1}}\psi)(h)}{h}X_3. \tag{29}$$

Therefore

$$\begin{aligned} H_{\bar{g}}{}^{\psi}(X_3, Y_3) &= \frac{(\nabla_{g_{M_1}}\psi)(h)}{h}\bar{g}(X_3, Y_3) \\ &= h^2 g_3(X_3, Y_3)\frac{(\nabla_{g_{M_1}}\psi)(h)}{h}. \end{aligned}$$

So,

$$H_{\bar{g}}{}^{\psi}(X_3, Y_3) = h(\nabla_{g_{M_1}}\psi)(h)g_{M_3}(X_3, Y_3). \tag{30}$$

Then putting the value of (30) in the equation (28) we derive

$$Ric^3(X_3, Y_3) = \omega g_3(X_3, Y_3),$$

where $\omega = (\mu + R\rho)h^2 - h(\nabla_{M_1}\psi)(h) + h\Delta(h) + (n_1 + n_2 - 1)|\nabla h|^2$. Hence the proof is completed.

Theorem 4.3 : Let $M = (M_1 \times_f M_2) \times_h M_3$ be a complete sequential warped product with the metric with the metric tensor $\bar{g} = (g_1 \oplus f^2 g_2) \oplus h^2 g_3$ where $f : M_1 \to (0, \infty)$ and $h : M_1 \times M_2 \to (0, \infty)$ are two smooth positive functions on M_1 and $M_1 \times M_2$ respectively and ϕ be a smooth function on M_1 so that $(M, \bar{g})$ is a sequential warped product gradient Ricci-Bourguignon soliton with the potential function $\psi = \bar{\phi}$, then
(i)

$$\begin{aligned} Ric^1(X_1, Y_1) &= (\mu + R\rho)g_1(X_1, Y_1) - H_{M_1}{}^{\phi}(X_1, Y_1) + \frac{n_2}{f}H_1^f(X_1, Y_1) \\ &+ \frac{n_3}{h}H^h(X_1, Y_1), \end{aligned}$$

(ii)

$$\begin{aligned} Ric^2(X_2, Y_2) &= (\mu + R\rho)f^2 g_2(X_2, Y_2) - H_{M_1}{}^{\phi}(X_2, Y_2) + f^2 g_2(X_2, Y_2)[\frac{\Delta^1 f}{f} \\ &+ (n_2 - 1)\frac{|\nabla^1 f|^2}{f^2}] + \frac{n_3}{h}H^h(X_2, Y_2), \end{aligned}$$

(iii)

$$Ric^3(X_3, Y_3) = \omega g_3(X_3, Y_3),$$

where $\omega = (\mu + R\rho)h^2 - h(\nabla_{M_1}\psi)(h) + h\Delta(h) + (n_1 + n_2 - 1)|\nabla h|^2$.

Proof : For $X_i, Y_i, Z_i \in \chi(M_i)$, from proposition 3.1 we have
(i) $\bar{Ric}(X_1, Y_1) = Ric^1(X_1, Y_1) - \frac{n_2}{f}H_1^f(X_1, Y_1) - \frac{n_3}{h}H^h(X_1, Y_1)$,
(ii)) $\bar{Ric}(X_2, Y_2) = Ric^2(X_2, Y_2) - f^2 g_2(X_2, Y_2)[\frac{\Delta^1 f}{f} + (n_2 - 1)\frac{|\nabla^1 f|^2}{f^2}] - \frac{n_3}{h}H^h(X_2, Y_2)$,
(iii) $\bar{Ric}(X_3, Y_3) = Ric^3(X_3, Y_3) - h^2 g_3(X_3, Y_3)[\frac{\Delta h}{h} + (n_1 + n_2 - 1)\frac{|\nabla h|^2}{h^2}]$.

