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(k, m, n)-partially isometric operators: A new generalization of partial isometries

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Abstract. A significant amount of elegant work has been accomplished in the study of partial isometries. In this article, we introduce a new class of operators, referred to as the (k, m, n)-partial isometries, which extends the concept of partial isometry. We delve into the most intriguing outcomes related to this class by extending previously established results for partial isometries and by exploring new results on partial isometries. We investigate the relationship of this new class of operators with classical notions of operators, such as partial isometries, power partial isometries, paranormal, semi-regular, and quasi-Fredholm. Additionally, we examine some fundamental properties and structure theorems of (k, m, n)-partial isometries. Furthermore, we provide spectral properties of (k, m, n)-partial isometries.

1. Introduction and Notations

Throughout this paper, we use the notation $\mathcal{H}$ to refer to a complex Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$ and the associated norm $\| \cdot \|$. We denote by $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators defined on $\mathcal{H}$. For an operator $T \in \mathcal{B}(\mathcal{H})$, we denote by $\mathsf{R}(T), \mathsf{N}(T)$ and T^* the range, the kernel and the adjoint operator of T, respectively. For a given subspace F of $\mathcal{H}$, the orthogonal subspace of F in $\mathcal{H}$ is denoted by $\mathsf{F}^\perp$. Let $T \in \mathcal{B}(\mathcal{H})$, we define T_F as the restriction of T to F viewed as a map from F into $\mathcal{H}$. If more F is invariant by T, we denote by $T_{|\mathsf{F}}$ the restriction of T viewed as a map from F into F. We denote by $\mathbb{D}$ the open unit disc of the complex plane, we use $\overline{\mathbb{D}}$ and $\partial\mathbb{D}$ to denote respectively the closure and boundary of $\mathbb{D}$.

An operator T on $\mathcal{H}$ is said to be an isometry if $T^*T = I$, where I is the identity operator on $\mathcal{H}$. It is also worth recalling that $T \in \mathcal{B}(\mathcal{H})$ is called a contraction if $\|T\| \leq 1$, a co-isometry if T^* is an isometry, and unitary if T is an invertible isometry. A partial isometry $T \in \mathcal{B}(\mathcal{H})$ is an isometry on the orthogonal complement of its kernel. It is evident that any partial isometry is a contraction, has a closed range, and its adjoint is also a

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partial isometry. Moreover, it is well known that T is a partial isometry if and only if $TT^*T = T$. Examples of partial isometries include orthogonal projections, unitaries, isometries, co-isometries, and their direct sums. Following Halmos, an operator $T \in \mathcal{B}(\mathcal{H})$ is called a power partial isometry if T^q is a partial isometry for every $q \geq 1$.

Partial isometries are a compelling class of operators that have applications in various fields, particularly in quantum physics [7, 28]. The concept of quantization, which we mentioned earlier, is a well-established concept in physics and shares similarities with the Jordan-Schwinger map. Instead of using creation-annihilation operators, a model of quantization employs a countable family of orthogonal partial isometries in a separable Hilbert space as its foundational elements (see [6]). In mathematics, partial isometries play a significant role as they provide a well-explored extension of isometries. They have been instrumental in operator theory, particularly in the theory of polar decomposition of operators and the dimension theory of von Neumann algebras. Noteworthy contributions to this field have been made by Erdélyi [12, 13], Halmos, and McLaughlin [23, 24], among others. For more comprehensive information on partial isometries in the infinite-dimensional case, interested readers are encouraged to consult [4, 9, 14, 15, 19, 25, 31, 32, 35–38]. Additionally, references [16, 20, 21] provide insights into the finite-dimensional case.

Recently, the literature has seen various developments concerning the generalization of the class of partial isometries in Hilbert spaces. Notable examples include $\mathcal{N}_A$-isometric operators [2], semi-generalized partial isometries [19], and m-partial isometries [33, 34] in Hilbert spaces. There have also been investigations on partial isometries from the perspective of C^*-algebras [4]. In this paper, we address a closely related problem of generalization and introduce a new class of operators with numerous properties. Specifically, we define the following class of operators: for $n,\ m \in \mathbb{N}$ and $k \in \mathbb{N}\backslash\{0\}$ such that $n + m \geq k$,

$$^{k}\!\Delta_{m,n} = \{T \in \mathcal{B}(\mathcal{H}) \ : \ T^mT^{*k}T^n = T^{m+n-k}\}.$$

An operator $T \in {}^{k}\!\Delta_{m,n}$ is referred to as an (k, m, n)-partial isometry.

We see that ${}^{1}\!\Delta_{0,1}$ (resp. ${}^{1}\!\Delta_{1,0}$) corresponds to the set of all isometries (resp. co-isometries). Thus, ${}^{1}\!\Delta_{1,0} \cap {}^{1}\!\Delta_{0,1}$ coincides with the set of unitary operators. Moreover, if we remove the condition $n + m \geq k$, then ${}^{1}\!\Delta_{0,0}$ will also correspond to the set of unitary operators. Additionally, we observe that ${}^{1}\!\Delta_{1,1}$ is the set of partial isometries. Clearly, for all $n \in \mathbb{N}\backslash\{0\}$ we have

$${}^{1}\!\Delta_{1,0} \cap {}^{1}\!\Delta_{0,1} \subsetneq {}^{1}\!\Delta_{0,1} \subsetneq {}^{1}\!\Delta_{1,1} \subsetneq {}^{1}\!\Delta_{1,n}$$

and

$${}^{1}\!\Delta_{1,0} \cap {}^{1}\!\Delta_{0,1} \subsetneq {}^{1}\!\Delta_{1,0} \subsetneq {}^{1}\!\Delta_{1,1} \subsetneq {}^{1}\!\Delta_{n,1}.$$

The contents of the paper are divided into four portions: an introduction and three sections. Section 2 provides some basic concepts and remarks that will be helpful in the subsequent sections. It includes fundamental results needed for later sections and studies the connection of our new class of operators with classical notions of operators, specifically partial isometries, isometries, co-isometries, unitary, normal, self-adjoint operators, operators that possess a suitable power as a partial isometry, paranormal, and semi-regular operators. This section provides a detailed introduction to (k, m, n)-partial isometries. Despite being easily proven, Theorem 2.9 is central because many subsequent results are based on it. In Section 3, presentations of decompositions are provided. The last section is devoted to describing the spectral picture of (k, m, n)-partial isometries, which is presented in detail after significant effort in Theorem 4.5 and Theorem 4.6. The paper concludes with a beautiful unfamiliar result on spectral properties of classical partial isometries. Several properties on (k, m, n)-partial isometries are proved, which result in a generalization of well-known assertions on partial isometries. The reader is warned of the following: several results are given for T^k. That is to say, by setting $k = n = m = 1$, well-known results on partial isometries can be rediscovered. This serves as evidence that the class of (k, m, n)-partial isometries effectively generalizes the class of partial isometries. It is also worth mentioning that we explore some new results on classical partial isometries.

2. Basic properties of (k, m, n)-partial isometries

This section begins by presenting several illustrative examples.

Example 2.1.

1) *If P is an orthogonal projection (i.e., $P^2 = P = P^*$), then for all $n, m \in \mathbb{N}$ and $k \in \mathbb{N}\backslash\{0\}$ such that $n + m \geq k + 1, P \in {}^k\Delta_{m,n}$.*

2) *If S is an orthogonal symmetry (i.e., $S^2 = I$ and $S = S^*$), then for all $n, m \in \mathbb{N}$ and $k \in \mathbb{N}\backslash\{0\}$ such that $n + m \geq k, S \in {}^k\Delta_{m,n}$.*

3) *If T is nilpotent of order $k \in \mathbb{N}\backslash\{0\}$, then $T \in {}^k\Delta_{m,n}$, for all $m, n \in \mathbb{N}$ such that $m + n \geq 2k$.*

4) *If T is nilpotent of order $n \in \mathbb{N}\backslash\{0\}$, then $T \in {}^k\Delta_{m,n}$, for all $m \in \mathbb{N}$ and $k \in \mathbb{N}\backslash\{0\}$ such that $m \geq k$.*

5) *If T is nilpotent of order $m \in \mathbb{N}\backslash\{0\}$, then $T \in {}^k\Delta_{m,n}$, for all $n \in \mathbb{N}$ and $k \in \mathbb{N}\backslash\{0\}$ such that $n \geq k$.*

6) *Let $\mathcal{H}$ be a separable Hilbert space with an orthonormal basis $(e_i)_{i\in\mathbb{N}}$. Define the operator T in $\mathcal{H}$ as follows:*

$$T(e_i) = e_{i+1}, \quad \forall\ i \in \mathbb{N}.$$

Clearly, $T^*(e_0) = 0$, $\ T^*(e_i) = e_{i-1}$, for all $i \geq 1$, and so

$$T^m T^{*k} T^n(e_i) = T^m T^{*k}(e_{i+n}) = T^m(e_{i+n-k}) = e_{i+m+n-k} = T^{m+n-k}(e_i),$$

for all $i, m \geq 0$ and $n \geq k \geq 1$. This implies that $T \in {}^k\Delta_{m,n}$, for all $m \geq 0$ and $n \geq k \geq 1$.

Example 2.2.

