Filomat 37:27 (2023), 9197–9214
https://doi.org/10.2298/FIL2327197S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Rate of Convergence of parametrically generalized bivariate Baskakov-Stancu operators

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Abstract. In the present work, we consider Stancu variant of a bivariate parametrically generalised Baskakov operator. We discuss the rate of convergence of these operators by means of partial moduli of continuity and modulus of continuity of second order. Also, we prove Vornovskaja type assymptotic theorem. Furthermore, the Generalized Boolean Sum (GBS) operators associated to Stancu variant of a-Baskakov operators are constructed and we study their order of convergence using mixed modulus of smoothness for Bögel continuous and Bögel differentiable functions. Some surface plotting illustrating the approximation for different values of Stancu variables and the error of approximation by the proposed operators are also given using MATLAB programming.

1. Introduction

Baskakov [8] introduced a sequence of positive linear operators, called Baskakov operators for suitable functions defined on an unbounded interval $[0, \infty)$. Later, Aral and Gupta considered q-analogues to Baskakov operators in [4]. Also, the same authors introduced another q-analogues to Baskakov operators [3] and studied some shape preserving properties and the convergence rate in weighted norm. In 2019, Aral and Erbay [2] considered a parametric generalization of Baskakov operators given as follows: For $r > 0$ and $\psi \in C_r(S) = \{\psi \in C(S) : \left|\psi(s)\right| \leq M(1 + s^r),\ for\ some\ M > 0\}$, where S = [0,∞), the a-Baskakov operators are given by

$$B_{n,a}(\psi; t) = \sum_{j=0}^{\infty} \psi\Big(\frac{j}{n}\Big) P^{a}_{n,j}(t) \tag{1}$$

where $n \geq 1,\ x \in S\ and$
$P^{a}_{n,j}(t) = \frac{t^{j-1}}{(1+t)^{n+j-1}}\left\{\frac{at}{1+t}\binom{n+j-1}{j} - (1-a)(1+t)\binom{n+j-3}{j-2} + (1-a)t\binom{n+j-1}{j}\right\}$ with $\binom{n-3}{-2} = \binom{n-2}{-1} = 0$ and represented these operators in terms of divided differences. Further, P. N. Agrawal et al. [1] investigated the degree of approximation for these operators and estabilished a Vornovskaya type assymptotic result for the bivariate extension of the operators given by (1).

2020 *Mathematics Subject Classification*. 41A25 ; 41A36 ; 47A58
Keywords. Modulus of continuity, Stancu operator, moment estimate, statistical convergence, error estimates, GBS operator
Received: 23 March 2023; Revised: 29 May 2023; Accepted: 02 June 2023
Communicated by Snežana Č. Živković-Zlatanović
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In 1969, Stancu [24] introduced the sequence of linear positive operators for $\psi \in C(S)$, *where* $S = [0, \infty)$ as

$$T_n^{(\alpha,\beta)}(\psi; t) = \sum_{k=0}^{n} p_{n,k}(t)\psi\Big(\frac{k+\alpha}{n+\beta}\Big),$$

where $p_{n,k}(t) = \binom{n}{i} t^k (1-t)^{n-k}$ *and* $0 \leq \alpha \leq \beta$. Inspired from their work, different authers [16, 17, 20–23, 25, 26] have studied Stancu-variant for different positive linear operators. Recently, Heshamuddin et al. [19] discussed approximation properties of α-Stancu-Schurer-Kantrovinch operators.

The present work have a close connection with the theory of sampling, machine learning and neural network (NN). The learning theory includes approximation of a function from its sample values. For more details, we refer the readers to [13]. Many researchers have worked with NN in order to reproduce the functions by their sample values. Cardalaiguet and Euvrard [12] developed the theory of NN operators. Motivated from above work, we construct the Stancu-variant of bivariate case of operators given by (1). This construction is given in the section 3.

Furthermore, Bögel [9] defined the concepts of Bögel continuous and differentiable functions. For more details on these notions we refer the readers to [10]. Dobrescu and Matei [15] introduced the uniform convergence of GBS of bivariate Bernstein operators to B-continuous functions. Further, a Korovkin type theorem and a quantitative variant of this theorem for GBS operators on B-continuous functions was given by Badea and Cottin [5] and Badea [6] respectively. Bărbosu and Muraru [7] study approximation of B-continuous functions by GBS of Bernstein-Schrurer-Stancu operators. Some definitions and notations are given as follows:

Let $Y_1,\ Y_2 \subset \mathbb{R}$. Then a function $\psi : Y_1 \times Y_2 \to \mathbb{R}$ is called B-continuous at a point $(t_1, t_2) \in Y_1 \times Y_2$ if

$$\lim_{(y_1,y_2)\to(t_1,t_2)} \Delta_{(y_1,y_2)}\psi(t_1, t_2) = 0,$$

where, $\Delta_{(y_1,y_2)}\psi(t_1, t_2) = \psi(t_1, t_2) - \psi(t_1, y_2) - \psi(y_1, t_2) + \psi(y_1, y_2)$ is the mixed difference and the function ψ is said to be B-continuous in $Y_1 \times Y_2$ if it is B-continuous at every point of the domain. Let $C_B(Y_1 \times Y_2)$ be collection of all B-continuous functions on $Y_1 \times Y_2$. The function ψ is called B-bounded if its mixed difference is bounded on $Y_1 \times Y_2$ i.e.

$$\left|\Delta_{(y_1,y_2)}\psi(t_1, t_2)\right| \leq M,\ \forall\ (t_1, t_2), (y_1, y_2) \in Y_1 \times Y_2$$

Let $B(Y_1 \times Y_2)$ be space of bounded funtions on $Y_1 \times Y_2$ with sup norm and $B_B(Y_1 \times Y_2)$ be the space of B-bounded functions ψ defined on $Y_1 \times Y_2$ endowed with the norm

$$\left\|\psi\right\|_B = \sup_{(y_1,y_2),(t_1,t_2)\in Y_1\times Y_2} \left|\Delta_{(y_1,y_2)}\psi(t_1, t_2)\right|$$

A function $\psi : Y_1 \times Y_2 \to \mathbb{R}$ is said to be uniformly B-continuous if for given $\epsilon > 0$, $\exists\, a\ \delta > 0$ such that

$$\left|\Delta_{(y_1,y_2)}\psi(t_1, t_2)\right| < \epsilon$$

whenever $\max\{\left|y_1 - t_1\right|, \left|y_2 - t_2\right|\} < \delta$ and $(t_1, t_2), (y_1, y_2) \in Y_1 \times Y_2$. Let the space of all uniformly B-continuous functions on $Y_1 \times Y_2$ is denoted by $\bar{C}_B(Y_1 \times Y_2)$. Also ψ is called B-differentiable at any point (t_1, t_2) if the limit

$$\lim_{(y_1,y_2)\to(t_1,t_2)} \frac{\Delta_{(y_1,y_2)}\psi(t_1, t_2)}{(y_1 - t_1)(y_2 - t_2)},$$

exists finitely. In this case the limit is B-differential of ψ at (t_1, t_2) and is denoted by $D_b(\psi; t_1, t_2)$.

Let

$$D_B(Y_1 \times Y_2) = \{\psi : Y_1 \times Y_2 \to \mathbb{R}/\ D_b(\psi; t_1, t_2)\ \textit{exists on}\ Y_1 \times Y_2\}.$$

The mixed modulus of smoothness $\omega_B(\psi; \zeta_1, \zeta_2)$ is defined by

$$\omega_B(\psi; \zeta_1, \zeta_2) = \sup\{\left|\delta_{(y_1,y_2)}\psi(t_1, t_2)\right| : \left|y_1 - t_1\right| < \zeta_1, \left|y_2 - t_2\right| < \zeta_2\}$$

. Here, supremum is taken over $Y_1 \times Y_2$. Also for any $k_1, k_2 > 0$,

$$\omega_B(\psi; k_1\zeta_1, k_2\zeta_2) \leq (1 + k_1)(1 + k_2)\omega_B(\psi; \zeta_1, \zeta_2) \tag{2}$$

2. Preliminaries

Lemma 2.1. *[1] The a-Baskakov operators $B_{n,a}(.;t)$, for $n \in \mathbb{N}$, satisfy the following identities:*

1. $B_{n,a}(1;t) = 1;$
2. $B_{n,a}(s;t) = t + \frac{2}{n}t(a-1);$
3. $B_{n,a}(s^2;t) = t^2 + \frac{1}{n}t^2(4a-3) + \frac{1}{n^2}t(n+4a-4);$
4. $B_{n,a}(s^3;t) = t^3 + \frac{1}{n^2}t^3(6na-3n+6a-4) + \frac{1}{n^2}t^2(18a+3n-15) + \frac{1}{n^3}t(8a+n-8);$
5. $B_{n,a}(s^4;t) = t^4 + \frac{1}{n^3}t^4\{(8a-2)n^2+(24a-13)n+(16a-10)\} + \frac{1}{n^3}t^3\{6n^2+(48a-30)n+(48a-36)\} + \frac{1}{n^3}t^2\{(4a+3)n+60a-53\} + \frac{1}{n^4}t(16a+n-16).$

