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Multiplicity of solutions for a singular Kirchhoff-type problem

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Abstract. This paper deals with some Kirchhoff-type problems driven by a non-local integrodifferential operator of singular elliptic type with combined nonlinearities which generalizes the fractional Laplacian operator. Our main result is to give and prove the existence of weak solutions for such problems with homogeneous Dirichlet boundary conditions. The proof is based on a variational method, precisely, we use the Nehari manifold method and the analysis of the fibering maps.

1. Introduction

During the past years, there has been considerable interest in the existence of solutions for problems involving fractional and non-local operators. This type of problem arises in a quite natural way in many different applications, such as continuum mechanics, phase transition phenomena, population dynamics, minimal surfaces, and game theory. For more details and applications, see for example ([1], [5], [6] and [16]).

This work presents a study related to the existence and the multiplicity of non-negative solutions for the following singular Kirchhoff-type problem, which is driven by a non-local integrodifferential operator of elliptic type

$$\begin{cases} M\Big(\int_{\mathbb{R}^{2N}} |u(x)-u(y)|^p K(x-y)dxdy + \int_{\mathbb{R}^N} V(x)|u|^p dx\Big)\Big(\mathcal{L}_K^p u + V(x)|u|^{p-2}u\Big) \\ = \lambda a_1(x)|u|^{-\alpha}u + a_2(x)|u|^{q-2}u \text{ in } \Omega, \\ u = 0 \text{ in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1}$$

where λ, μ are positive parameters, Ω is an open bounded subset of $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$, $N > ps$, $s \in (0,1)$, $1 < \alpha < 2$ and $1 < p < q < \frac{Np}{N-ps}$. The potential function $V : \Omega \longrightarrow (0,\infty)$ is continuous, and the Kirchhoff function defined on $[0,\infty)$ by $M : t \longrightarrow \mu + ct^m$, with $c > 0$, $0 < m < \frac{q}{p} - 1$. The weight functions a_1, a_2 are positive non-trivial functions satisfying the following conditions:

$(\mathbf{H}_1)$ $a_1 : \Omega \longrightarrow [0,\infty)$, is in $L^{\frac{p}{p-2+\alpha}}(\Omega)$.

$(\mathbf{H}_2)$ $a_2 : \Omega \longrightarrow [0,\infty)$, is in $L^{\infty}(\Omega)$.

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Here $\mathcal{L}_K^p$ is a non-local operator defined by

$$\mathcal{L}_K^p u(x) = 2\lim_{\varepsilon\longrightarrow 0}\Big(\int_{\mathbb{R}^N\setminus B_\varepsilon(x)} |u(x)-u(y)|^{p-2}(u(x)-u(y))K(x-y)dy\Big), \text{ for } x\in\mathbb{R}^N,$$

where K is a positive function defined in $\mathbb{R}^N\setminus\{0\}$ and satisfying the following property

($\mathbf{K}_1$) $K(x)=K(-x)$ and $\gamma K\in L^1(\mathbb{R}^N)$, where $\gamma(x)=\min(|x|^p,1)$.

($\mathbf{K}_2$) There exists $k_0>0$, such that $K(x)\geq k_0|x|^{-(N+ps)}$.

The special case where $K(x)=|x|^{-(N+ps)}$, the problem (1) becomes

$$\begin{cases} M\Big(\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy+\int_{\mathbb{R}^N}V(x)|u|^pdx\Big)((-\Delta)_p^s u+V(x)|u|^{p-2}u)\\ =\lambda a_1(x)|u|^{-\alpha}u+a_2(x)|u|^{q-2}u \text{ in } \Omega,\\ u=0 \text{ in } \mathbb{R}^N\setminus\Omega, \end{cases} \tag{2}$$

where $(-\Delta)_p^s$ is the fractional p-Laplace operator defined by

$$(-\Delta)_p^s u(x)=2\lim_{\varepsilon\longrightarrow 0}\Big(\int_{\mathbb{R}^N\setminus B_\varepsilon(x)}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}dy\Big), \text{ for } x\in\mathbb{R}^N.$$

Problems like (2) are extensively studied see [9, 13–15]. Moreover, according to different elliptic operators, several articles have been devoted to the study of problems of type (1), we refer to [2–4, 9] and the references therein. Namely, the special case where $p=2$ and $K(x)=|x|^{-(N+2s)}$, the authors in [9] considered the following problem

$$\begin{cases} M\Big(\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy\Big)(-\Delta)^s u=\lambda f(x,u)+|u|^{\frac{2N}{N-2s}-2}u \text{ in } \Omega,\\ u=0 \text{ in } \mathbb{R}^N\setminus\Omega. \end{cases} \tag{3}$$

Using the mountain pass theorem and under some suitable assumptions of the functions M and f, they established the existence of a non-negative solution to the problem (3) for any $0<\lambda_0<\lambda$, where λ_0 is an appropriate threshold. Later, for the degenerate problem case where M can be zero at zero, the authors in [2] obtained the existence and the asymptotic behavior of non-negative solutions to (3). Recently, when $p=2$ and $M\equiv 1$, the fractional Laplacian problem

$$\begin{cases} (-\Delta)^s u=f(x,u) \text{ in } \Omega,\\ u=0 \text{ in } \mathbb{R}^N\setminus\Omega, \end{cases}$$

has been studied by many authors, for interested readers, we cite the papers [7], [8], [17], [18] and [19]. Also, the Nehari manifold method is used in recent papers see for example [10–12].

This paper is motivated by a recent result stated in [20]. More precisely, the authors in [20] suppose that the Kirchhoff function M is a continuous function satisfying the following condition: there exist $m_1,m_2>0$ such that $M(t)\geq m_1$ and $\int_0^t M(\tau)d\tau\geq m_2tM(t)$, $\forall t\geq 0$. They proved that the following Kirchhoff-type problem with homogeneous Dirichlet boundary conditions

$$\begin{cases} M\Big(\int_{\mathbb{R}^{2N}}|u(x)-u(y)|^pK(x-y)dxdy\Big)\mathcal{L}_K^p u=f(x,u) \text{ in } \Omega,\\ u=0 \text{ in } \mathbb{R}^N\setminus\Omega, \end{cases}$$

has a non-trivial weak solution, where f is a Carathéodory function satisfying appropriate inequalities. Now, we are in a position to give the main result of this paper. 'Note that a weak solution of problem (1) satisfies the following definition

Definition 1.1. *A function $u \in W_0$ is called a weak solution of* (1) *if for any $\varphi \in W_0$, we have*

$$M(\|u\|_W^p)\left(\int_{\mathbb{R}^{2N}}|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)dxdy+\int_{\mathbb{R}^N}V(x)|u|^{p-2}u\varphi dx\right)$$
$$=\lambda\int_\Omega a_1(x)|u|^{-\alpha}u\varphi dx+\int_\Omega a_2(x)|u|^{q-2}u\varphi dx,$$

where W_0 will be introduced later in Section 2.

The main result of this paper is the following theorem.

