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Generalized Buzano inequality

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Abstract. If P is an orthogonal projection defined on an inner product space $\mathcal{H}$, then the inequality

$$|\langle Px, y\rangle| \leq \frac{1}{2}[\|x\|\|y\| + |\langle x, y\rangle|]$$

fulfills for any $x, y \in \mathcal{H}$ (see [10]). In particular, when P is the identity operator, then it recovers the famous Buzano inequality. We obtain generalizations of such classical inequality, which hold for certain families of bounded linear operators defined on $\mathcal{H}$. In addition, several new inequalities involving the norm and numerical radius of an operator are established.

1. Introduction

Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex numbers field $\mathbb{K}$. The following inequality is well known in literature as the Cauchy-Schwarz inequality

$$|\langle x, y\rangle| \leq \|x\|\|y\|, \tag{1}$$

for any $x, y \in \mathcal{H}$. The equality in (1) holds if and only if there exists a constant $\alpha \in \mathbb{K}$ such that $x = \alpha y$.

In [6], Maria Luisa Buzano gave the following extension of the celebrated Cauchy– Schwarz inequality in $\mathcal{H}$

$$|\langle x, z\rangle\langle z, y\rangle| \leq \frac{1}{2}(|\langle x, y\rangle| + \|x\|\|y\|)\|z\|^2, \tag{2}$$

for any $x, y, z \in \mathcal{H}$. Last inequality is called Buzano inequality.

The original proof of Buzano has it difficulty since it requires some facts about orthogonal decomposition of a complete inner product space.

In [8], Dragomir established a refinement of (1) which implies the Buzano inequality. Moreover, Fuji and Kubo [13] gave a simpler proof of (2) by using an orthogonal projection on a subspace of $\mathcal{H}$ and (1). Furthermore, they characterized when the equality holds.

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This paper aims to present new generalizations of Buzano inequality and it is organized as follows. Section 2 contains some definitions and usual results about bounded linear operators defined on a Hilbert space. In Section 3, we present and prove the $\frac{1}{\alpha}$-Buzano inequality (if $\alpha = 2$ gives the classical Buzano inequality) and it is devoted to describing different families of operators which fulfill such inequality for different values of the parameter α. Finally, in Section 4 relates the distinct inequalities previously obtained with the numerical radius, improving new bounds for the last one.

2. Preliminaries

As any pre-Hilbert space can be completed to a Hilbert space, from now on, we suppose that $\mathcal{H}$ is a Hilbert space. Let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators acting on a separable non trivial complex Hilbert space $\mathcal{H}$ with an inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$. The symbol I stands for the identity operator and $\mathcal{GL}(\mathcal{H})$ denotes the group of invertible operators on $\mathcal{H}$.

The range of every operator is denoted by $\mathcal{R}(T)$, its null space by $\mathcal{N}(T)$. If $T \in \mathcal{B}(\mathcal{H})$, we say that T is a positive operator, $T \geq 0$, whenever $\langle Tx, x \rangle \geq 0$ for all $x \in \mathcal{H}$ and we denote by $\mathcal{B}(\mathcal{H})^+$, the subset of all positive bounded linear operators definded on $\mathcal{H}$. The definition of positivity induces the order $T \geq S$ for self-adjoint operators if and only if $T - S \geq 0$. For any $T \in \mathcal{B}(\mathcal{H})^+$, there exists a unique positive $T^{1/2} \in \mathcal{B}(\mathcal{H})$ such that $T = (T^{1/2})^2$. Let T^* be the adjoint of T and $|T| = (T^*T)^{1/2}$.

The polar decomposition theorem asserts that for every operator $T \in \mathcal{B}(\mathcal{H})$ there is a partial isometry $V \in \mathcal{B}(\mathcal{H})$ such that can be written as the product $T = V|T|$. In particular, V satisfying $\mathcal{N}(V) = \mathcal{N}(T)$ exists and is uniquely determined.

For any $T \in \mathcal{B}(\mathcal{H})$, we denote by $\sigma(T)$ its spectrum and by $\sigma_{app}(T)$ its approximate point spectrum, that is

$$\sigma_{app}(T) = \{\lambda \in \mathbb{C} : \exists \{x_n\}_{n \in \mathbb{N}}, \|x_n\| = 1 \text{ and } \lim_{n \to \infty} \|Tx_n - \lambda x_n\| = 0\}.$$

For any $T \in \mathcal{B}(\mathcal{H})$, we define $m(T) = \inf\{\|Tx\| : x \in \mathcal{H}, \|x\| = 1\}$. Clearly, $m(T) \geq 0$ and $m(T) > 0$ if and only if $0 \notin \sigma_{app}(T)$ ([16]).

For a linear operator T on a Hilbert space $\mathcal{H}$, the numerical range $W(T)$ is the image of the unit sphere of $\mathcal{H}$ under the quadratic form $x \to \langle Tx, x \rangle$. More precisely,

$$W(T) = \{\langle Tx, x \rangle : x \in \mathcal{H}, \|x\| = 1\}.$$

The numerical range of an operator is a convex subset of the complex plane ([14]). Then, for any T in $\mathcal{B}(\mathcal{H})$ we define the numerical radius of T,

$$\omega(T) = \sup\{|\lambda| : \lambda \in W(T)\}.$$

It is well-known that $\omega(\cdot)$ defines a norm on $\mathcal{B}(\mathcal{H})$, and we have for all $T \in \mathcal{B}(\mathcal{H})$,

$$\frac{1}{2}\|T\| \leq \omega(T) \leq \|T\|. \tag{3}$$

Thus, the usual operator norm and the numerical radius are equivalent. Inequalities in (3) are sharp if $T^2 = 0$, then the first inequality becomes equality, while the second inequality becomes an equality if T is normal.

For any compact operator $T \in \mathcal{B}(\mathcal{H})$ and $j \in \mathbb{N}$, let $s_j(T) = \lambda_j(|T|)$, be the j-th singular value of T, i.e. the j-th eigenvalue of $|T|$ in decreasing order and repeated according to multiplicity. Let $\operatorname{tr}(\cdot)$ be the trace functional,

$$\operatorname{tr}(T) = \sum_{j=1}^{\infty} \langle Te_j, e_j \rangle,$$

where $\{e_j\}_{j=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}$. Note that this coincides with the usual definition of the trace if $\mathcal{H}$ is finite-dimensional.