3. Construction of Bivariate operators

Let $0 \le a_1 \le 1$ and $0 \le a_2 \le 1$ be parameters. For $r_1,\ r_2 > 0$ and $\psi \in C_{r_1,r_2}(I^2) = \{\psi \in C(I) : |\psi(y,t)| \le M_\psi(1+y^{r_1})(1+t^{r_2}),\ \textit{for some } M_\psi > 0\}$, where $I^2 = I \times I$, we consider Stancu type modification of bivariate a-Baskakov operators as follows:

$$\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(\psi;t_1,t_2) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\psi\Big(\frac{k_1+\alpha_1}{n_1+\beta_1}, \frac{k_2+\alpha_2}{n_2+\beta_2}\Big)P^{(a_1,a_2)}_{n_1,n_2,k_1,k_2}(t_1,t_2), \tag{3}$$

where $n_1, n_2 \in \mathbb{N},\ (t_1,t_2) \in I^2,\ 0 \le \alpha_i \le \beta_i$ and

$$P^{(a_1,a_2)}_{n_1,n_2,k_1,k_2}(t_1,t_2) = P^{a_1}_{n_1,k_1}(t_1)P^{a_2}_{n_2,k_2}(t_2),$$

where

$$P^{a_i}_{n_i,k_i}(t_i) = \frac{t_i^{k_i-1}}{(1+t_i)^{n_i+k_i-1}}\Bigg\{\frac{a_it_i}{1+t_i}\binom{n_i+k_i-1}{k_i} - (1-a_i)(1+t_i)\binom{n_i+k_i-3}{k_i-2} + (1-a_i)t_i\binom{n_i+k_i-1}{k_i}\Bigg\}$$

with $\binom{n_i-3}{-2} = \binom{n_i-2}{-1} = 0,\ i = 1,2.$

In order to validate the theoretical approach and to compare the convergence of (3) with GBS operators, we introduce some numerical example showing the approximation of proposed operators in section 3.

Let $e_{k_1,k_2} = t_1^{k_1}t_2^{k_2},\ k_1,k_2 \in \{0,1,2\}$. Now, we can obtain the following identities for the operator defined by (3).

Lemma 3.1. *For $a_1,\ a_2 \ge 0,\ n_1,\ n_2 \in \mathbb{N}$ and $0 \le \alpha_i \le \beta_i, i = 1,2,$ $\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(\psi;t_1,t_2)$ satisfies the followings:*

1. $\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(e_{00};t_1,t_2) = 1;$
2. $\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(e_{10};t_1,t_2) = \frac{n_1}{n_1+\beta_1}t_1 + \frac{2}{n_1+\beta_1}(a_1-1)t_1 + \frac{\alpha_1}{n_1+\beta_1};$
3. $\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(e_{01};t_1,t_2) = \frac{n_2}{n_2+\beta_2}t_2 + \frac{2}{n_2+\beta_2}(a_2-1)t_2 + \frac{\alpha_2}{n_2+\beta_2};$
4. $\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(e_{20};t_1,t_2) = \frac{n_1^2}{(n_1+\beta_1)^2}t_1^2 + \frac{n_1}{(n_1+\beta_1)^2}(4a_1-3)t_1^2 + \frac{(n_1+4a_1-4)}{(n_1+\beta_1)^2}t_1 + \frac{2\alpha_1n_1}{(n_1+\beta_1)^2}t_1 + \frac{4\alpha_1(a_1-1)}{(n_1+\beta_1)^2}t_1 + \frac{\alpha_1^2}{(n_1+\beta_1)^2};$
5. $\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(e_{02};t_1,t_2) = \frac{n_2^2}{(n_2+\beta_2)^2}t_2^2 + \frac{n_2}{(n_2+\beta_2)^2}(4a_2-3)t_2^2 + \frac{(n_2+4a_2-4)}{(n_2+\beta_2)^2}t_2 + \frac{2\alpha_2n_2}{(n_2+\beta_2)^2}t_2 + \frac{4\alpha_2(a_2-1)}{(n_2+\beta_2)^2}t_2 + \frac{\alpha_2^2}{(n_2+\beta_2)^2};$

Proof. Using the linearity of $\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(.;t_1,t_2)$, we have

$$\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(e_{10};t_1,t_2) = \frac{n_1}{n_1+\beta_1}B_{n_1,a_1}(e_{10};t_1)\frac{\alpha_1}{n_1+\beta_1}B_{n_1,a_1}(1;t_1)$$

By using preliminaries, part (2) is true. In a similar manner, readers can prove other parts of above lemma. □

Lemma 3.2. *For $a_1,\ a_2 \geq 0,\ n_1,\ n_2 \in \mathbb{N}$ and $0 \leq \alpha_i \leq \beta_i$ $(i = 1, 2)$, $\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(\psi; t_1, t_2)$ satisfies the followings:*

1. $\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}((y_i - t_i); t_1, t_2) = \left(\frac{n_i}{n_i+\beta_i} - 1\right)t_i + \frac{2}{n_i+\beta_i}(a_i - 1)t_i + \frac{\alpha_i}{n_i+\beta_i}$;
2. $\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}((y_i - t_i)^2; t_1, t_2) = (\frac{n_i^2}{(n_i+\beta_i)^2} - 1)t_i^2 + \frac{n_i}{(n_i+\beta_i)^2}(4a_i - 3)t_i^2 - 2(\frac{n_i}{n_i+\beta_i} - 1)t_i^2 - \frac{4}{n_i+\beta_i}(a_i - 1)t_i^2 + \frac{1}{(n_i+\beta_i)^2}(n_i(2\alpha_i + 1) + 4(a_i - 1) + 4\alpha_i(a_i - 1) - 2\alpha_i)t_i - \frac{2\alpha_i}{n_i+\beta_i}t_i + \frac{\alpha_i^2}{(n_i+\beta_i)^2}$;
3. $\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}((y_i - t_i)^4; t_1, t_2) = \frac{t_i^4}{(n_i+\beta_i)^4}\{n_i^4 + 2n_i^3(4a_i - 1) + n_i^2(24a_i - 13) + 2n_i(8a_i - 5)\} - \frac{t_i^4}{(n_i+\beta_i)^3}\{4n_i^3 + 12n_i^2(2a_i - 1) + 8n_i(3a_i - 2)\} + \frac{t_i^4}{(n_i+\beta_i)^2}\{6n_i^2 + 6n_i(4a_i - 3)\} - \frac{t_i^4}{(n_i+\beta_i)}\{4n_i + 8(a_i - 1)\} + t_i^4 + \frac{t_i^3}{(n_i+\beta_i)^4}\{2n_i^3(2\alpha_i + 3) + 6n_i^2[8a_i - 5 + 2\alpha_i(2a_i - 1)] + 4n_i[3(4a_i - 3) + 2\alpha_i(3a_i - 2)]\} - \frac{t_i^3}{(n_i+\beta_i)^3}\{12n_i^2(\alpha + 1) + 12n_i[(6a_i - 5) + \alpha_i(4a_i - 3)]\} + \frac{t_i^3}{(n_i+\beta_i)^2}\{6n_i(1 + 2\alpha_i) + 24(\alpha_i + 1)(a_i - 1)\} - \frac{t_i^3}{(n_i+\beta_i)}4\alpha_i + \frac{t_i^2}{(n_i+\beta_i)^4}\{n_i^2(4a_i + 3 + 12\alpha_i + 6\alpha_i^2) + n_i[60a_i - 53 + 12\alpha_i(6a_i - 5) + 6\alpha_i^2(4a_i - 3)]\} - \frac{t_i^2}{(n_i+\beta_i)^3}\{4n_i(1 + 3\alpha_i + 3\alpha_i^2) + 8n_i(4 - 6\alpha_i - 3\alpha_i^2)(a_i - 1)\} + \frac{t_i^2}{(n_i+\beta_i)^2}6\alpha_i^2 + \frac{t_i}{(n_i+\beta_i)^4}\{n_i(1 + 4\alpha_i + 6\alpha_i^2 + 4\alpha_i^3) + 8(a_i - 1)(2 + 4\alpha_i + 3\alpha_i^2 + 8\alpha_i^3)\} - \frac{t_i}{(n_i+\beta_i)^3}4\alpha_i^3 + \frac{\alpha_i^4}{(n_i+\beta_i)^4}$, *where i=1,2.*

Proof. Using the linear property of $\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(.; t_1, t_2)$ and lemma (3.1), readers can derive lemma (3.2). □

Corollary 3.3. *For $a_1,\ a_2 \geq 0,\ n_1,\ n_2 \in \mathbb{N}$ and $0 \leq \alpha_i \leq \beta_i$ $(i = 1, 2)$, we have following results:*

1. $\lim\limits_{n_i\to\infty} n_i \S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}((y_i - t_i); t_1, t_2) = (\alpha_i - \beta_i t_i) + 2t_i(a_i - 1)$;
2. $\lim\limits_{n_i\to\infty} n_i \S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}((y_i - t_i)^2; t_1, t_2) = t_i(t_i + 1)$;
3. $\lim\limits_{n_i\to\infty} n_i^2 \S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}((y_i - t_i)^4; t_1, t_2) = 3t_i^2(t_i + 1)^2 + 4t_i^2(a_i - 1)$.

