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On the central limit theorem for a conditional mode estimator in the single functional index modeling for functional time series data under random censorship

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Abstract. The main objective of this paper is to investigate the estimation of conditional density function based on the single-index model in the censorship model when the sample is considered as an dependent random variables. First of all, a kernel type estimator for the conditional density function (*cond-df*) is introduced. Afterwards, the asymptotic properties are stated when the observations are linked with a single-index structure. The pointwise almost complete convergence and the uniform almost complete convergence (with rate) of the kernel estimate of this model are established. As an application the conditional mode in functional single-index model is presented. Under general conditions, the asymptotic normality of the conditional density estimator is established. Simulation study is also presented to illustrate the validity and finite sample performance of the considered estimator. Finally, the estimation of the functional index via the pseudo-maximum likelihood method is discussed, but not tackled.

1. Introduction

Conditional density function estimator has been widely used to estimate some characteristic feature of the data set, such as the conditional mode, the conditional median, or the conditional quantiles. The Kernel estimation of the functional density with an application to conditional mode estimation have been presented by Ferraty *et al.* [14] and Ezzahrioui and Ould-Saïd [12] in the i.i.d case, the asymptotic normality has been studied in by Ezzahrioui and Ould-Saïd [13] when the variable are dependent. In the censoring case, Ould-Saïd and Cai [26] establish the strong uniform convergence (with rate) of kernel conditional mode estimator for i.i.d. random variables, while Ould-Saïd [27] constructed a kernel estimator of the conditional quantile and establish its strong uniforme convergence rate. Then, Khardani *et al.* [18] obtained the strong consistency with rate and asymptotic normality of the conditional mode. Later, Khardani *et al.* [19] established the strong consistency with rate of the conditional mode for the censored dependent case. Then, Khardani *et al.* [20] presented the asymptotic normality.

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The single index models have been used and studied in both statistical and econometric literatures. These models gave attracted the attention of many researchers as Aït-Saidi *et al.* [1, 2]. Bouchentouf *et al.* [6] established a nonparametric estimation of some characteristics of the the conditional cumulative distribution function and the successive derivatives of the conditional density of a scalar response variable Y given a Hilbertian random variable X when the observations are linked with a single-index structure. Attaoui *et al.* [3] studied the functional single-index model via its conditional density Kernel estimator, and established its pointwise and uniform almost complete convergence rates, their results were extended to dependent case by Attaoui [4]. Furthermore, Ling *et al.* [23] obtained the asymptotic normality of the conditional density estimator and the conditional mode estimator for the α-mixing dependence functional time series data.

In our infinite dimensional purpose, we use the terminology *functional nonparametric*, where the word *functional* referees to the infinite dimensionality of the data and where the word *nonparametric* referees to the infinite dimensionality of the model. Such *functional nonparametric* statistics is also called *doubly infinite dimensional* (see Ferraty and Vieu [16], for more details).

Inspired by the work of Rabhi *et al.* [28] under an i.i.d. censorship, our work in this paper aims to contribute to the research on functional nonparametric conditional model, by giving an alternative estimation of conditional mode estimation in the single functional index model with randomly right-censored data under α-mixing conditions whose definition is given below.

Recall that a process $(X_i, Y_i)_{i\geq 1}$ is called α-mixing or strongly mixing (see Lin and Lu [22]) for more details and examples, if

$$\sup_k \sup_{A\in\mathcal{F}_1^k} \sup_{B\in\mathcal{F}_{n+k}^{\infty}} |\mathbb{P}(A\cap B) - \mathbb{P}(A)\mathbb{P}(B)| = \alpha(n) \to 0 \text{ as } n \to \infty,$$

where $\mathcal{F}_j^k$ denotes the σ-field generated by the random variables $\{(X_i,\ Y_i),\ j \leq i \leq k\}$. The process $\{(X_i,\ Y_i),\ i \geq 1\}$ is said to be arithmetically α mixing with order $a > 0$, if $\exists C > 0,\ \alpha(n) \leq Cn^{-a}$.

The strong-mixing condition is reasonably weak and has many practical applications (see, e.g., Cai [7], Doukhan [11], Dedecker *et al.* [8] Ch. 1, for more details). In particular, Masry and Tøjstheim [25] proved that, both ARCH processes and nonlinear additive autoregressive models with exogenous variables, which are particularly popular in finance and econometrics, are stationary and α-mixing.

The main contribution of this work, is to establish the pointwise almost complete convergence and the uniform almost complete convergence (with rate) of the conditional density estimator in the single functional index model in strong mixing case under random censorship, this result will be applied to obtain the convergence rates of the conditional mode estimator. Moreover, we prove the asymptotic normality of the estimators of conditional density function and conditional mode. The layout of the paper is as follows, Section 2 presents the functional nonparametric framework, The asymptotic normality is given in Section 3. As then application, we study the asymptotic normality of the conditional mode in functional single-index model in Section 4. Section 5 illustrates those asymptotic properties through some simulations. Finally, the proofs of the main results are postponed to Section 6.

1.1. The functional nonparametric framework

Consider a random pair (X, T) where T is valued in $\mathbb{R}$ and X is valued in some infinite dimensional Hilbertian space $\mathcal{H}$ with scalar product $< \cdot, \cdot >$. Let $(X_i, T_i)_{i=1,\ldots,n}$ be the statistical sample of pairs which are identically distributed like (X, T), but not necessarily independent. X is called functional random variable *f.r.v.* . Assume that the conditional expectation of T given X is done through a fixed functional index θ in $\mathcal{H}$, such that

$$\mathbb{E}[T|X] = \mathbb{E}[T| < \theta, X >].$$

This model was introduced by Ferraty *et al.* [15] and we can refer to Attaoui *et al.* [3] for details. From this model, let $f(\theta, \cdot, x)$ be the conditional density of Y given $< X, \theta >=< x, \theta >$ for $x \in \mathcal{H}$, which also shows the relationship between X and Y but it often unknown.

Let $(T_i)_{i\geq 1}$ be a sequence of independent and identically distributed (i.i.d.) random variables, and assume that they form a strictly stationary sequence of lifetimes. Suppose that there exists a sample of i.i.d. censoring random variable (r.v) $(C_i)_{i\geq 1}$ with common unknown continuous distribution function (df).

In the censored framework, the observed random variables are the triplets (Y_i, δ_i, X_i) with

$$Y_i = \min\{T_i, C_i\} \text{ and } \delta_i = \mathbf{1}_{T_i \leq C_i},\ 1 \leq i \leq n,$$

where both of T_i and C_i are expected to exhibit some kind of dependence which ensures the identifiability of the model.

In biomedical case studies, it is assumed that C_i and (T_i, X_i) are independent, this condition is plausible whenever the censoring is independent of the patient's modality.

The Kernel estimator $f_n(\theta, \cdot, x)$ of $f(\theta, \cdot, x)$ is defined by :

$$f_n(\theta, t, x) = \frac{h_H^{-1} \sum_{i=1}^{n} K\left(h_K^{-1}(< x - X_i, \theta >)\right) H\left(h_H^{-1}(t - T_i)\right)}{\sum_{i=1}^{n} K\left(h_K^{-1}(< x - X_i, \theta >)\right)}, \tag{1}$$

where K and H are Kernel functions , and $h_K = h_{K,n}$ (resp. $h_H = h_{H,n}$) a sequence of positive real numbers.

The Kernel type estimator of the conditional density $f(\theta, \cdot, x)$ adapted for censorship model, can be reformulate from the expression (1) as follows :

$$\widetilde{f}(\theta, t, x) = \frac{h_H^{-1} \sum_{i=1}^{n} \frac{\delta_i}{\bar{G}(Y_i)} K\left(h_K^{-1}(< x - X_i, \theta >)\right) H\left(h_H^{-1}(t - Y_i)\right)}{\sum_{i=1}^{n} K\left(h_K^{-1}(< x - X_i, \theta >)\right)}. \tag{2}$$

In practice $\bar{G}(\cdot) = 1 - G(\cdot)$ is unknown, then using Kaplan and Meier (1958) estimator, $\bar{G}_n(\cdot)$ will be given as

$$\bar{G}_n(t) = 1 - G_n(t) = \begin{cases} \prod_{i=1}^{n} \left(1 - \frac{1 - \delta_{(i)}}{n - i + 1}\right)^{\mathbf{1}_{\{Y_{(i)} \leq t\}}}, & \text{if } t < Y_{(n)}; \\ 0, & \text{if } t \geq Y_{(n)}; \end{cases}$$

where $Y_{(1)} < Y_{(2)} < \ldots < Y_{(n)}$ are the order statistics of Y_i and $\delta_{(i)}$ is the non-censoring indicator corresponding to $Y_{(i)}$.

Therefore, estimator of the conditional density function $f(\theta, \cdot, x)$ is given by

$$\widehat{f}(\theta, t, x) = \frac{h_H^{-1} \sum_{i=1}^{n} \frac{\delta_i}{\bar{G}_n(Y_i)} K\left(h_K^{-1}(< x - X_i, \theta >)\right) H\left(h_H^{-1}(t - Y_i)\right)}{\sum_{i=1}^{n} K\left(h_K^{-1}(< x - X_i, \theta >)\right)}. \tag{3}$$

2. Asymptotic study

2.1. Pointwise almost complete rate of convergence

In the following, for any $x \in \mathcal{H}$, let N_x be a fixed neighborhood of x and $\mathcal{S}_{\mathbb{R}}$ is a fixed compact of $\mathbb{R}^+$.We denote by $B_\theta(x, h) = \{\chi \in \mathcal{H}/0 < | < x - \chi, \theta > | < h\}$ be a ball of center x and radius h, and let $d_\theta\,(x, X_i) = | < x - X_i, \theta > |$ denote a random variable such that its cumulative distribution function is given by $\phi_{\theta,x}(u) = \mathbb{P}(d_\theta(x, X_i) \leq u) = \mathbb{P}(X_i \in B_\theta(x,u))$. Assume that, $(C_i)_{i\geq 1}$ and $(T_i)_{i\geq 1}$ are independent and we assume that $\tau_G := \sup\{t : G(t) < 1\}$ and let τ be a positive real number such that $\tau < \tau_G$.

In order to establish the almost complete (a.co.) convergence of our estimator, we need some regular hypotheses as follows.

(H0) $(X_i, Y_i)_{i\in\mathbb{N}}$ is an α-mixing sequence whose the coefficients of mixture verify:

$$\exists a > 0,\ \exists c > 0 :\ \forall n \in \mathbb{N},\ \alpha(n) \leq cn^{-a}.$$

(H1) $0 < \sup_{i\neq j} \mathbb{P}\Big((X_i, X_j) \in B_\theta(x,h) \times B_\theta(x,h)\Big) = O\left(\frac{\big(\phi_{\theta,x}(h_K)\big)^{(a+1)/a}}{n^{1/a}}\right).$

(H2) The conditional density $f(\theta,t,x)$ satisfies the Hölder condition, i.e., $\forall(x_1,x_2) \in N_x \times N_x$, $\forall(t_1,t_2) \in \mathcal{S}^2_{\mathbb{R}}$,

$$|f(\theta,t_1,x_1) - f(\theta,t_2,x_2)| \leq C_{\theta,x}\left(\|x_1 - x_2\|^{b_1} + |t_1 - t_2|^{b_2}\right),\ \ b_1 > 0,\ b_2 > 0.$$

(H3) H is positive bounded function, such that $\forall(t_1,t_2) \in \mathbb{R}^2$, $|H(t_1) - H(t_2)| \leq C|t_1 - t_2|$, $\int H^2(t)dt < \infty$ and $\int |t|^{b_2} H(t)dt < \infty$ and $\lim_{n\to\infty} n^{\varsigma} h_H = \infty$ for some $\varsigma > 0$.

(H4) The kernel K is a positive bounded function supported on $[0,1]$ and is differentiable on $[0,1]$ with derivative such that: $\exists C_1,\ C_2,\ -\infty < C_1 < K'(t) < C_2 < 0$, for $0 < t < 1$.

(H5) The bandwidths h_K and h_H satisfy

(i) $\lim_{n\to\infty} h_K = 0$ and $\dfrac{\log n}{nh_H\,\phi_{\theta,x}(h_K)} \underset{n\to\infty}{\longrightarrow} 0.$

(ii) $\exists C > 0,\ h_H\phi_{\theta,x}(h_K) \geq C/n^{2/a+1}$ and $\left(\frac{\phi_{\theta,x}(h_K)}{n}\right)^{1/a} + \phi_{\theta,x}(h_K) = o\left(\frac{1}{n^{a/a-1}}\right).$

(H6) (X_i, Y_i) for $i = 1,\ldots,n$ are strongly mixing with arithmetic coefficient of order $a > 1$, and $\exists\beta > 2$ such that $s_{n,k}^{-(a+1)} = o\left(n^{-\beta}\right)$ for $k = 1,\ldots,6$.

- **Comments on the hypotheses**

The hypothesis (H0) specifies the asymptotic behavior of the α-mixing coefficients.

- $\phi_{\theta,x}(u)$ can be interpreted as a concentration hypothesis acting on the distribution of the *f.r.v.* X, while (H1) concerns the behavior of the joint distribution of the pairs (X_i, X_j). Indeed, this hypothesis is equivalent to assume that, for n large enough

$$\sup_{i\neq j} \frac{\mathbb{P}\Big((X_i, X_j) \in B_\theta(x,h_K) \times B_\theta(x,h_K)\Big)}{\mathbb{P}\left(X \in B_\theta(x,h_K)\right)} \leq C\left(\frac{\phi_{\theta,x}(h_K)}{n}\right)^{1/a}.$$

This is one way to control the local asymptotic ratio between the joint distribution and its margin. Remark that the upper bound increases with a. In other words, more the dependence is strong, more restrictive is (H1).