Let $T = x \otimes y$ be a rank one operator defined by $T(z) = \langle z, y\rangle x$ with $x, y, z \in \mathcal{H}$. Then, by Lemma 2.1 in [7] and using the well-known fact that $\mathrm{tr}(x \otimes y) = \langle x, y\rangle$, we obtain

$$\omega(x \otimes y) = \frac{1}{2}\left(|\mathrm{tr}(x \otimes y)| + \|x \otimes y\|\right) = \frac{1}{2}\left(|\langle x, y\rangle| + \|x\|\|y\|\right).$$

We remark that the numerical radius of the rank one operator $T = x \otimes y$ coincides with the upper bound of Buzano inequality. From this fact, we are able to give a new proof of inequality (2) using this fact. If $\|z\| = 1$, then $\langle Tz, z\rangle = \langle z, y\rangle\langle x, z\rangle \in W(T)$ and

$$|\langle x, z\rangle\langle z, y\rangle| = |\langle Tz, z\rangle| \leq \omega(T) = \frac{1}{2}\left(|\langle x, y\rangle| + \|x\|\|y\|\right).$$

For $T \in \mathcal{B}(\mathcal{H})$, we have, by definition,

$$dist(I, \mathbb{C}T) := \inf_{\gamma \in \mathbb{C}} \|\gamma T - I\| \text{ and } dist(T, \mathbb{C}I) := \inf_{\beta \in \mathbb{C}} \|T - \beta I\|.$$

Evidently there is at least one complex number $\gamma_0 \in \mathbb{C}$ such that $dist(I, \mathbb{C}T) = \|\gamma_0 T - I\|$ and in addition, if $m(T) > 0$ then the value γ_0 is unique. Following Stampfli [19], we call such scalar as the center of mass of T and we denote by $c(T)$. For $A, T \in \mathcal{B}(\mathcal{H})$ such that $m(T) > 0$ we consider

$$M_T(A) = \sup_{\|x\|=1}\left[\|Ax\|^2 - \frac{|\langle Ax, Tx\rangle|^2}{\|Tx\|^2}\right]^{1/2}. \tag{4}$$

In [17], Paul proved that $M_T(A) = dist(A, \mathbb{C}T)$.

Given $T, S \in \mathcal{B}(\mathcal{H})$ we said that T is Birkhoff-James orthogonal to S if and only if $\|T\| \leq \|T - \lambda S\|$ for every $\lambda \in \mathbb{C}$.

3. $\frac{1}{\alpha}$-Buzano inequality

In the last decades, several mathematicians presented different proofs of Buzano inequality. We start by presenting a new and simple proof of such inequality using a rank one operator.

Given $z \in \mathcal{H}$ with $\|z\| = 1$ and $\alpha \in \mathbb{C}$, we consider the rank one operator $T = z \otimes z$. Then, for any $u \in \mathcal{H}$, it holds

$$\|(\alpha T - I)u\|^2 = \|\alpha Tu - u\|^2 = (|\alpha - 1|^2 - 1)|\langle z, u\rangle|^2 + \|u\|^2 \leq \max\{1, |\alpha - 1|^2\}\|u\|^2.$$

Hence $\|\alpha T - I\| \leq \max\{1, |\alpha - 1|\}$ and for any $x, y \in \mathcal{H}$ we get

$$|\langle(\alpha T - I)\, x, y\rangle| \leq \|T - I\|\|x\|\|y\| \leq \max\{1, |\alpha - 1|\}\|x\|\|y\|.$$

In conclusion, we have

$$|\alpha\langle x, z\rangle\langle z, y\rangle - \langle x, y\rangle| \leq \max\{1, |\alpha - 1|\}\|x\|\|y\|, \tag{5}$$

for any $x, y, z \in \mathcal{H}$ with $\|z\| = 1$ and $\alpha \in \mathbb{C}$. If $\alpha \in \mathbb{C} - \{0\}$, then (5) is equivalent to

$$\left|\langle x, z\rangle\langle z, y\rangle - \frac{1}{\alpha}\langle x, y\rangle\right| \leq \frac{1}{|\alpha|}\max\{1, |\alpha - 1|\}\|x\|\|y\|.$$

From the continuity property of modulus for complex numbers, we obtain

$$|\langle x, z\rangle\langle z, y\rangle| \leq \frac{1}{|\alpha|}(|\langle x, y\rangle| + \max\{1, |\alpha - 1|\}\|x\|\|y\|),$$

for any $x, y, z \in \mathcal{H}$ with $\|z\| = 1$. The value $\alpha = 2$ gives Buzano inequality.

We note that the inequality (5) was previously obtained by Moslehian et al. ([15], Corollary 2.5) using properties of singular values.

The main idea in the previous proof was to obtain a bound for the distance between a rank one operator and the identity operator. On the other hand, Fujii and Kubo in [13] based their proof of Buzano inequality on the fact that $\|2P - I\| \leq 1$ where P is an orthogonal projection. Because of the above, we are in a position to prove our first result in this paper, which generalizes these previous ideas.

Proposition 3.1. *Let $T \in \mathcal{B}(\mathcal{H})$ and $\alpha \in \mathbb{C} - \{0\}$, with $\|\alpha T - I\| \leq 1$. Then, for any $x, y \in \mathcal{H}$*

$$\left|\langle Tx, y\rangle - \frac{1}{\alpha}\langle x, y\rangle\right| \leq \frac{1}{|\alpha|}\|x\|\|y\|,$$

and

$$|\langle Tx, y\rangle| \leq \left|\langle Tx, y\rangle - \frac{1}{\alpha}\langle x, y\rangle\right| + \frac{1}{|\alpha|}|\langle x, y\rangle| \leq \frac{1}{|\alpha|}(|\langle x, y\rangle| + \|x\|\|y\|). \tag{6}$$

On the other, if T fulfills

$$|\langle Tx, y\rangle| \leq \frac{1}{|\alpha|}(|\langle x, y\rangle| + \|x\|\|y\|), \tag{7}$$

for any $x, y \in \mathcal{H}$ and for some $\alpha \in \mathbb{C} - \{0\}$. Then, $dist(\alpha T, \mathbb{C}I) \leq 1$.