1) *Let $T = \frac{1}{2}\begin{pmatrix} 0 & \sqrt{3} & -1 \\ 0 & 1 & \sqrt{3} \\ 0 & 0 & 0 \end{pmatrix}$. Then for all $n \geq 1, m \geq 1$, $T \in {}^1\Delta_{m,n}$; but $T \notin {}^1\Delta_{1,0} \cup {}^1\Delta_{0,1}$.*

2) *Let $T = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then for all $n \geq 2, m \geq 2$, $T \in {}^2\Delta_{m,n}$; but $T \notin {}^1\Delta_{1,1}$.*

3) *Let A be an isometry and B be a partial isometry such that $B^*A = 0$. Put*

$$T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}.$$

Then for all $m, n \geq 1$ one can check that $T \in {}^n\Delta_{m,n}$. If more B is not an isometry, then $T \notin {}^1\Delta_{0,1}$.

We now offer several useful remarks.

Remark 2.3. *Let $T \in \mathcal{B}(\mathcal{H})$ and $n, m, k \in \mathbb{N}$ such that $n + m \geq k > 0$.*

1) ${}^k\Delta_{k,k} = \{T \in \mathcal{B}(\mathcal{H})\ :\ T^k T^{*k} T^k = T^k\} = \{T \in \mathcal{B}(\mathcal{H})\ :\ T^k$ *is a partial isometry*$\}$.
That is,

$$T \in {}^k\Delta_{k,k} \iff T^k \in {}^1\Delta_{1,1}.$$

2) *If $T \in {}^k\Delta_{m,n}$ and $\alpha \in \partial\mathbb{D}$, then $\alpha T \in {}^k\Delta_{m,n}$.*

3) *$T \in {}^k\Delta_{m,n}$ if and only if $T^* \in {}^k\Delta_{n,m}$.*

4) *${}^k\Delta_{m,n} \subset {}^k\Delta_{r,q}$, for all $r \geq m, q \geq n$. In particular, ${}^1\Delta_{1,1} \subset {}^1\Delta_{m,n}$, for all $m \geq 1, n \geq 1$.*

Remark 2.4.

1) *For $n \in \mathbb{N}\backslash\{0\}$, ${}^{n}\Delta_{0,n} = \{T \in \mathcal{B}(\mathcal{H}) \ : \ T^{*n}\, T^{n} = I\} = \{T \in \mathcal{B}(\mathcal{H}) \ : \ T^{n}$ is an isometry$\}$. Thus,*

$$T \in {}^{n}\Delta_{0,n} \Longleftrightarrow T^{n} \in {}^{1}\Delta_{0,1}.$$

On the other hand, we know that if T^n is an isometry, then T is an isometry if and only if T^{n+1} is an isometry. This leads to

$${}^{n}\Delta_{0,n} \cap {}^{n+1}\Delta_{0,n+1} = {}^{1}\Delta_{0,1}.$$

2) *For $m \in \mathbb{N}\backslash\{0\}$, ${}^{m}\Delta_{m,0} = \{T \in \mathcal{B}(\mathcal{H}) \ : \ T^{m}\, T^{*m} = I\} = \{T \in \mathcal{B}(\mathcal{H}) \ : \ T^{m}$ is a co-isometry$\}$. Thus,*

$$T \in {}^{m}\Delta_{m,0} \Longleftrightarrow T^{m} \in {}^{1}\Delta_{1,0},$$

and as $T \in {}^{m}\Delta_{m,0}$ if and only if $T^{*} \in {}^{m}\Delta_{0,m}$, using 1) *we infer that*

$${}^{m}\Delta_{m,0} \cap {}^{m+1}\Delta_{m+1,0} = {}^{1}\Delta_{1,0}.$$

3) *For $n \in \mathbb{N}\backslash\{0\}$,*

$$\begin{aligned} T \in {}^{n}\Delta_{0,n} \cap {}^{n}\Delta_{n,0} &\Longleftrightarrow T^{n} \text{ is unitary} \\ &\Longleftrightarrow T^{n} \in {}^{1}\Delta_{0,1} \cap {}^{1}\Delta_{1,0} = {}^{1}\Delta_{0,0}. \end{aligned}$$

4) *For n, $m \in \mathbb{N}\backslash\{0\}$, we note that if $T \in {}^{m+n}\Delta_{m,n} = \{T \in \mathcal{B}(\mathcal{H}) \ : \ T^{m}\,(T^{*})^{(m+n)}\,T^{n} = I\}$, then T is invertible and $T^{-(m+n)} = (T^{*})^{(m+n)}$ (i.e. T^{m+n} is unitary). Now, if T^{m+n} is unitary, then T is invertible and*

$$T^{m}\,(T^{*})^{(m+n)}\,T^{n} = T^{-n}\,T^{m+n}\,(T^{*})^{(m+n)}\,T^{n} = T^{-n}T^{n} = I.$$

Hence, $T \in {}^{m+n}\Delta_{m,n}$. That is,

$$\begin{aligned} T \in {}^{m+n}\Delta_{m,n} &\Longleftrightarrow T^{m+n} \text{ is unitary} \\ &\Longleftrightarrow T^{m+n} \in {}^{1}\Delta_{0,1} \cap {}^{1}\Delta_{1,0} = {}^{1}\Delta_{0,0}. \end{aligned}$$

5) *For n, $m \in \mathbb{N}\backslash\{0\}$, in view of the above we have*

$${}^{m+n}\Delta_{m,n} = {}^{m+n}\Delta_{0,m+n} \cap {}^{m+n}\Delta_{m+n,0}.$$

6) *Let $n \in \mathbb{N}\backslash\{0\}$. Denoting by $C = \{T \in \mathcal{B}(\mathcal{H}) \ : \|T\| \leq 1\}$ the set of all contractions, we can combine* 1) *and* 2) *along with [4, Remark 2.8], to obtain the following results:*

$${}^{n}\Delta_{0,n} \cap C = {}^{1}\Delta_{0,1}$$

and

$${}^{n}\Delta_{n,0} \cap C = {}^{1}\Delta_{1,0}.$$

Therefore, if a contraction T is both an $(n,0,n)$-partial isometry and an $(n,n,0)$-partial isometry, then T is unitary.

Recall that for $T \in \mathcal{B}(\mathcal{H})$, the ascent $\boldsymbol{a}(T)$ and descent $\boldsymbol{d}(T)$ are defined as follows: $\boldsymbol{a}(T) = \inf\{n \geq 0 : \mathsf{N}(T^{n}) = \mathsf{N}(T^{n+1})\}$ and $\boldsymbol{d}(T) = \inf\{n \geq 0 : \ \mathsf{R}(T^{n}) = \mathsf{R}(T^{n+1})\}$, respectively. If the sets in the infima are empty, we take the infimum to be ∞.

Remark 2.5. *Let $T \in \mathcal{B}(\mathcal{H})$ and $n, m, k \in \mathbb{N}$ such that $n + m \geq k > 0$.*

1) *If $T \in {}^{k}\Delta_{m,n}$, with $k > m$, then $\mathsf{N}(T^{n}) = \mathsf{N}(T^{m+n-k})$, and thus $\boldsymbol{a}(T) \leq n$.*

2) *If $T \in {}^{k}\Delta_{m,n}$, with $k > n$, then $\mathsf{R}(T^{m}) = \mathsf{R}(T^{m+n-k})$, and thus $\boldsymbol{d}(T) \leq m$.*

Here we investigate the relationship between (k, m, n)-partial isometries and classical isometries, co-isometries, partial isometries, and unitary operators.

Proposition 2.6.

1) *If $T \in \mathcal{B}(\mathcal{H})$ such that T^k is a partial isometry, then $T \in {}^k\Delta_{m,n}$, for all $n \geq k, m \geq k$. Thus,*

$$^k\Delta_{k,k} \subset \bigcap_{\substack{m,n\in\mathbb{N}\\ n\geq k, m\geq k}} {}^k\Delta_{m,n}.$$

2) *If T is an isometry, then $T \in {}^k\Delta_{m,n}$, for all $n \geq k$. Thus,*

$$^1\Delta_{0,1} \subset \bigcap_{\substack{m,n\in\mathbb{N}\\ n\geq k}} {}^k\Delta_{m,n}.$$

3) *If T is a co-isometry, then $T \in {}^k\Delta_{m,n}$, for all $m \geq k$. Thus,*

$$^1\Delta_{1,0} \subset \bigcap_{\substack{m,n\in\mathbb{N}\\ m\geq k}} {}^k\Delta_{m,n}.$$

4) *If T is unitary, then $T \in {}^k\Delta_{m,n}$. Thus,*

$$^1\Delta_{1,0} \cap {}^1\Delta_{0,1} \subset \bigcap_{m+n\geq k} {}^k\Delta_{m,n}.$$

5) *If $\mathsf{N}(T^*) \subset \mathsf{N}(T)$, then*

$$T \in {}^1\Delta_{0,2} \Longrightarrow T \text{ is a partial isometry.}$$

6) *If $\mathsf{N}(T) \subset \mathsf{N}(T^*)$, then*

$$T \in {}^1\Delta_{2,0} \Longrightarrow T \text{ is a partial isometry.}$$

Proof.
1) For all $n \geq k, m \geq k$, one has

$$T^m T^{*k} T^n = T^{m-k} T^k T^{*k} T^k T^{n-k} = T^{m-k} T^k T^{n-k} = T^{n+m-k}.$$

This implies that $T \in {}^k\Delta_{m,n}$, for all $n \geq k, m \geq k$.