- (H2) is a regularity conditions which characterize the functional space of our model and is needed to evaluate the bias term of our asymptotic results which have been adopted by Bouchentouf *et al.* (2014) for i.i.d case.

- Assumptions (H3) is technical conditions and are also similar to those done in Ferraty and Vieu (2006).

- Assumption (H4) is classical and permits to make the variance term negligible.
- Assumption (H5) concern the choice of the bandwidth which is closely linked to the small balls probability.
- (H6) is similar to analysis in Ferraty and Vieu [17], and it shows the influence of covariance structure on the convergence rate.

Proposition 2.1. *Under conditions (H0)-(H5) and assume that (H6)-(i) is satisfied, then we have as n goes to infinity,*

$$\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left|\widehat{f}(\theta,t,x)-f(\theta,t,x)\right| \;=\; O\left(h_K^{b_1}+h_H^{b_2}\right)+O_{a.co.}\left(\frac{\sqrt{s_n^2\log n}}{n}\right),$$

where $s_n^2=\max\{s_{n,1}^2;s_{n,2}^2\}$.

Proof. [Proof of Proposition 2.1]

Consider now, for $i=1,\dots,n$, in what follows, let's denote:

$$K_i(\theta,x)=K(h_K^{-1}(<x-X_i,\theta>)),\; H_i(t)=H\left(h_H^{-1}\left(t-Y_i\right)\right),\; \bar{G}_i=\bar{G}(Y_i),$$

$$\widehat{f_N}(\theta,t,x)=\frac{\sum_{i=1}^n\frac{\delta_i}{\bar{G}_n(Y_i)}K_i(\theta,x)H_i(t)}{n\,h_H\,\mathbb{E}(K_1(\theta,x))},\;\widetilde{f_N}(\theta,t,x)=\frac{\sum_{i=1}^n\frac{\delta_i}{\bar{G}(Y_i)}K_i(\theta,x)H_i(t)}{n\,h_H\,\mathbb{E}(K_1(\theta,x))},$$

$$\widehat{F_D}(\theta,x) \;=\; \frac{1}{n\,\mathbb{E}(K_1(\theta,x))}\sum_{i=1}^n K_i(\theta,x),\quad \Delta_i(x,\theta)=\frac{K(h_K^{-1}(<x-X_i,\theta>))}{\mathbb{E}K_1(\theta,x)}$$

and

$$s_{n,1}^2=\sum_{i=1}^n\sum_{j=1}^n\left|Cov\left(\Delta_i(x,\theta),\Delta_j(x,\theta)\right)\right|,\;\; s_{n,2}^2=\sum_{i=1}^n\sum_{j=1}^n\left|Cov\left(\frac{h_H^{-1}\delta_i}{\bar{G}(Y_i)}H_i(t)\Delta_i(x,\theta),\frac{h_H^{-1}\delta_j}{\bar{G}(Y_j)}H_j(t)\Delta_j(x,\theta)\right)\right|.$$

The proof is based on the following decomposition, valid for any $t\in\mathcal{S}_{\mathbb{R}}$,

$$\begin{aligned}\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left|\widehat{f}(\theta,t,x)-f(\theta,t,x)\right| \;\leq\;& \frac{1}{\widehat{F_D}(\theta,x)}\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left\{\left|\widehat{f_N}(\theta,t,x)-\widetilde{f_N}(\theta,t,x)\right|\right\}\\ &+\frac{1}{\widehat{F_D}(\theta,x)}\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left\{\left|\widetilde{f_N}(\theta,t,x)-\mathbb{E}\widetilde{f_N}(\theta,t,x)\right|\right\}\\ &+\frac{1}{\widehat{F_D}(\theta,x)}\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left\{\left|\mathbb{E}\widetilde{f_N}(\theta,t,x)-f(\theta,t,x)\right|\right\}\\ &+\sup_{t\in\mathcal{S}_{\mathbb{R}}}\frac{f(\theta,t,x)}{\widehat{F_D}(\theta,x)}\left|1-\widehat{F_D}(\theta,x)\right|. \end{aligned}\tag{4}$$

Finally, the proof of this proposition is a direct consequence of the following intermediate results.

Lemma 2.2. *Under hypotheses (H2)-(H5)-(i)) and if* $nh_H\phi_{\theta,x}(h_K)\longrightarrow\infty$, $\left(\frac{\log\log n}{n}\right)^{1/2}=o\left(\phi_{\theta,x}(h_K)\right)$, *we have,*

$$\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left\{\left|\widehat{f_N}(\theta,t,x)-\widetilde{f_N}(\theta,t,x)\right|\right\} \;=\; O_{a.s.}\left(\frac{\log\log n}{n}\right).$$

The lemmas below shows the asymptotic bias term of $\widetilde{f_N}(\theta,t,x)$ and $\widehat{F_D}(\theta,x)$ as n tends to infinity.

Lemma 2.3. *Under hypotheses (H2)-(H3), we have as $n \to \infty$,*

$$\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left|\mathbb{E}\left[\widetilde{f_N}(\theta,t,x)\right]-f(\theta,t,x)\right| = O\left(h_K^{b_1}+h_H^{b_2}\right).$$

The following result deals with the variance term of the right-hand side of (4) which is expressed by: $\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left\{\left|\widetilde{f_N}(\theta,t,x)-\mathbb{E}\widetilde{f_N}(\theta,t,x)\right|\right\}$. For $\widehat{F_D}(\theta,x)-\mathbb{E}\left[\widehat{F_D}(\theta,x)\right]$ the same arguments will be used with a slight difference.

Lemma 2.4. *Under conditions of the Proposition 2.1 and if $\left(\dfrac{\log\log n}{n}\right)^{1/2} = o\left(\phi_{\theta,x}(h_K)\right)$, we have as $n\to\infty$,*

(i) $\widehat{F_D}(\theta,x)-\mathbb{E}\widehat{F_D}(\theta,x) = O_{a.co.}\left(\dfrac{\sqrt{s_{n,1}^2\log n}}{n}\right)$

and

(ii) $\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left\{\left|\widetilde{f_N}(\theta,t,x)-\mathbb{E}\widetilde{f_N}(\theta,t,x)\right|\right\} = O_{a.co.}\left(\dfrac{\sqrt{s_{n,2}^2\log n}}{n}\right),$

furthermore, we have,

$$\sum_{n=1}^{\infty}\mathbb{P}\left(|\widehat{F_D}(\theta,x)\ \le 1/2\right)<\infty.$$

We conclude the proof of the Proposition 2.1 by making use the inequality (4), in conjunction with Lemmas 2.2-2.4.

□

The proof of these latter will be collected in Section 5.

3. Asymptotic Normality

In this section, the asymptotic normality of the conditional density and the conditional mode are established. Therefore, further assumptions are required. Assume that

(N1) The df of the censored random variable, G has a bounded first derivative G'.

(N2) There exists a function $\xi^{\theta,x}$, such that $\forall u\in[0,1]$

$$\lim_{h\to 0}\frac{\phi_{\theta,x}(uh)}{\phi_{\theta,x}(h)} = \lim_{h\to 0}\xi_h^{\theta,x}(u) = \xi_0^{\theta,x}(u).$$

(N3) The bandwidth h_H satisfies, as n goes to infinity,

- (i) $nh_H^3\phi_{\theta,x}(h_K)\longrightarrow\infty$ and $\frac{nh_H^3\phi_{\theta,x}(h_K)}{\log^2 n}\longrightarrow\infty.$
- (ii) $nh_H^3\phi_{\theta,x}^3(h_K)\longrightarrow 0.$

(N4) There exist sequences of integers (u_n) and (v_n) increasing to infinity such that $(u_n + v_n) \leq n$, satisfying as n goes to infinity

(i) $v_n = o((nh_H\phi_{\theta,x}(h_K))^{1/2})$ and $\left(\frac{n}{h_H\phi_{\theta,x}(h_K)}\right)^{1/2} \alpha(v_n) \to 0$.

(ii) $q_n v_n = o((nh_H\phi_{\theta,x}(h_K))^{1/2})$ and $q_n \left(\frac{n}{h_H\phi_{\theta,x}(h_K)}\right)^{1/2} \alpha(v_n) \to 0$,

where q_n is the largest integer such that $q_n(u_n + v_n) \leq n$.

(N5) The conditional density function $f(\theta, t, x)$ satisfies: $\exists \beta_0 > 0, \forall (t_1, t_2) \in \mathcal{S}_{\mathbb{R}}^2$,

$$|f^{(l)}(\theta, t_1, x) - f^{(l)}(\theta, t_2, x)| \leq C(|t_1 - t_2|^{\beta_0}), \quad \forall l = 1, 2.$$

(N6) H' and H'' are bounded respectively with $\int (H'(t))^2 dt < \infty$, $\int |t|^{\beta_0} H(t) dt < \infty$.

• **Comments**

Our hypotheses are very standard for the conditional density estimation in single functional index model, which have been adopted by Attatoui *et al.* [3].

- (N1) is classical in nonparametric estimation. Assumption (N2) is the concentration property of the explanatory variable in small balls under single-index topological structure. The function $\xi^{\theta,x}$ plays a fundamental role in all asymptotic, in particular for the variance term.

- Assumption (N3) is also classical in the functional estimation in finite or infinite dimension spaces, in particular, condition (N2)-(i) yields that $\lim_{n\to\infty} \dfrac{\log^2 n}{nh_H^3\phi_{\theta,x}(h_K)} = 0$ which implies $\lim_{n\to\infty} \dfrac{\log n}{nh_H^3\phi_{\theta,x}(h_K)} = 0$.

- For (N3)-(ii) is used to eliminate the term bias in the result of asymptotic normality

- To establish the asymptotic normality, dealing with strong mixing random variables (under (H1)), we use the well-known sectioning device introduced by Doob [10] in (N4).

- The conditions (N5)-(N6) are used to control the regularity of the functional space of our model and it is needed to evaluate the bias term of the convergence rates.

Theorem 3.1. *Under assumptions (H0)-(H5) and (N1)-(N4) for all $x \in \mathcal{H}$, we have as n goes to infinity,*

$$\sqrt{\frac{nh_H\phi_{\theta,x}(h_K)}{V(\theta,t,x)}} \left(\widehat{f}(\theta, t, x) - f(\theta, t, x)\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1),$$

where

$$V(\theta, t, x) = \frac{a_2(\theta, x) f(\theta, t, x)}{(a_1(\theta, x))^2 \bar{G}(t)} \int_{\mathbb{R}} H^2(u) du,$$

with

$$a_l(\theta, x) = K^l(1) - \int_0^1 (K^l)'(u) \xi_0^{\theta,x}(u) du, \; l = 1, 2.$$

" $\xrightarrow{\mathcal{D}}$ " means the convergence in distribution.

Proof. In order to establish the asymptotic normality of $\widehat{f}(\theta,t,x)$, we need further notations and definitions. First we consider the following decomposition

$$\begin{aligned}
\widehat{f}(\theta,t,x)-f(\theta,t,x) &= \frac{\widehat{f_N}(\theta,t,x)}{\widehat{F_D}(\theta,x)}-\frac{a_1(\theta,x)f(\theta,t,x)}{a_1(\theta,x)}\\
&= \frac{1}{\widehat{F_D}(\theta,x)}\left\{\left(\widehat{f_N}(\theta,t,x)-\mathbb{E}\widehat{f_N}(\theta,t,x)\right)-\left(a_1(\theta,x)f(\theta,t,x)-\mathbb{E}\widehat{f_N}(\theta,t,x)\right)\right\}\\
&\quad+\frac{f(\theta,t,x)}{\widehat{F_D}(\theta,x)}\left\{\left(a_1(\theta,x)-\mathbb{E}\widehat{F_D}(\theta,x)\right)-\left(\widehat{F_D}(\theta,x)-\mathbb{E}\widehat{F_D}(\theta,x)\right)\right\}\\
&= \frac{1}{\widehat{F_D}(\theta,x)}\left(A_n(\theta,t,x)+B_n(\theta,t,x)\right),
\end{aligned}$$

where

$$\begin{aligned}
A_n(\theta,t,x) &= \frac{1}{nh_H\mathbb{E}K_1(\theta,x)}\sum_{i=1}^{n}\left\{\left(\frac{\delta_i}{\bar{G}(Y_i)}H_i(t)-h_Hf(\theta,t,x)\right)K_i(\theta,x)-\mathbb{E}\left[\left(\frac{\delta_i}{\bar{G}(Y_i)}H_i(t)-h_Hf(\theta,t,x)\right)K_i(\theta,x)\right]\right\}\\
&= \frac{1}{nh_H\mathbb{E}K_1(\theta,x)}\sum_{i=1}^{n}N_i(\theta,t,x).
\end{aligned}$$

It follows that,

$$nh_H\phi_{\theta,x}(h_K)Var(A_n(\theta,t,x))=\frac{\phi_{\theta,x}(h_K)}{h_H(\mathbb{E}K_1(\theta,x))^2}Var(N_1(\theta,t,x))=V_n(\theta,t,x),$$

and

$$B_n(\theta,t,x)=a_1(\theta,x)f(\theta,t,x)-\mathbb{E}\widehat{f_N}(\theta,t,x)+f(\theta,t,x)(a_1(\theta,x)-\mathbb{E}\widehat{F_D}(\theta,x)).$$

Then, the proof of Theorem 3.1 can be deduced from the following Lemmas. □

Lemma 3.2. *Under hypotheses (H0)-(H1), (H4) and (N1)-(N2) as $n\to\infty$ we have,*

$$n\phi_{\theta,x}(h_K)Var\left(A_n(\theta,t,x)\right)\longrightarrow V(\theta,t,x),$$

where $V(\theta,t,x)$ is given in Theorem 3.1.