Proof. Let $x, y \in \mathcal{H}$ and $\alpha \in \mathbb{C} - \{0\}$, with $\|\alpha T - I\| \leq 1$. By (1), we have

$$\begin{aligned}\left|\langle Tx, y\rangle - \frac{1}{\alpha}\langle x, y\rangle\right| &= \left|\left\langle\left(T - \frac{1}{\alpha}I\right)x, y\right\rangle\right| \leq \frac{1}{|\alpha|}\,\|\alpha T - I\|\,\|x\|\|y\| \\ &\leq \frac{1}{|\alpha|}\|x\|\|y\|.\end{aligned}$$

Therefore, we obtain

$$|\langle Tx, y\rangle| \leq \left|\langle Tx, y\rangle - \frac{1}{\alpha}\langle x, y\rangle\right| + \frac{1}{|\alpha|}|\langle x, y\rangle| \leq \frac{1}{|\alpha|}(|\langle x, y\rangle| + \|x\|\|y\|).$$

If T satisfies (7) and we recall the formula which express the distance from T to the one-dimensional subspace $\mathbb{C}I$ (see [2]),

$$dist(T, \mathbb{C}I) = \sup\{|\langle Tx, y\rangle| : \|x\| = \|y\| = 1, \langle x, y\rangle = 0\},$$

then there exists $\beta \in \mathbb{C} - \{0\}$ such that $\left\|\alpha T - \beta I\right\| \leq 1$. □

The next example shows that not all bounded linear operator satisfies the hypothesis of Proposition 3.1.

Example 3.2. *Consider the unilateral shift operator, $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$, defined by $T(x_1, x_2, x_3, \cdots,) = (0, x_1, x_2, x_3, \cdots)$. Let $e_1 = (1, 0, 0, \cdots) \in l^2(\mathbb{N})$, then $\|e_1\| = 1$ and $\langle -Te_1, e_1\rangle = 0$, by Theorem 2.1 in [1], we have that for any $\alpha \in \mathbb{C} - \{0\}$ it holds*

$$\|I\|^2 + |\alpha|^2 m^2(-T) \leq \|I - \alpha T\|^2.$$

As $-T$ is a left invertible operator, then $m(-T) > 0$ and in consequence

$$1 < \|I\|^2 + |\alpha|^2 m^2(-T) \leq \|I - \alpha T\|^2.$$

For convenience, for any $\alpha \in \mathbb{C} - \{0\}$, we denote by

$$\mathcal{A}_\alpha = \{T \in \mathcal{B}(\mathcal{H}) : \|\alpha T - I\| \leq 1\},$$

the set of bounded operators that fulfills the hypothesis of Proposition 3.1. Since we have proved that each operator belonging to the set $\mathcal{A}_\alpha$ satisfies a $\frac{1}{\alpha}$ Buzano-type inequality, we will call to such set the $\frac{1}{\alpha}$ Buzano set.

Next, we collect some properties of $\mathcal{A}_\alpha$.

Proposition 3.3. *Let* $\alpha \in \mathbb{C} - \{0\}$, *then*

1. $\mathcal{A}_\alpha$ *is a non-empty convex and closed set.*

For any $T \in \mathcal{A}_\alpha$, *then*

2. $\|T\| \leq \frac{2}{|\alpha|}$.
3. $T^* \in \mathcal{A}_{\overline{\alpha}}$.
4. *If* $S \in \mathcal{A}_\alpha$, *then* $T + S \in \mathcal{A}_{\frac{\alpha}{2}}$.
5. *If* $\|\alpha T - I\| < 1$, *then* $T \in \mathcal{GL}(\mathcal{H})$.
6. *If* T *is self-adjoint, then* $T \geq 0$ *or* $-T \geq 0$.
7. *If* $T = h \otimes h$, $h \in \mathcal{H}$ *and* $h \neq 0$, *then* $\alpha = \frac{te^{i\theta}+1}{\|h\|^2}$ *for every* $\theta \in [0, 2\pi]$ *and* $t \in [-1, 1]$.
8. $d(T, \mathbb{C}I) \leq \frac{1}{|\alpha|}$ *and for every* $P \geq 0$ *with* $tr(P) = 1$

$$tr(|T|^2 P) - |tr(TP)|^2 \leq \frac{1}{|\alpha|}.$$

Proof. (1) Let $T, S \in \mathcal{A}_\alpha$ and $\lambda \in [0, 1]$, then

$$\begin{aligned}\|\alpha(\lambda T + (1-\lambda)S) - I\| &\leq \|\alpha\lambda T - \lambda I\| + \|\alpha(1-\lambda)S - (1-\lambda)I\| \\ &= \lambda\|\alpha T - I\| + (1-\lambda)\|\alpha S - I\| \\ &\leq \lambda + 1 - \lambda = 1.\end{aligned}$$

This shows that $\lambda T + (1-\lambda)S \in \mathcal{A}_\alpha$ and therefore, $\mathcal{A}_\alpha$ is convex.

Now, let $\{T_n\}$ be a sequence in $\mathcal{A}_\alpha$ such that converges to $T \in \mathcal{B}(\mathcal{H})$. We must show that $T \in \mathcal{A}_\alpha$. Then, we have for any $n \in \mathbb{N}$ that it hold

$$\|\alpha T - I\| = \|\alpha T - \alpha T_n + \alpha T_n - I\| \leq |\alpha|\|T_n - T\| + 1.$$

Taking limit when n tends to infinity we obtain $\|\alpha T - I\| \leq 1$, i.e. $T \in \mathcal{A}_\alpha$.

The proof of items (2), (3), (4), and (5) are trivial.