2) For all $n \geq k$, one has

$$T^m T^{*k} T^n = T^m T^{*k} T^k T^{n-k} = T^{m+n-k}.$$

This leads to $T \in {}^k\Delta_{m,n}$, for all $n \geq k$.

3) Evident from 2) and assertion 3) of Remark 2.3.

4) Since T is unitary, we have $T^{*k} T^k = I$, which implies that $T^m T^{*k} T^k T^n = T^{m+n}$. This equation, together with the invertibility of T, implies that $T \in {}^k\Delta_{m,n}$.

5) Since $T \in {}^1\Delta_{0,2}$, it follows that $T^{*2} T = T^*$. Hence, we have $\mathsf{R}(T^*T - I) \subset \mathsf{N}(T^*) \subset \mathsf{N}(T)$. Therefore, we obtain $TT^*T = T$.

6) As T is a partial isometry if and only if T^* is also a partial isometry, this assertion can be proven through duality using 5). Therefore, the proof is complete. □

Let $S, T \in \mathcal{B}(\mathcal{H})$. Recall that S and T are double commuting, or, equivalently, S, T is a double commuting pair, if $ST = TS$ and $S^*T = TS^*$. The reader will easily prove the next basic properties.

Proposition 2.7. *Let T, $S \in \mathcal{B}(\mathcal{H})$ and $n, m, k \in \mathbb{N}\backslash\{0\}$ such that $n + m \geq k$.*

1) *If S and T are double commuting,*

$$T \in {}^{k}\!\Delta_{m,n},\ S \in {}^{k}\!\Delta_{m,n} \quad \Longrightarrow ST \in {}^{k}\!\Delta_{m,n}.$$

2) *If S is an isometry,*

$$T \in {}^{k}\!\Delta_{m,n} \Longleftrightarrow STS^{*} \in {}^{k}\!\Delta_{m,n}.$$

3) *If S is a co-isometry,*

$$T \in {}^{k}\!\Delta_{m,n} \Longleftrightarrow S^{*}TS \in {}^{k}\!\Delta_{m,n}, \quad when \ \ n, m, k \geq 1 \ and \ n + m \geq k.$$

If furthermore S and T are double commuting,

$$T \in {}^{k}\!\Delta_{m,n} \Longrightarrow ST \in {}^{k}\!\Delta_{m,n},\ when\ m \geq k.$$

4) *If U is unitary,*

$$T \in {}^{k}\!\Delta_{m,n} \Longleftrightarrow U^{*}TU \in {}^{k}\!\Delta_{m,n}.$$

In the theorem that follows, we demonstrate that a normal contractive $(1, m, n)$-partial isometry, a compact self-adjoint (k, m, n)-partial isometry, or a contractive self-adjoint (k, m, n)-partial isometry is in fact a partial isometry.

Theorem 2.8. *Let $T \in \mathcal{B}(\mathcal{H})$, $n, m \in \mathbb{N}$ and $k \in \mathbb{N}\backslash\{0\}$ such that $n + m \geq k$.*

1) *If T is a normal contraction and $T \in {}^{k}\!\Delta_{m,n}$, then T^k is a partial isometry.*

2) *If T is self-adjoint and $T \in {}^{k}\!\Delta_{m,n}$, then $T^{2k+1} = T$. In particular, T^k is a partial isometry. If furthermore T is a contraction, then T is a partial isometry.*

3) *If T is a compact self-adjoint operator and $T \in {}^{k}\!\Delta_{m,n}$, then $T^3 = T$. In particular T is a partial isometry.*

4) *If T is skew-adjoint and $T \in {}^{k}\!\Delta_{m,n}$, then $(-1)^k T^{2k+1} = T$. In particular T^k and T^{2k} are partial isometries.*

5) *If T is a compact skew-adjoint operator and $T \in {}^{k}\!\Delta_{m,n}$, then $T^3 + T = 0$. In particular T and T^2 are partial isometries.*

Proof. Since $k \geq 1$, it follows that either $m \geq 1$ or $n \geq 1$. Note that without loss of generality, we can assume that $m \geq 1$.
1) Since T is a contraction, T^k is also a contraction. This implies that $I - T^{*k}T^k$ is positive. It follows that its positive square root $S := (I - T^{*k}T^k)^{\frac{1}{2}}$ exists. Since T is normal and $T \in {}^{k}\!\Delta_{m,n}$, we have $\mathsf{R}((T^{*k}T^k - I)T^n) \subset \mathsf{N}(T^m) = \mathsf{N}(T)$. Therefore, $T^{*n}(T^{*k}T^k - I)T^* = 0$. As T^* is normal, we obtain $TT^{*k}T^{k+1} = T^2$. Consequently, $T^*TS^2T^*T = T^*(TS^2T)T^* = 0$ and so $0 = \|T^*TS^2T^*T\| = \|T^*TS\|^2$. Hence, $TT^{*k}T^k = T$. Multipling this equation by T^{k-1} from the left side, we get that T^k is a partial isometry.

2) Since T is self-adjoint and $T \in {}^{k}\!\Delta_{m,n}$, $\mathsf{R}(T^{2k} - I) \subset \mathsf{N}(T^{m+n}) = \mathsf{N}(T)$. So, $T^{2k+1} = T$. This implies that T^k is a partial isometry. If more T is a contraction, then using [4, Corollary 2.5] we get that T is a partial isometry.

3) Since T is a compact self-adjoint operator, it is diagonalizable, and its eigenvalues are real (see [8, Theorem 6.11]). Let λ be an eigenvalue of T. From assertion 2), we have $\lambda^{2k+1} = \lambda$, which implies λ belongs to the set $\{0, 1, -1\}$. Consequently, we have $T^3 = T$.

4) and 5) Since T is skew-adjoint, iT and T^2 are self-adjoint. These assertions follow from 2) and 3). This completes the proof. □

The next theorem and its corollary provide very useful results on (k, m, n)-partial isometries.

Theorem 2.9. *Let $T \in \mathcal{B}(\mathcal{H})$ and $n, m \in \mathbb{N}$ and $k \in \mathbb{N}\backslash\{0\}$ such that $n + m \geq k$.*

1) *If T is injective or $k = m$, then*

$$T \in {}^k\!\Delta_{m,n} \iff T^kT^{*k}_{|\overline{\mathsf{R}(T^n)}} = I_{|\overline{\mathsf{R}(T^n)}}.$$

2) *If the range of T is dense or $k = n$, then*

$$T \in {}^k\!\Delta_{m,n} \iff T^{*k}T^k_{|\mathsf{N}(T^m)^\perp} = I_{|\mathsf{N}(T^m)^\perp}.$$

3) *If T is bijective, then*

$$T \in {}^k\!\Delta_{m,n} \iff T^k \text{ is unitary} \iff T^k \in {}^1\!\Delta_{0,1} \cap {}^1\!\Delta_{1,0} = {}^1\!\Delta_{0,0}.$$

4) *If $T \in {}^k\!\Delta_{m,n}$ with $k > m$, then $T^kT^{*k}_{|\overline{\mathsf{R}(T^n)}} = I_{|\overline{\mathsf{R}(T^n)}}$.*

5) *If $T \in {}^k\!\Delta_{m,n}$ with $k > n$, then $T^{*k}T^k_{|\mathsf{N}(T^m)^\perp} = I_{|\mathsf{N}(T^m)^\perp}$.*

Proof. 1) "$\Longrightarrow$" Let $T \in {}^k\!\Delta_{m,n}$. Then $T^mT^kT^{*k}T^n = T^{n+m}$. In the first case, when T is injective, we obtain $T^kT^{*k}T^n = T^n$. By continuity, we can conclude that $T^kT^{*k}_{|\overline{\mathsf{R}(T^n)}} = I_{|\overline{\mathsf{R}(T^n)}}$.

We again consider $T \in {}^k\!\Delta_{m,n}$. In the second case, where $k = m$, it can be observed directly that $T^kT^{*k}T^n = T^n$. Hence the result can be deduced as in the first case.
"$\Longleftarrow$" Let $x \in \mathcal{H}$. Then, $T^n(x) \in \mathsf{R}(T^n) \subset \overline{\mathsf{R}(T^n)}$. By hypothesis we get, $T^kT^{*k}T^n(x) = T^n(x)$. Hence, $T^kT^{*k}T^n = T^n$. If $k = m$, then by definition $T \in {}^k\!\Delta_{m,n}$. In the case where T is injective, by multiplication by T^m, we get $T^kT^mT^{*k}T^n = T^{m+n} = T^kT^{m+n-k}$. Thus, $T \in {}^k\!\Delta_{m,n}$.

2) From assertion 3) in Remark 2.3, together with 1) and duality, we can obtain the desired conclusion.

3) Follows directly from the previous assertions.

4) Using 4) in Remark 2.3, we can conclude $T \in {}^k\!\Delta_{k,n}$, which means $T^kT^{*k}T^n = T^n$. As in 1), we obtain that $\mathsf{R}(T^n)$ is a reducing subspace of T^kT^{*k}, and $T^kT^{*k}_{|\overline{R(T^n)}} = I_{|\overline{R(T^n)}}$.