4. Degree of Approximation of $\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(.; t_1, t_2)$

Let $I_{hk} = [0, h] \times [0, k]$ be compact subset of I^2 and $C_b(I^2)$ be collection of all bounded and continuous functions on I^2 endowed with the norm

$$\|\psi\| = \sup_{(t_1,t_2)\in I^2} |\psi(t_1, t_2)|$$

Theorem 4.1. *Let $\psi \in C_b(I^2)$, then*

$$\lim_{n_1,n_2\to\infty} \S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(\psi; t_1, t_2) = \psi(t)$$

uniformly on I_{hk}.

Proof. For $(k_1, k_2) \in \{(0, 0), (1, 0), (0, 1)\}$, using lemma (3.1), we have

$$\lim_{n_1,n_2\to\infty} \S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(e_{k_1,k_2}; t_1, t_2) = e_{k_1,k_2}(t_1, t_2),$$

and $\lim\limits_{n_1,n_2\to\infty} \S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(e_{20} + e_{02}; t_1, t_2) = (e_{20} + e_{02})(t_1, t_2)$ uniformly on I_{hk}. The required result follows immediately by theorem (2.1) [7]. □

Now denote $\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}((y_i - t_i)^r; t_1, t_2)$ by $\gamma^{\alpha_i,\beta_i}_{n_i,a_i,r}(t_i)$,i=1,2 and r=1,2,3,... Total modulus of continuity for $\psi \in \bar{C}_b(I^2) = \{\psi \in C_b(I^2) : \psi$ is uniformly continuous$\}$ and $\zeta > 0$ is defined as

$$\bar{\omega}(\psi; \zeta) = \sup_{(y_1,y_2),(t_1,t_2)\in I^2} \{|\psi(y_1, y_2) - \psi(t_1, t_2)| : \sqrt{(y_1 - t_1)^2 + (y_2 - t_2)^2} < \zeta\} \tag{4}$$

Also, the partial modulli of continuity w.r.t. t_1 and t_2 are defind as

$$\bar{\omega}_1(\psi;\zeta) = \sup\{|\psi(y_1,t) - \psi(y_2,t)| : \ t \in I,\ (y_1,y_2) \in I^2 \ \&\ |y_1 - y_2| \le \zeta\} \tag{5}$$

and

$$\bar{\omega}_2(\psi;\zeta) = \sup\{|\psi(y,t_1) - \psi(y,t_2)| : \ y \in I,\ (t_1,t_2) \in I^2 \ \&\ |t_1 - t_2| \le \zeta\} \tag{6}$$

Devore and Lorentz [14] derived some properties of continuity moduli for bi-variate case. Now the degree of approximation of $\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(.;t_1,t_2)$ via complete and partial moduli of continuity is given by the following theorems:

Theorem 4.2. *Let $\psi \in \bar{C}_b(I^2)$. Then we have the inequality*

$$|\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(\psi;t_1,t_2) - \psi(t_1,t_2)| \le 2\bar{\omega}\Big(\psi;\Big(\gamma^{\alpha_1,\beta_1}_{n_1,a_1,2}(t_1) + \gamma^{\alpha_2,\beta_2}_{n_2,a_2,2}(t_2)\Big)^{\frac{1}{2}}\Big).$$

Proof. By using the definition of $\bar{\omega}(\psi;\zeta)$, $\zeta > 0$ given by equation (4) and Cauchy-Schwartz inequality, we have

$$\begin{aligned}
|\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(\psi;t_1,t_2) - \psi(t_1,t_2)| &\le \S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(|\psi(y_1,y_2) - \psi(t_1,t_2)|;t_1,t_2) \\
&\le \S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(\bar{\omega}(\psi;\sqrt{(y_1-t_1)^2 + (y_2-t_2)^2});t_1,t_2) \\
&\le \bar{\omega}(\psi;\zeta)\Big\{1 + \frac{1}{\zeta}\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(\sqrt{(y_1-t_1)^2 + (y_2-t_2)^2};t_1,t_2)\Big\} \\
&\le \bar{\omega}(\psi;\zeta)\Big\{1 + \frac{1}{\zeta}\Big(\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}((y_1-t_1)^2 + (y_2-t_2)^2;t_1,t_2)\Big)^{\frac{1}{2}}\Big\} \\
&\le \bar{\omega}(\psi;\zeta)\Big\{1 + \frac{1}{\zeta}\Big(\gamma^{\alpha_1,\beta_1}_{n_1,a_1,2}(t_1) + \gamma^{\alpha_2,\beta_2}_{n_2,a_2,2}(t_2)\Big)^{\frac{1}{2}}\Big\}.
\end{aligned}$$

By taking $\zeta = \Big(\gamma^{\alpha_1,\beta_1}_{n_1,a_1,2}(t_1) + \gamma^{\alpha_2,\beta_2}_{n_2,a_2,2}(t_2)\Big)^{\frac{1}{2}}$, we will get the required result. □

Theorem 4.3. *For $\psi \in \bar{C}_b(I^2)$ and $(t_1,t_2) \in I^2$, we have*

$$|\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(\psi;t_1,t_2) - \psi(t_1,t_2)| \le 2\Big\{\bar{\omega}_1\Big(\psi;\Big(\gamma^{\alpha_1,\beta_1}_{n_1,a_1,2}(t_1)\Big)^{\frac{1}{2}}\Big) + \bar{\omega}_2\Big(\psi;\Big(\gamma^{\alpha_2,\beta_2}_{n_2,a_2,2}(t_2)\Big)^{\frac{1}{2}}\Big)\Big\}.$$

Proof. By using the definitions given by equations (5) & (6) and Cauchy-Schwartz inequality, we have

$$\begin{aligned}
|\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(\psi;t_1,t_2) - \psi(t_1,t_2)| &\le \S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(|\psi(y_1,y_2) - \psi(t_1,t_2)|;t_1,t_2) \\
&\le \S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(|\psi(y_1,y_2) - \psi(t_1,y_2)|;t_1,t_2) \\
&+ \S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(|\psi(t_1,t_2) - \psi(y_1,y_2)|;t_1,t_2) \\
&\le \S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(\bar{\omega}_1(\psi;|y_1-t_1|);t_1,t_2) \\
&+ \S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(\bar{\omega}_2(\psi;|y_2-t_2|);t_1,t_2) \\
&\le \bar{\omega}_1(\psi;\zeta_1)\Big\{\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(1;t_1,t_2) \\
&+ \frac{1}{\zeta_1}(\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}((y_1-t_1)^2;t_1,t_2))^{\frac{1}{2}}\Big\} \\
&\le \bar{\omega}_2(\psi;\zeta_2)\Big\{\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(1;t_1,t_2) \\
&+ \frac{1}{\zeta_2}(\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}((y_1-t_1)^2;t_1,t_2))^{\frac{1}{2}}\Big\}
\end{aligned}$$

The required result will follow by choosing $\zeta_1 = \left(\gamma^{\alpha_1,\beta_1}_{n_1,a_1,2}(t_1)\right)^{\frac{1}{2}}$ and $\zeta_2 = \left(\gamma^{\alpha_2,\beta_2}_{n_2,a_2,2}(t_2)\right)^{\frac{1}{2}}$ and lemma (3.1). □

Let $C^2_b(I^2) = \left\{\psi \in C_b(I^2) : \frac{\partial^j \psi}{\partial y_1^j}, \frac{\partial^j \psi}{\partial y_2^j}, \frac{\partial^2 \psi}{\partial y_1 \partial y_2} \in C_b(I^2),\ j = 1,2\right\}$ endowed with the norm

$$\left\|\psi\right\|_{C^2_b(I^2)} = \left\|\psi\right\| + \left\|\frac{\partial \psi}{\partial y_1}\right\| + \left\|\frac{\partial \psi}{\partial y_2}\right\| + \left\|\frac{\partial^2 \psi}{\partial y_1^2}\right\| + \left\|\frac{\partial^2 \psi}{\partial y_2^2}\right\| + \left\|\frac{\partial^2 \psi}{\partial y_1 \partial y_2}\right\|.$$

And corresponding Peetre's functional for $\psi \in \bar{C}_b(I^2)$ and $\zeta > 0$ is defined as

$$K(\psi;\zeta) = \inf_{g \in C^2_b(I^2)} \{\left\|\psi - g\right\| + \zeta \left\|g\right\|_{C^2_b(I^2)}.$$

From [11], there exists a constant M such that

$$K(\psi;\zeta) \leq M\tilde{\omega}_2(\psi;\sqrt{\zeta}), \tag{7}$$

where $\tilde{\omega}_2(\psi;\sqrt{\zeta})$ is 2^{nd} ordered modulus of continuity for bivariate case.