(6) If T is normal, then $(\alpha T - I)^*(\alpha T - I) = (\alpha T - I)(\alpha T - I)^*$, for every $\alpha \in \mathbb{C}$. Therefore, $\alpha T - I$ is normal and

$$r(\alpha T - I) = \omega(\alpha T - I) = \|\alpha T - I\|,$$

where $r(\alpha T - I) = \sup\{|\beta| : \beta \in \sigma(\alpha T - I)\}$ is the spectral radius. By the spectral theorem

$$\|\alpha T - I\| = r(\alpha T - I) = \sup\{|\alpha\lambda - 1| : \lambda \in \sigma(T)\}.$$

Thus, if T is selfadjoint and $T \in \mathcal{A}_\alpha$

$$|\alpha\lambda - 1| \leq 1,$$

for all $\lambda \in \sigma(T)$. Therefore, each $\alpha\lambda$ must lie in the unit disk in the complex plane centered in $z = 1$. Also,

$$\lambda \in \mathbb{R} \text{ for every } \lambda \in \sigma(T) \Rightarrow \begin{cases} Re(\lambda\alpha) = \lambda Re(\alpha) \in & [0, 2] \\ Im(\lambda\alpha) = \lambda Im(\alpha) \in & [-1, 1] \end{cases}$$

Suppose there exist $\lambda_j, \lambda_k \in \sigma(T)$ such that $\lambda_j < 0$ and $\lambda_k > 0$, then

$$Re(\lambda_j\alpha) \in [0, 2] \Rightarrow 0 \geq Re(\alpha) \geq \frac{2}{\lambda_j}$$

and

$$Re(\lambda_k\alpha) \in [0, 2] \Rightarrow 0 \leq Re(\alpha) \leq \frac{2}{\lambda_k}.$$

Thus, $Re(\alpha) = 0$, which means that $\lambda_j\alpha$ is pure imaginary and $\alpha = 0$ (because the unit disk in the complex plane centered in $z = 1$ intersects the imaginary axis only in $z = 0$). This is a contradiction, so λ_j and λ_k have the same sign.

(7) For any non-zero $h \in \mathcal{H}$, $T = h \otimes h$ is a compact, positive operator and its spectrum has only two eigenvalues, $\|h\|^2$ and 0. Then, by the proof of item (6)

$$\left|\alpha\|h\|^2 - 1\right| \leq 1$$

if and only if $\alpha\|h\|^2 = te^{i\theta} + 1$, with $|t| \leq 1$ and $\theta \in [0, 2\pi]$.

(8) $1 \geq \|\alpha T - I\| = |\alpha|\|T - \frac{1}{\alpha}I\| \geq |\alpha| d(T, \mathbb{C}I) \Rightarrow \frac{1}{|\alpha|} \geq d(T, \mathbb{C}I)$ and using Proposition 3.1 in [4] we obtain that

$$tr(|T|^2 P) - |tr(TP)|^2 \leq \frac{1}{|\alpha|}$$

for every $P \geq 0$ with $tr(P) = 1$. □

Next, we enumerate other properties of $\mathcal{A}_\alpha$ related to $dist(I, \mathbb{C}T)$ and the center of mass of T.

Lemma 3.4. *Let $T \in \mathcal{B}(\mathcal{H})$.*

1. *If $m(T) > 0$ and $c(T) \neq 0$, then $T \in \mathcal{A}_{c(T)}$.*
2. *If $m(T) > 0$ and $c(T) = 0$, then I is Birkhoff-James orthogonal to T. We deduce that $1 < \|\alpha T - I\|$ and $T \notin \mathcal{A}_\alpha$, for every $\alpha \in \mathbb{C} - \{0\}$.*
3. *If $dist(I, \mathbb{C}T) = 1$ and there exists $\alpha_0 \in \mathbb{C} - \{0\}$ such that*

$$1 = \|\alpha_0 T - I\| < \|\alpha T - I\|, \text{ for all } \alpha \in \mathbb{C} - \{\alpha_0, 0\},$$

then $0 \in \sigma_{app}(T)$ and $T \in \mathcal{A}_{\alpha_0}$.

Proposition 3.5. *Let $T \in \mathcal{B}(\mathcal{H})$ such that $m(T) > 0$, $c(T) \neq 0$ and $dist(T, \mathbb{C}I) = \|T\|$, then $\|I - c(T)T\| = 1$ and, in particular, $T \in \mathcal{A}_{c(T)}$.*

Proof. Let $x \in \mathcal{H}$ such that $\|x\| = 1$, then $|\langle Tx, x\rangle| \geq \|Tx\| \inf_{\|y\|=1} \frac{|\langle Ty,y\rangle|}{\|Ty\|}$ and

$$\left[\|Tx\|^2 - |\langle Tx, x\rangle|^2\right]^{1/2} \leq \left(1 - \inf_{\|y\|=1} \frac{|\langle Ty, y\rangle|^2}{\|Ty\|^2}\right)^{1/2} \|Tx\|.$$

Calculating the supremum of both sides, and using the equality (4), we get

$$\|T\| = dist(T, \mathbb{C}I) \leq dist(I, \mathbb{C}T)\|T\|.$$

Then $1 \leq dist(I, \mathbb{C}T) \leq \|I\| = 1$, i.e. $dist(I, \mathbb{C}T) = \|I - c(T)T\| = 1$. This completes the proof. □

As we have shown in Proposition 3.3, if $T \in \mathcal{A}_\alpha$ with $\|\alpha T - I\| < 1$, then $T \in \mathcal{GL}(\mathcal{H})$ and T verifies Proposition 3.1. Now, we obtain a generalization of such statement for any invertible operator. In order to prove it, we need the following result.

Lemma 3.6 (Corollary 3.7, [5]). *If $T \in \mathcal{GL}(\mathcal{H})$, then there exists a unitary operator $U \in \mathcal{B}(\mathcal{H})$ and a non-zero complex number β such that $\|\beta T^{-1} - U^*\| < 1$.*

Observe that in the value β in Lemma 3.6 satisfies that $|\beta| < \frac{2}{\|T^{-1}\|}$.

Theorem 3.7. *Let $T \in \mathcal{GL}(\mathcal{H})$, then there exists a unitary operator $U \in \mathcal{B}(\mathcal{H})$ and a non- zero complex number β as in Lemma 3.6 such that for any $x, y \in \mathcal{H}$*

$$\left|\langle T^{-1}x, y\rangle - \frac{1}{\beta}\langle U^*x, y\rangle\right| \leq \frac{1}{|\beta|}\|x\|\|y\|,$$

and

$$|\langle T^{-1}x, y\rangle| \leq \left|\langle T^{-1}x, y\rangle - \frac{1}{\beta}\langle U^*x, y\rangle\right| + \frac{1}{|\beta|}|\langle U^*x, y\rangle| \leq \frac{1}{|\beta|}(|\langle U^*x, y\rangle| + \|x\|\|y\|).$$

Proof. It is analogous to the proof of Proposition 3.1, so we omit it. □

3.1. Bounded Linear operators which belong to $\mathcal{A}_\alpha$

We begin this subsection showing when a normal operator belongs to $\mathcal{A}_\alpha$.