5) The desired result follows from 4) and assertion 3) in Remark 2.3. Therefore, the theorem is proven. □

Corollary 2.10. *Let $T \in \mathcal{B}(\mathcal{H})$ and $n, m \in \mathbb{N}$ and $k \in \mathbb{N}$ such that $n + m \geq k$.*

1) *Suppose that $\mathsf{R}(T^n)$ is a reducing subspace of T. If T is injective or $k = m$, then*

$$T \in {}^k\!\Delta_{m,n} \iff T^k_{|\overline{\mathsf{R}(T^n)}} \text{ is a co-isometry.}$$

2) *Suppose that $\mathsf{N}(T^m)$ is a reducing subspace of T. If the range of T is dense or $k = n$, then*

$$T \in {}^k\!\Delta_{m,n} \iff T^k_{|\mathsf{N}(T^m)^\perp} \text{ is an isometry.}$$

3) *If $\mathsf{R}(T^n)$ is a reducing subspace of T, then*

$$T \in {}^k\!\Delta_{m,n} \text{ with } k > m \Longrightarrow T^k_{|\overline{\mathsf{R}(T^n)}} \text{ is a co-isometry.}$$

4) *If $\mathsf{N}(T^m)$ is a reducing subspace of T, then*

$$T \in {}^k\!\Delta_{m,n} \text{ with } k > n \Longrightarrow T^k_{|\mathsf{N}(T^m)^\perp} \text{ is an isometry.}$$

As stated in the introduction, it is known that if T is a partial isometry, then

$$\|Tx\| = \|x\|, \ \forall x \in \mathsf{N}(T)^{\perp}$$

and

$$\|T^*x\| = \|x\|, \ \forall x \in \mathsf{R}(T).$$

Now, we extend this result to (k, m, n)-partial isometries, where $n, m \in \mathbb{N}$ and $k \in \mathbb{N}\backslash\{0\}$ such that $n + m \geq k$. This is achieved through the following corollary.

Corollary 2.11. *Let $T \in \mathcal{B}(\mathcal{H})$ and $n, m \in \mathbb{N}$ and $k \in \mathbb{N}\backslash\{0\}$ such that $n + m \geq k$.*

1) *Suppose that T is injective or $k \geq m$, then*

$$T \in {}^{k}\!\Delta_{m,n} \Longrightarrow \|T^{*k}x\| = \|x\|, \ \ \forall\, x \in \overline{\mathsf{R}(T^{kn})}.$$

2) *Suppose that the range of T is dense or $k \geq n$, then*

$$T \in {}^{k}\!\Delta_{m,n} \Longrightarrow \|T^{k}x\| = \|x\|, \ \ \forall\, x \in \mathsf{N}(T^{kn})^{\perp}.$$

Proof. 1) From Theorem 2.9, the subspace $\mathsf{R}(T^{kn})$ is a reducing subspace of T^kT^{*k} and

$$\|T^{*k}x\|^2 = \langle T^kT^{*k}x\,,\, x\rangle = \langle x\,,\, x\rangle = \|x\|^2, \ \ \forall\, x \in \overline{\mathsf{R}(T^{kn})}.$$

2) follows from 1) and assertion 3) of Remark 2.3 by duality. This completes the proof. □

The class of power partial isometries has received significant attention in the literature. A detailed description of the structure of power partial isometries is provided in [14, 24]. It is easy to observe that T^k is a partial isometry if and only if T is a (k, k, k)-partial isometry. In the following corollary, we present a necessary and sufficient condition to ensure that T^k is a partial isometry when T is a (k, m, n)-partial isometry.

Corollary 2.12. *Let $T \in \mathcal{B}(\mathcal{H})$ and $n, m \in \mathbb{N}$ and $k \in \mathbb{N}\backslash\{0\}$ such that $n + m \geq k$.*

1) *If $T \in {}^{k}\!\Delta_{m,n}$ with $k \geq m$, then*

$$T^k \text{ is a partial isometry } \Longleftrightarrow \|T^{*k}x\| = \|x\|, \ \ \forall\, x \in \mathsf{R}(T^n)^{\perp} \cap \overline{\mathsf{R}(T^k)}.$$

2) *$T \in {}^{k}\!\Delta_{m,n}$ with $k \geq n$, then*

$$T^k \text{ is a partial isometry } \Longleftrightarrow \|T^{k}x\| = \|x\|, \ \ \forall\, x \in \mathsf{N}(T^m) \cap \mathsf{N}(T^k)^{\perp}.$$

Proof. 1) The direct implication follows from the fact that T^k is a partial isometry if and only if T^{*k} is also a partial isometry. Now assume that $\|T^{*k}x\| = \|x\|$ for all $x \in \mathsf{R}(T^n)^{\perp} \cap \overline{\mathsf{R}(T^k)}$. Since $T \in {}^{k}\!\Delta_{m,n}$ and $k \geq m$, according to Theorem 2.9, we have $T^kT^{*k}T^n = T^n$, which implies $\mathsf{R}(T^n) \subset \mathsf{R}(T^k)$. Using the fact that $\mathcal{H} = \mathsf{R}(T^n)^{\perp} \oplus \overline{\mathsf{R}(T^n)}$, we obtain the decomposition

$$\overline{\mathsf{R}(T^k)} = \mathsf{R}(T^n)^{\perp} \cap \overline{\mathsf{R}(T^k)} \oplus \overline{\mathsf{R}(T^n)}.$$

It is clear that $\overline{\mathsf{R}(T^k)}$, $\mathsf{R}(T^n)^{\perp} \cap \overline{\mathsf{R}(T^k)}$ and $\overline{\mathsf{R}(T^n)}$ are reducing subspaces of T^kT^{*k}. Writing $x = y + z \in \mathsf{R}(T^n)^{\perp} \cap \overline{\mathsf{R}(T^k)} \oplus \overline{\mathsf{R}(T^n)} = \overline{\mathsf{R}(T^k)}$, we have

$$\begin{array}{rcl}\|T^{*k}x\|^2 & = & \langle T^kT^{*k}y\,,\, y\rangle + \langle T^kT^{*k}z\,,\, z\rangle \\ & = & \|T^{*k}y\|^2 + \|T^{*k}z\|^2 \\ & = & \|y\|^2 + \|z\|^2 = \|x\|^2.\end{array}$$

Therefore T^{*k} (and hence T^k) is a partial isometry.
2) This assertion follows from 1) and assertion 3) of Remark 2.3. This completes the proof. □

Thanks to [30, Corollary 3.2], we know that a non-zero $(1, 1, 1)$-partial isometry has a closed range and a norm of 1. In the following propositions, we extend these results to T^k when T is a (k, m, n)-partial isometry.

Proposition 2.13. *Let $T \in \mathcal{B}(\mathcal{H})$ and $n, m \in \mathbb{N}$ and $k \in \mathbb{N}\backslash\{0\}$ such that $n + m \geq k$.*

1) *Assume that $T \in {}^k\Delta_{m,n}$, with $k \geq m$. Then*

$$T^{*k}(\overline{\mathsf{R}(T^n)}) \perp T^{*k}(\mathsf{R}(T^n)^{\perp}).$$

If furthermore $T^n \neq 0$, then

$$\|T^k\| = \max\{1, \|T^{*k}_{\mathsf{R}(T^n)^{\perp}}\|\}.$$

2) *Assume that $T \in {}^k\Delta_{m,n}$, with $k \geq n$. Then*

$$T^k(\mathsf{N}(T^m)^{\perp}) \perp T^k(\mathsf{N}(T^m)).$$

If furthermore $T^m \neq 0$, then

$$\|T^k\| = \max\{1, \|T^k_{\mathsf{N}(T^m)}\|\}.$$

Proof. 1) Consider $x \in \overline{\mathsf{R}(T^n)}$, $y \in \mathsf{R}(T^n)^{\perp}$ and $z = x + y$. By Theorem 2.9, we know that $\mathsf{R}(T^n)$ is a reducing subspace of T^kT^{*k}. This leads to

$$\begin{aligned} \|T^{*k}(x+y)\|^2 &= \langle T^kT^{*k}x, x\rangle + \langle T^kT^{*k}y, y\rangle \\ &= \|T^{*k}x\|^2 + \|T^{*k}y\|^2. \end{aligned}$$

Therefore, we infer that $T^{*k}(\overline{\mathsf{R}(T^n)}) \perp T^{*k}(\mathsf{R}(T^n)^{\perp})$. Furthermore, we have

$$\|T^{*k}z\|^2 = \|x\|^2 + \|T^{*k}y\|^2 \leq \max\{1, \|T^{*k}_{\mathsf{R}(T^n)^{\perp}}\|^2\}\|z\|^2.$$

Hence, $\max\{1, \|T^{*k}_{\mathsf{R}(T^n)^{\perp}}\|\} \geq \|T^{*k}\| = \|T^k.\|$

On the other hand, we have

$$\|T^n\| = \|T^kT^{*k}T^n\| \leq \|T^n\|\|T^k\|^2.$$

Therefore, if $T^n \neq 0$; then $1 \leq \|T^k\|$. Thus, $\max\{1, \|T^{*k}_{\mathsf{R}(T^n)^{\perp}}\|\} \leq \|T^k\|$.
2) The result follows from 1) and assertion 3) of Remark 2.3. Thus, the proof is complete. □

Remark 2.14. *If $T \in {}^k\Delta_{m,n}$ and T is a non-zero hyponormal operator, then $\|T\| \geq 1$. Indeed, since $T^mT^kT^{*k}T^n = T^{n+m}$ and T is hyponormal, we can deduce that $\|T\|^{m+n} = \|T^{m+n}\| \leq \|T\|^{m+n}\|T\|^{2k}$. Therefore, the result follows.*

Proposition 2.15. *Let $T \in \mathcal{B}(\mathcal{H})$ and $n, m, k \in \mathbb{N}$ such that $n + m \geq k$.*

1) *If $T \in {}^k\Delta_{m,n}$ with $k \geq m$, then $T^{*k}(\overline{\mathsf{R}(T^n)})$ is closed and*

$$\mathsf{R}(T^k) \text{ is closed } \Longleftrightarrow \mathsf{R}(T^{*k}_{\overline{\mathsf{R}(T^n)}^{\perp}}) \text{ is closed.}$$

In particular, if codim $\overline{\mathsf{R}(T^n)} < +\infty$, then T^k is right semi-Fredholm.