Theorem 1.2. *Let a_1, a_2 be two non trivial positive functions satisfying hypotheses* ($\mathbf{H}_1$)-($\mathbf{H}_2$). *Assume that K satisfies conditions* ($\mathbf{K}_1$)-($\mathbf{K}_2$)*, then there exist $\lambda_0 > 0$ and $\mu_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$ and $\mu \in (0, \mu_0)$, problem (1) has at least two weak non trivial solutions.*

Remark 1.3. *The result of Theorem 1.2, extends a result established in [20] to the singular case $f(x,u) = \lambda a_1(x)|u|^{-\alpha}u+ a_2(x)|u|^{q-2}u$.*

2. Preliminaries and Neharie manifold analysis

In this section, we begin by presenting some preliminaries which are used in the second part, to manipulate the Nehari manifold and fibering maps analysis. First, we begin this section by giving some notations that will be used in the sequel. We define the fractional Sobolev space $W^{s,p}(\Omega)$ by

$$W^{s,p}(\Omega) = \{u \in L^p(\Omega);\ [u]_{s,p} < \infty\},$$

where $[u]_{s,p}$ denotes the following Gagliardo semi-norm

$$[u]_{s,p} = \Big(\int_{\Omega^2}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy\Big)^{\frac{1}{p}}.$$

$W^{s,p}(\Omega)$ is equipped with the norm

$$\|u\|_{W^{s,p}(\Omega)} = (\|u\|_p^p + [u]_{s,p}^p)^{\frac{1}{p}},$$

where and hereafter we denote by $\|.\|_p$ the norm on the Lebesgue space $L^p(\Omega)$. For a detailed account of the properties of $W^{s,p}(\Omega)$, we refer to [6]. We denote also by $L^p(\mathbb{R}^N, V)$ the Lebesgue space of real-valued functions, with $V(x)|u|^p \in L^1(\mathbb{R}^N)$, equipped with the norm

$$\|u\|_{p,V} = \left(\int_{\mathbb{R}^N}V(x)|u|^p dx\right)^{\frac{1}{p}}.$$

Let $W_V^{s,p}(\Omega)$ denote the completion of $C_0^\infty(\Omega)$, with respect the norm

$$\|u\|_{W_V^{s,p}(\Omega)} = ([u]_{s,p}^p + \|u\|_{p,V}^p)^{\frac{1}{p}}.$$

We stress that the embedding $W_V^{s,p}(\Omega) \hookrightarrow L^\nu(\Omega)$ is continuous for any $p \le \nu \le \frac{Np}{N-ps}$, (see [6] Theorem 6.7). Namely, there exists a positive constant c_ν such that

$$\|u\|_\nu \le c_\nu\|u\|_{W_V^{s,p}(\Omega)},\ \text{for all } u \in W_V^{s,p}(\Omega). \tag{4}$$

Let W be a space of Lebesgue measurable functions from $\mathbb{R}^N$ to $\mathbb{R}$ such that the restriction to Ω of any function u in W belongs to $L^p(\Omega)$ and

$$\int_Q|u(x)-u(y)|^pK(x-y)dxdy+\int_{\mathbb{R}^N}V(x)|u|^pdx<\infty,$$

where $Q = (\mathbb{R}^N \times \mathbb{R}^N) \setminus ((\mathbb{R}^N \setminus \Omega) \times (\mathbb{R}^N \setminus \Omega))$. The space W is equipped with the norm

$$\|u\|_W = \left(\int_Q |u(x) - u(y)|^p K(x-y)dxdy + \int_{\mathbb{R}^N} V(x)|u|^p dx\right)^{\frac{1}{p}}.$$

We consider the subspace

$$W_0 = \left\{u \in W, u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\right\}.$$

Next, we give some properties of W and W_0.

Lemma 2.1. *The following statements hold.*

i) If $u \in W$, then $u \in W_V^{s,p}(\Omega)$. Moreover,

$$\|u\|_{W_V^{s,p}(\Omega)} \leq \max(1, k_0^{\frac{-1}{p}})\|u\|_W.$$

ii) If $u \in W_0$, then $u \in W_V^{s,p}(\mathbb{R}^N)$. Moreover,

$$\|u\|_{W_V^{s,p}(\Omega)} \leq \|u\|_{W_V^{s,p}(\mathbb{R}^N)} \leq \max(1, k_0^{\frac{-1}{p}})\|u\|_W.$$

Proof. Let $u \in W$, then we have

$$\begin{aligned}\|u\|_{W_V^{s,p}(\Omega)} &= \left(\int_{\Omega^2} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}}dxdy + \int_{\mathbb{R}^N} V(x)|u|^p dx\right)^{\frac{1}{p}} \\ &\leq \left(\frac{1}{k_0}\int_Q |u(x) - u(y)|^p K(x-y)dxdy + \int_{\mathbb{R}^N} V(x)|u|^p dx\right)^{\frac{1}{p}} \\ &\leq \max(k_0^{-p}, 1)\|u\|_W^p.\end{aligned}$$

So, assertion i) is proved.
Now, let $u \in W$ such that $u = 0$ a.e. in $\mathbb{R}^N \setminus \Omega$, then

$$\begin{aligned}\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}}dxdy + \int_{\mathbb{R}^N} V(x)|u|^p dx &= \int_Q \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}}dxdy + \int_{\mathbb{R}^N} V(x)|u|^p dx \\ &\leq \frac{1}{k_0}\int_Q |u(x) - u(y)|^p K(x-y)dxdy + \int_{\mathbb{R}^N} V(x)|u|^p dx \\ &< \infty.\end{aligned}$$

Hence $u \in W_V^{s,p}(\mathbb{R}^N)$ and

$$\|u\|_{W_V^{s,p}(\Omega)} \leq \|u\|_{W_V^{s,p}(\mathbb{R}^N)} \leq \max(1, k_0^{\frac{-1}{p}})\|u\|_W.$$

□

In the sequel, we define the singular energy functional associated with the problem (1) $I : W_0 \to \mathbb{R}$ by

$$I(u) = \frac{1}{p}\bar{M}\left(\|u\|_W^p\right) - \frac{\lambda}{2-\alpha}\int_\Omega a_1(x)|u|^{2-\alpha}dx - \frac{1}{q}\int_\Omega a_2(x)|u|^q dx,$$

where $\bar{M}(t) = \int_0^t M(s)ds$. We remark that

$$M(t) \geq \mu \text{ and } \bar{M}(t) \geq \frac{t}{m+1}M(t), \text{ for all } t \geq 0. \tag{5}$$

It is important to mention that I is well defined but not differentiable due to the singular term.
Next, we have for $u \in W_0$, $I(0u) = 0$ and for $t > 0$,

$$\frac{d}{dt}I(tu) = t^{p-1}\|u\|_W^p M\Big(t^p\|u\|_W^p\Big) - \lambda t^{1-\alpha}\int_\Omega a_1(x)|u|^{2-\alpha}dx - t^{q-1}\int_\Omega a_2(x)|u|^q dx$$

and

$$\begin{aligned}\frac{d^2}{dt^2}I(tu) &= (p-1)t^{p-2}\|u\|_W^p M\Big(t^p\|u\|_W^p\Big) + pt^{2(p-1)}\|u\|_W^{2p}M'\Big(t^p\|u\|_W^p\Big)\\ &\quad -\lambda(1-\alpha)t^{-\alpha}\int_\Omega a_1(x)|u|^{2-\alpha}dx - (q-1)t^{q-2}\int_\Omega a_2(x)|u|^q dx.\end{aligned}$$