Theorem 3.8. *Let T be a normal operator in $\mathcal{B}(\mathcal{H}) - \{0\}$, such that $\sigma(T)$ is fully included into an arc of the disk of radius $\|T\|$ and centered in the origin, with central angle less than π.*
Then, $T \in \mathcal{A}_\alpha$ for every $\alpha \in \mathbb{C}$ such that

$$\arg(\alpha) + \arg(\lambda) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \text{ for all } \lambda \in \sigma(T) \tag{8}$$

and

$$|\alpha| \leq \frac{2}{\|T\|} \min_{\lambda \in \sigma(T)} \cos(\arg(\alpha) + \arg(\lambda)). \tag{9}$$

Proof. Since T is normal, $\alpha T - I$ is a normal operator, then

$$\|\alpha T - I\| = r(\alpha T - I) = \sup\{|\alpha\lambda - 1| : \lambda \in \sigma(T)\} \leq 1$$

if and only if $|\alpha\lambda - 1| \leq 1$ for every $\lambda \in \sigma(T)$. This is equivalent to find if there exists $\alpha \in \mathbb{C}$ such that

$$|\alpha\lambda - 1|^2 \leq 1 \text{ for every } \lambda \in \sigma(T). \tag{10}$$

Taking $\alpha = |\alpha|e^{i\theta}$ and $\lambda = |\lambda|e^{i\varphi_\lambda}$, with $|\lambda| \leq \|T\|$,

$$\begin{aligned} \left||\alpha||\lambda|e^{i(\theta+\varphi_\lambda)} - 1\right|^2 &= (|\alpha||\lambda|\cos(\theta + \varphi_\lambda) - 1)^2 + (|\alpha||\lambda|\sin(\theta + \varphi_\lambda))^2 \\ &= |\alpha|^2|\lambda|^2 - 2|\alpha||\lambda|\cos(\theta + \varphi_\lambda) + 1 \end{aligned}$$

and we can rewrite (10) as follows

$$|\alpha||\lambda|\,(|\alpha||\lambda| - 2\cos(\theta + \varphi_\lambda)) \leq 0.$$

Then, we arrive to the following condition

$$|\alpha||\lambda| - 2\cos(\theta + \varphi_\lambda) \leq 0. \tag{11}$$

Take an $\alpha \in \mathbb{C}$ that satisfies (8) and (9). For $\lambda = 0$ it is immediate that α satisfies condition (11). Consider $\lambda \neq 0$, then $\cos(\arg(\alpha) + \varphi_\lambda) > 0$ for every $\lambda \in \sigma(T)$ and

$$|\alpha| \leq \frac{2}{\|T\|} \min_{\lambda \in \sigma(T)} \cos(\arg(\alpha) + \arg(\lambda)) \leq \frac{2}{|\lambda|} \cos(\arg(\alpha) + \arg(\lambda)), \lambda \neq 0.$$

Therefore, α fulfills the condition (11) for every $\lambda \in \sigma(T)$ and we conclude that $\|\alpha T - I\| \leq 1$ ($T \in \mathcal{A}_\alpha$). □

In order to fulfill (11) and $\cos(\arg(\alpha) + \varphi_\lambda) \geq 0$, it is a necessary condition that the spectrum of T lies in into an arc of the disk of radius $\|T\|$ and centered in the origin, with central angle less than π. Otherwise, it is not possible to fix any α such that the property holds. Additionally, we exclude $\arg(\alpha) + \varphi_\lambda = \pm\frac{\pi}{2}$, since $|\alpha||\lambda| = 0$ if and only if $\lambda = 0$ or $\alpha = 0$.

For example, if T is Hermitian $\varphi_\lambda \in \{0, \pi\}$ it can be seen that there is no α that (11) holds, unless $\lambda \geq 0$ for every $\lambda \in \sigma(T)$ ($\varphi_\lambda = 0$), or $\lambda \leq 0$ for every $\lambda \in \sigma(T)$ ($\varphi_\lambda = \pi$). Thus, the unique Hermitian operators T that can reach (11) are semidefinite positive or semidefinite negative, as we show in item 6 of Proposition 3.3.

In particular, for positive operators, we arrive to the following result.

Corollary 3.9. *If $T \in \mathcal{B}(\mathcal{H})^+$, then $T \in \mathcal{A}_\alpha$ for every $\alpha \in \mathbb{C}$ such that $|\alpha| \leq \frac{a}{\|T\|} \leq \frac{2}{\|T\|}$ and $\cos(\arg(\alpha)) \geq \frac{a}{2}$.*

Proof. As we mention before, in this case $\varphi_\lambda = 0$ for every $\lambda \in \sigma(T)$. Then,

$$|\alpha||\lambda| - 2\cos(\arg(\alpha)) \le \frac{a}{\|T\|}|\lambda| - 2\cos(\arg(\alpha)) \le a - 2\cos(\arg(\alpha)) \le 0.$$

□

The next result is a generalization of Buzano inequality for any bounded linear operator.

Theorem 3.10. *Let $T \in \mathcal{B}(\mathcal{H}) - \{0\}$. Then, for any $x, y \in \mathcal{H}$*

$$\left|\langle Tx, Ty\rangle - \frac{\|T\|^2}{2}\langle x, y\rangle\right| \le \frac{\|T\|^2}{2}\|x\|\|y\|,$$

and

$$\left|\langle Tx, Ty\rangle\right| \le \left|\langle Tx, Ty\rangle - \frac{\|T\|^2}{2}\langle x, y\rangle\right| + \frac{\|T\|^2}{2}|\langle x, y\rangle| \le \frac{\|T\|^2}{2}(|\langle x, y\rangle| + \|x\|\|y\|). \tag{12}$$

Proof. By Corollary 3.9, if $S \in \mathcal{B}(\mathcal{H})^+ - \{0\}$, then $S \in \mathcal{A}_{\frac{2}{\|S\|}}$. In particular, if we consider the positive operator $S = T^*T$, then we conclude that $T^*T \in \mathcal{A}_{\frac{2}{\|T\|^2}}$ and the proof is complete as a consequence of Proposition 3.1 □

The constant $\frac{\|T\|^2}{2}$ is best possible in (12). Now, if we assume that (12) holds with a constant $C > 0$, i.e.

$$\left|\langle Tx, Ty\rangle\right| \le C(|\langle x, y\rangle| + \|x\|\|y\|),$$

for any $T \in \mathcal{B}(\mathcal{H})$. So, if we choose $x = y$, then $\|Tx\|^2 \le 2C\|x\|^2$ and we deduce that $2C \ge \|T\|^2$. Thus, (12) is an improvement and refinement of

$$\left|\langle Tx, Ty\rangle\right| \le \frac{\|T\|^2}{2}(|\langle x, y\rangle| + \|x\|\|y\|),$$

which was obtained in a different way earlier by Dragomir in [11] using a non-negative Hermitian form on a Hilbert space.