2) *If $T \in {}^k\Delta_{m,n}$ with $k \geq n$, then $T^k(\mathsf{N}(T^m)^{\perp})$ is closed and*

$$\mathsf{R}(T^k) \text{ is closed } \Longleftrightarrow \mathsf{R}(T^k_{\mathsf{N}(T^m)}) \text{ is closed.}$$

In particular, if $\dim \mathsf{N}(T^m) < +\infty$, then T^k is left semi-Fredholm.

Proof. 1) First, it should be noted that it is well-known that an operator has a closed range if and only if its adjoint also has a closed range. Since $\overline{\mathsf{R}(T^n)}$ is closed and based on the fact that $\|T^{*k}(x)\| = \|x\|$ for every $x \in \overline{\mathsf{R}(T^n)}$, it follows that $T^{*k}(\overline{\mathsf{R}(T^n)})$ is also closed. By using the fact that

$$\mathsf{R}(T^{*k}) = T^{*k}(\overline{\mathsf{R}(T^n)}) + T^{*k}(\mathsf{R}(T^n)^{\perp}),$$

we obtain the equivalence

$$\mathsf{R}(T^{*k}) \text{ is closed } \Longleftrightarrow \mathsf{R}(T^{*k}_{\mathsf{R}(T^n)^{\perp}}) \text{ is closed.}$$

Notice that if $\operatorname{codim} \overline{\mathsf{R}(T^n)} < +\infty$, then we can use the fact that $\mathsf{R}(T^n) \subset \mathsf{R}(T^k)$ to conclude that $\operatorname{codim} \overline{\mathsf{R}(T^k)} < +\infty$ and $\mathsf{R}(T^{*k})$ is closed. Consequently, T^k is right semi-Fredholm (see [27, Theorem 5.1]).
2) By duality, we obtain this assertion. Therefore, the proof is complete. □

3. Structure theorems of (k, m, n)-partial isometries

For $n, m \in \mathbb{N}$ and $k \in \mathbb{N}\backslash\{0\}$ such that $n + m \geq k$, we begin this section by a decomposition theorem for T^k when T is a (k, m, n)-partial isometry.

Theorem 3.1. *Let $n, m \in \mathbb{N}$ and $k \in \mathbb{N}\backslash\{0\}$ such that $n + m \geq k$.*

1) *Let T be a (k, m, n)-partial isometry. Assume that T is injective or $k \geq m$. Then there exist $\mathcal{P} \in \mathcal{B}(\mathcal{H})$, a partial isometry, and $\mathcal{N} \in \mathcal{B}(\mathcal{H})$, a nilpotent operator of degree n, such that*

$$T^k = \mathcal{P} + \mathcal{N},\ \mathcal{N}\mathcal{P}^* = \mathcal{N}\mathcal{P} = \mathcal{N}^*\mathcal{P} = 0.$$

2) *Let T be a (k, m, n)-partial isometry. Assume that the range of T is dense or $k \geq n$. Then there exist $\mathcal{P} \in \mathcal{B}(\mathcal{H})$, a partial isometry, and $\mathcal{N} \in \mathcal{B}(\mathcal{H})$, a nilpotent operator of degree m, such that*

$$T^k = \mathcal{P} + \mathcal{N},\ \mathcal{P}\mathcal{N} = \mathcal{N}^*\mathcal{P} = \mathcal{N}\mathcal{P}^* = 0.$$

Proof. 1) Consider the orthogonal projection $Q \in \mathcal{B}(\mathcal{H})$ onto $\mathsf{R}(T^{kn})^{\perp}$. Let $\mathcal{P} = T^{*k}(I - Q)$ and $\mathcal{N} = T^{*k}Q$. From Theorem 2.9, we have $T^k T^{*k}_{|\overline{\mathsf{R}(T^{kn})}} = I_{|\overline{\mathsf{R}(T^{kn})}}$. It follows that

$$\mathcal{N}^*\mathcal{P} = \mathcal{P}\mathcal{N} = \mathcal{N}\mathcal{P}^* = 0,$$

$$QT^{*k}Q = T^{*k}Q$$

and therefore

$$\mathcal{N}^n = T^{*kn}Q = 0.$$

Moreover, we have

$$\mathcal{P}\mathcal{P}^*\mathcal{P} = T^{*k}(I - Q)T^k T^{*k}(I - Q) = T^{*k}(I - Q) = \mathcal{P}.$$

Since $\mathcal{N}$ is a nilpotent operator of degree n (respectively, $\mathcal{P}$ is a partial isometry) if and only if $\mathcal{N}^*$ a nilpotent operator of degree n (respectively, $\mathcal{P}^*$ is a partial isometry), we obtain the first assertion.
2) The result follows from 1) and assertion 3) of Remark 2.3. This completes the proof. □

Following Apostol (see [3]), an operator T on a Hilbert space $\mathcal{H}$ is said to be quasi-commuting if

$$\lim_{n\to+\infty} \|T^*T^n - T^nT^*\|^{\frac{1}{n}} = 0.$$

If this condition holds, it is easy to see that T^* also satisfies the same condition. Furthermore, T satisfies an even weaker point-wise condition:

$$\lim_{n\to+\infty} \|(T^*T^n - T^nT^*)x\|^{\frac{1}{n}} = 0, \ \ \forall\, x \in \mathcal{H}.$$

Consider the set

$$\mathcal{H}_0 := \Big\{x \in \mathcal{H} : \lim_{n\to+\infty} \|T^n x\|^{\frac{1}{n}} = 0\Big\}.$$

It is straightforward to observe that $\mathcal{H}_0$ is a subspace of $\mathcal{H}$ and is hyperinvariant for T.

The subsequent property represents one of the key characteristics of $\mathcal{H}_0$.

Lemma 3.2 ([3, 13]). *Let $T \in \mathcal{B}(\mathcal{H})$ be a quasi-commuting operator. Then $\mathcal{H}_0$ is a reducing subspace of T and $T_{|\mathcal{H}_0{}^\perp}$ is normal.*

It is evident that every partial isometry for which its kernel is a reducing subspace can be written as a direct sum of an isometry and a zero operator. In the following, we extend this result to (k, m, n)-partial isometries.

Theorem 3.3. *Let $n, m \in \mathbb{N}$ and $k \in \mathbb{N}\backslash\{0\}$ such that $n + m \geq k \geq n$. Let $T \in \mathcal{B}(\mathcal{H})$ such that $\mathsf{N}(T^m)$ is a reducing subspace of T. If T is a (k, m, n)-partial isometry, then T is decomposed by $\mathsf{N}(T^m)^\perp$ and $\mathsf{N}(T^m)$ in the direct sum*

$$T = \mathcal{S} \oplus \mathcal{N},$$

where $\mathcal{S}^k$ is an isometry and $\mathcal{N}$ is nilpotent of degree m. If furthermore T is a quasi-commuting operator, then $\mathcal{S}^k$ is unitary.

Proof. It is clear that $\mathcal{N} = T_{|\mathsf{N}(T^m)}$ is a nilpotent operator of degree m, and if we consider $\mathcal{S} = T_{|\mathsf{N}(T^m)^\perp}$, then by Corollary 2.10, $\mathcal{S}^k$ is an isometry. Now, let us prove that $\mathcal{H}_0 = \mathsf{N}(T^m)$. Let $x \in \mathcal{H}_0$ and write $x = x_0 + x_1$ with $x_0 \in \mathsf{N}(T^m)$ and $x_1 \in \mathsf{N}(T^m)^\perp$. As $\mathcal{N}$ is nilpotent and $\mathcal{S}^{kq}$ is an isometry for all $q \in \mathbb{N}$, we have

$$\lim_{q\to+\infty} \|x_1\|^{\frac{1}{q}} = \lim_{q\to+\infty} \|\mathcal{S}^{kq}x_1\|^{\frac{1}{q}} = \lim_{q\to+\infty} (\|T^{kq}x\|^{\frac{1}{kq}})^k = 0.$$

Therefore $x_1 = 0$ and so $\mathcal{H}_0 = \mathsf{N}(T^m)$.

If T is also quasi-commuting, then Lemma 3.2 implies that $\mathcal{S}^k$ is unitary. Thus, the proof is complete. □

As is customary, the next corollary can be obtained by duality.

Corollary 3.4. *Let $n, m \in \mathbb{N}$ and $k \in \mathbb{N}\backslash\{0\}$ such that $n + m \geq k \geq m$. Let $T \in \mathcal{B}(\mathcal{H})$ such that $\mathsf{R}(T^n)$ is a reducing subspace of T. If T is a (k, m, n)-partial isometry, then T is decomposed by $\mathsf{R}(T^n)$ and $\mathsf{R}(T^n)^\perp$ in the direct sum*

$$T = \mathcal{S} \oplus \mathcal{N},$$

where $\mathcal{S}^k$ is a co-isometry and $\mathcal{N}$ is nilpotent of degree n. If more T is a quasi-commuting operator, then $\mathcal{S}^k$ is unitary.