Since the functional energy is not coercive, we will study in the subset N which is called the Nehari manifold and is defined as follows

$$N := \left\{u \in W_0 \setminus \{0\},\ \frac{d}{dt}I(tu)\Big|_{t=1} = 0\right\},$$

Lemma 2.2. *I is coercive and bounded on N.*

Proof. Let $2-\alpha < 1 < p < q < \frac{Np}{N-ps}$, $0 < m < \frac{q}{p} - 1$ and $u \in N$. Then from hypotheses $(\mathbf{H}_1)$-$(\mathbf{H}_2)$, Lemma 2.1, the Hölder inequality, and Equations (4), (5), we have

$$\begin{aligned}I(u) &= \frac{1}{p}\bar{M}\Big(\|u\|_W^p\Big) - \frac{\lambda}{2-\alpha}\int_\Omega a_1(x)|u|^{2-\alpha}dx - \frac{1}{q}\int_\Omega a_2(x)|u|^q dx\\ &= \frac{1}{p}\bar{M}\Big(\|u\|_W^p\Big) - \frac{\lambda}{2-\alpha}\int_\Omega a_1(x)|u|^{2-\alpha}dx - \frac{1}{q}\|u\|_W^p M(\|u\|_W^p) + \frac{\lambda}{q}\int_\Omega a_1(x)|u|^{2-\alpha}dx\\ &= \frac{1}{p}\bar{M}\Big(\|u\|_W^p\Big) - \frac{1}{q}\|u\|_W^p M(\|u\|_W^p) - \lambda(\frac{1}{2-\alpha} - \frac{1}{q})\int_\Omega a_1(x)|u|^{2-\alpha}dx.\\ &\geq (\frac{1}{p(m+1)} - \frac{1}{q})\|u\|_W^p M(\|u\|_W^p) - \lambda(\frac{1}{2-\alpha} - \frac{1}{q})\int_\Omega a_1(x)|u|^{2-\alpha}dx\\ &\geq \mu(\frac{1}{p(m+1)} - \frac{1}{q})\|u\|_W^p - \lambda(\frac{1}{2-\alpha} - \frac{1}{q})\Big(\int_\Omega |a_1(x)|^{\frac{p}{p-2+\alpha}}dx\Big)^{\frac{p-2+\alpha}{p}}\Big(\int_\Omega |u|^p dx\Big)^{\frac{2-\alpha}{p}}\\ &\geq \mu(\frac{q-p-pm}{q(m+1)p})\|u\|_W^p - \lambda(\frac{1}{2-\alpha} - \frac{1}{q})\|a_1\|_{\frac{p}{p-2+\alpha}}\|u\|_p^{2-\alpha}\\ &\geq \mu(\frac{q-p-pm}{q(m+1)p})\|u\|_W^p - c_p^{2-\alpha}\lambda(\frac{1}{2-\alpha} - \frac{1}{q})\|a_1\|_{\frac{p}{p-2+\alpha}}\|u\|_{W_V^{s,p}(\Omega)}^{2-\alpha}\\ &\geq \mu(\frac{q-p-pm}{q(m+1)p})\|u\|_W^p - \lambda\Big(c_p\max(1, k_0^{\frac{-1}{p}})\Big)^{2-\alpha}(\frac{1}{2-\alpha} - \frac{1}{q})\|a_1\|_{\frac{p}{p-2+\alpha}}\|u\|_W^{2-\alpha}.\end{aligned}$$

Since $2-\alpha < p$ and $q-p-pm > 0$, we deduce that

$$I(u) \longrightarrow +\infty, \text{ as } \|u\|_W \longrightarrow +\infty.$$

Which completes the proof of Lemma 2.2. □

Note that $u \in N$ if and only if

$$\|u\|_W^p M(\|u\|_W^p) - \lambda\int_\Omega a_1(x)|u|^{2-\alpha}dx - \int_\Omega a_2(x)|u|^q dx = 0. \tag{6}$$

If $u \in N$, we obtain by the above equality that

$$\frac{d^2}{dt^2}I(tu)\Big|_{t=1} = (p-1)\|u\|_W^p M\Big(\|u\|_W^p\Big) + p\|u\|_W^{2p}M'\Big(\|u\|_W^p\Big) \tag{7}$$

$$\begin{aligned}
&-\lambda(1-\alpha)\int_\Omega a_1(x)|u|^{2-\alpha}dx-(q-1)\int_\Omega a_2(x)|u|^q dx\\
=\;& (p-2+\alpha)\|u\|_W^p M\Big(\|u\|_W^p\Big)+p\|u\|_W^{2p}M'\Big(\|u\|_W^p\Big)\\
&-(q-2+\alpha)\int_\Omega a_2(x)|u|^q dx\\
=\;& (p-q)\|u\|_W^p M\Big(\|u\|_W^p\Big)+p\|u\|_W^{2p}M'\Big(\|u\|_W^p\Big)\\
&-\lambda(2-\alpha-q)\int_\Omega a_1(x)|u|^{2-\alpha}dx\\
=\;& p\|u\|_W^{2p}M'\Big(\|u\|_W^p\Big)-\lambda(2-\alpha-p)\int_\Omega a_1(x)|u|^{2-\alpha}dx\\
&-(q-p)\int_\Omega a_2(x)|u|^q dx.
\end{aligned}$$

We split N into the following three subsets

$$N^0=\left\{u\in N,\ \frac{d^2}{dt^2}I(tu)\Big|_{t=1}=0\right\},$$

$$N^+=\left\{u\in N,\ \frac{d^2}{dt^2}I(tu)\Big|_{t=1}>0\right\},$$

and

$$N^-=\left\{u\in N,\ \frac{d^2}{dt^2}I(tu)\Big|_{t=1}<0\right\}.$$

Put

$$\lambda_1:=\frac{c(q-p-pm)\left(\frac{(q-2+\alpha)(c_q\max(1,k_0^{\frac{-1}{p}}))^q}{c(p-2+\alpha+pm)}\|a_2\|_\infty\right)^{\frac{p-2+\alpha+pm}{p-q+pm}}}{(q+\alpha-2)\left(c_p\max(1,k_0^{\frac{-1}{p}})\right)^{2-\alpha}\|a_1\|_{\frac{p}{p-2+\alpha}}}.$$

Lemma 2.3. *If $0<\lambda<\lambda_1$, then $N^0=\emptyset$.*

Proof. Suppose otherwise that for $0<\lambda<\lambda_1$, we have $N^0\neq\emptyset$. So, there exists $u_0\in N$ such that $\frac{d^2}{dt^2}I(tu_0)\Big|_{t=1}=0$. Then by Equation (7), we obtain

$$(p-q)\|u_0\|_W^p M\Big(\|u_0\|_W^p\Big)+p\|u_0\|_W^{2p}M'\Big(\|u_0\|_W^p\Big)=\lambda(2-\alpha-q)\int_\Omega a_1(x)|u_0|^{2-\alpha}dx, \tag{8}$$

and

$$(p-2+\alpha)\|u_0\|_W^p M\Big(\|u_0\|_W^p\Big)+p\|u_0\|_W^{2p}M'\Big(\|u_0\|_W^p\Big)=(q-2+\alpha)\int_\Omega a_2(x)|u_0|^q dx. \tag{9}$$