Using (1) and the fact ${H_M}^{\psi}(X_i, Y_i) = {H_{M_1}}^{\phi}(X_i, Y_i)$ for $i = 1, 2$ we obtain
(i) $Ric^1(X_1, Y_1) = (\mu + R\rho)g_1(X_1, Y_1) - {H_{M_1}}^{\phi}(X_1, Y_1) + \frac{n_2}{f} H_1^f(X_1, Y_1) + \frac{n_3}{h} H^h(X_1, Y_1)$,
(ii) $Ric^2(X_2, Y_2) = (\mu + R\rho)f^2 g_2(X_2, Y_2) - {H_{M_1}}^{\phi}(X_2, Y_2) + f^2 g_2(X_2, Y_2)[\frac{\Delta^1 f}{f} + (n_2 - 1)\frac{|\nabla^1 f|^2}{f^2}] + \frac{n_3}{h} H^h(X_2, Y_2)$.
Since we are assuming $\psi = \tilde{\phi}$, the proof of (iii) is the same as in Theorem 4.2.

Theorem 4.4 : Let $M = ({M_1}^{n_1} \times_f {M_2}^{n_2}) \times_h {M_3}^{n_3}$, $n_2, n_3 > 1$ be a gradient Ricci-Bourguignon soliton sequential warped product with smooth function ϕ on M_1 satisfying the condition $d(\mu + R\rho)\phi + \frac{1}{2}|\nabla\phi|^2 - \Delta\phi = \frac{n_2}{f^2}\nabla\phi(f)df + \nabla\phi(h)dh + \frac{n_1 n_2}{fh}\nabla hf - \frac{1}{2}n_1\rho dR$.
(i) If $n_2(\mu + R\rho)f^2 + n_2(n_2 - 1)|\nabla f|^2 + n_2(f\Delta f) - n_2 f\nabla\phi f = c_1$, c_1 being constant and h reaches both maximum and minimum with $\mu \leq -R\rho$ then h is a constant function on $M_1 \times M_2$.
(ii) If $n_3(\mu + R\rho)h^2 + n_3(n_3 - 1)|\nabla h|^2 + n_3(h\Delta h) - n_2 h\nabla\phi h = c_2$, c_2 being constant and f reaches both maximum and minimum with $\mu \leq -R\rho$ then f is a constant function on M_1.

Proof : Taking the trace of Proposition 3.1 (i) we get

$$r^1 = n_1(\mu + R\rho) - \Delta\phi + \frac{n_2}{f}\Delta f + \frac{n_3}{h}\Delta h, \tag{31}$$

where r^1 is the scalar curvature of (M_1, g_{M_1}). Thus from the equation (31) we have

$$dr^1 = -d(\Delta\phi) - \frac{n_2}{f^2}(df\Delta f) + \frac{n_2}{f}d(\Delta f) - \frac{n_3}{h}(dh\Delta h) + \frac{n_3}{h}(\Delta h) + (\rho dRn_1)$$

From Bianchi second identity we have

$$dr^1 = 2\text{div}(Ric^1)$$

$$\text{div}(Ric^1) = -\text{div}(\nabla^2\phi) + n_2\text{div}(\frac{\nabla_1^2 f}{f}) + n_3\text{div}(\frac{\nabla^2 h}{h}).$$

We know that

$$\text{div}(\nabla^2\phi)(X) = Ric(\nabla\phi, X) + d(\Delta\phi)(X),$$

and

$$\frac{1}{2}|\nabla\phi|^2(X) = \nabla^2(\nabla\phi, X).$$

$$\begin{aligned} \therefore \text{div}(Ric^1)(X) &= -Ric^1(\nabla\phi, X) - d(\Delta\phi)(X) + \frac{n_2}{f}Ric^1(\nabla_1 f, x) + \frac{n_2}{f}\Delta(df)(X) \\ &- \frac{n_2}{2f^2}d|\nabla^1 f|^2(X) + \frac{n_3}{h}Ric^1(\nabla h, X) + \frac{n_3}{h}\Delta(dh)(X) \\ &- \frac{n_3}{2h^2}d(|\nabla h|^2)(X). \end{aligned} \tag{32}$$