Recall that an operator $T \in \mathcal{B}(\mathcal{H})$ is said to be semi-regular if

$$\mathsf{N}^\infty(T) \subset \mathsf{R}^\infty(T) \text{ and } \mathsf{R}(T) \text{ is closed},$$

where $\mathsf{N}^\infty(T) = \bigcup_{n\in\mathbb{N}} \mathsf{N}(T^n)$ and $\mathsf{R}^\infty(T) = \bigcap_{n\in\mathbb{N}} \mathsf{R}(T^n)$ are respectively the generalized kernel and generalized range of T. For more information on these concepts, we refer to the book [1]. In particular, the following statements are very useful.

Lemma 3.5. *Let $T \in \mathcal{B}(\mathcal{H})$ be a semi-regular operator. Then*

1) T^* *is also semi-regular.*

2) $\mathsf{R}^\infty(T)$ *is closed.*

3) $\mathsf{R}^\infty(T)^\perp = \overline{\mathsf{N}^\infty(T^*)}$, $T(\overline{\mathsf{N}^\infty(T)}) = \overline{\mathsf{N}^\infty(T)}$ *and* $T(\mathsf{R}^\infty(T)) = \mathsf{R}^\infty(T)$.

We will use the next lemma in the proof of Theorem 3.7.

Lemma 3.6. *Let* $T \in \mathcal{B}(\mathcal{H})$ *and* $n, m \in \mathbb{N}$ *and* $k \in \mathbb{N}\backslash\{0\}$ *such that* $n + m \geq k$.

1) *Assume that* $T \in {}^k\!\Delta_{m,n}$, *with* T *is surjective or* $k \geq n$. *If* T *is semi-regular, then* $\mathsf{R}^\infty(T)$ *and* $\overline{\mathsf{N}^\infty(T)}$ *are reducing subspaces for* T^k.

2) *Assume that* $T \in {}^k\!\Delta_{m,n}$, *with* T *is injective or* $k \geq m$. *If* T *is semi-regular, then* $\mathsf{R}^\infty(T)$ *and* $\overline{\mathsf{N}^\infty(T)}$ *are reducing subspaces for* T^k.

The proof of Lemma 3.6 -a delightful exercise- is left to the reader.

Theorem 3.7. *Let* $T \in \mathcal{B}(\mathcal{H})$ *be a semi-regular operator. Let* $n, m \in \mathbb{N}$ *and* $k \in \mathbb{N}\backslash\{0\}$ *such that* $n + m \geq k$.

1) *Let* T *be a* (k, m, n)*-partial isometry. Assume that* T *is surjective or* $k \geq n$. *Then* T^k *is decomposed following* $\mathcal{H} = \mathsf{N}^\infty(T)^\perp \oplus \overline{\mathsf{N}^\infty(T)}$ *(resp.* $\mathcal{H} = \mathsf{R}^\infty(T)^\perp \oplus \mathsf{R}^\infty(T)$*) on the direct sum*

$$T^k = \mathcal{S} \oplus \mathcal{V},$$

where $\mathcal{S}$ *is an isometry and* $\mathcal{V}$ *is surjective. Furthermore,*

$$T^k \textit{ is a partial isometry if and only if } \mathcal{V} \textit{ is a co-isometry.}$$

In this case, T^k *is a semi-regular power partial isometry.*

2) *Let* T *be a* (k, m, n)*-partial isometry. Assume that* T *is injective or* $k \geq m$. *Then* T^k *is decomposed following* $\mathcal{H} = \mathsf{R}^\infty(T) \oplus \mathsf{R}^\infty(T)^\perp$ *(resp.* $\mathcal{H} = \overline{\mathsf{N}^\infty(T)} \oplus \mathsf{N}^\infty(T)^\perp$*) on the direct sum*

$$T^k = \mathcal{S} \oplus \mathcal{V},$$

where $\mathcal{S}$ *is co-isometry and* $\mathcal{V}$ *is injective. Furthermore,*

$$T^k \textit{ is a partial isometry if and only if } \mathcal{V} \textit{ is an isometry.}$$

In this case, T^k *is a semi-regular power partial isometry.*

Proof. First note that as $\mathsf{R}^\infty(T)$ is closed, then we can write

$$\mathcal{H} = \mathsf{R}^\infty(T) \oplus \mathsf{R}^\infty(T)^\perp = \overline{\mathsf{N}^\infty(T)} \oplus \mathsf{N}^\infty(T)^\perp.$$

Consider $\mathsf{M} = \mathsf{R}^\infty(T)$ and $\mathsf{N} = \mathsf{R}^\infty(T)^\perp$ (resp. $\mathsf{M} = \overline{\mathsf{N}^\infty(T)}$ and $\mathsf{N} = \mathsf{N}^\infty(T)^\perp$).

1) From Lemma 3.6, M and N are reducing subspaces for T^k. Let $\mathcal{S} = T^k_{|\mathsf{N}}$ and $\mathcal{V} = T^k_{|\mathsf{M}}$. By Lemma 3.5, it follows that $\mathcal{V}$ is surjective. Also, since $\mathsf{N} \subset \mathsf{N}(T^m)^\perp$, according to Theorem 2.9, we have $\mathcal{S}^*\mathcal{S} = I$. Now if T^k is a partial isometry, then

$$\mathcal{S} \oplus \mathcal{V} = T^k = T^k T^{*k} T^k = \mathcal{S}\mathcal{S}^*\mathcal{S} \oplus \mathcal{V}\mathcal{V}^*\mathcal{V} = \mathcal{S} \oplus \mathcal{V}\mathcal{V}^*\mathcal{V}.$$

This implies that $\mathcal{V}$ is a partial isometry, and the surjectivity of $\mathcal{V}$ forces that $\mathcal{V}$ is a co-isometry. If $\mathcal{V}$ is a co-isometry, then

$$T^{kp}T^{*kp}T^{kp} = \mathcal{S}^p\mathcal{S}^{*p}\mathcal{S}^p \oplus \mathcal{V}^p\mathcal{V}^{*p}\mathcal{V}^p = \mathcal{S}^p \oplus \mathcal{V}^p = T^{kp}, \ \ \forall\, p \geq 1.$$

We claim that $\mathsf{R}(T^k)$ is closed because the range of any partial isometry is closed. Moreover,

$$\mathsf{N}^\infty(T^k) = \mathsf{N}^\infty(T) \subset \mathsf{R}^\infty(T) = \mathsf{R}^\infty(T^k).$$

Hence, T^k is semi-regular.

2) As is customary, we obtain this assertion by duality from 1) and assertion 3) of Remark 2.3. This completes the proof of the theorem. □

As mentioned in the introduction, familiar examples of partial isometries include isometries, co-isometries, and their direct sums. Taking $k = m = n = 1$ in the previous theorem is sufficient to derive the following corollary.

Corollary 3.8. *If $T \in \mathcal{B}(\mathcal{H})$ is a semi-regular operator, then T is a partial isometry if and only if $T = \mathcal{S} \oplus \mathcal{V}$ for some isometric operator $\mathcal{S}$ and co-isometric operator $\mathcal{V}$.*

Labrousse in [29] has introduced the class of quasi-Fredholm operators which contains many operators already studied in the litterature, such semi-Fredholm operators. An operator T is said to be quasi-Fredholm of degree d if the following conditions are satisfied:

(1) For all n greater than d, $\mathsf{R}\left(T^n\right) \cap \mathsf{N}(T) = \mathsf{R}\left(T^d\right) \cap \mathsf{N}(T)$;

(2) $\mathsf{N}(T) \cap \mathsf{R}\left(T^d\right)$ is closed in $\mathcal{H}$;

(3) $\mathsf{R}(T) + \mathsf{N}\left(T^d\right)$ is closed in $\mathcal{H}$.

Following Labrousse, an operator $T \in \mathcal{B}(\mathcal{H})$ is called paranormal if $\lim\limits_{n\to+\infty} \|T_n\|^{\frac{1}{n}} = 0$ where $(T_n)_{n\in\mathbb{N}}$ is defined as follows

$$T_0 = T^* \text{ and } T_{j+1} = i(TT_k - T_kT), \text{ for all } k \in \mathbb{N}.$$

The following theorem describes the structure of T^k when T is a (k, m, n)-partial isometry that is both paranormal and quasi-Fredholm.

Theorem 3.9. *Let $n, m \in \mathbb{N}, k \in \mathbb{N}\backslash\{0\}$ such that $n+m \geq k \geq n$ and $T \in {}^k\!\Delta_{m,n}$. If T is paranormal and quasi-Fredholm of degree d, then it can be decomposed as*

$$T^k = \mathcal{S} \oplus \mathcal{R} \oplus \mathcal{N},$$

where $\mathcal{S}$ is an isometry, $\mathcal{R}$ is surjective and $\mathcal{N}$ is nilpotent of degree d_k, with $kd_k \geq d$.