From ($\mathbf{H}_1$), inequality (4), Lemma 2.1, and the Hölder inequality, we get

$$\begin{aligned}
\int_\Omega a_1(x)|u_0|^{2-\alpha}dx &\le \left(\int_\Omega |a_1(x)|^{\frac{p}{p-2+\alpha}}dx\right)^{\frac{p-2+\alpha}{p}}\left(\int_\Omega |u_0|^p dx\right)^{\frac{2-\alpha}{p}}\\
&\le \|a_1\|_{\frac{p}{p-2+\alpha}}\|u_0\|_p^{2-\alpha}\\
&\le c_p^{2-\alpha}\|a_1\|_{\frac{p}{p-2+\alpha}}\|u_0\|_{W_V^{s,p}(\Omega)}^{2-\alpha}\\
&\le \left(c_p\max(1,k_0^{\frac{-1}{p}})\right)^{2-\alpha}\|a_1\|_{\frac{p}{p-2+\alpha}}\|u_0\|_W^{2-\alpha}.
\end{aligned}$$

So from (8), we obtain

$$\|u_0\|_W^{p-2+\alpha}\Big(M(\|u_0\|_W^p) + \frac{p}{p-q}\|u_0\|_W^p M'(\|u_0\|_W^p)\Big) \leq \lambda \frac{q+\alpha-2}{q-p}\left(c_p \max(1, k_0^{\frac{-1}{p}})\right)^{2-\alpha} \|a_1\|_{\frac{p}{p-2+\alpha}}.$$

According to the explicit expression of the function M, we obtain

$$\begin{aligned} c(1+\frac{pm}{p-q})\|u_0\|_W^{p-2+\alpha+pm} &\leq \|u_0\|_W^{p-2+\alpha}\Big(\mu + c(1+\frac{pm}{p-q})\|u_0\|_W^{pm}\Big) \\ &\leq \lambda \frac{q+\alpha-2}{q-p}\left(c_p \max(1, k_0^{\frac{-1}{p}})\right)^{2-\alpha} \|a_1\|_{\frac{p}{p-2+\alpha}} \end{aligned}$$

Since $p-q+pm<0$, and $p-2+\alpha+pm>0$, we get

$$\|u_0\|_W \leq \left[\lambda \frac{2-\alpha-q}{c(p-q+pm)}\left(c_p \max(1, k_0^{\frac{-1}{p}})\right)^{2-\alpha} \|a_1\|_{\frac{p}{p-2+\alpha}}\right]^{\frac{1}{p-2+\alpha+pm}}. \tag{10}$$

On the other hand, by the same arguments, we obtain

$$\begin{aligned} \frac{q-2+\alpha}{p-2+\alpha}\int_\Omega a_2(x)|u_0|^q dx &= \|u_0\|_W^p\Big(M(\|u_0\|_W^p) + \frac{p}{p-2+\alpha}\|u_0\|_W^p M'(\|u_0\|_W^p)\Big) \\ &\leq \frac{q-2+\alpha}{p-2+\alpha}\|a_2\|_\infty \|u_0\|_q^q \\ &\leq \frac{q-2+\alpha}{p-2+\alpha} c_q^q \|a_2\|_\infty \|u_0\|^q_{W_V^{s,p}(\Omega)} \\ &\leq \frac{q-2+\alpha}{p-2+\alpha}\left(c_q \max(1, k_0^{\frac{-1}{p}})\right)^q \|a_2\|_\infty \|u_0\|_W^q. \end{aligned}$$

So from (9), one has

$$\|u_0\|_W^{p-q}\left(M(\|u_0\|_W^p) + \frac{p}{p-2+\alpha}\|u_0\|_W^p M'(\|u_0\|_W^p)\right) \leq \frac{q-2+\alpha}{p-2+\alpha}\left(c_q \max(1, k_0^{\frac{-1}{p}})\right)^q \|a_2\|_\infty.$$

Again from the definition of the function M, we obtain

$$\begin{aligned} c(1+\frac{pm}{p-2+\alpha})\|u_0\|_W^{p-q+pm} &\leq \|u_0\|_W^{p-q}\Big(\mu + c(1+\frac{pm}{p-2+\alpha})\|u_0\|_W^{pm}\Big) \\ &\leq \frac{q-2+\alpha}{p-2+\alpha}\left(c_q \max(1, k_0^{\frac{-1}{p}})\right)^q \|a_2\|_\infty. \end{aligned}$$

Also, from the fact that $p-q+pm<0$, and $p-2+\alpha+pm>0$, we deduce

$$\|u_0(x)\|_W \geq \left[\frac{q-2+\alpha}{c(p-2+\alpha+pm)}\left(c_q \max(1, k_0^{\frac{-1}{p}})\right)^q \|a_2\|_\infty\right]^{\frac{1}{p-q+pm}}. \tag{11}$$

By combining Equation (10) with Equation (11), we get $\lambda_1 \leq \lambda$, which is a contradiction. □

Remark 2.4. *From Lemma 2.3, if* $0<\lambda<\lambda_1$, *then we can write*

$$N = N^+ \cup N^-.$$

Moreover, we will prove in the following lemma that each of the subsets is nonempty.

Lemma 2.5. *Let* $u \in W_0 \setminus \{0\}$, *there exist* $\lambda_0>0$ *and* $\mu_0>0$, *for* $\lambda \in (0,\lambda_0)$ *and* $\mu \in (0,\mu_0)$, *then there exist unique positive numbers* t_1 *and* t_2 *such that*

$$\frac{d}{dt}I(tu)\Big|_{t=t_1} = \frac{d}{dt}I(tu)\Big|_{t=t_2} = 0.$$

That is $t_1u \in N^+$ *and* $t_2u \in N^-$.