Theorem 4.4. *Let $\psi \in \bar{C}_b(I^2)$. Then $\forall\ (t_1,t_2) \in I^2$, following inequality holds for $\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}$*

$$\begin{aligned}\left|\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(\psi;t_1,t_2) - \psi(t_1,t_2)\right| &\leq M\tilde{\omega}_2\Big(\psi; \frac{1}{2}\sqrt{\Gamma^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(t_1,t_2)}\Big) \\ &\quad + \bar{\omega}\Big(\psi; \sqrt{\left(\gamma^{\alpha_1,\beta_1}_{n_1,a_1,1}(t_1)\right)^2 + \left(\gamma^{\alpha_2,\beta_2}_{n_2,a_2,1}(t_2)\right)^2}\Big),\end{aligned}$$

where $\Gamma^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(t_1,t_2) = \frac{1}{2}\Big\{\Big(\left|\gamma^{\alpha_1,\beta_1}_{n_1,a_1,1}(t_1)\right| + \left|\gamma^{\alpha_2,\beta_2}_{n_2,a_2,1}(t_2)\right|\Big)^2 + \Big(\sqrt{\gamma^{\alpha_1,\beta_1}_{n_1,a_1,2}(t_1)} + \sqrt{\gamma^{\alpha_2,\beta_2}_{n_2,a_2,2}(t_2)}\Big)^2\Big\}.$

Proof. We first define an auxiliary operator as

$$\begin{aligned}\mathring{A}^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(\psi;t_1,t_2) &= \S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(\psi;t_1,t_2) + \psi(t_1,t_2) \\ &\quad - \psi\Big(\gamma^{\alpha_1,\beta_1}_{n_1,a_1,1}(t_1), \gamma^{\alpha_2,\beta_2}_{n_2,a_2,1}(t_2)\Big)\end{aligned} \tag{8}$$

Using lemma (3.1), we can write

$$\mathring{A}^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(1;t_1,t_2) = 1,\ \mathring{A}^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}((y_i - t_i);t_1,t_2) = 0,\ i = 1,2. \tag{9}$$

Also,

$$|\mathring{A}^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(\psi;t_1,t_2)| \leq 3\left\|\psi\right\| \tag{10}$$

Applying operator $\mathring{A}^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(.;t_1,t_2)$ to Taylor expansion for $g \in C^2_b(I^2)$ given by

$$\begin{aligned}g(y_1,y_2) - g(t_1 - t_2) &= (y_1 - t_1)\frac{\partial g(t_1,t_2)}{\partial t_1} + \int_{t_1}^{y_1}(y_1 - p)\frac{\partial^2 g(p,t_2)}{\partial p^2}dp \\ &\quad + (y_2 - t_2)\frac{\partial g(t_1,t_2)}{\partial t_2} + \int_{t_2}^{y_2}(y_2 - q)\frac{\partial^2 g(t_1,q)}{\partial q^2}dq \\ &\quad + \int_{t_1}^{y_1}\int_{t_2}^{y_2}\frac{\partial^2 g(p,q)}{\partial p \partial q}dpdq,\end{aligned}$$

we have

$$
\begin{aligned}
&|\mathring{A}^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(g(y_1,y_2);t_1,t_2)-g(t_1-t_2)| \\
&\qquad \le \mathring{A}^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}\left(\left|\int_{t_1}^{y_1}(y_1-p)\frac{\partial^2 g(p,t_2)}{\partial p^2}dp\right|;t_1,t_2\right) \\
&\qquad + \mathring{A}^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}\left(\left|\int_{t_2}^{y_2}(y_2-q)\frac{\partial^2 g(t_1,q)}{\partial q^2}dq\right|;t_1,t_2\right) \\
&\qquad + \mathring{A}^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}\left(\left|\int_{t_1}^{y_1}\int_{t_2}^{y_2}\frac{\partial^2 g(p,q)}{\partial p\partial q}dpdq\right|;t_1,t_2\right)
\end{aligned}
$$

Equation (8) led us to

$$
\begin{aligned}
&|\mathring{A}^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(g(y_1,y_2);t_1,t_2)-g(t_1-t_2)| \\
&\qquad \le \S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}\left(\left|\int_{t_1}^{y_1}(y_1-p)\frac{\partial^2 g(p,t_2)}{\partial p^2}dp\right|;t_1,t_2\right) \\
&\qquad + \left|\int_{t_1}^{\frac{t_1(2a_1-2+n_1)+\alpha_1}{n_1+\beta_1}}\left(\frac{t_1(2a_1-2+n_1)+\alpha_1}{n_1+\beta_1}-p\right)\frac{\partial^2 g(p,t_2)}{\partial p^2}dp\right| \\
&\qquad + \S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}\left(\left|\int_{t_2}^{y_2}(y_2-q)\frac{\partial^2 g(t_1,q)}{\partial q^2}dq\right|;t_1,t_2\right) \\
&\qquad + \left|\int_{t_2}^{\frac{t_2(2a_2-2+n_2)+\alpha_2}{n_2+\beta_2}}\left(\frac{t_2(2a_2-2+n_2)+\alpha_2}{n_2+\beta_2}-q\right)\frac{\partial^2 g(t_1,q)}{\partial q^2}dq\right| \\
&\qquad + \S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}\left(\left|\int_{t_1}^{y_1}\int_{t_2}^{y_2}\frac{\partial^2 g(p,q)}{\partial p\partial q}dpdq\right|;t_1,t_2\right) \\
&\qquad + \left|\int_{t_1}^{\frac{t_1(2a_1-2+n_1)+\alpha_1}{n_1+\beta_1}}\int_{t_2}^{\frac{t_2(2a_2-2+n_2)+\alpha_2}{n_2+\beta_2}}\frac{\partial^2 g(p,q)}{\partial p\partial q}dpdq\right|
\end{aligned}
$$

Now,

$$\left|\int_{t_1}^{y_1}(y_1-p)\frac{\partial^2 g(p,t_2)}{\partial p^2}dp\right| \le \|g\|_{C_b^2(I^2)}\frac{(y_1-t_1)^2}{2},$$

$$\left|\int_{t_2}^{y_2}(y_2-q)\frac{\partial^2 g(t_1,q)}{\partial q^2}dq\right| \le \|g\|_{C_b^2(I^2)}\frac{(y_2-t_2)^2}{2},$$

$$\left|\int_{t_1}^{\frac{t_1(2a_1-2+n_1)+\alpha_1}{n_1+\beta_1}}\left(\frac{t_1(2a_1-2+n_1)+\alpha_1}{n_1+\beta_1}-p\right)\frac{\partial^2 g(p,t_2)}{\partial p^2}dp\right| \le \frac{1}{2}\|g\|_{C_b^2(I^2)}\left(\gamma^{\alpha_1,\beta_1}_{n_1,a_1,1}(t_1)\right)^2,$$

$$\left|\int_{t_2}^{\frac{t_2(2a_2-2+n_2)+\alpha_2}{n_2+\beta_2}}\left(\frac{t_2(2a_2-2+n_2)+\alpha_2}{n_2+\beta_2}-q\right)\frac{\partial^2 g(t_1,q)}{\partial q^2}dq\right| \le \frac{1}{2}\|g\|_{C_b^2(I^2)}\left(\gamma^{\alpha_2,\beta_2}_{n_2,a_2,1}(t_2)\right)^2$$

and

$$\left|\int_{t_1}^{\frac{t_1(2a_1-2+n_1)+\alpha_1}{n_1+\beta_1}}\int_{t_2}^{\frac{t_2(2a_2-2+n_2)+\alpha_2}{n_2+\beta_2}}\frac{\partial^2 g(p,q)}{\partial p\partial q}dpdq\right| \le \|g\|_{C_b^2(I^2)}\left|\gamma^{\alpha_1,\beta_1}_{n_1,a_1,1}(t_1)\right|\left|\gamma^{\alpha_2,\beta_2}_{n_2,a_2,1}(t_2)\right|.$$

So we can conclude

$$
\begin{aligned}
|\mathring{A}^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}&(g(y_1,y_2);t_1,t_2)-g(t_1,t_2)| \\
&\le \frac{1}{2}\left\|g\right\|_{C^2_B(I^2)}\left\{\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}\Big((y_1-t_1)^2;t_1,t_2\Big)+\Big(\gamma^{\alpha_1,\beta_1}_{n_1,a_1,1}(t_1)\Big)^2\right. \\
&+\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}\Big((y_2-t_2)^2;t_1,t_2\Big)+\Big(\gamma^{\alpha_2,\beta_2}_{n_2,a_2,1}(t_2)\Big)^2 \\
&+2\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(|y_1-t_1||y_2-t_2|;t_1,t_2) \\
&\left.+2\Big|\gamma^{\alpha_1,\beta_1}_{n_1,a_1,1}(t_1)\Big|\Big|\gamma^{\alpha_2,\beta_2}_{n_2,a_2,1}(t_2)\Big|\right\}
\end{aligned}
$$

$$
|\mathring{A}^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(g(y_1,y_2);t_1,t_2)-g(t_1,t_2)| \le \left\|g\right\|_{C^2_B(I^2)}\Gamma^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(t_1,t_2) \tag{11}
$$