Proof. Referring to [29, Theorem 3.2.1, Proposition 5.4.4] we can write $T = \mathcal{V} \oplus \mathcal{A}$, for some semi-regular operator $\mathcal{V}$ and some nilpotent operator $\mathcal{A}$ of degree d. It is no difficult to see that $\mathcal{V}$ is a (k, m, n)-partial isometry. Thus, by Theorem 3.7, we infer that $\mathcal{V}^k = \mathcal{S} \oplus \mathcal{R}$, with $\mathcal{S}$ is an isometry and $\mathcal{R}$ is surjective. This completes the proof. □

Corollary 3.10. *Let $T \in \mathcal{B}(\mathcal{H})$ be a paranormal and quasi-Fredholm operator of degree d. If T is a partial isometry, then it can be decomposed as*

$$T = \mathcal{S} \oplus \mathcal{R} \oplus \mathcal{N},$$

where $\mathcal{S}$ is an isometry, $\mathcal{R}$ is a surjective operator, and $\mathcal{N}$ is nilpotent of degree d.

By combining [29, Proposition 3.3.5, Remark in page 245], Theorem 3.9 and Corollary 3.10, we can obtain the following corollaries by duality:

Corollary 3.11. *Let $n, m \in \mathbb{N}$, $k \in \mathbb{N}\backslash\{0\}$ such that $n + m \geq k \geq m$ and $T \in {}^{k}\Delta_{m,n}$. If T is paranormal and quasi-Fredholm of degree d, then*

$$T^k = \mathcal{S} \oplus \mathcal{R} \oplus \mathcal{N},$$

where $\mathcal{S}$ is a co-isometry, $\mathcal{R}$ is injective and $\mathcal{N}$ is nilpotent of degree d_k, with $kd_k \geq d$.

Corollary 3.12. *Let $T \in \mathcal{B}(\mathcal{H})$ be a paranormal and quasi-Fredholm operator of degree d. If T is a partial isometry, then it can be decomposed as*

$$T = \mathcal{S} \oplus \mathcal{R} \oplus \mathcal{N},$$

where $\mathcal{S}$ is a co-isometry, $\mathcal{R}$ is an injective operator and $\mathcal{N}$ is nilpotent of degree d.

4. Spectral properties of (k, m, n)-partial isometries

The spectrum $\sigma(T)$ of an operator $T \in \mathcal{B}(\mathcal{H})$ is the set of all scalars λ in $\mathbb{C}$ for which the operator $\lambda I - T$ fails to be an invertible element of the algebra $\mathcal{B}(\mathcal{H})$. And so,

$$\sigma(T) = \{\lambda \in \mathbb{C} : \mathsf{N}(\lambda I - T) \neq \{0\} \text{ or } \mathsf{R}(\lambda I - T) \neq \mathcal{H}\}.$$

The set $\sigma_p(T)$ of those λ for which the operator $\lambda I - T$ is not injective is the point spectrum of T,

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : \mathsf{N}(\lambda I - T) \neq \{0\}\}.$$

Thus the point spectrum of T is precisely the set of all eigenvalues of T. There is another overlapping part of the spectrum which is commonly used, namely the approximate point spectrum $\sigma_{ap}(T)$, which is defined by

$$\sigma_{ap}(T) = \Big\{\lambda \in \mathbb{C} : \ \exists\, (x_n)_{n\in\mathbb{N}} \in \mathcal{H},\ \|x_n\| = 1,\ (\lambda I - T)x_n \underset{n\to+\infty}{\longrightarrow} 0\Big\}.$$

A fundamental property of the spectrum of an operator on a complex Hilbert space is that it is a non-empty compact subset of $\mathbb{C}$. Additionally, the approximate point spectrum $\sigma_{ap}(T)$ is a non-empty closed subset of $\mathbb{C}$ that contains the boundary $\partial\sigma(T)$ of the spectrum $\sigma(T)$. Furthermore, we have $\overline{\sigma_p(T)} \subset \sigma_{ap}(T) \subset \sigma(T)$.

Of course if T is a nilpotent (k, m, n)-partial isometry such that $k \geq m$ or $k \geq n$, then

$$\sigma_{ap}(T) = \sigma_p(T) = \sigma(T) = \{0\}.$$

If $k \geq m$ or $k \geq n$, from Proposition 2.13 the norm of a non-nilpotent (k, m, n)-partial isometry is greater than or equal to 1. In the sequel, let T be a non-nilpotent (k, m, n)-partial isometry such that $k \geq m$ or $k \geq n$. We denote by

$$\Gamma_T = \Big\{\lambda \in \mathbb{C} : \ \frac{1}{\|T\|} \leq |\lambda| \leq \|T\|\Big\}.$$

Here are some useful Lemmas.

Lemma 4.1. *Let $T \in \mathcal{B}(\mathcal{H})$ be a quasi-commuting operator. Then for all $q \in \mathbb{N}$, T^q is also a quasi-commuting operator.*

Proof. We have the identity

$$T^{*2}T^{2n} - T^{2n}T^{*2} = T^*(T^*T^{2n} - T^{2n}T^*) + (T^*T^{2n} - T^{2n}T^*)T^*,$$

which implies that

$$\|T^{*2}T^{2n} - T^{2n}T^{*2}\| \leq 2\|T\|\|T^*T^{2n} - T^{2n}T^*\|.$$

Therefore, T^2 is quasi-commuting.

Similarly, we have

$$T^{*3}T^{3n} - T^{3n}T^{*3} = T^{*2}(T^*T^{3n} - T^{3n}T^*) + (T^*T^{3n} - T^{3n}T^*)T^{*2} + T^*(T^*T^{3n} - T^{3n}T^*)T^*,$$

which leads to

$$\|T^{*3}T^{3n} - T^{3n}T^{*3}\| \leq 3\|T\|^2\|T^*T^{3n} - T^{3n}T^*\|.$$

Hence, T^3 is quasi-commuting. More generally, we can see that

$$T^{*q}T^{qn} - T^{qn}T^{*q} = \sum_{k=0}^{q-1} T^{*k}(T^*T^{qn} - T^{qn}T^*)(T^*)^{(q-1-k)}.$$

Consequently, we have $\|T^{*q}T^{pn} - T^{qn}T^{*q}\| \leq q\|T\|^{q-1}\|T^*T^{qn} - T^{qn}T^*\|$. This proves that T^q is quasi-commuting. Therefore, the proof is complete. □

Lemma 4.2. *Let $n, m \in \mathbb{N}$, $k \in \mathbb{N}\backslash\{0\}$ such that $n + m \geq k \geq n$, and let $T \in \mathcal{B}(\mathcal{H})$ be a quasi-commuting operator. If $T \in {}^k\Delta_{m,n}$, then $\sigma_p(T) \subset \Gamma_T \cup \{0\}$.*

Proof. Let $\lambda \in \sigma_p(T)\backslash\{0\}$ and x be a non-zero vector such that $Tx = \lambda x$. Since $k \geq n$, we have $T^iT^{*k}T^k = T^i$, for all $i \geq m$, as shown in Theorem 2.9. Furthermore, since T is quasi-commuting, it follows from Lemma 4.1 that T^k is also quasi-commuting. This leads to

$$\begin{aligned} 0 = \lim_{\ell\to+\infty} \|T^{*k}T^{k\ell}x - T^{k\ell}T^{*k}x\|^{\frac{1}{\ell}} &= \lim_{\ell\to+\infty} \|\lambda^{k\ell}T^{*k}x - \lambda^{-k}T^{k\ell}T^{*k}T^kx\|^{\frac{1}{\ell}} \\ &= \lim_{\ell\to+\infty} \|\lambda^{k\ell}T^{*k}x - \lambda^{-k}T^{k\ell}x\|^{\frac{1}{\ell}} \\ &= |\lambda|^k \lim_{\ell\to+\infty} \|T^{*k}x - \lambda^{-k}x\|^{\frac{1}{\ell}}. \end{aligned}$$

We can infer that $T^{*k}x = \lambda^{-k}x$, thus $\lambda^{-k} \in \sigma(T^{*k})$. Hence, $|\lambda| \leq \|T\|$ and $|\lambda^{-k}| \leq \|T^{*k}\| \leq \|T^*\|^k = \|T\|^k$. This implies that $\lambda \in \Gamma_T$. Therefore, the proof is complete. □

Lemma 4.3. *Let $n, m \in \mathbb{N}$, $k \in \mathbb{N}\backslash\{0\}$ such that $n + m \geq k \geq n$, and let $T \in \mathcal{B}(\mathcal{H})$ be a quasi-commuting operator. If $T \in {}^k\Delta_{m,n}$, then $\sigma_{ap}(T) \subset \Gamma_T \cup \{0\}$.*

Proof. In [5], S. K. Berberian constructed a certain extension of a Hilbert space $\mathcal{H}$ to a Hilbert space $\mathcal{L}$, which reduces the problem of the approximate point spectrum of an operator T on $\mathcal{H}$ to the point spectrum problem of T' on $\mathcal{L}$. It is evident to see that the properties of quasi-commutativity and (k, m, n)-partial isometry are inherited by T'. Hence, Lemma 4.2 and [5, Theorem 1] imply that $\sigma_{ap}(T) = \sigma_p(T') \subset \Gamma'_T \cup \{0\} = \Gamma_T \cup \{0\}$. This completes the proof. □

The following result provides a new structure theorem for (k, m, n)-partial isometries.