Lemma 3.3. *Under conditions of Theorem 3.1, we have,*

$$\sqrt{nh_H\phi_{\theta,x}(h_K)}A_n(\theta,t,x)\xrightarrow{\mathcal{D}}\mathcal{N}(0,V(\theta,t,x)).$$

Lemma 3.4. *Under assumptions (H1)-(H5) and (N1)-(N2), we have,*

$$\sqrt{nh_H\phi_{\theta,x}(h_K)}B_n(\theta,t,x)\underset{n\to\infty}{\longrightarrow}0\ \text{ in probability.}$$

4. Application: The conditional mode in functional single-index model

The main objective of this section is to establish the asymptotic normality a of the kernel estimator of the conditional mode of Y given $<X,\theta>=<x,\theta>$ denoted by $M_\theta(x)$. We will consider the problem of the estimation of the conditional mode in the functional single-index model, denoted by $M_\theta(x)$. For this, we assume that $M_\theta(x)$ satisfies the following uniqueness property:

(H7) $\forall\varepsilon_0>0,\exists\eta>0,\forall\varphi$:

$$|M_\theta(x)-\varphi(x)|\geq\varepsilon_0\Longrightarrow|f(\theta,\varphi(x),x)-f(\theta,M_\theta(x),x)|\geq\eta,$$

where $M_\theta(x) = \arg\sup_{t\in\mathcal{S}_\mathbb{R}} f(\theta,t,x)$.

We estimate the conditional mode $M_\theta(x)$ with a random variable $\widehat{M}_\theta(x)$ such that,

$$\widehat{M}_\theta(x) = \arg\sup_{t\in\mathcal{S}_\mathbb{R}} \widehat{f}(\theta,t,x). \tag{5}$$

The difficulty of the problem is naturally linked with the flatness of the function $f(\theta,t,x)$ around the mode M_θ. This flatness can be controlled by the number of vanishing derivatives at point M_θ, and this parameter will also have a great influence on the asymptotic rates of our estimates. More precisely, we introduce the following additional smoothness condition.

(H8) There exists some integer $j > 1$ such that $\forall x \in \mathcal{S}_\mathcal{H}$, the function $f(\theta,\cdot,x)$ is j times continuously differentiable w.r.t t on $\mathcal{S}_\mathbb{R}$ with,

$$f^{(l)}(\theta, M_\theta(x), x) = 0, \quad if \quad 1 \le l < j$$

and $f^{(j)}(\theta,\cdot,x)$ is uniformly continuous on $\mathcal{S}_\mathbb{R}$ such that,

$$f^{(j)}(\theta, M_\theta(x), x) \neq 0,$$

where $f^{(j)}(\theta,\cdot,x)$ is the j^{th} order derivative of the conditional density $f(\theta,\cdot,x)$.

Theorem 4.1. *Put $s_n'^2 = \max\{s_{n,1}^2; s_{n,2}^2\}$, under hypotheses of Proposition 2.1 and if the conditional density $f(\theta,\cdot,x)$ satisfies (H7) and (H8), then we get,*

$$|\widehat{M}_\theta(x) - M_\theta(x)| = O\left(\left(h_K^{b_1} + h_H^{b_2}\right)^{\frac{1}{j}}\right) + O_{a.co}\left(\left(\frac{s_n'^2 \log n}{n^2}\right)^{\frac{1}{2j}}\right).$$

Proof. [Proof of Theorem 4.1] By the Taylor expansion of $f(\theta,t,x)$ in neighborhood of $M_\theta(x)$, we get,

$$\widehat{f}(\theta,\widehat{M}_\theta(x),x) = f(\theta,M_\theta(x),x) + \frac{f^{(j)}(\theta,M_\theta^*(x),x)}{j!}(\widehat{M}_\theta(x) - M_\theta(x))^j,$$

where $M_\theta^*(x)$ is between $M_\theta(x)$ and $\widehat{M}_\theta(x)$.

Combining the last equality with the fact that

$$|\widehat{f}(\theta,\widehat{M}_\theta(x),x) - f(\theta,M_\theta(x),x)| \le 2\sup_{t\in\mathcal{S}_\mathbb{R}} |\widehat{f}(\theta,t,x) - f(\theta,t,x)|,$$

allow to write:

$$|\widehat{M}_\theta(x) - M_\theta(x)|^j \le \frac{j!}{f^{(j)}(\theta,M_\theta^*,x)} \sup_{t\in\mathcal{S}_\mathbb{R}} |\widehat{f}(\theta,t,x) - f(\theta,t,x)|.$$

Using the second part of (H8) we obtain that,

$$\exists c > 0, \quad \sum_{n=1}^{\infty} \mathbb{P}\left(f^{(j)}(\theta,M_\theta^*,x) < c\right) < \infty.$$

So, we would have

$$|\widehat{M}_\theta(x) - M_\theta(x)|^j = O_{a.co.}\left(\sup_{t\in\mathcal{S}_\mathbb{R}} \left|\widehat{f}(\theta,t,x) - f(\theta,t,x)\right|\right).$$

Finally, Theorem 4.1 can be deduced from proposition 2.1. □

Theorem 4.2. *Under the hpotheses of Proposition 2.1, thus we have,*

$$\widehat{M}_\theta(x) - M_\theta(x) \underset{n\to\infty}{\longrightarrow} 0, \ a.co.$$

Proof. [Proof of Theorem 4.2] Because the continuity of the function $f(\theta,t,x)$, we have, for all $\varepsilon>0$, $\exists\eta(\varepsilon)>0$ such that:

$$|f(\theta,t,x)-f(\theta,M_\theta(x),x)| \le \eta(\varepsilon) \Longrightarrow |t-M_\theta(x)| \le \varepsilon.$$

Therefore, for $t=\widehat{M}_\theta(x)$,

$$\mathbb{P}\left(|\widehat{M}_\theta(x)-M_\theta(x)|>\varepsilon\right) \le \mathbb{P}\left(|f(\theta,\widehat{M}_\theta(x),x)-f(\theta,M_\theta(x),x)|>\eta(\varepsilon)\right).$$

Then, according to theorem, $\widehat{M}_\theta - M_\theta$ go almost completely to 0, as n goes to infinity. □

Theorem 4.3. *If the assumptions (H1)-(H8) as well as (N1)-(N5) hold, then, we have,*

$$\sqrt{\frac{nh_H^3\phi_{\theta,x}(h_K)}{\sigma_1^2(\theta,x)}}(\widehat{M}_\theta(x)-M_\theta(x)) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1), \quad as \quad n\longrightarrow\infty, \tag{6}$$

where

$$\sigma_1^2(\theta,t,x) = \frac{a_2(\theta,x)f(\theta,M_\theta(x),x)}{(a_1(\theta,x)f^{(2)}(\theta,M_\theta(x),x))^2\bar{G}(t)}\int_{\mathbb{R}} H'^2(u)du.$$

Proof. Firstly, by (5) and (H8), it follows that $f^{(1)}(\theta,M_\theta(x),x)=0$.

Writing the first order Taylor expansion for $f^{(1)}(\theta,y,x)$ at point $M_\theta(x)$ leads to the existence of some $M_\theta^*(x)$ between $\widehat{M}_\theta(x)$ and $M_\theta(x)$ such that,

$$\sqrt{nh_H^3\phi_{\theta,x}(h_K)}(\widehat{M}_\theta(x)-M_\theta(x)) = \frac{-\sqrt{nh_H^3\phi_{\theta,x}(h_K)}\widehat{f}^{(1)}(\theta,M_\theta(x),x)}{\widehat{f}^{(2)}(\theta,M_\theta^*(x),x)}.$$

In order to prove (6), we only need to show that,

$$-\sqrt{nh_H^3\phi_{\theta,x}(h_K)}\widehat{f}^{(1)}(\theta,M_\theta(x),x) \xrightarrow{\mathcal{D}} \mathcal{N}(0,\sigma_0^2(\theta,x)) \tag{7}$$

and

$$\widehat{f}^{(2)}(\theta,M_\theta^*(x),x) \longrightarrow \widehat{f}^{(2)}(\theta,M_\theta(x),x) \neq 0, \text{ in probability}, \tag{8}$$

where,

$$\sigma_0^2(\theta,x) = \frac{a_2(\theta,x)f(\theta,M_\theta(x),x)}{\left(a_1(\theta,x)\right)^2\bar{G}(t)}\int,x)\Big)^2\bar{G}(t)\int (H'(u))^2du.$$

In fact, because the continuity of the function $f(\theta,t,x)$ and by (H7) and the definitions of $\widehat{M}_\theta(x)$ and $M_\theta(x)$, we have, for all $\varepsilon>0$, $\exists\eta(\varepsilon)>0$ such that:

$$\begin{aligned}\mathbb{P}\left(|\widehat{M}_\theta(x)-M_\theta(x)|\ge\varepsilon\right) &\le \mathbb{P}\left(|f(\theta,M_\theta(x),x)-\widehat{f}(\theta,M_\theta(x),x)|\ge\frac{\eta(\varepsilon)}{2}\right)\\ &+ \mathbb{P}\left(|\widehat{f}(\theta,\widehat{M}_\theta(x),x)-f(\theta,\widehat{M}_\theta(x),x)|\ge\frac{\eta(\varepsilon)}{2}\right).\end{aligned} \tag{9}$$

Thus, similar to [17], by (H0)-(H6), we have, $\widehat{f}(\theta,t,x)\longrightarrow f(\theta,t,x)$ in probability, which implies that $\widehat{M}_\theta(x)\longrightarrow M_\theta(x)$ in probability by (9) as $n\to\infty$. Similarly, the methodology can be also applied to obtain

$\widehat{f}^{(2)}(\theta,t,x) \longrightarrow f^{(2)}(\theta,t,x)$ in probability as $n \to \infty$ by (H0)-(H1), (H4), (N1), (N3) and (N5)-(N6). Therefore, (8) is valid by the fact that $f^{(2)}(\theta,t,x)$ is uniformly continuous with respect to y on $\mathcal{S}_{\mathbb{R}}$. Next, we prove (7).

In fact, since,

$$\begin{aligned}\widehat{f}^{(1)}(\theta,M_\theta(x),x) &= \frac{1}{\widehat{f_D}(\theta,x)}\left(\widehat{f_N}^{(1)}(\theta,M_\theta(x),x) - \mathbb{E}\widehat{f_N}^{(1)}(\theta,M_\theta(x),x)\right)\\ &- \frac{1}{\widehat{f_D}(\theta,x)}\left(f^{(1)}(\theta,M_\theta(x),x) - \mathbb{E}\widehat{f_N}^{(1)}(\theta,M_\theta(x),x)\right).\end{aligned} \tag{10}$$

By (H8), (N3)-(i), (N5)-(N6) and (10), similar to the proof of lemmas, Lemma 3.1 and Lemma 3.4 respectively, (7) follows directly. Then, the proof of Theorem 4.3 is completed. □

4.1. Application and Confidence bands

The asymptotic variances $V(\theta,t,x)$ and $\sigma_1^2(\theta,t,x)$ in Theorem 3.1 and Theorem 4.3 depend on some unknown quantities including a_1, a_2, $\phi(u)$, $M_\theta(x)$ and $f(\theta,M_\theta(x),x)$. So, $M_\theta(x)$, and $f(\theta,M_\theta(x),x)$ should be replaced by their respective estimators $\widehat{M}_\theta(x)$, and $\widehat{f}(\theta,M_\theta(x),x)$.

Because the unknown functions $a_j := a_j(\theta,x)$ and $f(\theta,t,x)$ intervening in the expression of the variance. So we need to estimate the quantities $a_1(\theta,x)$, $a_2(\theta,x)$ and $f(\theta,t,x)$, respectively.

By the assumptions (H0)-(H4) we know that $a_j(\theta,x)$ can be estimated by $\widehat{a_j}(\theta,x)$ which is defined as:

$$\widehat{a_j}(\theta,x) = \frac{1}{n\widehat{\phi}_{\theta,x}(h)}\sum_{i=1}^{n} K_i^j(\theta,x), \text{ where } \widehat{\phi}_{\theta,x}(h) = \frac{1}{n}\sum_{i=1}^{n}\mathbf{1}_{\{|<x-X_i,\theta>|<h\}},$$

with $\mathbf{1}_{\{\cdot\}}$ being the indicator function.

By applying the kernel estimator of $f(\theta,t,x)$ given above, the quantity $V(\theta,t,x)$ can be estimated finally by:

$$\widehat{V}(\theta,t,x) = \frac{\widehat{a_2}(\theta,x)}{\widehat{a}_1^2(\theta,x)}\frac{\widehat{f}(\theta,t,x)}{\bar{G}_n(t)}\int H'^2(u)du.$$

So, we can derive the following corollary.