Using equations (9), (10) and (11), from (8) we can write

$$
\begin{aligned}
|\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(\psi;t_1,t_2)-\psi(t_1,t_2)| &\le |\mathring{A}^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(\psi;t_1,t_2)-\psi(t_1,t_2)| \\
&+\left|\psi\Big(\gamma^{\alpha_1,\beta_1}_{n_1,a_1,1}(t_1),\gamma^{\alpha_2,\beta_2}_{n_2,a_2,1}(t_2)\Big)-\psi(t_1,t_2)\right| \\
&\le |\mathring{A}^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(\psi-g;t_1,t_2)|+|\psi(t_1,t_2)-g(t_1,t_2)| \\
&+|\mathring{A}^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(g;t_1,t_2)-g(t_1,t_2)| \\
&+\left|\psi\Big(\gamma^{\alpha_1,\beta_1}_{n_1,a_1,1}(t_1),\gamma^{\alpha_2,\beta_2}_{n_2,a_2,1}(t_2)\Big)-\psi(t_1,t_2)\right| \\
&\le 4\left\|\psi-g\right\|+\left\|g\right\|_{C^2_b(I^2)}\Gamma^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(t_1,t_2) \\
&+\bar{\omega}\left(\psi;\sqrt{\Big(\gamma^{\alpha_1,\beta_1}_{n_1,a_1,1}(t_1)\Big)^2+\Big(\gamma^{\alpha_2,\beta_2}_{n_2,a_2,1}(t_2)\Big)^2}\right)
\end{aligned}
$$

On taking infimum to RHS over all $g \in C^2_b(I^2)$, we obtain

$$
\begin{aligned}
\left|\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(\psi;t_1,t_2)-\psi(t_1,t_2)\right| &\le 4K\left(\psi;\frac{\Gamma^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(t_1,t_2)}{4}\right) \\
&+\bar{\omega}\left(\psi;\sqrt{\Big(\gamma^{\alpha_1,\beta_1}_{n_1,a_1,1}(t_1)\Big)^2+\Big(\gamma^{\alpha_2,\beta_2}_{n_2,a_2,1}(t_2)\Big)^2}\right)
\end{aligned}
$$

Using equation (7), we will reach the required result. □

Further, we give the degree of approximation by the operator given by (3) on bivariate Lipschitz class $Lip^{(r_1,r_2)}_M$, $r_1,r_2 \in (0,1]$ *and* $M>0$ such that

$$
Lip^{(r_1,r_2)}_M=\{f:\ \left|f(y_1,y_2)-f(t_1,t_2)\right| \le M\left|y_1-t_1\right|^{r_1}\left|y_2-t_2\right|^{r_2},\ (y_1,y_2),(t_1,t_2)\in I^2\}
$$

Theorem 4.5. *Let* $\psi \in Lip^{(r_1,r_2)}_M$. *Then following inequality holds for* $\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}$

$$
\left|\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(\psi;t_1,t_2)-\psi(t_1,t_2)\right| \le M\Big(\gamma^{\alpha_1,\beta_1}_{n_1,a_1,2}(t_1)\Big)^{\frac{r_1}{2}}\Big(\gamma^{\alpha_2,\beta_2}_{n_2,a_2,2}(t_2)\Big)^{\frac{r_2}{2}}.
$$

Proof. For $\psi \in Lip_M^{(r_1,r_2)}$, we can write

$$\begin{aligned}\left|\S_{n_1,n_2,a_1,a_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(\psi;t_1,t_2)-\psi(t_1,t_2)\right| &\leq \S_{n_1,n_2,a_1,a_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(\left|\psi(y_1,y_2)-\psi(t_1,t_2)\right|;t_1,t_2)\\ &\leq \S_{n_1,n_2,a_1,a_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(M\left|y_1-t_1\right|^{r_1}\left|y_2-t_2\right|^{r_2};t_1,t_2)\\ &\leq M\S_{n_1,n_2,a_1,a_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(\left|y_1-t_1\right|^{r_1};t_1,t_2)\\ &\times \S_{n_1,n_2,a_1,a_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(\left|y_2-t_2\right|^{r_2};t_1,t_2)\end{aligned}$$

Now Hölder's inequality for $p_i = \frac{2}{r_i}$ & $q_i = \frac{2}{2-r_i}$ implies that

$$\begin{aligned}\left|\S_{n_1,n_2,a_1,a_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(\psi;t_1,t_2)-\psi(t_1,t_2)\right| &\leq M\left\{\S_{n_1,n_2,a_1,a_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}((y_1-t_1)^2;t_1,t_2)\right\}^{\frac{r_1}{2}}\\ &\times\left\{\S_{n_1,n_2,a_1,a_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(e_{00};t_1,t_2)\right\}^{\frac{2-r_1}{2}}\\ &\times\left\{\S_{n_1,n_2,a_1,a_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}((y_2-t_2)^2;t_1,t_2)\right\}^{\frac{r_2}{2}}\\ &\times\left\{\S_{n_1,n_2,a_1,a_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(e_{00};t_1,t_2)\right\}^{\frac{2-r_2}{2}}\end{aligned}$$

In the view of lemma (3.1), we are led to required result. □

Theorem 4.6. *For $\psi \in C_b^1(I^2)$, we have*

$$\begin{aligned}\left|\S_{n_1,n_2,a_1,a_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(\psi;t_1,t_2)-\psi(t_1,t_2)\right| &\leq \left\|\frac{\partial\psi(t_1,t_2)}{\partial t_1}\right\|\sqrt{\gamma_{n_1,a_1,2}^{\alpha_1,\beta_1}(t_1)}\\ &+\left\|\frac{\partial\psi(t_1,t_2)}{\partial t_2}\right\|\sqrt{\gamma_{n_2,a_2,2}^{\alpha_2,\beta_2}(t_2)}\end{aligned}$$

Proof. For $\psi \in C_b^1(I^2)$, we can write

$$\psi(y_1,y_2)-\psi(t_1,t_2)=\int_{t_1}^{y_1}\frac{\partial\psi(p,y_2)}{\partial p}dp+\int_{t_2}^{y_2}\frac{\partial\psi(y_2,q)}{\partial q}dq.$$

Now operating above equation by $\S_{n_1,n_2,a_1,a_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(.;t_1,t_2)$

$$\begin{aligned}\left|\S_{n_1,n_2,a_1,a_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(\psi;t_1,t_2)-\psi(t_1,t_2)\right| &\leq \S_{n_1,n_2,a_1,a_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(\int_{t_1}^{y_1}\left|\frac{\partial\psi(p,y_2)}{\partial p}\right|dp;t_1,t_2)\\ &+\S_{n_1,n_2,a_1,a_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(\int_{t_2}^{y_2}\left|\frac{\partial\psi(y_2,q)}{\partial q}\right|dq;t_1,t_2)\\ &\leq\left\|\frac{\partial\psi(t_1,t_2)}{\partial t_1}\right\|\S_{n_1,n_2,a_1,a_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(\left|y_1-t_1\right|;t_1,t_2)\\ &+\left\|\frac{\partial\psi(t_1,t_2)}{\partial t_2}\right\|\S_{n_1,n_2,a_1,a_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(\left|y_2-t_2\right|;t_1,t_2)\end{aligned}$$

Using Cauchy-Schwartz inequality & lemma (3.1), we are led to the desired inequality. □

Next, we establish a Vornovskaya type Asymptotic result.

Theorem 4.7. *Let $\psi \in C_b^1(I^2)$. Then*

$$\lim_{n\to\infty} n\Big\{ \S_{n,n,a_1,a_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(\psi;t_1,t_2) - \psi(t_1,t_2)\Big\} = (\alpha_1 - \beta_1 t_1 + 2t_1(a_1-1))\frac{\partial\psi}{\partial y_1}(t_1,t_2) + (\alpha_2 - \beta_2 t_2 + 2t_2(a_2-1))\frac{\partial\psi}{\partial y_2}(t_1,t_2) + \frac{1}{2}\Big\{t_1(t_1+1)\frac{\partial^2\psi}{\partial y_1^2}(t_1,t_2) + t_2(t_2+1) \times \frac{\partial^2\psi}{\partial y_2^2}(t_1,t_2)\Big\}$$

uniformly on I_{hk}.