Proof. Let $u \in W_0 \setminus \{0\}$, we note that $\int_\Omega a_1(x)|u|^{2-\alpha}dx > 0$ and $\int_\Omega a_2(x)|u|^q dx > 0$. For $t \in (0,\infty)$, we put

$$\psi_1(t) := ct^{p-1+pm}\|u\|_W^{p+pm} - t^{q-1}\int_\Omega a_2(x)|u|^q dx$$

and

$$\psi_2(t) := \mu t^{p-1}\|u\|_W^p - \lambda t^{2-\alpha-1}\int_\Omega a_1(x)|u|^{2-\alpha}dx.$$

It is not difficult to see that

$$\begin{aligned}\frac{d}{dt}I(tu) &= t^{p-1}\|u\|_W^p M(t^p\|u\|_W^p) - \lambda t^{2-\alpha-1}\int_\Omega a_1(x)|u|^{2-\alpha}dx - t^{q-1}\int_\Omega a_2(x)|u|^q dx\\ &= \mu t^{p-1}\|u\|_W^p + ct^{p-1+pm}\|u\|_W^{p+pm} - \lambda t^{2-\alpha-1}\int_\Omega a_1(x)|u|^{2-\alpha}dx - t^{q-1}\int_\Omega a_2(x)|u|^q dx\\ &= \psi_1(t) + \psi_2(t).\end{aligned}$$

On the other hand, ψ_1 has a unique maximum point $t_0 > 0$, which is given by

$$t_0 = \Big(\frac{c(p-1+pm)\|u\|_W^{p+pm}}{(q-1)\int_\Omega a_2(x)|u|^q dx}\Big)^{\frac{1}{q-p-pm}},$$

moreover, its table of variations is as follows

t	0		t_0		∞
$\psi_1'(t)$		$+$	0	$-$	
$\psi_1(t)$	0	$\nearrow$	$\psi_1(t_0)$	$\searrow$	$-\infty$

We remark that there exists $T \in (0, t_0)$, such that $\frac{\psi_1}{\psi_2}$ is strictly decreasing in $(0,T)$.
Using the fact that $\lim_{t\longrightarrow 0} -\psi_2(t) = +\infty$, we get the existence of $T_1 \in (0,T)$ such that $\psi_1(T_1) < -\psi_2(T_1)$. That is

$$\frac{d}{dt}I(tu)\Big|_{t=T_1} = \psi_1(T_1) + \psi_2(T_1) < 0. \tag{12}$$

Moreover, for each $\lambda > 0$ sufficiently small, there exists $T_2 \in (0,T)$ such that

$$\frac{d}{dt}I(tu)\Big|_{t=T_2} = \psi_1(T_2) + \psi_2(T_2) > 0. \tag{13}$$

Combining (12) and (13), we deduce by the intermediate value theorem that there exists $t_1 \in (0,T)$ such that

$$\frac{d}{dt}I(tu)\Big|_{t=t_1} = 0.$$

It is clear to see that ψ_2 is strictly increasing in $(0,\infty)$ and we deduce by the table of variations of ψ_1 that $t \longmapsto \frac{d}{dt}I(tu)$ is strictly increasing in $(0,t_0)$. So, t_1 is unique in $(0,t_0)$.
For $t \in (t_1,T)$, we have $\frac{\psi_1(t)}{\psi_2(t)} < \frac{\psi_1(t_1)}{\psi_2(t_1)} = -1$. That is $\psi_1(t)+\psi_2(t) > 0$ for all $t \in (t_1,T)$. Moreover, we can fix λ_2 such that, for all $\lambda \in (0,\lambda_2)$, we have

$$\frac{d}{dt}I(tu) = \psi_1(t) + \psi_2(t) > 0 \text{ for all } t \in [T, t_0).$$

Then for all $\lambda \in (0,\lambda_2)$, we get

$$\frac{d}{dt}I(tu) > 0 \text{ on } (t_1,t_0). \tag{14}$$

Using the fact that $\lim_{t\longrightarrow+\infty}\frac{d}{dt}I(tu)=-\infty$, we obtain that there exists $T_3\in(t_0,\infty)$ such that

$$\frac{d}{dt}I(tu)\Big|_{t=T_3}<0. \tag{15}$$

Combining (14) and (15), we deduce by the intermediate value theorem for $\lambda\in(0,\lambda_2)$, that there exists $t_2\in(t_0,\infty)$ such that

$$\frac{d}{dt}I(tu)\Big|_{t=t_2}=0.$$

Since ψ_1 is strictly decreasing in (t_0,∞), we can fix $(\lambda_3,\mu_0)\in(0,\infty)\times(0,\infty)$ such that for $\lambda\in(0,\lambda_3)$ and $\mu\in(0,\mu_0)$, the function $t\longmapsto\frac{d}{dt}I(tu)$ is strictly decreasing in (t_0,∞). So, t_2 is unique in (t_0,∞). Since $t\longmapsto\frac{d}{dt}I(tu)$ is strictly increasing in $(0,t_0)$ and $\frac{d}{dt}I(tu)\Big|_{t=t_1}=0$, we deduce that $\frac{d}{dt}I(tu)<0$ in $(0,t_1)$ and $\frac{d}{dt}I(tu)>0$ in (t_1,t_0).
Put

$$\lambda_0:=\min(\lambda_1,\lambda_2,\lambda_3),$$

then for $\lambda\in(0,\lambda_0)$, we obtain by Lemma 2.3 that $N^0=\emptyset$ and we conclude that $t\longmapsto I(tu)$ attains a local minimum at t_1 and $\frac{d^2}{dt^2}I(tu)\Big|_{t=t_1}>0$. That is $t_1u\in N^+$.
Using the same arguments and the fact that $\frac{d}{dt}I(tu)\Big|_{t=t_2}=0$, $\frac{d}{dt}I(tu)>0$ in (t_0,t_2) and $\frac{d}{dt}I(tu)<0$ in (t_2,∞), we deduce that $t\longmapsto I(tu)$ attains a local maximum at t_2 and $\frac{d^2}{dt^2}I(tu)\Big|_{t=t_2}<0$. That is $t_2u\in N^-$. □

Lemma 2.6. *If* $0<\lambda<\lambda_1$, *then* $\inf_{u\in N^+}I<0$.

Proof. Let $u\in N^+\subset N$. So, we deduce by equality (7) that

$$\begin{aligned}0 &< \frac{d^2}{dt^2}I(tu)\Big|_{t=1}\\ &= (p-2+\alpha)\|u\|_W^pM\left(\|u\|_W^p\right)+p\|u\|_W^{2p}M'\left(\|u\|_W^p\right)-(q-2+\alpha)\int_\Omega a_2(x)|u|^qdx.\end{aligned}$$

That is

$$\int_\Omega a_2(x)|u|^qdx<\frac{p-2+\alpha}{q-2+\alpha}\|u\|_W^pM\left(\|u\|_W^p\right)+\frac{p}{q-2+\alpha}\|u\|_W^{2p}M'\left(\|u\|_W^p\right).$$

According to the explicit expression of the function M, we deduce by (6) that

$$\begin{aligned}I(u) &= \frac{1}{p}\bar{M}\left(\|u\|_W^p\right)-\frac{1}{2-\alpha}\|u\|_W^pM(\|u\|_W^p)-\left(\frac{1}{q}-\frac{1}{2-\alpha}\right)\int_\Omega a_2(x)|u|^qdx.\\ &< \frac{1}{p}\bar{M}\left(\|u\|_W^p\right)+\frac{p-q-2+\alpha}{q(2-\alpha)}\|u\|_W^pM(\|u\|_W^p)+\frac{p}{q(2-\alpha)}\|u\|_W^{2p}M'(\|u\|_W^p)\\ &= \frac{\mu}{p}\|u\|_W^p+\frac{c}{p(m+1)}\|u\|_W^{pm+p}+\mu\frac{p-q-2+\alpha}{q(2-\alpha)}\|u\|_W^p\\ &\quad +c\frac{p-q-2+\alpha}{q(2-\alpha)}\|u\|_W^{pm+p}+\frac{cmp}{q(2-\alpha)}\|u\|_W^{pm+p}\\ &= -\mu\frac{(p-2+\alpha)(q-p)}{qp(2-\alpha)}\|u\|_W^p-c\frac{(q-pm-p)(p-2+\alpha+pm)}{pq(2-\alpha)}\|u\|_W^{pm+p}.\end{aligned}$$

Since $0<2-\alpha<1<p<q$ and $q-pm-p>0$, we conclude that for all $u\in N^+$, $I(u)<0$. That is $\inf_{u\in N^+}I(u)<0$. □

3. Proof of the main result

In this section, we will prove the main result of this paper (Theorem 1.2). So we assume that all hypotheses of Theorem 1.2 are satisfied, and λ_0, μ_0 are given by Section 2. The proof is divided into two lemmas.