Corollary 4.4. *Under the assumptions of Theorem 3.1, we have as $n \to \infty$,*

$$\sqrt{\frac{nh_H\widehat{\phi}_{\theta,x}(h_K)}{\widehat{V}(\theta,t,x)}}\left(\widehat{f}(\theta,t,x) - f(\theta,t,x)\right)\xrightarrow{\mathcal{D}} \mathcal{N}(0,1).$$

Proof. Observe that,

$$\begin{aligned}\Sigma &= \frac{\widehat{a_1}(\theta,x)}{\sqrt{\widehat{a_2}(\theta,x)}}\sqrt{\frac{nh_H\widehat{\phi}_{\theta,x}(h_K)}{\widehat{f}(\theta,t,x)}}\left(\widehat{f}(\theta,t,x) - f(\theta,t,x)\right)\\ &= \frac{\widehat{a_1}(\theta,x)\sqrt{a_2(\theta,x)}}{a_1(\theta,x)\sqrt{\widehat{a_2}(\theta,x)}}\sqrt{\frac{nh_H\widehat{\phi}_{\theta,x}(h_K)f(\theta,t,x)}{\widehat{f}(\theta,t,x)nh_H\phi_{\theta,x}(h_K)}}\times\frac{a_1(\theta,x)}{\sqrt{a_2(\theta,x)}}\sqrt{\frac{nh_H\phi_{\theta,x}(h_K)}{f(\theta,t,x)}}\left(\widehat{f}(\theta,t,x) - f(\theta,t,x)\right).\end{aligned}$$

Via Theorem 3.1, we have,

$$\frac{a_1(\theta,x)}{\sqrt{a_2(\theta,x)}}\sqrt{\frac{nh_H\phi_{\theta,x}(h_K)}{f(\theta,t,x)}}\left(\widehat{f}(\theta,t,x) - f(\theta,t,x)\right) \longrightarrow \mathcal{N}(0,1).$$

Next, by [21], we can prove that,

$$\widehat{a_1}(\theta,x) \xrightarrow{\mathbb{P}} a_1(\theta,x),\ \widehat{a_2}(\theta,x) \xrightarrow{\mathbb{P}} a_2(\theta,x), \text{ and } \frac{\widehat{\phi}_{\theta,x}(h_K)}{\phi_{\theta,x}(h_K)} \xrightarrow{\mathbb{P}} 1, \quad \text{as} \quad n \to \infty.$$

Therefore, we obtain,

$$\frac{\widehat{a_1}(\theta,x)\sqrt{a_2(\theta,x)}}{a_1(\theta,x)\sqrt{\widehat{a_2}(\theta,x)}}\sqrt{\frac{nh_H\widehat{\phi}_{\theta,x}(h_K)f(\theta,t,x)}{\widehat{f}(\theta,t,x)nh_H\phi_{\theta,x}(h_K)}} \longrightarrow 1, \text{ as } n \to \infty.$$

This yields the proof of Corollary 4.4.
□

Finally, in order to show the asymptotic $(1-\xi)$ confidence interval of $M_\theta(x)$, we need to consider the estimator of $\sigma_1^2(\theta,x)$ as follows:

$$\widehat{\sigma}_1^2(\theta,x) = \frac{\widehat{a_2}(\theta,x)\widehat{f}(\theta,\widehat{M}_\theta(x),x)}{\left(\widehat{a_1}(\theta,x)\widehat{f}^{(2)}(\theta,\widehat{M}_\theta(x),x)\right)^2}\int (H'(u))^2 du.$$

Thus, the following corollary is obtained.

Corollary 4.5. *Under conditions of Theorem 4.3, as $n \to \infty$ we have,*

$$\sqrt{\frac{nh_H^3\widehat{\phi}_{\theta,x}(h_K)}{\widehat{\sigma}_1^2(\theta,x)}}(\widehat{M}_\theta(x) - M_\theta(x)) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1).$$

Proof. Observe that

$$\begin{aligned}
\Sigma' &= \frac{\widehat{a_1}(\theta,x)\widehat{f}^{(2)}(\theta,\widehat{M}_\theta(x),x)}{\sqrt{\widehat{a_2}(\theta,x)}}\sqrt{\frac{nh_H^3\widehat{\phi}_{\theta,x}(h_K)}{\widehat{f}(\theta,\widehat{M}_\theta(x),x)}}\left(\widehat{M}_\theta(x) - M_\theta(x)\right)\\
&= \frac{\widehat{a_1}(\theta,x)\sqrt{a_2(\theta,x)}}{a_1(\theta,x)\sqrt{\widehat{a_2}(\theta,x)}}\sqrt{\frac{nh_H^3\widehat{\phi}_{\theta,x}(h_K)f(\theta,M_\theta(x),x)}{\widehat{f}(\theta,\widehat{M}_\theta(x),x)nh_H^3\phi_{\theta,x}(h_K)}}\frac{\widehat{f}^{(2)}(\theta,\widehat{M}_\theta(x),x)}{f^{(2)}(\theta,M_\theta(x),x)}\\
&\quad\times\frac{a_1(\theta,x)}{\sqrt{a_2(\theta,x)}}\sqrt{\frac{nh_H^3\phi_{\theta,x}(h_K)}{f(\theta,M_\theta(x),x)}}f^{(2)}(\theta,M_\theta(x),x)\left(\widehat{M}_\theta(x) - M_\theta(x)\right).
\end{aligned}$$

Making use of Theorem 4.3, we obtain,

$$\frac{a_1(\theta,x)}{\sqrt{a_2(\theta,x)}}\sqrt{\frac{nh_H^3\phi_{\theta,x}(h_K)}{f(\theta,M_\theta(x),x)}}f^{(2)}(\theta,M_\theta(x),x)\left(\widehat{M}_\theta(x) - M_\theta(x)\right) \longrightarrow \mathcal{N}(0,\,1)\,.$$

Further, by considering Lemma 3.4, (8) and (9), we obtain

$$\frac{\widehat{a_1}(\theta,x)\sqrt{a_2(\theta,x)}}{a_1(\theta,x)\sqrt{\widehat{a_2}(\theta,x)}}\sqrt{\frac{nh_H^3\widehat{\phi}_{\theta,x}(h_K)f(\theta,M_\theta(x),x)}{\widehat{f}(\theta,\widehat{M}_\theta(x),x)nh_H^3\phi_{\theta,x}(h_K)}}\frac{\widehat{f}^{(2)}(\theta,\widehat{M}_\theta(x),x)}{f^{(2)}(\theta,M_\theta(x),x)} \xrightarrow[n\to\infty]{\mathbb{P}} 1.$$

Hence, the proof is completed. □

Remark 4.6. *Thus, following the corollaries, Corollary 4.4 and Corollary 4.5, the asymptotic* $(1-\xi)$ *confidence interval of* $f(\theta,t,x)$ *and* $M_\theta(x)$ *are given by:*

$$\widehat{f}(\theta,t,x) \pm \tau_{\xi/2} \times \sqrt{\frac{\widehat{V}(\theta,t,x)}{nh_H\widehat{\phi}_{\theta,x}(h_K)}} \text{ and } \widehat{M}_\theta(x) \pm \tau_{\xi/2} \times \frac{\widehat{\sigma}_1(\theta,x)}{\sqrt{nh_H^3\widehat{\phi}_{\theta,x}(h_K)}},$$

where $\tau_{\xi/2}$ *is the upper* $\xi/2$ *quantile of standard Normal* $\mathcal{N}(0,1)$.

Corollary 4.7. *If the assumptions (H1)-(H7) as well as (N1)-(N5) hold, then, we have,*

$$\sqrt{\frac{nh_H^3\phi_{\theta,x}(h_K)}{\sigma_1^2(\theta,x)}}(\widehat{M}_\theta(x)-M_\theta(x)) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1), \text{ as } n\to\infty,$$

where,

$$\sigma_1^2(\theta,t,x) = \frac{a_2(\theta,x)f(\theta,M_\theta(x),x)}{(a_1(\theta,x)f^{(2)}(\theta,M_\theta(x),x))^2\bar{G}(t)}\int_{\mathbb{R}} H^{'2}(u)du.$$

5. Simulation study

To study the behavior of our conditional mode estimator, we consider in this part two examples of simulation. In the first one, we compare our model CFSIM (functional single index model with censored data) with that of CNPFDA (censored non-parametric functional data analysis) and in the latter, knowing the distribution of the regression model (the distribution is known and usual), we look to the behavior of our estimator of the conditional density function with respect to this distribution. Therefore, the best way to know the behavior of the estimator of conditional density is to compute its mean square error. So, in this part of paper we compare between the conditional density estimation in the CFSIM which is our model and the conditional density estimation in the CNPFDA defined in (11).

$$\widehat{f_n}(x|y) = \frac{\sum_{i=1}^{n}\frac{\delta_i}{\bar{G}_n(Y_i)}K\left(h_K^{-1}d(x,X_i)\right)H\left(h_H^{-1}(t-Y_i)\right)}{\sum_{i=1}^{n}K\left(h_K^{-1}d(x,X_i)\right)}. \tag{11}$$

So, we have to compare their respective conditional density estimators by computing and comparing their respective mean square errors for some values of the scalar response T.

In the following, our purpose consists in assessing the performance, in terms of prediction, of $\widehat{M}_\theta(x)$ and $\widehat{M}(x)$. For each given predictor $(X_j)_{j\in\mathcal{J}}$ in the testing subsample, we are interested in the prediction of the response variable $(Y_j)_{j\in\mathcal{J}}$ via the single functional index conditional mode $\widehat{M}_\theta(x)$ and the fully nonparametric conditional mode $\widehat{M}(x)$ so as to compare the finite-sample behavior of the estimator. As assessment tool we consider the mean square error (MSE) defined as follows:

$$SSR = \frac{1}{|\mathcal{J}|}\sum_{j\in\mathcal{J}}\left(Y_j-\widehat{Y}_j\right)^2, \tag{12}$$

where $\widehat{Y}_j$ is a predictor of Y_j obtained either semi-parametrically by $\widehat{M}_\theta(x)$ or nonparametrically via $\widehat{M}(x)$.

Furthermore, some tuning parameters have to be specified. The kernel $K(\cdot)$ is chosen to be the quadratic function defined as $K(u) = \frac{3}{2}\left(1-u^2\right)\mathbf{1}_{[0,1]}$ and the cumulative df $H(u) = \int_{-\infty}^{u}\frac{3}{4}\left(1-z^2\right)\mathbf{1}_{[-1,1]}(z)\,dz$.

The semi-metric $d(\cdot,\cdot)$ will be specified according to the choice of the functional space $\mathcal{H}$ discussed in the scenarios below. It is well-known that one of the crucial parameters in semi-parametric models is the smoothing parameters which are involved in defining the shape of the link function between the response and the covariate.

Using the result given in Theorem 4.3, the variance of our estimator is obtained as,

$$CV = \frac{a_2(\theta,x) f(\theta, M_\theta(x), x)}{n h_H^3 \phi_{\theta,x}(h_K)\Big(a_1(\theta,x) f^{(2)}(\theta, M_\theta(x), x)\Big)^2 \bar{G}(t)}.$$

The idea is to choose the parameters h_K and h_H so that the variance is minimal. Since the variance (CV) depends on several unknown parameters that must be estimated, the calculus becomes tedious. Thus, by replacing the unknown parameters by their respective estimators $\widehat{a_1}(\theta,x)$, $\widehat{a_2}(\theta,x)$, $\widehat{M}_\theta(x)$, $\widehat{f}$, and $\widehat{\phi}_{\theta,x}(h_K)$, we obtain,

$$(h_K, h_H) = \arg\min_{h_K,h_H} CV(h_K,h_H) = \arg\min_{h_K,h_H} \frac{\widehat{a_2}(\theta,x)\widehat{f}(\theta,\widehat{M}_\theta(x),x)}{n h_H^3 \widehat{\phi}_{\theta,x}(h_K)\Big(\widehat{a_1}(\theta,x)\widehat{f^{(2)}}(\theta,\widehat{M}_\theta(x),x)\Big)^2 \bar{G}_n(t)}.$$

Now for simplifying the implementation of our methodology, we take the bandwidths $h_H \sim h_K = h$, where h will be chosen by the cross-validation method on the k-nearest neighbors (see [17]), p. 102).

5.1. Simulation 1: case of smooth curves

Let us consider the following regression model, where the covariate is a curve and the response is a scalar:

$$T_i = R(X_i) + \epsilon_i,\ i = 1,\dots,n,$$

where ϵ_i is the error supposed to be generated by an autoregressive model defined by:

$$\epsilon_i = \frac{1}{\sqrt{2}}\epsilon_{i-1} + \eta_i,\ i = 1,\dots,n,$$

with $(\eta_i)_i$ a sequence of i.i.d. random variables normally distributed with a variance equal to 0.1.

The functional covariate X is assumed to be a diffusion process defined on [0, 1] and generated by the following equation:

$$X(t) = a\cos(b + \pi W t) + c\sin(d + \pi W t) + (1 - A)\sin(\pi t W),\ t \in [0, 1],$$

where W is an a process generated by $W_i = \frac{2}{9} + \epsilon$, $i = 1,\dots,200$, b and d are independent of normal distributions respectively $\rightsquigarrow \mathcal{N}(0, 0.03)$ and $\rightsquigarrow \mathcal{N}(0, 0.05)$. The variables a and c are Bernoulli's laws Bernoulli $\mathcal{B}(0.5)$. Figure 1 depicts a sample of 200 curves representing a realization of the functional random variable X.