Proof. Let $(t_1,t_2) \in I_{hk}$. Taylor's series expansion for $\psi \in C_b^2(I^2)$ is given as

$$\psi(y_1,y_2) - \psi(t_1,t_2) = (y_1-t_1)\frac{\partial\psi}{\partial y_1}(t_1,t_2) + (y_2-t_2)\frac{\partial\psi}{\partial y_2}(t_1,t_2) + \frac{1}{2!}\Big\{(y_1-t_1)^2 \times \frac{\partial^2\psi}{\partial y_1^2}(t_1,t_2) + 2(y_1-t_1)(y_2-t_2)\frac{\partial^2\psi}{\partial y_1\partial y_2}(t_1,t_2) + (y_2-t_2)^2 \times \frac{\partial^2\psi}{\partial y_2^2}(t_1,t_2)\Big\} + \epsilon(y_1,y_2;t_1,t_2)\sqrt{(y_1-t_1)^4+(y_2-t_2)^4}, \tag{12}$$

where $\epsilon \in C_b(I^2)$ and $\epsilon \to 0$ as $(y_1,y_2) \to (t_1,t_2)$. Applying $\S_{n,n,a_1,a_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(.;t_1,t_2)$ to both sides of above equation, we obtain

$$\begin{aligned}
&\S_{n,n,a_1,a_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(\psi;t_1,t_2) - \psi(t_1,t_2) \\
&= \S_{n,n,a_1,a_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}((y_1-t_1);t_1,t_2)\frac{\partial\psi}{\partial y_1}(t_1,t_2) \\
&+ \S_{n,n,a_1,a_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}((y_2-t_2);t_1,t_2)\frac{\partial\psi}{\partial y_2}(t_1,t_2) \\
&+ \frac{1}{2!}\Big\{\S_{n,n,a_1,a_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}((y_1-t_1)^2;t_1,t_2)\frac{\partial^2\psi}{\partial y_1^2}(t_1,t_2) \\
&+ 2\S_{n,n,a_1,a_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}((y_1-t_1)(y_2-t_2);t_1,t_2)\frac{\partial^2\psi}{\partial y_1\partial y_2}(t_1,t_2) \\
&+ \S_{n,n,a_1,a_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}((y_2-t_2)^2;t_1,t_2)\frac{\partial^2\psi}{\partial y_2^2}(t_1,t_2)\Big\} \\
&+ \S_{n,n,a_1,a_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(\epsilon(y_1,y_2;t_1,t_2)\sqrt{(y_1-t_1)^4+(y_2-t_2)^4};t_1,t_2)
\end{aligned} \tag{13}$$

Now as a result of Cauchy-Schwartz inequality, we obtain

$$\lim_{n\to\infty} n\S_{n,n,a_1,a_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(\epsilon(y_1,y_2;t_1,t_2)\sqrt{(y_1-t_1)^4+(y_2-t_2)^4};t_1,t_2) = 0.$$

The required result follows from equation (13) in the view of corollary (3.3). □

5. Generalized Boolean Sum (GBS) operators

We construct GBS operators for $\psi \in C_B(I^2)$ associated with the operators given by equation (3) as follows:

$$G_{n,n,a_1,a_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(\psi;t_1,t_2) = \S_{n,n,a_1,a_2}^{\alpha_1,\beta_1,\alpha_2,\beta_2}(\psi(y_1,t_2) + \psi(t_1,y_2) - \psi(y_1,y_2);t_1,t_2) \tag{14}$$

Now, we give error estimation while approximation of B-continuous function by GBS operators defined in equation (14) in terms of mixed modulus of continuity.

Theorem 5.1. *For $\psi \in \bar{C}_B(I^2)$ and $(t_1, t_2) \in I^2$, we have*

$$\left|G^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n,n,a_1,a_2}(\psi; t_1, t_2) - \psi(t_1, t_2)\right| \le 4\omega_B\left(\psi;\ \sqrt{\gamma^{\alpha_1,\beta_1}_{n_1,a_1,2}(t_1)},\ \sqrt{\gamma^{\alpha_2,\beta_2}_{n_2,a_2,2}(t_2)}\right),$$

Proof. By using definition of mixed modulus $\omega_B(\psi : \zeta_1, \zeta_2)$ of smoothness, equation (2) and (14) for $\zeta_1, \zeta_2 > 0$ & $(y_1, y_2),\ (t_1, t_2) \in I^2$, we have

$$\begin{aligned}\left|\triangle_{(y_1,y_2)}\psi(t_1, t_2)\right| &\le \omega_B(\psi;\ \left|y_1 - t_1\right|, \left|y_2 - t_2\right|)\\ &\le \omega_B(\psi;\ \zeta_1, \zeta_2)\left(1 + \frac{\left|y_1 - t_1\right|}{\zeta_1}\right)\left(1 + \frac{\left|y_2 - t_2\right|}{\zeta_2}\right)\\ &\le \omega_B(\psi;\ \zeta_1, \zeta_2)\left(1 + \frac{\left|y_1 - t_1\right|}{\zeta_1} + \frac{\left|y_2 - t_2\right|}{\zeta_2} + \frac{\left|y_1 - t_1\right|\left|y_2 - t_2\right|}{\zeta_1\zeta_2}\right).\end{aligned}$$

By using definition of mixed difference and Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}\left|G^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n,n,a_1,a_2}(\psi; t_1, t_2) - \psi(t_1, t_2)\right| \le\ & \omega_B(\psi; \zeta_1, \zeta_2)\Big\{1 + \frac{1}{\zeta_1}\sqrt{\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}((y_1 - t_1)^2; t_1, t_2)}\\ &+ \frac{1}{\zeta_2}\sqrt{\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}((y_2 - t_2)^2; t_1, t_2)}\\ &+ \frac{1}{\zeta_1\zeta_2}\sqrt{\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}((y_1 - t_1)^2; t_1, t_2)}\\ &\times\sqrt{\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}((y_2 - t_2)^2; t_1, t_2)}\Big\}\end{aligned}$$

Now, taking $\zeta_1 = \sqrt{\gamma^{\alpha_1,\beta_1}_{n_1,a_1,2}(t_1)}$ and $\zeta_2 = \sqrt{\gamma^{\alpha_2,\beta_2}_{n_2,a_2,2}(t_2)}$, leads us to the required result. □

Lipschitz class of B-continuous functions: Lipschitz class of B-continuous functions for $r_1, r_2 \in (0, 1]$ and $M > 0$ is defined as follows

$$Lip^B_M(r_1, r_2) = \{\psi \in C_B(I^2) : \left|\triangle_{(y_1,y_2)}\psi(t_1, t_2)\right| \le M\left|y_1 - t_1\right|^{r_1}\left|y_2 - t_2\right|^{r_2}\},$$

$\forall\ (y_1, y_2), (t_1, t_2) \in I^2$. Error estimation over the Lipschitz class of B-continuous functions is given by the following theorem.

Theorem 5.2. *Let $\psi \in Lip^B_M(r_1, r_2),\ 0 < r_1, r_2 \le 1$. Then for $(t_1, t_2) \in I^2$, error estimation of $G^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n,n,a_1,a_2}$ is given by*

$$\left|G^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n,n,a_1,a_2}(\psi; t_1, t_2) - \psi(t_1, t_2)\right| \le M\left(\gamma^{\alpha_1,\beta_1}_{n_1,a_1,2}(t_1)\right)^{\frac{r_1}{2}}\left(\gamma^{\alpha_2,\beta_2}_{n_2,a_2,2}(t_2)\right)^{\frac{r_2}{2}}.$$

Proof. For $\psi \in L^B_M(r_1, r_2)$, from equation (14), we may write

$$\begin{aligned}\left|G^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n,n,a_1,a_2}(\psi; t_1, t_2) - \psi(t_1, t_2)\right| &\le \S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n,n,a_1,a_2}(\left|\triangle_{(y_1,y_2)}\psi(t_1, t_2)\right|; t_1, t_2)\\ &\le M\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n,n,a_1,a_2}(\left|y_1 - t_1\right|^{r_1}\left|y_2 - t_2\right|^{r_2}; t_1, t_2)\\ &\le M\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n,n,a_1,a_2}(\left|y_1 - t_1\right|^{r_1}; t_1, t_2)\\ &\times\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n,n,a_1,a_2}(\left|y_2 - t_2\right|^{r_2}; t_1, t_2).\end{aligned}$$

In the view of lemma (3.1) and using Hölder's inequality for $p_i = \frac{2}{r_i}$ & $q_i = \frac{2}{2-r_i}$, we get

$$\left|G^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n,n,a_1,a_2}(\psi;t_1,t_2) - \psi(t_1,t_2)\right| \leq M\left(\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n,n,a_1,a_2}(\left|y_1-t_1\right|^2;t_1,t_2)\right)^{\frac{r_1}{2}} \times \left(\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n,n,a_1,a_2}(\left|y_2-t_2\right|^2;t_1,t_2)\right)^{\frac{r_2}{2}}.$$

This completes the result. □

Next, Mixed modulus of smoothness (cf.[18]) for $\psi \in B_B(Y_1 \times Y_2)$ is defined by

$$\omega_B(\psi;\zeta_1,\zeta_2) = \sup\{\left|\Delta_{(y_1,y_2)}\psi(t_1,t_2)\right| : \left|y_1-t_1\right| < \zeta_1,\ \left|y_2-t_2\right| < \zeta_2\},$$

$\forall\ (t_1,t_2),(y_1,y_2) \in Y_1 \times Y_2$ and $\zeta_1,\zeta_2 > 0$. From [6] and [5], it is known that $\omega_B(\psi;\zeta_1,\zeta_2) \to 0$ as $\zeta_1,\zeta_2 \to 0$ iff ψ is uniformly B-continuous on $Y_1 \times Y_2$. Further, for positive numbers λ_1 *and* λ_2, we have

$$\omega_B(\psi;\lambda_1\zeta_1,\lambda_2\zeta_2) \leq (1+\lambda_1)(1+\lambda_2)\omega_B(\psi;\zeta_1,\zeta_2). \tag{15}$$

Now the following result yields us the rate of convergence of GBS operator $G^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n,n,a_1,a_2}$ in terms of mixed modulus $\omega_B(\psi;\zeta_1,\zeta_2)$ of smoothness of B-derivative of ψ.