Lemma 3.1. *If $0 < \lambda < \lambda_0$ and $0 < \mu < \mu_0$, then I achieves its minimum on N^+. That is there exists $v \in N^+$, such that $I(v) = \inf_{u \in N^+} I(u)$.*

Proof. Since $N^+ \subset N$ and by Lemma 2.2, we obtain that I is bounded on N^+ and so there exists a minimizing sequence $(u_n)_{n\in\mathbb{N}}$ on N^+ such that $\lim_{n\to\infty} I(u_n) = \inf_{u\in N^+} I(u)$.
I is coercive on N, then $(u_n)_{n\in\mathbb{N}}$ is bounded in W_0 and so,

$$\begin{cases} u_n \rightharpoonup v, & \text{weakly in } W_0, \\ u_n \to v, & \text{strongly in } L^{2-\alpha}, \\ u_n \to v, & \text{strongly in } L^{\nu}(\Omega), \text{ for } 1 < \nu < \frac{Np}{N-ps}. \end{cases}$$

Since $u_n \rightharpoonup v$ in W_0 and using the compact embedding Theorem, we obtain the strongly convergence in L^q. That is

$$\lim_{n\to\infty} \int_\Omega a_2(x)|u_n|^q dx = \int_\Omega a_2(x)|v|^q dx.$$

Moreover, we have

$$\int_\Omega |v|^{2-\alpha} dx - \int_\Omega |u_n - v|^{2-\alpha} dx \le \int_\Omega |u_n|^{2-\alpha} dx \le \int_\Omega |v|^{2-\alpha} dx + \int_\Omega |u_n - v|^{2-\alpha} dx.$$

Using the Hölder inequality, we obtain

$$\int_\Omega |v|^{2-\alpha} dx - c\|u_n - v\|_2^{2-\alpha} \le \int_\Omega |u_n|^{2-\alpha} dx \le \int_\Omega |v|^{2-\alpha} dx + c\|u_n - v\|_2^{2-\alpha}.$$

We deduce by passing to the limit $n \to \infty$ that

$$\int_\Omega |v|^{2-\alpha} dx - \circ(1) \le \int_\Omega |u_n|^{2-\alpha} dx \le \int_\Omega |v|^{2-\alpha} dx + \circ(1).$$

Which implies that

$$\lim_{n\to\infty} \int_\Omega a_1(x)|u_n|^{2-\alpha} dx = \int_\Omega a_1(x)|v|^{2-\alpha} dx.$$

We claim that $v \in W_0 \setminus \{0\}$. If not then, we deduce by (5) and (6) that

$$\begin{aligned} I(u_n) &= \frac{1}{p}\bar{M}\Big(\|u_n\|_W^p\Big) - \frac{1}{q}\|u_n\|_W^p M(\|u_n\|_W^p) - \lambda\Big(\frac{1}{2-\alpha} - \frac{1}{q}\Big)\int_\Omega a_1(x)|u_n|^{2-\alpha} dx \\ &\ge \frac{q - pm - p}{qp(m+1)}\|u_n\|_W^p M\Big(\|u_n\|_W^p\Big) - \lambda\Big(\frac{1}{2-\alpha} - \frac{1}{q}\Big)\int_\Omega a_1(x)|u_n|^{2-\alpha} dx. \end{aligned}$$

Since $q - pm - p > 0$, we obtain by passing to the limit that $\lim_{n\to\infty} I(u_n) = \inf_{u\in N^+} I(u) \ge 0$.
This is a contradiction by the Lemma 2.6. So, $v \in W_0 \setminus \{0\}$.
We claim that $u_n \to v$. Suppose this is not true, (i.e. $u_n \nrightarrow v$). Then

$$\bar{M}\Big(\|v\|_W^p\Big) < \liminf_{n\to\infty} \bar{M}\Big(\|u_n\|_W^p\Big).$$

Hence

$$\begin{aligned}\lim_{n\to\infty} I(u_n) &= \lim_{n\to\infty}\inf\Big(\frac{1}{p}\bar{M}(\|u_n\|_W^p) - \frac{\lambda}{2-\alpha}\int_\Omega a_1(x)|u_n|^{2-\alpha}dx - \frac{1}{q}\int_\Omega a_2(x)|u_n|^q dx\Big)\\ &> \frac{1}{p}\bar{M}(\|v(x)\|_W^p) - \frac{\lambda}{2-\alpha}\int_\Omega a_1(x)|v|^{2-\alpha}dx - \frac{1}{q}\int_\Omega a_2(x)|v|^q dx\\ &= I(v).\end{aligned}$$

From Lemma 2.5, there exist $t_1 > 0$ such that $t_1 v \in N^+$. Using again $u_n \nrightarrow v$, we get

$$\|v\|_W^p M\Big(t_1^p\|v\|_W^p\Big) < \lim_{n\to\infty}\inf \|u_n\|_W^p M\Big(t_1^p\|u_n\|_W^p\Big).$$

Hence

$$\begin{aligned}\lim_{n\to\infty}\frac{d}{dt}I(tu_n)\Big|_{t=t_1} &= \lim_{n\to\infty}\inf\Big(t_1^{p-1}\|u_n\|_W^p M(t_1^p\|u_n\|_W^p) - \lambda t_1^{r-1}\int_\Omega a_1(x)|u_n|^r dx\\ &\quad - t_1^{q-1}\int_\Omega a_2(x)|u_n|^q dx\Big)\\ &> t_1^{p-1}\|v(x)\|_W^p M\Big(t_1^p\|v(x)\|_W^p\Big) - \lambda t_1^{1-\alpha}\int_\Omega a_1(x)|v|^{2-\alpha}dx - t_1^{q-1}\int_\Omega a_2(x)|v|^q dx\\ &= \frac{d}{dt}I(tv)\Big|_{t=t_1} = 0.\end{aligned}$$

We deduce for sufficiently large n that

$$\frac{d}{dt}I(tu_n)\Big|_{t=t_1} > 0. \tag{16}$$

On the other hand, $u_n \in N^+$ for $n \in \mathbb{N}$, that is

$$\frac{d}{dt}I(tu_n)\Big|_{t=1} = 0 \text{ and } \frac{d^2}{dt^2}I(tu_n)\Big|_{t=1} > 0.$$