Take into account of the smoothness of the curves $X_i(t)$ (see Figure 1), we choose the distance $deriv_1$ (the semi-metric based on the first derivatives of the curves) in $\mathcal{H}$ as:

$$d(\chi_1, \chi_2) = \left(\int_0^1 \left(\chi_1'(t) - \chi_2'(t)\right)^2 dt\right)^{1/2},$$

as semi-metric.

Then, we consider a nonlinear regression function defined as

$$R(X) = 4\log\left\{1/\left(\int_0^1 (X'(t))^2 dt + \left[\int_0^1 X'(t)dt\right]^2\right)\right\}.$$

On the other hand, n i.i.d. random variables $(C_i)_i$ are simulated through the exponential distribution $\mathcal{E}(1.5)$.

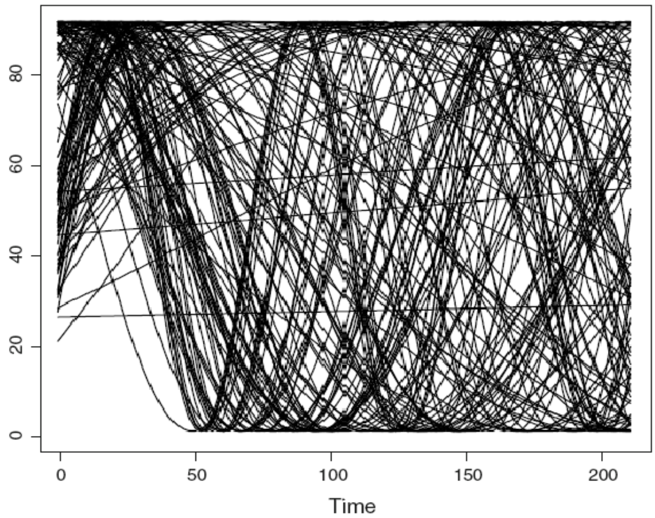

Figure 1: A sample of 200 curves $X_{i=1,\dots,200}(t_j)$, $t_{j=1,\dots,200} \in [0,1]$

Given $X = x$, $T \rightsquigarrow \mathcal{N}(R(x), 0.2)$, and thus, the conditional median, the conditional mode and the conditional mean functions will coincide and will be equal to $R(x)$, for any fixed x. The computation of our estimator is based on the observed data $(X_i, Y_i, \delta_i)_{i=1,\dots,n}$ where $Y_i = min(T_i, C_i)$; $\delta_i = \mathbb{I}_{\{T_i \leq C_i\}}$ and the single index θ which is unknown and has to be estimated.

In practice this parameter can be selected by cross-validation approach (see [2]). In this passage it may be that one can select the real-valued function $\theta(t)$ among the eigenfunctions of the covariance operator $\mathbb{E}[(X' - \mathbb{E}X') < X', \cdot >_{\mathcal{H}}]$ where $X(t)$ is a diffusion processes defined on a real interval $[a, b]$ and $X'(t)$ its first derivative (see [5]). So for a chosen training sample $\mathcal{L}$, by applying the principal component analysis (PCA) method, the computation of the eigenvectors of the covariance operator estimated by its empirical covariance operator: $\frac{1}{|\mathcal{L}|}\sum_{i\in\mathcal{L}}(X'_i - \mathbb{E}X')^t(X'_i - \mathbb{E}X')$, will be the one best approximation of our functional parameter θ. Now, let us denote $\theta^\star$ the first eigenfunction corresponding to the first higher eigenvalue of the empirical covariance operator, which will replace θ during the simulation step.

In the following graphs, the covariance operator for $\mathcal{L} = \{1, \dots, 200\}$ gives the discretization of the first eigenfunction θ (presented by a continuous curve) and all the eigenfunctions $\theta_i(t)$ (Figure 2 and 3). In this simulation part, we divide our sample of size 200 into two parts. The first one from 1 to 125 will be used to make the simulation and the second from 126 to 200 will serve us for the prediction.

We follow the following steps:

Step 1. Compute the inner product: $< \theta^*, X_1 >, \dots, < \theta^*, X_{200} >$, generate independently the variables $\varepsilon_1, \dots, \varepsilon_{200}$, then simulate the response variables $Y_i = r(< \theta^*, X_i >) + \varepsilon_i$, where $r(< \theta^*, X_i >) = \exp(10(< \theta^*, X_i > -0.05))$ and generate independently the variables $\varepsilon_1, \dots, \varepsilon_{200}$.

Step 2. For each k in the test sample $\mathcal{J} = \{126, \dots, 200\}$,we compute: $\widehat{Y_k} = \widehat{M}_{\theta^\star}(X_k)$ and $\widehat{Y_k} = \widehat{M}(X_k)$,

where
$$M(x) = \arg\sup_{y\in\mathcal{S}_\mathbb{R}} f(x|y) \ \text{ and } \ \widehat{M}(x) = \arg\sup_{y\in\mathcal{S}_\mathbb{R}} \widehat{f_n}(x|y).$$

Finally, we present the results by plotting the predicted values versus the true values and compute the sum of squared residuals (SSR) defined by (12).

We see that the sum of squared residuals (SSR) of our method Functional-Single-Index-Model with Censored Data (CFSIM) is less than the one of the Censored Non-Parametric-Functional-Data-Analysis (CNPFDA). This is confirmed by the following graphs, when we compare the conditional mode by (CFSIM)

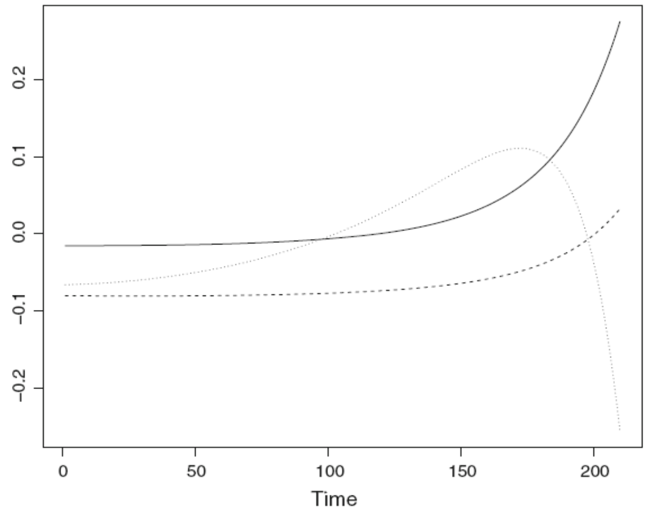

Figure 2: The curves $\theta_{i=1,2,3}(t_j)$, $t_{j=1,\dots,200} \in [0, 1]$

against the conditional mode by (CNPFDA) (Fig. 4). Our estimator is so acceptable. As intuitively expected, it is well observed that the mean square errors of our estimator are smaller than that of CNPFDA. Thus, again, the CFSIM model produces much more accurate estimation accuracies than CNPFDA model in all criteria.

In order to construct conditional confidence bands we proceed by the following algorithm:

Step 1. $< \theta^\star, X_1 >, \dots, < \theta^\star, X_{200} >$, generate independently the variables $\varepsilon_1, \dots, \varepsilon_{200}$, then simulate the response variables $Y_i = r(< \theta^\star, X_i >) + \varepsilon_i$, where $r(< \theta^\star, X_i >) = \exp(10(< \theta^\star, X_i > -0.05))$ and generate independently the variables $\varepsilon_1, \dots, \varepsilon_{200}$.

Step 2. For each i in the training sample, we calculate the estimator: $\widehat{Y_i} = \widehat{M}_{\theta^\star}(X_i)$.

Step 3. For each X_j in the test sample $\mathcal{J} = 126, \dots, 200$, we set: $j_\star := \arg\min_{i \in \mathcal{L}} d_\theta(X_i, X_j)$.

Step 4. For each j in the test sample $\mathcal{J} = 126, \dots, 200$, we define the confidence bands by:

$$\Big[\widehat{M}_{\theta^\star}(X_{j_\star}) - \tau_{0.975} \times \Big(\frac{\widehat{\nu}(\theta^\star, X_{j_\star})}{\sqrt{\mathcal{L} h_H^3 \widehat{\phi}_{\theta^\star,x}(h_K)}}\Big), \widehat{M}_{\theta^\star}(X_{j_\star}) + \tau_{0.975} \times \Big(\frac{\widehat{\nu}(\theta^\star, X_{j_\star})}{\sqrt{\mathcal{L} h_H^3 \widehat{\phi}_{\theta^\star,x}(h_K)}}\Big)\Big].$$

We obtain the following figure which gathers asymptotic confidence bands study.

6. Proofs of technical lemmas

Proof. [Proof of Lemma 2.2] The proof is similar to that of Lemma 5.2 in [18]. From Equations (2) and (3), we have,

$$\begin{aligned}
|\widehat{f_N}(\theta, t, x) - \widetilde{f_N}(\theta, t, x)| &\leq \frac{h_H^{-1}}{n \mathbb{E} K_1(\theta, x)} \sum_{i=1}^n \Big| \frac{\delta_i}{\bar{G}_n(Y_i)} K_i(\theta, x) H_i(t) - \frac{\delta_i}{\bar{G}(Y_i)} K_i(\theta, x) H_i(t) \Big| \\
&\leq \frac{h_H^{-1}}{n \mathbb{E} K_1(\theta, x)} \sum_{i=1}^n |\delta_i K_i(\theta, x) H_i(t)| \Big| \frac{1}{\bar{G}_n(Y_i)} - \frac{1}{\bar{G}(Y_i)} \Big| \\
&\leq \frac{h_H^{-1}}{\phi_{\theta,x}(h_K)} \frac{C}{\bar{G}_n(\tau_G) \bar{G}(\tau_G)} \sup_{t \in \mathbb{R}} |\bar{G}_n(t) - \bar{G}(t)| \frac{1}{n} \sum_{i=1}^n |K_i(\theta, x) H_i(t)|.
\end{aligned}$$

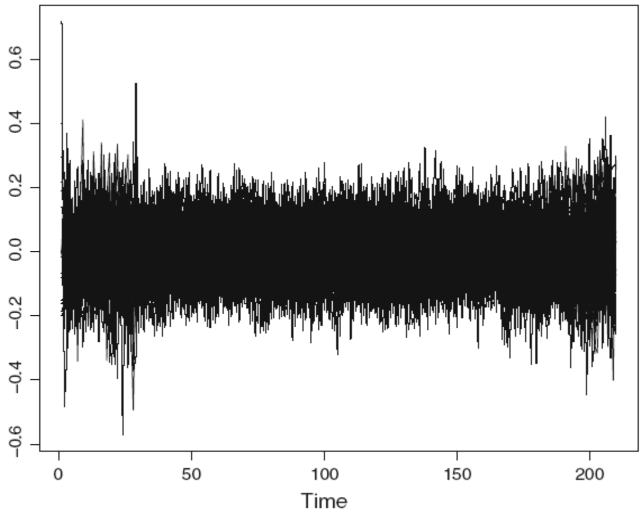

Figure 3: The curves $\theta_{i=1,\dots,200}(t_j),\ t_{j=1,\dots,200} \in [0, 1]$

Since $\bar{G}(\tau_G) > 0$, together with the SLLN and the LIL on the censoring law (see formula (4.28) in Deheuvels and Einmahl [9]), we obtain

$$\sup_{t\le\tau_G}\left|\bar{G}_n(t) - \bar{G}(t)\right| = O_{a.s.}\left(\frac{\log\log n}{n}\right).$$

We achieve the proof by considering the conditions (H3) and (H4). □

Proof. [Proof of Lemma 2.3] We have,

$$\begin{aligned}
\mathbb{E}\widetilde{f_N}(\theta,t,x) - f(\theta,t,x) &= \frac{h_H^{-1}}{\mathbb{E}K_1(x,\theta)}\mathbb{E}\left(\frac{\delta_i}{\bar{G}(Y_i)}K_i(x,\theta)H_i(t)\right) - f(\theta,t,x) \\
&= \frac{h_H^{-1}}{\mathbb{E}K_1(x,\theta)}\mathbb{E}\left(K_i(x,\theta)\left[\mathbb{E}\left(\frac{\delta_i}{\bar{G}(Y_i)}H_i(t)| < X_1,\theta >\right) - h_H f(\theta,t,x)\right]\right). \qquad (13)
\end{aligned}$$

Using the fact that H is a *cdf* and the use a double conditioning with respect to T_1, we can easily get

$$\begin{aligned}
I &= \mathbb{E}\left(\frac{\delta_i}{\bar{G}(Y_i)}H_i(t)| < X_1,\theta >\right) \\
&= \mathbb{E}\left(\mathbb{E}\left[\frac{\mathbf{1}_{T_1\le C_1}}{\bar{G}(T_1)}H\left(\frac{t-T_1}{h_H}\right)| < X_1,\theta >, T_1\right]\right) \\
&= \mathbb{E}\left(\frac{1}{\bar{G}(T_1)}H\left(\frac{t-T_1}{h_H}\right)\mathbb{E}\left[\mathbf{1}_{T_1\le C_1}|T_1\right]| < X_1,\theta >\right) \\
&= \mathbb{E}\left[H\left(\frac{t-T_1}{h_H}\right)| < X_1,\theta >\right] \\
&= \int_{\mathbb{R}} H\left(\frac{t-u}{h_H}\right)f(\theta,u,X_1)du, \\
&= h_H\int_{\mathbb{R}} H(v)f(\theta,t-vh_H,X_1)dv, \\
&= h_H\int_{\mathbb{R}} H(v)\left(f(\theta,t-vh_H,X_1) - f(\theta,t,x)\right)dv + h_H f(\theta,t,x)\int_{\mathbb{R}} H(v)dv,
\end{aligned}$$