Theorem 5.3. *Let $\psi \in D_B(I^2)$ such that $D_B\psi \in \bar{C}_B(I^2) \cap B(I^2)$. Then for each $(t_1,t_2) \in I^2$, $G^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n,n,a_1,a_2}$ satisfies*

$$\left|G^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n,n,a_1,a_2}(\psi;t_1,t_2) - \psi(t_1,t_2)\right| \leq \frac{C}{\sqrt{n_1n_2}}\left\{\left\|D_B\psi\right\|_\infty + \omega_B\left(D_B\psi;\ \frac{1}{\sqrt{n_1}},\frac{1}{\sqrt{n_2}}\right)\right\}.$$

Proof. From Mean value theorem for $\psi \in D_B(I^2)$, we have

$$\Delta_{(y_1,y_2)}\psi(t_1,t_2) = (y_1-t_1)(y_2-t_2)D_B\psi(\eta_1,\eta_2),$$

for some $\eta_1 \in (t_1,y_1)$ & $\eta_2 \in (t_2,y_2)$. For $D_B\psi \in \bar{C}_B(I^2) \cap B(I^2)$, using the definition of mixed difference, we obtain

$$\begin{aligned}
&|\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n,n,a_1,a_2}(\Delta_{(y_1,y_2)}\psi(t_1,t_2);t_1,t_2)| \\
&\quad = \left|\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n,n,a_1,a_2}((y_1-t_1)(y_2-t_2)D_B\psi(\eta_1,\eta_2);t_1,t_2)\right| \\
&\quad \leq \S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n,n,a_1,a_2}(|y_1-t_1||y_2-t_2|(|\Delta_{(y_1,y_2)}D_B\psi(t_1,t_2)| \\
&\quad + |D_B\psi(\eta_1,t_2)| + |D_B\psi(t_1,\eta_2)| \\
&\quad + |D_B\psi(t_1,t_2)|);t_1,t_2) \\
&\quad \leq \S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n,n,a_1,a_2}(|y_1-t_1||y_2-t_2|\omega_B(D_B\psi;|\eta_1-t_1|,|\eta_2-t_2|);t_1,t_2) \\
&\quad + 3\,\|D_B\|_\infty\,\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n,n,a_1,a_2}(|y_1-t_1||y_2-t_2|;t_1,t_2)
\end{aligned}$$

Now for any $\zeta_1,\zeta_2 > 0$ taking into account

$$\begin{aligned}
\omega_B(D_B\psi;|\zeta_1-t_1|,|\zeta_2-t_2|) &\leq \omega_B(D_B\psi;\left|y_1-t_1\right|,\left|y_2-t_2\right|) \\
&\leq \left(1+\frac{\left|y_1-t_1\right|}{\zeta_1}\right)\left(1+\frac{\left|y_2-t_2\right|}{\zeta_2}\right)\omega_B(D_B\psi;\zeta_1,\zeta_2)
\end{aligned}$$

and applying Cauchy-Schwartz inequality, we are led to

$$\begin{aligned}\left|G^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n,n,a_1,a_2}(\psi;t_1,t_2)-\psi(t_1,t_2)\right| &\le 3\left\|D_B\psi\right\|_\infty \sqrt{\gamma^{\alpha_1,\beta_1}_{n_1,a_1,2}(t_1)}\sqrt{\gamma^{\alpha_2,\beta_2}_{n_2,a_2,2}(t_2)}\\ &+\omega_B(D_B\psi;\zeta_1,\zeta_2)\Big\{\sqrt{\gamma^{\alpha_1,\beta_1}_{n_1,a_1,2}(t_1)}\sqrt{\gamma^{\alpha_2,\beta_2}_{n_2,a_2,2}(t_2)}\\ &+\frac{1}{\zeta_1}\sqrt{\gamma^{\alpha_1,\beta_1}_{n_1,a_1,4}(t_1)}\sqrt{\gamma^{\alpha_2,\beta_2}_{n_2,a_2,2}(t_2)}\\ &+\frac{1}{\zeta_2}\sqrt{\gamma^{\alpha_1,\beta_1}_{n_1,a_1,2}(t_1)}\sqrt{\gamma^{\alpha_2,\beta_2}_{n_2,a_2,4}(t_2)}\\ &+\frac{1}{\zeta_1\zeta_2}\gamma^{\alpha_1,\beta_1}_{n_1,a_1,2}(t_1)\gamma^{\alpha_2,\beta_2}_{n_2,a_2,2}(t_2)\Big\}\end{aligned}$$

We reach the required result by taking $\zeta_1 = n_1^{-\frac{1}{2}}$ and $\zeta_2 = n_2^{-\frac{1}{2}}$ and using lemma (3.2). □

6. Graphical Analysis

Now, we present some numerical results to show the convergence behavior of $\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(\psi;t_1,t_2)$ to $\psi(t)$ by fixing the parameters a_1, a_2, n_1, n_2 and by varying the values of α_1, α_2 & β_1, β_2 by using MATLAB.

Example 6.1. *Let $\psi(t_1,t_2) = t_1^3 t_2 + t_1^2 t_2$ and, $a_1 = a_2 = 0.8$, $\alpha_1 = \alpha_2 = 0$ and $(\beta_1,\beta_2) \in \{0, 0.5, 0.6, 0.7\}$. Figure 1 and 3 illustrate the convergence behavior of $\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(\psi;t_1,t_2)$ to $\psi(t)$ on I^2 at $n_1 = n_2 = 10$ and 20 respectively for different values of (β_1,β_2). Whereas the respective absolute error in the approximation are shown in figure 2 and 4. Table 1 and 2 gives the absolute errors in the approximation $E^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(\psi;t_1,t_2) = \left|\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(\psi;t_1,t_2) - \psi(t_1,t_2)\right|$ at some points in I^2 for $n_1 = n_2 = 10$ and 20 respectively. From the figures, it is clear that for $\beta_1 = \beta_2 = 0.7$, the operator $\S^{\alpha_1,\beta_1,\alpha_2,\beta_2}_{n_1,n_2,a_1,a_2}(\psi;t_1,t_2)$ gives best approximation to $\psi(t)$ in comparison to other values of $\beta_1 = \beta_2$.*

Further, figure 5 and 7 illustrates the approximation by GBS operator $G^{0,\beta_1,0,\beta_2}_{n_1,n_2,0.8,0.8}(\psi;t_1,t_2)$ to ψ for $n_1 = n_2 = 10$ and 20 respectively. Also the respective errors are shown in figures 6 and 8. From these figures and tables 3 and 4, it is clear that upper bound of errors for GBS operator is smaller for $\beta_1 = \beta_2 = 0.7$ than that for other values of β_1 and β_2.

By comparing table 1 with table 3 and table 2 with table 4, we observe that errors in the approximation by GBS operators are much smaller as compared to that by Stancu type a-Baskakov operators.

Table 1: Error of approximation $E^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{10,10,0.8,0.8}(\psi;t)$ for $\alpha_1 = \alpha_2 = 0$ & $\beta_1,\beta_2 \in \{0, 0.5, 0.6, 0.7\}$.

(t_1,t_2)	$E^{(0,0,0,0)}_{10,10,0.8,0.8}$	$E^{(0,0.5,0,0.5)}_{10,10,0.8,0.8}$	$E^{(0,0.6,0,0.6)}_{10,10,0.8,0.8}$	$E^{(0,0.7,0,0.7)}_{10,10,0.8,0.8}$
(1,1)	0.4983	0.0993	0.0296	0.0370
(0.9,0.9)	0.3608	0.0836	0.0352	0.0112
(0.8,0.8)	0.2531	0.0678	0.0354	0.0044
(0.7,0.7)	0.1706	0.0526	0.0319	0.0122
(0.4,0.4)	0.0357	0.0167	0.0133	0.0101
(0.2,0.2)	0.0062	0.0038	0.0034	0.0030
(0.2,0.7)	0.0216	0.0135	0.0120	0.0106
(0.7,0.2)	0.0488	0.0150	0.0091	0.0035

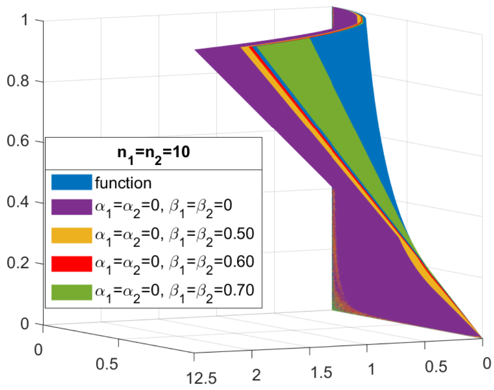

Figure 1: Approximation by $\S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{10,10,0.8,0.8}(\psi;t_1,t_2)$.

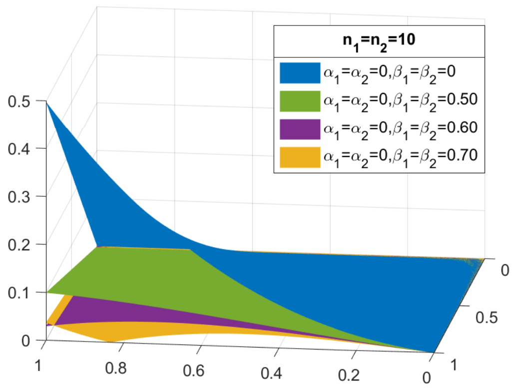

Figure 2: Error approximation of $\S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{10,10,0.8,0.8}(\psi;t_1,t_2)$.