Which implies by Lemma 2.5 that $\frac{d}{dt}I(tu_n) < 0$ in $(0,1)$ and we deduce by (16) that $t_1 > 1$. Using the fact that $t_1 v \in N^+$, we deduce by Lemma 2.5 that $\frac{d}{dt}I(tv) < 0$ in $(0,t_1)$, that is $t \to I(tv)$ is strictly decreasing in $(0,t_1)$. Then

$$I(t_1 v) < I(v) \le \lim_{n\to\infty} I(u_n) = \inf_{u\in N^+} I(u).$$

Which is a contradiction with the fact that $t_1 v \in N^+$. So $u_n \to v$, strongly in W_0. Since $u_n \in N^+$ for $n \in \mathbb{N}$, that is $\frac{d}{dt}I(tu_n)\Big|_{t=1} = 0$ and $\frac{d^2}{dt^2}I(tu_n)\Big|_{t=1} > 0$. Passing to the limit $n \to \infty$, we get $\frac{d}{dt}I(tv)\Big|_{t=1} = 0$ and $\frac{d^2}{dt^2}I(tv)\Big|_{t=1} \ge 0$. Since $N^0 = \emptyset$ by Lemma 2.3, we deduce that $\frac{d^2}{dt^2}I(tv)\Big|_{t=1} > 0$ and so $v \in N^+$. □

Lemma 3.2. *If $0 < \lambda < \lambda_0$ and $0 < \mu < \mu_0$, then I achieves its minimum on N^-. That is there exists $w \in N^-$, such that $I(w) = \inf_{u\in N^-} I(u)$.*

Proof. We obtain by the fact that $N^- \subset N$ and Lemma 2.2 that I is bounded on N^- and so there exists a minimizing sequence $(u_n)_{n\in\mathbb{N}}$ on N^- such that $\lim_{n\to\infty} I(u_n) = \inf_{u\in N^-} I(u)$.
Since I is coercive on N, then $(u_n)_{n\in\mathbb{N}}$ is bounded in W_0 and so,

$$\begin{cases} u_n \rightharpoonup w, & \text{weakly in } W_0,\\ u_n \to w, & \text{strongly in } L^{2-\alpha},\\ u_n \to w, & \text{strongly in } L^{\nu}(\Omega), \text{ for } 1 < \nu < \frac{Np}{N-ps}.\end{cases}$$

Since $u_n \rightharpoonup w$ in W_0 and using the compact embedding Theorem, we obtain the strongly convergence in L^q. That is

$$\lim_{n\to\infty}\int_\Omega a_2(x)|u_n|^q dx = \int_\Omega a_2(x)|w|^q dx.$$

Moreover, we have

$$\int_\Omega |w|^{2-\alpha}dx - \int_\Omega |u_n - w|^{2-\alpha}dx \le \int_\Omega |u_n|^{2-\alpha}dx \le \int_\Omega |w|^{2-\alpha}dx + \int_\Omega |u_n - w|^{2-\alpha}dx$$

Using the Hȯlder inequality, we obtain

$$\int_\Omega |w|^{2-\alpha}dx - c\|u_n - w\|_2^{2-\alpha} \le \int_\Omega |u_n|^{2-\alpha}dx \le \int_\Omega |w|^{2-\alpha}dx + c\|u_n - w\|_2^{2-\alpha}.$$

We deduce by passing to the limit $n \to \infty$ that

$$\int_\Omega |w|^{2-\alpha}dx - \circ(1) \le \int_\Omega |u_n|^{2-\alpha}dx \le \int_\Omega |w|^{2-\alpha}dx + \circ(1).$$

Which implies that

$$\lim_{n\to\infty}\int_\Omega a_1(x)|u_n|^{2-\alpha}dx = \int_\Omega a_1(x)|w|^{2-\alpha}dx.$$

We claim that $u_n \to w$. Suppose this is not true, (i.e. $u_n \nrightarrow w$). Then

$$\bar{M}\Big(\|w\|_W^p\Big) < \lim_{n\to\infty}\inf \bar{M}\Big(\|u_n\|_W^p\Big).$$

From Lemma 2.5, there exist $t_2 > t_0 > 0$ such that $t_2 w \in N^-$. Using again $u_n \nrightarrow w$, we get

$$\|w\|_W^p M\Big(t_2^p\|w\|_W^p\Big) < \lim_{n\to\infty}\inf \|u_n\|_W^p M\Big(t_2^p\|u_n\|_W^p\Big).$$

Hence

$$\begin{aligned}
\lim_{n\to\infty} I(t_2u_n) &= \lim_{n\to\infty}\inf\Big(\frac{1}{p}\bar{M}(t_2^p\|u_n\|_W^p) - \frac{\lambda t_2^{2-\alpha}}{2-\alpha}\int_\Omega a_1(x)|u_n|^{2-\alpha}dx - \frac{t_2^q}{q}\int_\Omega a_2(x)|u_n|^q dx\Big)\\
&> \frac{1}{p}\bar{M}(t_2^p\|w\|_W^p) - \frac{\lambda t_2^{2-\alpha}}{2-\alpha}\int_\Omega a_1(x)|w|^{2-\alpha}dx - \frac{t_2^q}{q}\int_\Omega a_2(x)|w|^q dx\\
&= I(t_2w)
\end{aligned}$$

and

$$\begin{aligned}
\lim_{n\to\infty}\frac{d}{dt}I(tu_n)\Big|_{t=t_2} &= \lim_{n\to\infty}\inf\Big(t_2^{p-1}\|u_n\|_W^p M(t_2^p\|u_n\|_W^p) - \lambda t_2^{1-\alpha}\int_\Omega a_1(x)|u_n|^{2-\alpha}dx\\
&\quad -t_2^{q-1}\int_\Omega a_2(x)|u_n|^q dx)\Big)\\
&> t_2^{p-1}\|w\|_W^p M\Big(t_2^p\|w\|_W^p\Big) - \lambda t_2^{1-\alpha}\int_\Omega a_1(x)|w|^{2-\alpha}dx - t_2^{q-1}\int_\Omega a_2(x)|w|^q dx\\
&= \frac{d}{dt}I(tw)\Big|_{t=t_2} = 0.
\end{aligned}$$

We deduce for sufficiently large n that

$$\frac{d}{dt}I(tu_n)\Big|_{t=t_2} > 0. \tag{17}$$

On the other hand, we have $u_n \in N^-$ for $n \in \mathbb{N}$, that is

$$\frac{d}{dt}I(tu_n)\Big|_{t=1} = 0 \text{ and } \frac{d^2}{dt^2}I(tu_n)\Big|_{t=1} < 0.$$

Which implies by Lemma 2.5 that $\frac{d}{dt}I(tu_n) < 0$ on $(1,\infty)$ and we obtain by (17) that $t_2 < 1$. Using the fact that $t_2 w \in N^-$, we deduce by Lemma 2.5 that $\frac{d}{dt}I(tw) > 0$ in (t_0, t_2) and $\frac{d}{dt}I(tw) < 0$ in (t_2,∞), that is $t \to I(tw)$ is strictly decreasing in (t_2,∞). We have by Lemma (2.5) that $\frac{d}{dt}I(tu_n) > 0$ on $(t_0, 1)$ in particular on $(t_2, 1)$. Then

$$I(t_2w) < \lim_{n\to\infty} I(t_2u_n) \leq \lim_{n\to\infty} I(u_n) = \inf_{u\in N^-} I(u).$$