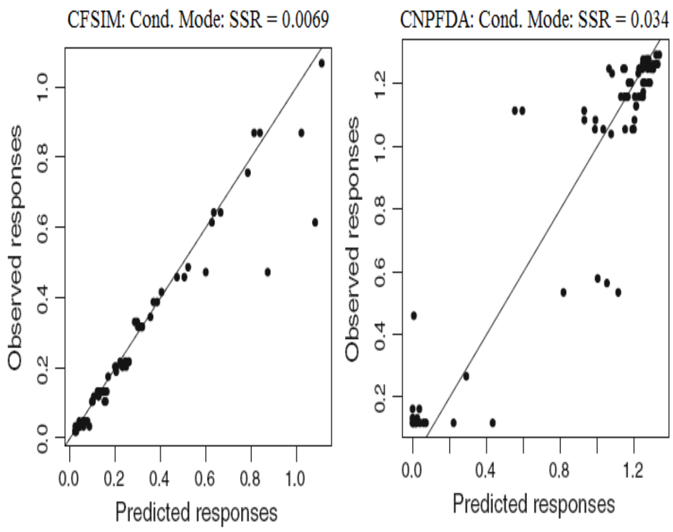

Figure 4: Prediction via the conditional mode by CFSIM with error SSR = 0.0069 against CNPFDA with error SSR = 0.034, with CR∼ 18%

we can write, because of (H2) and (H3):

$$\begin{aligned} I &\leq h_H\, C_{x,\theta} \int_{\mathbb{R}} H(v) \left(h_K^{b_1} + |v|^{b_2} h_H^{b_2}\right) dv + h_H\, f(\theta, t, x) \\ &= O\left(h_K^{b_1} + h_H^{b_2}\right) + h_H\, f(\theta, t, x). \end{aligned}$$

Combining this last result with (13) allows us to achieve the proof. □

Proof. [Proof of Lemma 2.4]

(i) Similar to the proof of Lemma 3 in Attaoui [4], it can be completed easily. Here we omit its proof.

(ii) Using the compactness of $\mathcal{S}_{\mathbb{R}}$, we can write that, $\mathcal{S}_{\mathbb{R}} \subset \bigcup_{k=1}^{\tau_n} (z_k - l_n, z_k + l_n)$ with l_n and τ_n can be chosen such that $l_n = C\tau_n^{-1} \sim Cn^{-\varsigma-1/2}$. Taking $k_t = \arg \min_{\{z_1,\ldots,z_{\tau_n}\}} |t - z_k|$.

Thus, we have the following decomposition:

$$\begin{aligned} T &= \frac{1}{\widehat{F}_D(\theta, x)} \sup_{t \in \mathcal{S}_{\mathbb{R}}} \left| \widetilde{f_N}(\theta, t, x) - \mathbb{E}\widetilde{f_N}(\theta, t, x) \right| \\ &\leq \frac{1}{\widehat{F}_D(\theta, x)} \sup_{t \in \mathcal{S}_{\mathbb{R}}} \left| \widetilde{f_N}(\theta, t, x) - \widehat{f_N}(\theta, t_k, x) \right| \\ &\quad + \frac{1}{\widehat{F}_D(\theta, x)} \sup_{t \in \mathcal{S}_{\mathbb{R}}} \left| \widehat{f_N}(\theta, t_k, x) - \mathbb{E}\widehat{f_N}(\theta, t_k, x) \right| \\ &\quad + \frac{1}{\widehat{F}_D(\theta, x)} \sup_{t \in \mathcal{S}_{\mathbb{R}}} \left| \mathbb{E}\widehat{f_N}(\theta, t_k, x) - \mathbb{E}\widetilde{f_N}(\theta, t, x) \right| \\ &\leq T_1 + T_2 + T_3. \end{aligned}$$

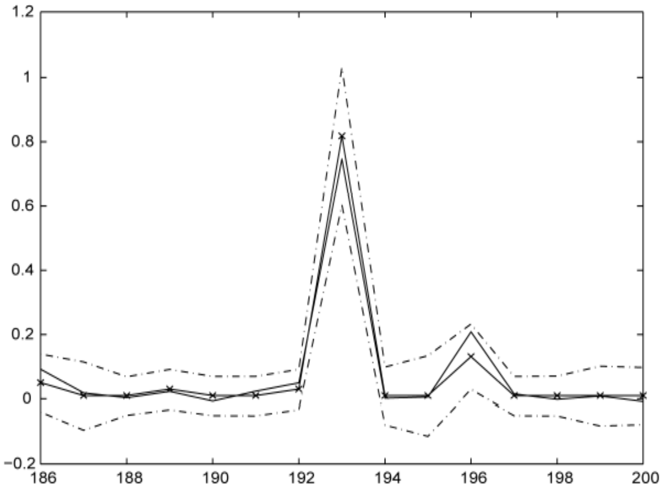

Figure 5: The 95% conditional predictive bands. The solid curve connects the true values. The crossed curve joins the predicted values. The dashed curves connects the lower and upper predicted values with CR~ 3%

On the one hand, as the first and the third terms can be treated in the same manner, we deal only with first term. Making use of (H3) we get

$$\begin{aligned} T_1' &= \sup_{t\in\mathcal{S}_{\mathbb{R}}}\left|\widetilde{f_N}(\theta,t,x)-\widehat{f_N}(\theta,t_k,x)\right| \\ &\leq \frac{1}{nh_H\mathbb{E}K_1(\theta,x)}\sup_{t\in\mathcal{S}_{\mathbb{R}}}\sum_{i=1}^{n}\left|\frac{\delta_i}{\bar{G}(Y_i)}H_i(t)-\frac{\delta_i}{\bar{G}_n(Y_i)}H_i(t_k)\right||K_i(\theta,x)| \\ &\leq \frac{1}{nh_H\mathbb{E}K_1(\theta,x)}\sup_{t\in\mathcal{S}_{\mathbb{R}}}\sum_{i=1}^{n}\left|\frac{\delta_i}{\bar{G}(Y_i)}H_i(t)-\frac{\delta_i}{\bar{G}_n(Y_i)}H_i(t_k)\right||K_i(\theta,x)| \\ &\leq \frac{C}{nh_H\mathbb{E}K_1(\theta,x)}\sup_{t\in\mathcal{S}_{\mathbb{R}}}\frac{|t-t_k|}{h_H}\times\left(\sum_{i=1}^{n}K_i(\theta,x)\left(\frac{1}{\bar{G}(Y_i)}-\frac{1}{\bar{G}_n(Y_i)}\right)\right) \\ &\leq \frac{Cl_n}{h_H^2\bar{G}_n(\tau_G)\bar{G}(\tau_G)}\sup_{t\in\mathcal{S}_{\mathbb{R}}}|G_n(t)-G(t)|\widehat{F}_D(\theta,x). \end{aligned}$$

Using $l_n = n^{-\varsigma-1/2}$ we obtain

$$T_1 \leq \frac{Cn^{-\varsigma-1/2}}{h_H^2\bar{G}_n(\tau_G)\bar{G}(\tau_G)}\left(\frac{\log n\log n}{n}\right)^{1/2}$$

and note that, because of (H3), we have,

$$\frac{l_n}{h_H^2} = o\left(\sqrt{\frac{\log n}{nh_H\phi_{\theta,x}(h_K)}}\right).$$

Thus, for n large enough, we have,

$$T_1 = O_{a.co}\left(\sqrt{\frac{\log n}{nh_H\phi_{\theta,x}(h_K)}}\right).$$

Following similar arguments, we can write

$$T_3 \;\leq\; T_1.$$

Concerning T_2, let us consider $\varepsilon = \epsilon_0 \sqrt{\frac{s_{n,2}^2 \log n}{n^2}}$. Since for all $\epsilon_0 > 0$, we have that,

$$\begin{aligned}
B &= \mathbb{P}\left(\sup_{t\in\mathcal{S}_{\mathbb{R}}} \left|\widehat{f_N}(\theta, t_k, x) - \mathbb{E}\widehat{f_N}(\theta, t_k, x)\right| > \varepsilon)\right) \\
&\leq \mathbb{P}\left(\max_{k\in\{1\ldots\tau_n\}} \left|\widehat{f_N}(\theta, t_k, x) - \mathbb{E}\widehat{f_N}(\theta, t_k, x)\right| > \varepsilon\right) \\
&\leq \tau_n \mathbb{P}\left(\left|\widehat{f_N}(\theta, t_k, x) - \mathbb{E}\widehat{f_N}(\theta, t_k, x)\right| > \varepsilon\right).
\end{aligned}$$

The application of Fuk-Nagaev's inequality (see Proposition A.11-ii of Ferraty and Vieu [17]) with $r = (\log n)^2$ and $q = a + 1$, we get that,

$$\begin{aligned}
\mathbb{P}\left(\left|\widehat{f_N}(\theta, t_k, x) - \mathbb{E}\widehat{f_N}(\theta, t_k, x)\right| > \varepsilon\right) &\leq \left(1 + \frac{\epsilon_0^2}{(\log n)^2}\right)^{-(\log n)^2/2} + n(\log n)^{-2}\left(\frac{\sqrt{\log n}}{\epsilon_0 s_{n,2}}\right)^{a+1} \\
&\leq C_{\theta,x} B_1 + C'_{\theta,x} B_2.
\end{aligned}$$

Finally, the use that, $s_{n,2}^2 = O(nh_H\phi_{\theta,x}(h_K))$, allows to get directly that there exist some $\eta > 0$ such that

$$B_1 + B_2 \;\leq\; Cn^{-1-\eta}.$$

Finally, we arrive at,

$$\sup_{t\in\mathcal{S}_{\mathbb{R}}} \left|\widehat{f_N}(\theta, t_k, x) - \mathbb{E}\widehat{f_N}(\theta, t_k, x)\right| = O_{a.co.}\left(\frac{\sqrt{s_{n,2}^2 \log n}}{n}\right).$$

□

Proof. [Proof of Lemma 3.2]

$$\begin{aligned}
V_n(\theta, t, x) &= \frac{h_H^{-1}\phi_{\theta,x}(h_K)}{(\mathbb{E}K_1(\theta, x))^2} \mathbb{E}\left[K_1^2(\theta, x)\left(\frac{\delta_1}{\bar{G}(Y_1)} H_1(t) - h_H f(\theta, t, x)\right)^2\right] \\
&= \frac{h_H^{-1}\phi_{\theta,x}(h_K)}{(\mathbb{E}K_1(\theta, x))^2} \mathbb{E}\left\{K_1^2(\theta, x)\mathbb{E}\left[\left(\frac{\delta_1}{\bar{G}(Y_1)} H_1(t) - h_H f(\theta, t, x)\right)^2 \mid <\theta, X_1>\right]\right\}. \qquad (14)
\end{aligned}$$

Using the definition of conditional variance, we have,

$$\mathbb{E}\left[\left(\frac{\delta_1}{\bar{G}(Y_1)} H_1(t) - h_H f(\theta, t, x)\right)^2 \mid <\theta, X_1>\right] = J_{1n} + J_{2n},$$

where

$$J_{1n} = Var\left(\frac{\delta_1}{\bar{G}(Y_1)} H_1(t) | <\theta, X_1>\right)$$

and

$$J_{2n} = \left[\mathbb{E}\left(\frac{\delta_1}{\bar{G}(Y_1)} H_1(t) | <\theta, X_1>\right) - h_H f(\theta, t, x)\right]^2.$$

• Concerning J_{1n}

$$J_{1n} = \mathbb{E}\left(\frac{\delta_1}{\bar{G}^2(Y_1)}H_1^2(t)| < \theta, X_1 >\right) - \mathbb{E}\left(\frac{\delta_1}{\bar{G}(Y_1)}H_1(t)| < \theta, X_1 >\right)^2 = J_1 + J_2.$$

As for J_1, by the property of double conditional expectation and by changing variables, we get that,

$$\begin{aligned}
J_1 &= \mathbb{E}\left[\mathbb{E}\left(\frac{\delta_1}{\bar{G}^2(Y_1)}H_1^2\left(\frac{t-Y_1}{h_H}\right)| < \theta, X_1 >, T_1\right)\right] \\
&= \mathbb{E}\left(\frac{1}{\bar{G}^2(T_1)}H_1^2\left(\frac{t-T_1}{h_H}\right)\mathbb{E}[\mathbf{1}_{T_1\leq C_1}|T_1]| < \theta, X_1 >\right) \\
&= \mathbb{E}\left(\frac{1}{\bar{G}(T_1)}H_1^2\left(\frac{t-T_1}{h_H}\right)| < \theta, X_1 >\right) \\
&= \int_{\mathbb{R}} \frac{1}{\bar{G}(v)}H_1^2\left(\frac{t-v}{h_H}\right)f(\theta, v, X_1)dv \\
&= \int_{\mathbb{R}} \frac{1}{\bar{G}(t-uh_H)}H_1^2(u)dF(\theta, t-uh_H, X_1). \qquad (15)
\end{aligned}$$

By the first order Taylor's expansion of the function $\bar{G}^{-1}(\cdot)$ around zero, one gets

$$J_1 = \int_{\mathbb{R}} \frac{1}{\bar{G}(t)}H_1^2(u)dF(\theta, t-uh_H, X_1) + \frac{h_H^2}{\bar{G}(t)^2}\int_{\mathbb{R}} uH_1^2(u)\bar{G}^{(1)}(t^*)f(\theta, t-uh_H, X_1)du + o(1),$$

where t^* is between t and $t - uh_H$.