Table 2: Error of approximation $E^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{20,20,0.8,0.8}(\psi;t)$ for $\alpha_1 = \alpha_2 = 0$ & $\beta_1,\beta_2 \in \{0, 0.5, 0.6, 0.7\}$.

(t_1,t_2)	$E^{(0,0,0,0)}_{20,20,0.8,0.8}$	$E^{(0,0.5,0,0.5)}_{20,20,0.8,0.8}$	$E^{(0,0.6,0,0.6)}_{20,20,0.8,0.8}$	$E^{(0,0.7,0,0.7)}_{20,20,0.8,0.8}$
(1,1)	0.2548	0.0662	0.0310	0.0035
(0.9,0.9)	0.1846	0.0542	0.0298	0.0060
(0.8,0.8)	0.1296	0.0429	0.0267	0.0108
(0.7,0.7)	0.0874	0.0326	0.0223	0.0123
(0.4,0.4)	0.0183	0.0098	0.0082	0.0066
(0.2,0.2)	0.0032	0.0022	0.0020	0.0018
(0.2,0.7)	0.0111	0.0077	0.0070	0.0064
(0.7,0.2)	0.0250	0.0093	0.0064	0.0035

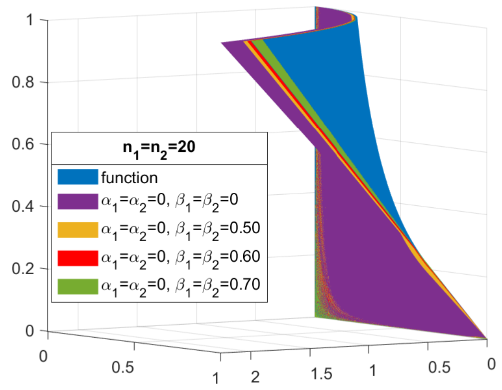

Figure 3: Approximation by $\mathbb{S}_{20,20,0.8,0.8}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(\psi;t_1,t_2)$.

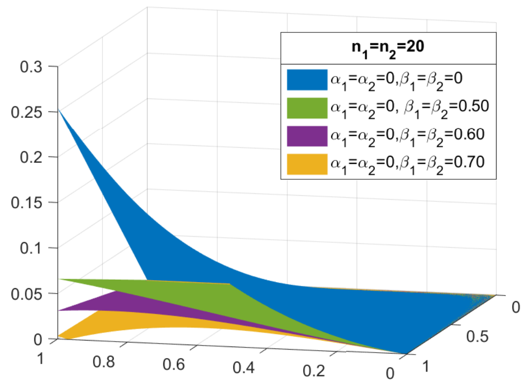

Figure 4: Error approximation of $\mathbb{S}_{20,20,0.8,0.8}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(\psi;t_1,t_2)$.

Table 3: Error of approximation $E_{10,10,0.8,0.8}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(\psi;t)$ for $\alpha_1=\alpha_2=0$ & $\beta_1,\beta_2\in\{0,0.5,0.6,0.7\}$.

(t_1,t_2)	$E_{10,10,0.8,0.8}^{(0,0,0,0)}$	$E_{10,10,0.8,0.8}^{(0,0.5,0,0.5)}$	$E_{10,10,0.8,0.8}^{(0,0.6,0,0.6)}$	$E_{10,10,0.8,0.8}^{(0,0.7,0,0.7)}$
(1,1)	0.0241	0.0254	0.0227	0.0193
(0.9,0.9)	0.0173	0.0190	0.0173	0.0150
(0.8,0.8)	0.0121	0.0138	0.0127	0.0114
(0.7,0.7)	0.0081	0.0096	0.0091	0.0083
(0.4,0.4)	0.0016	0.0023	0.0023	0.0022
(0.2,0.2)	2.7315e-04	4.3742e-04	4,5192e-04	4.6132e-04
(0.2,0.7)	9.5603e-04	0.0015	0.0016	0.0016
(0.7,0.2)	0.0023	0.0027	0.0026	0.0024

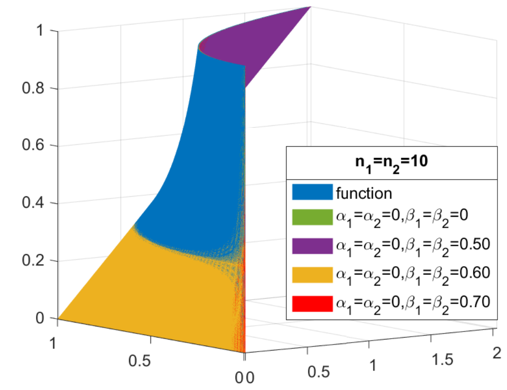

Figure 5: Approximation by GBS of $\S_{10,10,0.8,0.8}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(\psi;t_1,t_2)$.

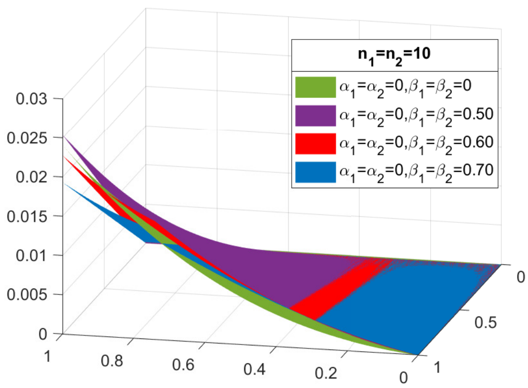

Figure 6: Error in the approximation by GBS of $\S_{10,10,0.8,0.8}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(\psi;t_1,t_2)$.

Table 4: Error of approximation $E_{20,20,0.8,0.8}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(\psi;t)$ for $\alpha_1 = \alpha_2 = 0$ & $\beta_1,\beta_2 \in \{0, 0.5, 0.6, 0.7\}$.

(t_1, t_2)	$E_{20,20,0.8,0.8}^{(0,0,0,0)}$	$E_{20,20,0.8,0.8}^{(0,0.5,0,0.5)}$	$E_{20,20,0.8,0.8}^{(0,0.6,0,0.6)}$	$E_{20,20,0.8,0.8}^{(0,0.7,0,0.7)}$
(1,1)	0.0060	0.0071	0.0065	0.0058
(0.9,0.9)	0.0043	0.0053	0.0050	0.0045
(0.8,0.8)	0.0030	0.0038	0.0036	0.0034
(0.7,0.7)	0.0020	0.0027	0.0026	0.0024
(0.4,0.4)	4.1094e-04	6.3051e-04	6.4009e-04	6.3925e-04
(0.2,0.2)	6.8544e-04	1.1982e-04	1.2603e-04	1.3102e-04
(0.2,0.7)	2.3990e-04	4.1936e-04	4.4111e-04	4.5855e-04
(0.7,0.2)	5.7772e-04	7.6312e-04	7.3784e-04	6.9370e-04

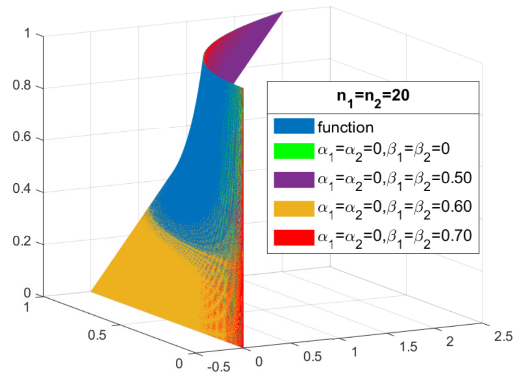

Figure 7: Approximation by GBS of $\S_{20,20,0.8,0.8}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(\psi; t_1, t_2)$.

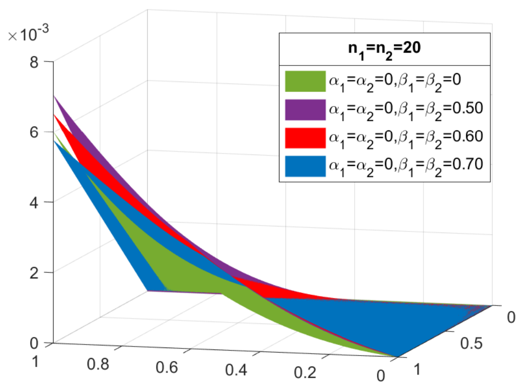

Figure 8: Error in the approximation by GBS of $\S_{20,20,0.8,0.8}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(\psi; t_1, t_2)$.

7. Conclusion

The Stancu type modification of a-Baskakov operators yields better approximation than a-Baskakov operators. The Stancu parameters $(\alpha_1, \beta_1, \alpha_2, \beta_2)$ involving in the modification provides flexibility to proposed operators. This can be seen from the figures 2 and 4. Further from the error tables 1,2,3 and 4, it can be observed that the rate of convergence of GBS operators is much better than that of bivariate a-Baskakov-Stancu operators.

Acknowledgement

The second author is very grateful to the University Grant Commission (UGC), India for financial support to carry out this research work.

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