Again we have

$$\begin{aligned} Ric^1(\nabla\phi, X) &= (\mu + \rho R)d\phi(X) + \frac{n_2}{f}((\nabla_1)^2 f)(\nabla\phi, x) + \frac{n_3}{h}(\nabla^2 h)(\nabla\phi, X) \\ &- \frac{1}{2}d(|\nabla\phi|^2)(X), \end{aligned} \tag{33}$$

$$\begin{aligned} Ric^1(\nabla f, X) &= (\mu + \rho R)df(X) + \frac{n_2}{2f}d(|\nabla f|^2)(X) + \frac{n_3}{h}(\nabla^2 h)(\nabla f, X) \\ &- (\nabla^2\phi)(\nabla f, X), \end{aligned} \tag{34}$$

and

$$\begin{aligned} Ric^1(\nabla h, X) &= (\mu + \rho R)dh(X) + \frac{n_2}{f}\nabla^2 f(\nabla h, x) + \frac{n_3}{2h}d(|\nabla h|^2)(X) \\ &- (\nabla^2\phi)(\nabla h, X). \end{aligned} \tag{35}$$

Putting these equations (33), (34), (35) in (32) we obtain

$$\begin{aligned} \mathrm{div}(Ric^1)(X) &= -(\mu + \rho R)d\phi(X) - \frac{n_2}{f}(\nabla_1^2 f)(\nabla\phi, X) - \frac{n_3}{h}(\nabla^2 h)(\nabla\phi, X) \\ &+ \frac{1}{2}d(|\nabla\phi_1|^2)(X) + \frac{n_2}{f}[(\mu + \rho R)df(X) + \frac{n_2}{2f}d(|\nabla f|^2)(X) + \frac{n_3}{h}(\nabla^2 h)(\nabla f, X) \\ &- (\nabla^2\phi)(\nabla f, X)] + \frac{n_2}{f}\Delta(df)(X) - \frac{n_2}{2f^2}d(|\nabla f|^2)(X) + \frac{n_3}{h}[(\mu + \rho R)dh(X) \\ &+ \frac{n_2}{f}\nabla^2 f(\nabla h, X) + \frac{n_3}{2h}d(|\nabla h|^2)(X) - (\nabla^2\phi)(\nabla h, X)] + \frac{n_3}{h}\Delta(dh)(X) \\ &- \frac{n_3}{2h^2}d(|\nabla h|^2)(X). \end{aligned} \tag{36}$$

We know that

$$d(\nabla\phi(f))(X) = (\nabla^2\phi)(\nabla f, X) + (\nabla^2 f)(\nabla\phi, X). \tag{37}$$

Putting the equation (37) in (36) we obtain

$$\begin{aligned} \mathrm{div}(Ric^1)(X) &= -(\mu + \rho R)(d\phi)(X) + \frac{1}{2}d(|\nabla\phi|^2)(X) - \Delta(d\phi)(X) + \frac{n_2}{f}(\mu + \rho R)(df)(X) \\ &+ \frac{n_2^2}{2f^2}d(|\nabla f|^2)(X) - \frac{n_2}{f}d(|\nabla\phi|)(X) + \frac{n_2 n_3}{fh}d(\nabla h(f))(X) \\ &+ \frac{n_3}{h}(\mu + \rho R)(dh)(X) + \frac{n_3^2}{2h^2}d(|\nabla h|^2)(X) - \frac{n_3}{h}d(\nabla\phi(h))(X) \\ &+ \frac{n_2}{f}\Delta(df)(X) - \frac{n_2}{2f^2}d(|\nabla f|^2)(X) + \frac{n_3}{h}\Delta(dh)(X) \\ &- \frac{n_3}{2h^2}d(|\nabla h|^2)(X). \end{aligned} \tag{38}$$

The equation (38) implies that

$$\begin{aligned} 0 &= [-(\mu + \rho R)d\phi(X) + \frac{1}{2}d(|\nabla\phi|^2)(X) - \frac{1}{2}d(\Delta\phi)(X) - \frac{n_2}{f}d(\nabla\phi(f))(X) - \frac{n_3}{h}d(\nabla\phi(h))(X)] \\ &+ \frac{n_2 n_3}{fh}(\nabla h(f)(X)) + [\frac{n_2}{f}(\mu + \rho R)df(X) + \frac{n_2^2 - n_2}{2f^2}d(|\nabla f|^2)(X) + \frac{n_2}{2f}\Delta(df)(X) \\ &+ \frac{n_2}{2f^2}(df)(\Delta f)(X)] + [\frac{n_3}{h}(\mu + \rho R)dh(X) + \frac{n_3^2 - n_3}{2h^2}d(|\nabla h|^2)(X) + \frac{n_3}{2h}\Delta(dh)(X) \\ &+ \frac{n_3}{2h^2}(dh)(\Delta h)(X)] - \frac{1}{2}\mu dR(X)n_1. \end{aligned} \tag{39}$$