Under assumptions (N1) and using hypothesis (H2), we get

$$\frac{h_H^2}{\bar{G}^2(t)}\int_{\mathbb{R}} uH_1^2(u)\bar{G}^{(1)}(t^*)f(\theta, t-uh_H, X_1)du = o(h_H^2).$$

Indeed

$$\begin{aligned}
J_1^{'} &= \frac{h_H^2}{\bar{G}^2(t)}\int_{\mathbb{R}} uH_1^2(u)\bar{G}^{(1)}(t^*)f(\theta, t-uh_H, X_1)du \\
&\leq h_H^2\left(\sup_{u\in\mathbb{R}}|G^{'}(u)||\bar{G}^2(t)\right)\int_{\mathbb{R}} uf(\theta, t-yh_H, x)du.
\end{aligned}$$

On the other hand, by applying (H2) and (H3), we have,

$$\begin{aligned}
\int_{\mathbb{R}} \frac{1}{\bar{G}(t)}H_1^2(u)dF(\theta, t-uh_H, X_1) &= h_H\int_{\mathbb{R}} \frac{1}{\bar{G}(t)}H_1^2(u)f(\theta, t-uh_H, X_1)du \\
&\leq \frac{h_H}{\bar{G}(t)}\left(\int_{\mathbb{R}} H_1^2(u)(f(\theta, t-uh_H, X_1) - f(\theta, t, x))du + \int_{\mathbb{R}} H_1^2(u)f(\theta, t, x)du\right) \\
&\leq \frac{h_H}{\bar{G}(t)}C_{x,\theta}\left(\int_{\mathbb{R}} H^2(u)\left(h_K^{b_1} + |v|^{b_2}h_H^{b_2}\right)du + f(\theta, t, x)\int_{\mathbb{R}} H^2(u)du\right) \\
&= O\left(h_k^{b_1} + h_H^{b_2}\right) + \frac{h_H f(\theta, t, x)}{\bar{G}(t)}\int_{\mathbb{R}} H^2(u)du. \qquad (16)
\end{aligned}$$

As for J_2,

$$\begin{aligned} J_2^{'} &= \mathbb{E}\left(\frac{\delta_1}{\bar{G}(Y_1)}H_1(t)| < \theta, X_1 >\right) \\ &= \mathbb{E}\left[\mathbb{E}\left(\frac{\delta_1}{\bar{G}(Y_1)}H_1\left(\frac{t-Y_1}{h_H}\right)| < \theta, X_1 >, T_1\right)\right] \\ &= \mathbb{E}\left(\frac{1}{\bar{G}(T_1)}H_1\left(\frac{t-T_1}{h_H}\right)\mathbb{E}[\mathbf{1}_{T_1\leq C_1}|T_1]| < \theta, X_1 >\right) \\ &= \mathbb{E}\left(H_1\left(\frac{t-T_1}{h_H}\right)| < \theta, X_1 >\right) \\ &= \int_{\mathbb{R}} H^{(1)}\left(\frac{t-v}{h_H}\right)f(\theta, t, X_1)dv. \end{aligned}$$

Moreover, we have by changing variables,

$$J_2^{'} = h_H\int_{\mathbb{R}} H(u)(f(\theta, t-uh_H, X_1 - f(\theta, t, x))du + h_H f(\theta, t, x)\int_{\mathbb{R}} H(u)du,$$

the last equality is due to the fact that H is a probability density. Thus, we have,

$$J_2^{'} = O\left(h_k^{b_1} + h_H^{b_2}\right) + h_H f(\theta, t, x). \tag{17}$$

Finally we get $J_2 \underset{n\to\infty}{\longrightarrow} 0$. As for J_{2n}, by (H2)-(H3), we obtain that $J_{2n} \longrightarrow 0$, as $n \to \infty$.
Meanwhile, by (H2)-(H3) and (N1), it follows that

$$\frac{\phi_{\theta,x}(h_K)\mathbb{E}K_1^2(\theta, x)}{\mathbb{E}^2K_1(\theta, x)} \underset{n\to\infty}{\longrightarrow} \frac{a_2(\theta, x)}{(a_1(\theta, x))^2},$$

which leads to combining equations (14)-(17).

$$V_n(\theta, t, x) \underset{n\to\infty}{\longrightarrow} \frac{a_2(\theta, x)}{(a_1(\theta, x))^2}\frac{f(\theta, t, x)}{\bar{G}(t)}\int_{\mathbb{R}} H^2(u)du. \tag{18}$$

Secondly, by the boundness of H and conditioning on $(< \theta, X_i >, < \theta, X_j >)$, we have,

$$\begin{aligned} \mathbb{E}\left(|N_iN_j|\right) &= \mathbb{E}\left[(\Omega_i)\left(\Omega_j\right)K_i(\theta, x)K_j(\theta, x)\right] \\ &= \mathbb{E}\Big(\mathbb{E}\Big[(\Omega_i)\left(\Omega_j\right)| < \theta, X_i >, < \theta, X_j > \Big]K_i(\theta, x)K_j(\theta, x)\Big) \\ &\leq \left(h_H + \frac{1}{\bar{G}(\tau_F)}\right)^2 \mathbb{E}(K_i(\theta, x)K_j(\theta, x)) \\ &\leq Ch_H^2\mathbb{P}\left((X_i, X_j) \in B_\theta(x, h) \times B_\theta(x, h)\right) \\ &\leq Ch_H^2\left(\left(\frac{\phi_{\theta,x}(h_K)}{n}\right)^{1/a}\phi_{\theta,x}(h_K)\right), \end{aligned}$$

where $\Omega_i = \dfrac{\delta_i}{\bar{G}_i(t)}H_i(t) - h_H f(\theta, t, x)$. Then, taking

$$\begin{aligned} \frac{\phi_{\theta,x}(h_K)}{n\mathbb{E}^2K_1(x,\theta)}\sum_{|i-j|>0}^{n}\sum^{n} Cov(N_i, N_j) &= \frac{\phi_{\theta,x}(h_K)}{nh_H\mathbb{E}^2K_1(x,\theta)}\left(\sum_{0<|i-j|\leq m_n}^{n} Cov(N_i, N_j) + \sum_{|i-j|>m_n}^{n} Cov(N_i, N_j)\right) \\ &= K_{1n} + K_{2n}. \end{aligned}$$

Therefore

$$K_{1n} \leq Cm_n h_H \left\{ \left(\frac{\phi_{\theta,x}(h_K)}{n} \right)^{1/a} \right\}, \quad \forall i \neq j.$$

For K_{2n}: since the variable $(\Delta_i)_{1\leq i\leq n}$ is bounded (i.e, $\|\Delta_i\|_\infty < \infty$, we can use the Davydov-Rio's inequality. So, we have for all $i \neq j$,

$$|Cov(\Delta_i, \Delta_j)| \leq Ca(|i-j|).$$

By the fact, $\displaystyle\sum_{k\geq m_n+1} k^{-a} \leq \int_{m_n}^{\infty} v^{-a} dv = \frac{m_n^{-a+1}}{a-1}$, we get by applying (H0),

$$K_{2n} \leq \sum_{|i-j|\geq m_n+1} |i-j|^{-a} \leq \frac{n m_n^{-a+1}}{a-1}.$$

Thus,

$$K_{1n} + K_{2n} \leq Cn \left(m_n h_H \left(\frac{\phi_{\theta,x}(h_K)}{n} \right)^{1/a} + \frac{m_n^{-a+1}}{a-1} \right).$$

Choosing $m_n = h_H^{-1} \left(\frac{\phi_{\theta,x}(h_K)}{n} \right)^{-1/a}$, we get $K_{1n} = o(h_H)$ and $K_{2n} = o(1)$.
Finally by

$$\frac{\phi_{\theta,x}(h_K)}{n\mathbb{E}^2 K_1(x,\theta)} \sum_{|i-j|>0} \sum^{n} Cov(N_i, N_j) = o(1), \tag{19}$$

this complete the proof of lemma. □

Proof. [Proof of Lemma 3.3]
We will establish the asymptotic normality of $A_n(\theta, t, x)$ suitably normalized. We have,

$$\begin{aligned}
\sqrt{nh_H\phi_{\theta,x}(h_K)} A_n(\theta,t,x) &= \frac{\sqrt{nh_H\phi_{\theta,x}(h_K)}}{nh_H \mathbb{E}K_1(\theta,x)} \sum_{i=1}^{n} N_i(\theta,t,x) \\
&= \frac{\sqrt{\phi_{\theta,x}(h_K)}}{\sqrt{nh_H}\mathbb{E}K_1(\theta,x)} \sum_{i=1}^{n} N_i(\theta,t,x) \\
&= \frac{1}{\sqrt{nh_H}} \sum_{i=1}^{n} \Xi_i(\theta,t,x) = \frac{1}{\sqrt{nh_H}} S_n.
\end{aligned}$$

Now we can write, $\Xi_i = \dfrac{\sqrt{h_H\phi_{\theta,x}(h_K)}}{\mathbb{E}K_1(\theta,x)} N_i$, we have,

$$Var(\Xi_i) = \frac{\phi_{\theta,x}(h_K)}{h_H \mathbb{E}^2 K_1(\theta,x)} Var(N_i) = V_n(\theta,t,x).$$

Note that by (18), we have $Var(\Xi_i) \longrightarrow V(\theta,t,x)$ as n goes to infinity and by (19), we have,

$$\sum_{|i-j|>0} |Cov(\Xi_i, \Xi_j)| = \frac{\phi_{\theta,x}(h_K)}{h_H \mathbb{E}^2 K_1(x,\theta)} \sum_{|i-j|>0}^{n} |Cov(N_i, N_j)| = o(nh_H\phi_{\theta,x}(h_K)). \tag{20}$$

Obviously, we have,

$$\sqrt{\frac{n\phi_{\theta,x}(h_K)}{V(\theta,t,x)}} \left(A_n(\theta,t,x)\right) = \left(nh_H V(\theta,t,x)\right)^{-1/2} S_n.$$

Thus, the asymptotic normality of $(nV(\theta,t,x))^{-1/2}\, S_n$, is sufficient to show the proof of this Lemma. This last is shown by the blocking method, where the random variables Ξ_i are grouped into blocks of different sizes defined.

We consider the classical big- and small-block decomposition. We split the set $\{1,2,\ldots,n\}$ into $2k_n+1$ subsets with large blocks of size u_n and small blocks of size v_n and put

$$k_n \;:=\; \left[\frac{n}{u_n+v_n}\right].$$

Now by Assumption (H10)-(ii) allows us to define the large block size by

$$u_n \;=:\; \left[\left(\frac{nh_H\phi_{\theta,x}(h_K)}{q_n}\right)^{1/2}\right].$$

Using Assumption (H10) and simple algebra allows us to prove that

$$\frac{v_n}{u_n}\to 0,\quad \frac{u_n}{n}\to 0,\quad \frac{u_n}{\sqrt{n\phi_{\theta,x}(h_K)}}\to 0,\quad \text{and}\quad \frac{n}{u_n}a(v_n)\to 0. \tag{21}$$

Now, let Υ_j, Υ_j' and Υ_j'' be defined as follows:

$$\Upsilon_j(\theta,t,x) \;=\; \Upsilon_j \;=\; \sum_{i=j(u+v)+1}^{j(u+v)+u}\Xi_i(\theta,t,x),\quad 0\le j\le k-1,$$

$$\Upsilon_j'(\theta,t,x) \;=\; \Upsilon_j' \;=\; \sum_{i=j(u+v)+u+1}^{(j+1)(u+v)}\Xi_i(\theta,t,x),\quad 0\le j\le k-1,$$

$$\Upsilon_j''(\theta,t,x) \;=\; \Upsilon_j'' \;=\; \sum_{i=k(u+v)+1}^{n}\Xi_i(\theta,t,x),\quad 0\le j\le k-1.$$

Clearly, we can write

$$\begin{aligned} S_n(\theta,t,x) \;=\; S_n \quad &= \quad \sum_{j=1}^{k-1}\Upsilon_j+\sum_{j=1}^{k-1}\Upsilon_j'+\Upsilon_k'' \\ &=: \quad \Psi_n(\theta,t,x)+\Psi_n'(\theta,t,x)+\Psi_n''(\theta,t,x) \\ &=: \quad \Psi_n+\Psi_n'+\Psi_n''. \end{aligned}$$

We prove that

$$(i)\ \frac{1}{n}\mathbb{E}(\Psi_n')^2 \longrightarrow 0,\quad (ii)\ \frac{1}{n}\mathbb{E}(\Psi_n'')^2 \longrightarrow 0, \tag{22}$$

$$\left|\mathbb{E}\left\{\exp\left(izn^{-1/2}\Psi_n\right)\right\}-\prod_{j=0}^{k-1}\mathbb{E}\left\{\exp\left(izn^{-1/2}\Upsilon_j\right)\right\}\right| \longrightarrow 0, \tag{23}$$

$$\frac{1}{n}\sum_{j=0}^{k-1}\mathbb{E}\left(\Upsilon_j^2\right) \longrightarrow V(\theta,t,x), \tag{24}$$

$$\frac{1}{n}\sum_{j=0}^{k-1}\mathbb{E}\left(\Upsilon_j^2\mathbf{1}_{\{|\Upsilon_j|>\varepsilon\sqrt{nV(\theta,t,x)}\}}\right)\longrightarrow 0, \tag{25}$$

for every $\varepsilon > 0$.