Which is a contradiction with the fact that $t_2 w \in N^-$. So $u_n \to w$, strongly in W_0.
Since $u_n \in N^-$ for $n \in \mathbb{N}$, that is $\frac{d}{dt}I(tu_n)\Big|_{t=1} = 0$ and $\frac{d^2}{dt^2}I(tu_n)\Big|_{t=1} < 0$. Passing to the limit $n \to \infty$, we get $\frac{d}{dt}I(tw)\Big|_{t=1} = 0$ and $\frac{d^2}{dt^2}I(tw)\Big|_{t=1} \leq 0$. Since $N^0 = \emptyset$ by Lemma 2.3, we deduce that $\frac{d^2}{dt^2}I(tw)\Big|_{t=1} < 0$ and so $w \in N^-$. Moreover, w is non-trivial, else $w \in N^0$ which contradicted with the fact that $N^0 = \emptyset$. □

Lemma 3.3. *Let $u \in N^+$ (respectively N^-). Then for any $\varphi \in W_0$, there exist a continuous positive function f and $\xi > 0$ such that for all $s \in \mathbb{R}$ with $|s| < \xi$, we have*

$$f(0) = 1 \text{ and } f(s)(u + s\varphi) \in N^+ \text{ (respectively } N^-).$$

Proof. Let $u \in N^+$. For $s, t \in \mathbb{R}$ and $\varphi \in W_0$, we recall that

$$\begin{aligned}\frac{d}{dt}I(t(u+s\varphi)) &= t^{p-1}\|u+s\varphi\|_W^p M\Big(t^p\|u+s\varphi\|_W^p\Big) - \lambda t^{1-\alpha}\int_\Omega a_1(x)|u+s\varphi|^{2-\alpha}dx \\ &\quad -t^{q-1}\int_\Omega a_2(x)|u+s\varphi|^q dx.\end{aligned}$$

Using the fact that $u \in N^+$, we obtain that

$$\begin{aligned}\frac{d}{dt}I(t(u+s\varphi))\Big|_{(s,t)=(0,1)} &= \|u(x)\|_W^p M\Big(\|u(x)\|_W^p\Big) - \lambda\int_\Omega a_1(x)|u|^{2-\alpha}dx - \int_\Omega a_2(x)|u|^q dx \\ &= 0\end{aligned}$$

and

$$\begin{aligned}\frac{d^2}{dt^2}I(t(u+s\varphi))\Big|_{(s,t)=(0,1)} &= (p-1)\|u(x)\|_W^p M\Big(\|u(x)\|_W^p\Big) + p\|u(x)\|_W^{2p} M'(\|u(x)\|_W^p) \\ &\quad -\lambda(1-\alpha)\int_\Omega a_1(x)|u|^{2-\alpha}dx - (q-1)\int_\Omega a_2(x)|u|^q dx \\ &> 0.\end{aligned}$$

Then by applying the implicit function theorem to the function $(s,t) \longrightarrow \frac{d}{dt}I(t(u+s\varphi))$ at the point $(0,1)$, there exists $\varepsilon > 0$ and a continuous function f satisfying

$$f(0) = 1,\ \frac{d}{dt}I(t(u+s\varphi)) = 0, \text{ for } |s| < \varepsilon \text{ and } t = f(s).$$

That is for $|s| < \varepsilon$, we have

$$\begin{aligned} 0 &= \frac{d}{dt} I(t(u+s\varphi))\Big|_{t=f(s)} = (f(s))^{p-1}\|u+s\varphi\|_W^p M\Big(\|f(s)(u+s\varphi)\|_W^p\Big) \\ &\quad -\lambda(f(s))^{1-\alpha}\int_\Omega a_1(x)|u+s\varphi|^{2-\alpha}dx - (f(s))^{q-1}\int_\Omega a_2(x)|u+s\varphi|^q dx \\ &= \frac{1}{f(s)}\Big(\|f(s)(u+s\varphi)\|_W^p M\Big(\|f(s)(u+s\varphi)\|_W^p\Big) \\ &\quad -\lambda\int_\Omega a_1(x)|f(s)(u+s\varphi)|^{2-\alpha}dx - \int_\Omega a_2(x)|f(s)(u+s\varphi)|^q dx\Big). \end{aligned}$$

So, $f(s)(u+s\varphi) \in N$ for $|s| < \varepsilon$. Moreover, since $u \in N^+$, we can choose $\xi \in (0,\varepsilon)$ small enough such that for $|s| < \xi$

$$\begin{aligned} \frac{d^2}{dt^2} I(t(u+s\varphi))\Big|_{t=f(s)} &= \frac{1}{(f(s))^2}\Big((p-1)\|f(s)(u+s\varphi)\|_W^p M(\|f(s)(u+s\varphi)\|_W^p) \\ &\quad +p\|f(s)(u+s\varphi)\|_W^{2p} M'(\|f(s)(u+s\varphi)\|_W^p) \\ &\quad -\lambda(1-\alpha)\int_\Omega a_1(x)|f(s)(u+s\varphi)|^{2-\alpha}dx - (q-1)\int_\Omega a_2(x)|f(s)(u+s\varphi)|^q dx\Big) \\ &> 0. \end{aligned}$$

Hence

$$f(s)(u+s\varphi) \in N^+ \text{ for } |s| < \xi.$$

Finally, by the same arguments, we obtain the proof for $u \in N^-$. □

Proof of Theorem 1. From Lemma 3.1, there exists $v \in N^+$ such that $I(v) = \inf_{u\in N^+} I(u)$. By applying Lemma 3.3, we deduce for any $\varphi \in W_0$ that there exist a continuous positive function f and $\xi > 0$ such that for $|s| < \xi$, we have

$$f(0) = 1 \text{ and } f(s)(v+s\varphi) \in N^+.$$

This implies that there exists $\xi_0 \in (0,\xi)$ such that for $|s| < \xi_0$, we have

$$I(v) \le I(v+s\varphi).$$

Dividing by $s > 0$ and passing to the limit $s \to 0$, we obtain

$$\begin{aligned} &M(\|v\|_W^p)\Big(\int_{\mathbb{R}^{2N}} |v(x)-v(y)|^{p-2}(v(x)-v(y))(\varphi(x)-\varphi(y))K(x-y)dxdy \\ &+\int_{\mathbb{R}^N} V(x)|v|^{p-2}v\varphi dx\Big) - \int_\Omega (\lambda a_1(x)|v|^{r-2}v + a_2(x)|v|^{q-2}v)\varphi dx \ge 0. \end{aligned}$$

If we replace φ by $-\varphi$, the previous inequality remains true. Hence v is a weak solution of problem (1). Moreover, from Lemma 2.6, the function v is non-trivial. Using Lemma 3.2 and Lemma 3.3, we conclude by the same way that w is a weak solution of the problem (1), moreover, since $w \in N^-$ we see that w is nontrivial. Finally, Remark 2.4, implies that v and w are distinct. This completes the proof.

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