Theorem 4.4. *Let $n, m \in \mathbb{N}$, $k \in \mathbb{N}\backslash\{0\}$ such that $n + m \geq k$ and either $k \geq n$ or $k \geq m$. Let $T \in \mathcal{B}(\mathcal{H})$ be a quasi-commuting operator. If $T \in {}^k\Delta_{m,n}$, then $T = U \oplus N$, where U is an operator such that U^k is unitary and N is quasinilpotent.*

Proof. Without loss of generality we may assume that $k \geq n$. Let $N = T_{|\overline{\mathcal{H}_0}}$ and $U = T_{|\mathcal{H}_0^\perp}$.

Since $\mathcal{H}_0$ is a reducing subspace for T, the properties of of quasi-commutativity and (k, m, n)-partial isometry are inherited by N. By applying [13, Theorem 6] and Lemma 4.3, we obtain that $\sigma(N) \subset \Gamma_N \cup \{0\}$. Therefore, 0 is an isolated point of the spectrum, and using [13, Theorem 2], we conclude that N is quasinilpotent.

On the other hand, Lemma 3.2 states that know U is normal. Moreover, since $\mathcal{H}_0^\perp \subset \mathsf{N}(T^p)^\perp$ for all $p \in \mathbb{N}$, it follows from Theorem 2.9 that U^k is unitary. □

What might the spectrum of a partial isometry be? Since a partial isometry is a contraction, its spectrum is included in the closed unit disc. If we moreover impose to a partial isometry to be quasi-commuting, then its approximate point spectrum (and so its point spectrum) is a subset of union of the unit circle and the singleton $\{0\}$ (see [13, Theorems 3 and 4]). The goal of the following theorem is to demonstrate that this outcome stays substantial for (k, m, n)-partial isometries.

Theorem 4.5. *Let $n, m \in \mathbb{N}$, $k \in \mathbb{N}\backslash\{0\}$ such that $n + m \geq k$ and either $k \geq n$ or $k \geq m$. Let $T \in \mathcal{B}(\mathcal{H})$ be a quasi-commuting operator. If $T \in {}^k\Delta_{m,n}$, then*

1) $\sigma(T) \subset \overline{\mathbb{D}}$,

2) $\sigma_{ap}(T) \subset \partial\mathbb{D} \cup \{0\}$. *In particular,* $\overline{\sigma_p(T)} \subset \partial\mathbb{D} \cup \{0\}$.

Proof. By Theorem 4.4, we can express T as $T = U \oplus N$, where N is quasinilpotent and U^k is unitary. Since $U^k \in C$ and $\sigma(N^k) = \{0\}$, we have

$$\{\lambda^k : \ \lambda \in \sigma(T)\} = \sigma(T^k) = \sigma(U^k) \cup \sigma(N^k) \subset \overline{\mathbb{D}}.$$

This implies that

$$\sigma(T) \subset \overline{\mathbb{D}}.$$

To prove the second assertion, observe that:

$$\sigma_{ap}(T^k) = \sigma_{ap}(U^k) \cup \sigma_{ap}(N^k).$$

Since $\|T^*T^n - T^nT^*\| \geq \|U^*U^n - U^nU^*\|$, it follows that U is quasi-commuting, and therefore U^k is as well. Referring to [13, Theorems 3 and 4], we conclude that

$$\sigma_{ap}(U^k) \subset \partial\mathbb{D}.$$

As $\sigma_{ap}(N^k) = \{0\}$, we have:

$$\sigma_{ap}(T^k) \subset \partial\mathbb{D} \cup \{0\}.$$

This implies that $\sigma_{ap}(T) \subset \partial\mathbb{D} \cup \{0\}$, which completes the proof. □

Here we present an improved version of the previous result under a new condition.

Theorem 4.6. *Let $n, m \in \mathbb{N}$, $k \in \mathbb{N}\backslash\{0\}$ such that $n + m \geq k$. Let $T \in \mathcal{B}(\mathcal{H})$ such that* $\mathsf{N}(T^m) \perp \mathsf{R}(T^n)$.

1) *Assume that $T \in {}^k\Delta_{m,n}$, with $k \geq n$. Then*

   a) $\sigma_{ap}(T) \subset \partial\mathbb{D} \cup \{0\}$. *In particular,* $\overline{\sigma_p(T)} \subset \partial\mathbb{D} \cup \{0\}$ *and* $\sigma(T) \subset \overline{\mathbb{D}}$.

   b) *If $\alpha \in \sigma_p(T^k)$, then $\bar{\alpha} \in \sigma_p\left(T^{*k}\right)$.*

   c) *If $\alpha \in \sigma_{ap}(T^k)$, then $\bar{\alpha} \in \sigma_{ap}\left(T^{*k}\right)$.*

   d) *The eigenspaces corresponding to distinct non-zero eigenvalues of T^k are orthogonal.*

2) *Assume that $T \in {}^k\Delta_{m,n}$ with $k \geq m$. Then*

a) $\sigma_{ap}(T^*) \subset \partial\mathbb{D} \cup \{0\}$. *In particular,* $\overline{\sigma_p(T^*)} \subset \partial\mathbb{D} \cup \{0\}$ *and* $\sigma(T^*) \subset \overline{\mathbb{D}}$.

b) *If* $\alpha \in \sigma_p(T^{*k})$, *then* $\bar{\alpha} \in \sigma_p\left(T^k\right)$.

c) *If* $\alpha \in \sigma_{ap}(T^{*k})$, *then* $\bar{\alpha} \in \sigma_{ap}\left(T^k\right)$.

d) *The eigenspaces corresponding to distinct non-zero eigenvalues of* T^{*k} *are orthogonal.*

Proof. 1) First, note that if $T \in {}^k\Delta_{m,n}$ with $k \geq n$, then by Theorem 2.9 we have $T^mT^{*k}T^k = T^m$. So, $\mathsf{R}(T^{*k}T^k - I) \subset \mathsf{N}(T^m) \subset \mathsf{R}(T^n)^\perp \subset \mathsf{R}(T^k)^\perp$. Thus, $T^{*k}T^{2k} = T^k$.
a) Let $\alpha \in \sigma_{ap}(T)\backslash\{0\}$. Choose a sequence $(x_n)_{n\in\mathbb{N}}$ of unit vectors such that $(T-\alpha I)x_n \xrightarrow[n\to+\infty]{} 0$. Then, $(T^k - \alpha^k I)x_n \xrightarrow[n\to+\infty]{} 0$. And so,

$$\begin{aligned} 0 &= \left\langle T^{*k}T^{2k}(x_n), x_n\right\rangle - \langle T^k(x_n), x_n\rangle \\ &= \langle T^{*k}T^k(T^k-\alpha^kI)(x_n), x_n\rangle + \alpha^k\,\|Tx_n\|^2 - \langle (T^k-\alpha^kI)x_n, x_n\rangle - \alpha^k. \end{aligned}$$

As $(T^k - \alpha^k I)x_n \to 0$ and $(T-\alpha I)x_n \to 0$, we obtain $\alpha^k(\|Tx_n\|-1) \xrightarrow[n\to+\infty]{} 0$. Hence, $\|Tx_n\| \xrightarrow[n\to+\infty]{} 1$. Since $(T-\alpha I)x_n \xrightarrow[n\to+\infty]{} 0$, $|\alpha| = 1$.
b) Let $\alpha \in \sigma_p(T^k)$. Suppose first that $\alpha = 0$. If $0 \in \mathbb{C}\backslash\sigma_p\left(T^{*k}\right)$, then from $T^{*2k}T^k = T^{*k}$ we get that T^k is an isometry. But this contradicts the fact that $0 \in \sigma_p(T^k)$. If α is non-zero, choose a non-zero vector x such that $T^kx = \alpha x$. Since $T^{*2k}T^k = T^{*k}$, $T^{*k}x = \alpha T^{*2k}x$. If $T^{*k}x = 0$, then $0 = \left\langle x, T^{*k}x\right\rangle = \langle T^kx, x\rangle = \alpha\langle x, x\rangle$, which leads to a contraction. Hence, $\frac{1}{\alpha} = \bar{\alpha} \in \sigma_p\left(T^{*k}\right)$.
c) is proven similarly.
d) Let α and β be distinct nonzero eigenvalues of T^k. If $T^kx = \alpha x$ and $T^ky = \beta y$ then $0 = \left\langle T^{2k}x, T^{2k}y\right\rangle - \langle T^kx, T^ky\rangle = \alpha\bar{\beta}(\alpha\bar{\beta}-1)\langle x, y\rangle$. Since $\alpha \neq 0, \beta \neq 0, |\beta| = 1$ and $\alpha \neq \beta$, it follows that $\langle x, y\rangle = 0$.

2) As is customary, the second assertion follows by duality. This completes the proof of the theorem. □

We end up with an elegant result on spectral properties of some partial isometries.

Corollary 4.7. *Let* $T \in \mathcal{B}(\mathcal{H})$ *be a partial isometry such that* $\mathsf{N}(T) \perp \mathsf{R}(T)$. *Then*

1) $\sigma_{ap}(T) \subset \partial\mathbb{D} \cup \{0\}$. *In particular,* $\overline{\sigma_p(T)} \subset \partial\mathbb{D} \cup \{0\}$ *and* $\sigma(T) \subset \overline{\mathbb{D}}$.

2) *If* $\alpha \in \sigma_p(T)$, *then* $\bar{\alpha} \in \sigma_p(T^*)$.

3) *If* $\alpha \in \sigma_{ap}(T)$, *then* $\bar{\alpha} \in \sigma_{ap}(T^*)$.

4) *The eigenspaces corresponding to distinct non-zero eigenvalues of* T *are orthogonal.*

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