Expression (22) show that the terms Ψ_n' and Ψ_n'' are asymptotically negligible, while Equations (23) and (24) show that the Υ_j are asymptotically independent, verifying that the sum of their variances tends to $V(\theta,t,x)$. Expression (25) is the Lindeberg-Feller's condition for a sum of independent terms. Asymptotic normality of S_n is a consequence of Equations (22)-(25).

- **Proof of (22)** Because $\mathbb{E}(\Xi_j) = 0,\ \forall j$, we have that,

$$\mathbb{E}(\Psi_n')^2 = Var\left(\sum_{j=1}^{k-1}\Upsilon_j'\right) = \sum_{j=1}^{k-1}Var\left(\Upsilon_j'\right) + \sum_{|i-j|>0}^{k-1}Cov\left(\Upsilon_i',\Upsilon_j'\right) := \Pi_1 + \Pi_2.$$

By the second-order stationarity and (20) we get

$$\begin{aligned}
Var\left(\Upsilon_j'\right) &= Var\left(\sum_{i=j(u_n+v_n)+u_n+1}^{(j+1)(u_n+v_n)}\Xi_i(\theta,t,x)\right)\\
&= v_nVar(\Xi_1(x)) + \sum_{|i-j|>0}^{v_n}Cov\left(\Xi_i(\theta,t,x),\Xi_j(\theta,t,x)\right)\\
&= v_nVar(\Xi_1(x)) + o(v_n).
\end{aligned}$$

Then

$$\begin{aligned}
\frac{\Pi_1}{n} &= \frac{kv_n}{n}Var(\Xi_1(\theta,t,x)) + \frac{1}{n}\sum_{j=0}^{k-1}\sum_{i\neq j}^{v_n}Cov\left(\Xi_i(\theta,t,x),\Xi_j(\theta,t,x)\right)\\
&\leq \frac{kv_n}{n}\left\{\frac{\phi_{\theta,x}(h_K)}{h_H\mathbb{E}^2K_1(x)}Var\left(\Xi_1(\theta,t,x)\right)\right\} + \frac{1}{n}\sum_{i\neq j}^{n}\left|Cov\left(\Xi_i(\theta,t,x),\Xi_j(\theta,t,x)\right)\right|\\
&\leq \frac{kv_n}{n}\left\{\frac{1}{h_H\phi_{\theta,x}(h_K)}Var\left(\Xi_1(\theta,t,x))\right)\right\} + \frac{1}{n}\sum_{i\neq j}^{n}\left|Cov\left(\Xi_i(\theta,t,x),\Xi_j(\theta,t,x)\right)\right|\\
&\leq \frac{kv_n}{n}\left\{\frac{1}{\phi_{\theta,x}(h_K)}Var\left(\Xi_1(x)\right)\right\} + \frac{k}{n}o(v_n).
\end{aligned}$$

Simple algebra gives us

$$\frac{kv_n}{n} \cong \left(\frac{n}{u_n+v_n}\right)\frac{v_n}{n} \cong \frac{v_n}{u_n+v_n} \cong \frac{v_n}{u_n} \longrightarrow 0 \quad \text{as } n\to\infty.$$

Using Equation (19) we have,

$$\lim_{n\to\infty}\frac{\Pi_1}{n} = 0. \tag{26}$$

Now, let us turn to Π_2/n. We have,

$$\frac{\Pi_2}{n} = \frac{1}{n}\sum_{|i-j|>0}^{k-1} Cov\left(\Upsilon_i(x), \Upsilon_j(x)\right) = \frac{1}{n}\sum_{|i-j|>0}^{k-1}\sum_{l_1=1}^{v_n}\sum_{l_2=1}^{v_n} Cov\left(\Xi_{m_j+l_1}, \Xi_{m_j+l_2}\right),$$

with $m_i = i(u_n + v_n) + u_n + 1$. As $i \neq j$, we have, $|m_i - m_j + l_1 - l_2| \geq u_n$. It follows that

$$\frac{\Pi_2}{n} \leq \frac{1}{n}\sum_{|i-j|\geq u_n}^{n} Cov\left(\Xi_i(x), \Xi_j(x)\right) = o(1),$$

then

$$\lim_{n\to\infty} \frac{\Pi_2}{n} = 0. \tag{27}$$

By Equations (26) and (27) we get Part(i) of the Equation(22).
We turn to (ii), we have,

$$\begin{aligned} \frac{1}{n}\mathbb{E}\,(\Psi_n'')^2 &= \frac{1}{n}Var\left(\Upsilon_k''\right) \\ &= \frac{\vartheta_n}{n}Var\,(\Xi_1(x)) + \frac{1}{n}\sum_{|i-j|>0}^{\vartheta_n} Cov\left(\Xi_i(x), \Xi_j(x)\right), \end{aligned}$$

where $\vartheta_n = n - k_n(u_n + v_n)$; by the definition of k_n, we have, $\vartheta_n \leq u_n + v_n$.
Then,

$$\frac{1}{n}\mathbb{E}\,(\Psi_n'')^2 \leq \frac{u_n + v_n}{n}Var\,(\Xi_1(x)) + \frac{1}{n}\sum_{|i-j|>0}^{\vartheta_n} Cov\left(\Xi_i(x), \Xi_j(x)\right),$$

and by the definition of u_n and v_n we achieve the proof of (ii) of Equation (22).

- **Proof of (23)** We make use of Volkonskii and Rozanov's lemma (see the appendix in Masry [24]) and the fact that the process $(X_i, X_j)$is strong mixing.

  Note that Υ_a is $\mathcal{F}_{i_a}^{j_a}$-mesurable with $i_a = a(u_n + v_n) + 1$ and $j_a = a(u_n + v_n) + u_n$; hence, with $V_j = \exp\left(izn^{-1/2}\Psi_n\right)$ we have,

$$\left|\mathbb{E}\left\{V_j\right\} - \prod_{j=0}^{k-1}\mathbb{E}\left\{\exp\left(izn^{-1/2}\Upsilon_j\right)\right\}\right| \leq 16k_n a(v_n + 1) \cong \frac{n}{v_n}a(v_n + 1),$$

  which goes to zero by the last part of Equation (21). Now we establish Equation (24).

- **Proof of (24)** Note that $Var(\Psi_n) \longrightarrow V(\theta, t, x)$ by equation (22) (by the definition of the Ξ_i). Then because

$$\mathbb{E}\,(\Psi_n)^2 = Var\,(\Psi_n) = \sum_{j=0}^{k-1} Var\left(\Upsilon_j\right) + \sum_{i=0}^{k-1}\sum_{i\neq j}\sum_{j=1}^{k-1} Cov\left(\Upsilon_i, \Upsilon_j\right),$$

  all, we have to prove is that the double sum of covariances in the last equation tends to zero. Using the same arguments as those previously used for Π_2 in the proof of first term of Equation (22)we obtain by replacing v_n by u_n we get

$$\frac{1}{n}\sum_{j=1}^{k-1}\mathbb{E}\left(\Upsilon_j^2\right) = \frac{ku_n}{n}Var\,(\Xi_1) + o(1).$$

As $Var(\Xi_1) \longrightarrow V(\theta,t,x)$ and $\frac{ku_n}{n} \longrightarrow 1$, we get the result.

Finally, we prove Equation (25).

- **Proof of (25)** Recall that

$$\Upsilon_j = \sum_{i=j(u_n+v_n)+1}^{j(u_n+v_n)+u_n} \Xi_i.$$

Finally for establish (25) it suffices to show for n large enough that the set $\{|\Upsilon_j| > \varepsilon\sqrt{nV(\theta,t,x)}\}$ is empty
.

Making use Assumptions (H1), (H3) and (H4), we have,

$$\left|\Xi_i\right| \leq C\left(h_H\phi_{\theta,x}(h_K)\right)^{-1/2},$$

therefore,

$$\left|\Upsilon_j\right| \leq Cu_n\left(h_H\phi_{\theta,x}(h_K)\right)^{-1/2},$$

which goes to zero as n goes to infinity by Equation (21).

Then for n large enough, the set $\left\{|\Upsilon_j| > \varepsilon\,(nV(\theta,t,x))^{-1/2}\right\}$ becomes empty, this completes the proof and therefore that of the asymptotic normality of $(nV(\theta,t,x))^{-1/2}S_n$ and the Lemma 3.3. □

Proof. [Proof of Lemma 3.4] We have,

$$\sqrt{nh_H\phi_{\theta,x}(h_K)}B_n(\theta,t,x) = \frac{\sqrt{nh_H\phi_{\theta,x}(h_K)}}{\widehat{F}_D(\theta,x)}\left\{\mathbb{E}\widehat{f_N}(\theta,t,x) - a_1(\theta,x)f(\theta,t,x) + f(\theta,t,x)\left(a_1(\theta,x) - \mathbb{E}\widehat{F}_D(\theta,x)\right)\right\}.$$

Firstly, observed that the results below

$$\frac{1}{\phi_{\theta,x}(h_K)}\mathbb{E}\left[K^l\left(\frac{< x - X_i,\theta >}{h_K}\right)\right]\underset{n\to\infty}{\longrightarrow} a_l(\theta,x),\ \text{for } l = 1,2,\quad \mathbb{E}\left[\widehat{F}_D(\theta,x)\right]\underset{n\to\infty}{\longrightarrow} a_1(\theta,x)$$

and

$$\mathbb{E}\left[\widehat{f_n}(\theta,t,x)\right] \longrightarrow a_1(\theta,x)f(\theta,t,x),\ \text{ as } n\to\infty,$$

can be proved in the same way as in Ezzahrioui and Ould Saïd [12] corresponding to their Lemmas 5.1-5.2, and then their proofs omitted.

Secondly, on the one hand, making use of, we have,

$$\mathbb{E}\widehat{f_N}(\theta,t,x) - a_1(\theta,x)f(\theta,t,x) + f(\theta,t,x)\left(a_1(\theta,x) - \mathbb{E}\widehat{F}_D(\theta,x)\right)\underset{n\to\infty}{\longrightarrow} 0.$$

On other hand,

$$\frac{\sqrt{nh_H\phi_{\theta,x}(h_K)}}{\widehat{F}_D(\theta,x)} = \frac{\sqrt{nh_H\phi_{\theta,x}(h_K)}\widetilde{f}(\theta,t,x)}{\widehat{F}_D(\theta,x)\widetilde{f}(\theta,t,x)} = \frac{\sqrt{nh_H\phi_{\theta,x}(h_K)}\widetilde{f}(\theta,t,x)}{\widetilde{f_N}(\theta,t,x)}.$$

Then using, it suffices to show that, $\frac{\sqrt{nh_H\phi_{\theta,x}(h_K)}}{\widetilde{f_N}(\theta,t,x)}$ tend to zero as n goes to infinity. Indeed,

$$\widetilde{f_N}(\theta,t,x) = \frac{1}{nh_H\mathbb{E}K_1(\theta,x)}\sum_{i=1}^{n}\frac{\delta_i}{\bar{G}(Y_i)}K\left(\frac{< x - X_i,\theta >}{h_K}\right)H\left(\frac{t - Y_i}{h_H}\right).$$

Because $K(\cdot)H(\cdot)$ is continuous with support on [0,1], then by (H3) and (H4) $\exists m = \inf_{[0,1]}K(t)H(t)$ if follows that,

$$\widetilde{f_N}(\theta,t,x) \geq \frac{m}{h_H\phi_{\theta,x}(h_K)},\ \text{ which gives }\ \frac{\sqrt{nh_H\phi_{\theta,x}(h_K)}}{\widetilde{f_N}(\theta,t,x)} \leq \frac{\sqrt{nh_H^3\phi_{\theta,x}^3(h_K)}}{m}.$$

Finally, using (N2), completes the proof of Lemma. 3.4 □

7. Conclusion

This paper focused on nonparametric estimation of conditional mode in the single functional index model for dependent data under random censorship. Both the asymptotic normality as well as a confidence interval of the resulted estimator are derived. Our prime aim was to improve the performance of the single-index model for the conditional mode with a scalar response variable conditioned by a functional Hilbertian regressor under the dependent property. The nonparametric aspect is well exploited in the first two sections by the given hypotheses. The proposed estimators are consistent and asymptotically distributed under appropriate conditions. Note that this approach is more significant in the presence of a simple single functional index. The dimensionality of the model is the bias part while the dimensionality of the functional space of the explanatory variable is in the dispersion part.

Research in the nonparametric field remains an open question which will be the subject of several future studies in order to improve and highlight the results obtained in this work. Extend our study of estimation of the conditionals mode to the estimation of the conditional models of a MAR (missing at random) response to the independent case and the dependent case. Another type of dependency could be considered such as the quasi-associated data. Develop the asymptotic properties of a kernel estimator of the k-nearest neighbors. Generalize the results obtained by using other families of semi-metrics in order to improve the prediction performance of our estimators where the choice of the smoothing window is important.

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