From the polar decomposition of any bounded linear operator and the main idea used in the proof of Theorem 3.10, we have the following statement.

Corollary 3.11. *Let $T \in \mathcal{B}(\mathcal{H})$ and $x, y \in \mathcal{H}$. Then,*

$$\begin{aligned} |\langle Tx, y\rangle| &= |\langle |T|x, V^*y\rangle| \le \left|\langle Tx, y\rangle - \frac{\|T\|}{2}\langle x, V^*y\rangle\right| + \frac{\|T\|}{2}|\langle x, V^*y\rangle| \\ &\le \frac{\|T\|}{2}(|\langle x, V^*y\rangle| + \|x\|\|V^*y\|) \\ &\le \frac{\|T\|}{2}(|\langle x, V^*y\rangle| + \|x\|\|y\|). \end{aligned} \tag{13}$$

where $T = V|T|$ is the polar decomposition of T.

Remark 3.12. *Inequality* (13) *is an improvement and refinement of a result recently obtained by Sababheh et al. (see [18, Remark 3.1]).*

Recall that T is called a positive contraction if $0 \le T \le I$. As a consequence of Corollary 3.9, we conclude that $T \in \mathcal{A}_\alpha$ for every $\alpha \in [0, 2]$.

Now, we obtain a refinement of the classical Cauchy-Schwarz inequality, using positive contractions. The idea of the proof is based in [11, Theorem 2.1]. Recently, in [18] the same result was obtained with a different proof.

Theorem 3.13. *Let $T \in \mathcal{B}(\mathcal{H})$ be a positive contraction and $x, y \in \mathcal{H}$, then*

$$|\langle x, y\rangle| + \langle Tx, x\rangle^{1/2}\langle Ty, y\rangle^{1/2} - |\langle Tx, y\rangle| \leq \|x\|\|y\|.$$

Proof. For any $x, y \in \mathcal{H}$ and the elementary inequality $(ac - bd)^2 \geq (a^2 - b^2)(c^2 - d^2)$, which holds for any real numbers a, b, c, d, we have

$$\begin{aligned}\left(\|x\|\|y\| - \langle Tx, x\rangle^{1/2}\langle Ty, y\rangle^{1/2}\right)^2 &\geq (\|x\|^2 - \langle Tx, x\rangle)(\|y\|^2 - \langle Ty, y\rangle) \\ &= \langle (I - T)x, x\rangle\langle (I - T)y, y\rangle. \qquad (14)\end{aligned}$$

As T is a positive contraction, then $I - T \in \mathcal{B}(\mathcal{H})^+$. By Cauchy-Schwarz inequality for positive operators

$$\langle (I - T)x, x\rangle\langle (I - T)y, y\rangle \geq |\langle (I - T)x, y\rangle|^2 = |\langle x, y\rangle - \langle Tx, y\rangle|^2. \qquad (15)$$

Now, by (14) and (15),

$$\left(\|x\|\|y\| - \langle Tx, x\rangle^{1/2}\langle Ty, y\rangle^{1/2}\right)^2 \geq |\langle x, y\rangle - \langle Tx, y\rangle|^2, \qquad (16)$$

for any $x, y \in \mathcal{H}$. Since $\|x\| \geq \langle Tx, x\rangle^{1/2}$ and $\|y\| \geq \langle Ty, y\rangle^{1/2}$, by taking the square root, (16) is equivalent to

$$\|x\|\|y\| - \langle Tx, x\rangle^{1/2}\langle Ty, y\rangle^{1/2} \geq |\langle x, y\rangle - \langle Tx, y\rangle|. \qquad (17)$$

On making use of the triangle inequality for the modulus, we have

$$\begin{aligned}\|x\|\|y\| - \langle Tx, x\rangle^{1/2}\langle Ty, y\rangle^{1/2} &\geq |\langle x, y\rangle - \langle Tx, y\rangle| \\ &\geq |\langle x, y\rangle| - |\langle Tx, y\rangle|, \qquad (18)\end{aligned}$$

and this completes the proof. □

Remark 3.14. *Recall that any orthogonal projection $P = P^2 = P^*$ is a positive contraction with $\|P\| = 1$ and $P \in \mathcal{A}_2$. Then, for any $x, y \in \mathcal{H}$*

$$|\langle Px, y\rangle| \leq \left|\langle Px, y\rangle - \frac{1}{2}\langle x, y\rangle\right| + \frac{1}{2}|\langle x, y\rangle| \leq \frac{1}{2}(|\langle x, y\rangle| + \|x\|\|y\|), \qquad (19)$$

and

$$\left|\langle Px, y\rangle - \langle x, y\rangle\right| \leq \frac{1}{2}(|\langle x, y\rangle| + \|x\|\|y\|). \qquad (20)$$

In (19) *and* (20) *we reach, with a new proof, an improvement and refinement of different statements previously obtained by Dragomir in [10].*

Motivated by the previous inequalities valid for orthogonal projections, we establish some vector inequalities for particular projections. Let $T = z \otimes z$, with $z \in \mathcal{H}$ and $\|z\| = 1$, as T is an orthogonal projection then by using inequality (6) *we get*

$$|\langle x, z\rangle\langle z, y\rangle| \leq \left|\langle x, z\rangle\langle z, y\rangle - \frac{1}{2}\langle x, y\rangle\right| + \frac{1}{2}|\langle x, y\rangle| \leq \frac{1}{2}(|\langle x, y\rangle| + \|x\|\|y\|),$$

for any $x, y \in \mathcal{H}$. This inequality refines the classical Buzano inequality.