Using the given condition

$$d(\mu + R\rho)\phi + \frac{1}{2}|\nabla\phi|^2 - \Delta\phi = \frac{n_2}{f^2}\nabla\phi(f)df + \nabla\phi(h)dh + \frac{n_1 n_2}{fh}\nabla hf - \frac{1}{2}\rho dRn_1,$$

we have

$$\begin{aligned} 0 &= \frac{1}{f^2}d[n_2(\mu + \rho R)(f^2)(X) + (n_2^2 - n_2)|\nabla f|^2(X) + n_2(f\Delta f)(X) - n_2 f\nabla\phi(f)(X)] \\ &+ \frac{1}{h^2}d[n_3(\mu + \rho R)(h^2)(X) + (n_3^2 - n_3)|\nabla h|^2(X) + n_3(h\Delta h)(X) - n_3 hf\nabla\phi(h)(X)]. \end{aligned}$$

Case 1 : If $n_2(\mu + R\rho)f^2 + n_2(n_2 - 1)|\nabla f|^2 + n_2(f\Delta f) - n_2 f\nabla\phi f = c_1$, c_1 being constant then we obtain

$$n_3(\mu + R\rho)h^2 + n_3(n_3 - 1)|\nabla h|^2 + n_3(h\Delta h) - n_2 h\nabla\phi h = \upsilon, \tag{40}$$

where υ is constant.
Let $(p_1, q_1), (p_2, q_2) \in M_1 \times M_2$ be the points where h attains its maximum and minimum in $M_1 \times M_2$. Then

$$\nabla h(p_1, q_1) = 0 = \nabla h(p_2, q_2)$$

and

$$\Delta h(p_1, q_1) \leq 0 \leq \Delta h(p_2, q_2).$$

Since $h > 0$ and $\mu < -R\rho$ we have

$$-(\mu + R\rho)h^2(p_1, q_1) \geq -(\mu + R\rho)h^2(p_2, q_2).$$

From the equation we have

$$0 \geq h(p_1, q_1)\Delta h(p_1, q_1) = \upsilon - (\mu + R\rho)h^2(p_1, q_1) \geq \upsilon - (\mu + R\rho)h^2(p_2, q_2) = h(p_2, q_2)\Delta h(p_2, q_2) \geq 0.$$

This implies

$$\upsilon - (\mu + R\rho)h^2(p_1, q_1) = (\mu + R\rho)h^2(p_2, q_2)$$

Hence we get h is constant on $M_1 \times M_2$ when $\mu < -R\rho$.
For $\mu = -R\rho$ we obtain that $\upsilon = 0$.
The equation (40) reduces to

$$\Delta h - \nabla\phi(h) = \frac{1}{h}(1 - n_3)|\nabla h|^2 \leq 0.$$

Therefore, h is constant on $M_1 \times M_2$ by the strong maximum principle.

Case 2 : If $n_3(\mu + R\rho)h^2 + n_3(n_3 - 1)|\nabla h|^2 + n_3(h\Delta h) - n_2 h\nabla\phi h = c_2$, c_2 being constant then we obtain

$$n_2(\mu + R\rho)f^2 + n_2(n_2 - 1)|\nabla f|^2 + n_2(f\Delta f) - n_2 f\nabla\phi f = \tau, \tag{41}$$

where τ is constant.
Similarly, it can be shown that f is constant on M_1 by the strong maximum principle.

Conclusion : Sequential warped product comprise an important topic in Differential Geometry. Solitons are the natural extension of the Einstein's metric. We have done applications of some vector fields on Ricci-Bourguignon soliton. Gradient Ricci-Bourguignon soliton are natural generalization of Einstein manifold. We have characterized that the warping functions are constants for sequential warped product gradient Ricci-Bourguignon soliton. The paper brings new ideas on the geometry of the manifold.

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