On the other hand, using inequality (17)

$$|\langle x, y\rangle| \leq |\langle x, y\rangle - \langle x, z\rangle\langle z, y\rangle| + |\langle x, z\rangle\langle z, y\rangle| \leq \|x\|\|y\|,$$

for any $x, y \in \mathcal{H}$. This refinement of (1) *was also obtained in [8].*

Now we are in position to obtain a Buzano type inequality for the sum of two orthogonal projections. It is well-known that given two orthogonal projections on $\mathcal{H}$, P and Q, then

$$\|P+Q\| = 1 + \|PQ\|. \tag{21}$$

This is usually called Duncan-Taylor equality, and its proof can be found in [12].

Proposition 3.15. *Let P, Q be orthogonal projections on $\mathcal{H}$. Then, $P+Q \in \mathcal{A}_\alpha$ for $|\alpha| \le \frac{2}{1+\|PQ\|}$.*

Proof. Note that $P+Q \in \mathcal{B}(\mathcal{H})^+$ and by (21), $\|P+Q\| = 1+\|PQ\|$. Thus, using Corollary 3.9, the proof is complete. □

Throughout, $\mathcal{S}$ and $\mathcal{T}$ denote two closed subspaces of $\mathcal{H}$. The mininal angle or angle of Dixmier between $\mathcal{S}$ and $\mathcal{T}$ is the angle $\theta_0(\mathcal{S},\mathcal{T}) \in [0, \frac{\pi}{2}]$ whose cosine is defined by

$$c_0(\mathcal{S},\mathcal{T}) = \sup\{|\langle x, y\rangle| : x \in \mathcal{S}, y \in \mathcal{T}; \|x\|, \|y\| \le 1\}.$$

A linear operator defined on $\mathcal{H}$, such that $Q^2 = Q$ is called a projection. Such operators are not necessarily bounded, since on every infinite-dimensional Hilbert space there exist unbounded examples of projections (see [3]). The operator $Q_{\mathcal{M}//\mathcal{N}}$ is an oblique projection along (or parallel to) its null space $\mathcal{N} = \mathcal{N}(Q)$ onto its range $\mathcal{M} = \mathcal{R}(Q)$.

Theorem 3.16. *Let H be a Hilbert space such that is the direct sum of closed subspaces $\mathcal{M}$ and $\mathcal{N}$. Let $Q_{\mathcal{M}//\mathcal{N}}$, be the bounded projection with range $\mathcal{M}$ and null space $\mathcal{N}$, and $\theta_0(\mathcal{M},\mathcal{N})$, be the minimal angle between $\mathcal{M}$ and $\mathcal{N}$. Then, for any $x, y \in \mathcal{H}$*

$$\left|\langle Qx, y\rangle - \frac{1}{2}\langle x, y\rangle\right| \le \frac{\cot(\alpha_0)}{2}\|x\|\|y\|,$$

and

$$|\langle Qx, y\rangle| \le \frac{\cot(\alpha_0)}{2}(|\langle x, y\rangle| + \|x\|\|y\|),$$

where $\alpha_0 = \frac{\theta_0(\mathcal{M},\mathcal{N})}{2}$.

Proof. By Theorem 2 in [3] we have that $\|Q\| = \csc(\theta_0(\mathcal{M},\mathcal{N}))$ and $\|2Q-I\| = \cot(\alpha_0)$. From the boundness of Q we can assert that $0 < \theta_0(\mathcal{M},\mathcal{N}) \le \frac{\pi}{2}$ and $\cot(\alpha_0) \ge 1$. From these facts and mimicking the proof of Proposition 3.1, we have that for any $x, y \in \mathcal{H}$

$$\left|\langle Qx, y\rangle - \frac{1}{2}\langle x, y\rangle\right| \le \frac{\cot(\alpha_0)}{2}\|x\|\|y\|,$$

and

$$\begin{aligned} |\langle Qx, y\rangle| &\le \left|\langle Qx, y\rangle - \frac{1}{2}\langle x, y\rangle\right| + \frac{1}{2}|\langle x, y\rangle| \\ &\le \frac{\cot(\alpha_0)}{2}\|x\|\|y\| + \frac{1}{2}|\langle x, y\rangle| \\ &\le \frac{\cot(\alpha_0)}{2}(|\langle x, y\rangle| + \|x\|\|y\|). \end{aligned}$$

This completes the proof. □

We finish this section by showing that any operator whose real part is greater than sI for some $s > 0$, is invertible and its inverse belongs to $\mathcal{A}_{2s}$.

Theorem 3.17. *Let $T \in \mathcal{B}(\mathcal{H}) - \{0\}$ with $Re(T) = \frac{T+T^*}{2} \geq sI$ for some $s > 0$. Then, $T^{-1} \in \mathcal{A}_{2s}$.*

Proof. First, we show that T is invertible. The hypothesis $Re(T) \geq sI$ implies that

$$W(T) \subseteq \{z \in \mathbb{C} : Re(z) \geq s\},$$

since if $z \in W(T)$ then

$$\begin{aligned} Re(z) &= \frac{z+\overline{z}}{2} = \frac{\langle Tx,x\rangle + \overline{\langle Tx,x\rangle}}{2} = \frac{\langle Tx,x\rangle + \langle T^*x,x\rangle}{2} \\ &= \langle Re(T)x,x\rangle \geq s. \end{aligned}$$

Thus $\sigma(T) \subseteq \overline{W(T)} \subseteq \{z \in \mathbb{C} : Re(z) \geq s\}$ and, in particular, we have that $0 \notin \sigma(T)$, which means $T \in \mathcal{GL}(\mathcal{H})$. If $T + T^* \geq 2sI$ and $T \in \mathcal{GL}(\mathcal{H})$, then $2sT^{-1}(T + T^* - 2sI)(T^*)^{-1} \geq 0$ and

$$I \geq I - 2sT^{-1}(T + T^* - 2sI)(T^*)^{-1} = (I - 2sT^{-1})(I - 2sT^{-1})^*,$$

which is equivalent to $\|2sT^{-1} - I\| \leq 1$. Hence, $T^{-1} \in \mathcal{A}_{2s}$ and the result follows. □

4. Bounds for the numerical radius using Buzano inequality

In this section, we use (13) to obtain a refinement of the classical inequality $\omega(T) \leq \|T\|$ and an upper bound for $\omega(T) - \frac{1}{2}\|T\|$.

Proposition 4.1. *Let $T \in \mathcal{B}(\mathcal{H})$ with polar decomposition $T = V|T|$. Then,*

$$\omega(T) \leq \frac{\|T\|}{2}(1 + \omega(V)) \leq \|T\| \tag{22}$$

and

$$\omega(T) - \frac{\|T\|}{2} \leq \frac{\|T\|}{2}\omega(V). \tag{23}$$

Proof. Taking $x = y$ in (13), and the supremum over all $x \in \mathcal{H}$ with $\|x\| = 1$, we obtain $\omega(T) \leq \frac{\|T\|}{2}(1 + \omega(V))$ and this completes the proof.
□

It is important to note that inequalities (22) and (23) are not trivial, since $\omega(V)$ may be less than one, depending on the partial isometry V. For instance, let

$$T = \begin{bmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} = V|T|,$$

where V is a partial isometry in $\mathbb{C}^2$ with $\ker(T) = \ker(V) = \text{span}\{(0,1)\}$ and $\ker(V)^\perp = \text{span}\{(1,0)\}$. Then, for any $x = (x_1, x_2) \in \mathbb{C}$ with $\|x\| = \sqrt{|x_1|^2 + |x_2|^2} = 1$, we have as a consequence of the arithmetic-geometric mean inequality

$$|\langle Vx,x\rangle| = |x_1\overline{x_2}| = |x_1||x_2| \leq \frac{|x_1|^2 + |x_2|^2}{2} = \frac{1}{2}.$$

Therefore, $W(V) = \{z \in \mathbb{C} : |z| \leq \frac{1}{2}\}$ and $\omega(V) = \frac{1}{2} < 1$.

Proposition 4.2. *Let $T \in \mathcal{B}(\mathcal{H}) - \{0\}$ with polar decomposition $T = V|T|$ such that $\omega(T) = \|T\|$, then $\omega(V) = \|V\| = 1$.*

Proof. From inequality (22), we get

$$\omega(T) \leq \frac{\|T\|}{2}(1+\omega(V)) \leq \|T\|.$$

Thus, if $\omega(T) = \|T\|$, then $1 + \omega(V) = 2$, and hence $\omega(V) = 1$. As V is a nonzero partial isometry, therefore $\|V\| = 1$ and thus $\omega(V) = \|V\| = 1$, as required. □

It should also be mentioned here that the converse of Proposition 4.2 is not true. To see this, consider

$$T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = V|T|.$$

where $V|T|$ is a polar decomposition of T. Then, $\omega(V) = \|V\| = 1$, but $\omega(T) = \frac{1}{\sqrt{2}} < 1 = \|T\|$.

In order to estimate how close the numerical radius is from the operator norm, the following reverse inequalities have been obtained under appropriate conditions for the involved operator $T \in \mathcal{B}(\mathcal{H})$. If $T \in \mathcal{A}_\alpha$, then by (3) and Proposition 3.3 we have that

$$0 \leq \|T\| - \omega(T) \leq \|T\| - \frac{\|T\|}{2} \leq \frac{1}{|\alpha|}.$$

Motivated by the above inequality, we establish a new upper bound for the non-negative quantity $\|T\| - \omega(T)$.

Theorem 4.3. *Let $T \in \mathcal{A}_\alpha$. Then,*

$$\|T\| - \omega(T) \leq \frac{1}{2|\alpha|}.$$

Proof. For $x \in \mathcal{H}$ with $\|x\| = 1$, we have

$$\|\alpha Tx - x\|^2 = |\alpha|^2\|Tx\|^2 - 2Re(\alpha\langle Tx, x\rangle) + 1 \leq 1$$

giving

$$|\alpha|^2\|Tx\|^2 + 1 \leq 1 + 2Re(\alpha\langle Tx, x\rangle) \leq 2|\alpha||\langle Tx, x\rangle| + 1.$$

By arithmetic-geometric mean inequality, we deduce

$$2|\alpha|\|Tx\| \leq |\alpha|^2\|Tx\|^2 + 1 \leq 1 + 2Re(\alpha\langle Tx, x\rangle) \leq 2|\alpha||\langle Tx, x\rangle| + 1. \tag{24}$$

Now, taking the supremum over $x \in \mathcal{H}$, $\|x\| = 1$ in (24), we obtain

$$2|\alpha|\|T\| - 2|\alpha|\omega(T) \leq 1.$$

□

Now, we derive upper bounds for the numerical radius of products of bounded linear operators.

Theorem 4.4. *Let $R, S, T \in \mathcal{B}(\mathcal{H})$ such that $T \in \mathcal{A}_\alpha$. Then,*

$$\omega(STR) \leq \frac{1}{|\alpha|}\left(\|R\|\|S\| + \omega(SR)\right). \tag{25}$$

Proof. From inequality (6), we have

$$|\langle STRx, y\rangle| = |\langle TRx, S^*y\rangle| \leq \frac{1}{|\alpha|}(|\langle Rx, S^*y\rangle| + \|Rx\|\|S^*y\|).$$

Taking $y = x$ and the supremum over $x \in \mathcal{H}$ with $\|x\| = 1$, yields the desired inequality.
□

We note that the previous result is a generalization of Theorem 3.6 in [11]. In particular, for the sum of two orthogonal projections, we obtain the following result.

Corollary 4.5. *Let $P, Q, R, S \in \mathcal{B}(\mathcal{H})$ with P, Q be orthogonal projections. Then,*

$$\omega(R(P+Q)S) \leq \frac{1+\|PQ\|}{2}\left(\|S\|\|R\| + \omega(RS)\right).$$

Proof. As we have already mentioned, $P+Q \in \mathcal{A}_{\frac{2}{1+\|PQ\|}}$. Then, the statement is a consequence of Theorem 4.4. □

On the other hand, in [9], Dragomir obtained, utilizing Buzano's inequality, the following inequality for the numerical radius

$$\omega(S)^2 \leq \frac{1}{2}\left(\|S\|^2 + \omega(S^2)\right),$$

combining with the following power inequality for the numerical radius, $w(S^n) \leq w(S)^n$ for any natural number n, we have

$$w(S^2) \leq \frac{1}{2}\left(\|S\|^2 + \omega(S^2)\right). \tag{26}$$

The following corollary, which is an immediate consequence of Theorem 4.4 considering $R = S$, gives a generalization of (26).

Corollary 4.6. *Let $S, T \in \mathcal{B}(\mathcal{H})$ with $T \in \mathcal{A}_\alpha$. Then,*

$$\omega(STS) \leq \frac{1}{|\alpha|}\left(\|S\|^2 + \omega(S^2)\right).$$

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