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The projectively Hurewicz property is t-invariant

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Abstract. A space X is *projectively Hurewicz* provided every separable metrizable continuous image of X is Hurewicz.

In this paper we prove that the projectively Hurewicz property is t-invariant, i.e., if $C_p(X)$ is homeomorphic to $C_p(Y)$ and X is projectively Hurewicz, then Y is projectively Hurewicz, too.

1. Introduction

Let $\mathcal{P}$ be a topological property. A.V. Arhangel'skii calls X *projectively* $\mathcal{P}$ if every second countable continuous image of X is $\mathcal{P}$ [1, 3]. The projective selection principles were introduced and first time considered in [5]. Lj.D.R. Kočinac characterized the classical covering properties of Menger, Rothberger, Hurewicz and Gerlits-Nagy in term of continuous images in $\mathbb{R}^\omega$. Characterizations of the classical covering properties in terms a selection principle restricted to countable covers by cozero sets are given in [4]. In [8, 9] we obtained the functional characterizations of all projective versions of the selection properties in the Scheepers Diagram.

Let us recall that a topological space is *Hurewicz* if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X, there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every n, $\mathcal{V}_n$ is a finite subfamily of $\mathcal{U}_n$ and every point of X is contained in $\bigcup \mathcal{V}_n$ for all but finitely many n's.

Recall that if $C_p(X)$ and $C_p(Y)$ are homeomorphic (linearly homeomorphic, uniform homeomorphic), we say that the spaces X and Y are *t-equivalent (l-equivalent, u-equivalent)*. The properties preserved by t-equivalence (l-equivalence, u-equivalence) we call *t-invariant (l-invariant, u-invariant)* [2].

The following interesting results were obtained:

- (Lj.D.R. Kočinac) A space is Hurewicz if and only if it is Lindelöf and projectively Hurewicz [5].
- (L. Zdomskyy) The Hurewicz property is l-invariant (Corollary 7 in [12]).
- (N.V. Velichko) The Lindelöf property is l-invariant [11].
- (M. Krupski) The projectively Hurewicz property is l-invariant (Theorem 1.5 in [7]).

In this paper we prove that the projectively Hurewicz property is t-invariant.

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2. Main definitions and notation

Throughout this paper, all spaces are assumed to be Tychonoff. The set of positive integers is denoted by $\mathbb{N}$. Let $\mathbb{R}$ be the real line, we put $\mathbb{I} = [0, 1] \subset \mathbb{R}$, and let $\mathbb{Q}$ be the rational numbers. For a space X, we denote by $C_p(X)$ the space of all real-valued continuous functions on X with the topology of pointwise convergence. The symbol $\mathbf{0}$ stands for the constant function to 0. Since $C_p(X)$ is a homogenous space we may always consider the point $\mathbf{0}$ when studying local properties of this space.

A basic open neighborhood of $\mathbf{0}$ is of the form $[F, (-\epsilon, \epsilon)] = \{f \in C(X) : f(F) \subset (-\epsilon, \epsilon)\}$, where F is a finite subset of X and $\epsilon > 0$.

We recall that a subset of X that is the complete preimage of zero for a certain function from $C(X)$ is called a zero-set. A subset $O \subseteq X$ is called a cozero-set (or functionally open) of X if $X \setminus O$ is a zero-set.

Many topological properties are characterized in terms of the following classical selection principles. Let $\mathcal{A}$ and $\mathcal{B}$ be sets consisting of families of subsets of an infinite set X. Then:

$S_{fin}(\mathcal{A}, \mathcal{B})$ is the selection hypothesis: for each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ of finite sets such that for each n, $B_n \subseteq A_n$, and $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.

$U_{fin}(\mathcal{A}, \mathcal{B})$ is the selection hypothesis: whenever $\mathcal{U}_1, \mathcal{U}_2, ... \in \mathcal{A}$ and none contains a finite subcover, there are finite sets $\mathcal{F}_n \subseteq \mathcal{U}_n$, $n \in \mathbb{N}$, such that $\{\bigcup \mathcal{F}_n : n \in \mathbb{N}\} \in \mathcal{B}$.

In this paper, by a cover we mean a nontrivial one, that is, $\mathcal{U}$ is a cover of X if $X = \bigcup \mathcal{U}$ and $X \notin \mathcal{U}$.

An open cover $\mathcal{U}$ of a space X is:

- an *ω-cover* if every finite subset of X is contained in a member of $\mathcal{U}$.
- a *γ-cover* if it is infinite and each $x \in X$ belongs to all but finitely many elements of $\mathcal{U}$.
- *γ_F-shrinkable* if $\mathcal{U}$ is a cozero γ-cover and there exists a γ-cover $\{F_U : U \in \mathcal{U}\}$ of zero-sets of X with $F_U \subset U$ for every $U \in \mathcal{U}$.
- *ω-groupable* if there is a partition of the cover into finite parts such that for each finite set $F \subseteq X$ and all but finitely many parts $\mathcal{P}$ of the partition, there is a set $U \in \mathcal{P}$ with $F \subseteq U$ [6].

For a topological space X we denote:

- $\mathcal{O}$ — the family of all open covers of X;
- $\mathcal{O}^{\omega}_{cz}$ — the family of all countable cozero covers of the space X;
- Γ — the family of all open γ-covers of the space X;
- Γ_{cz} — the family of all cozero γ-covers of the space X;
- Ω^{gr} — the family of open ω-groupable covers of the space X;
- Γ_F — the family of all cozero γ_F-shrinkable covers of the space X.

Since any infinite part of the γ-cover is also a γ-cover, we further assume that all γ_F-shrinkable covers are countable.

Let us recall that a topological space X is *Hurewicz* if X has the property $U_{fin}(\mathcal{O}, \Gamma)$.

3. The projectively Hurewicz property

A space X is *projectively Hurewicz* provided every separable metrizable continuous image of X is Hurewicz.

In ([4], Theorem 30), M. Bonanzinga, F. Cammaroto, M. Matveev proved

Theorem 3.1. *The following conditions are equivalent for a space X:*

1. *X is projectively $U_{fin}(\mathcal{O}, \Gamma)$ [projectivelyHurewicz];*
2. *Every Lindelöf continuous image of X is Hurewicz;*
3. *for every continuous mapping $f : X \mapsto \mathbb{R}^{\omega}$, $f(X)$ is Hurewicz;*
4. *for every continuous mapping $f : X \mapsto \mathbb{R}^{\omega}$, $f(X)$ is bounded;*
5. *X satisfies $U_{fin}(\mathcal{O}^{\omega}_{cz}, \Gamma)$.*

Proposition 3.2. (Proposition 31 in [4])

1. *Every σ-pseudocompact space is projectively Hurewicz.*
2. *Every space of cardinality less than $\mathfrak{b}$ is projectively Hurewicz.*
3. *The projectively Hurewicz property is preserved by continuous images, by countably unions, by C^*-embedded zero-sets, and by cozero sets.*

Definition 3.3. Let $\mathcal{S} = \{S_n : n \in \mathbb{N}\}$ be a family of subsets of a space X and $x \in X$. Then $\mathcal{S}$ *weakly converges* to x if for every neighborhood W of x there is a sequence $(s_n : n \in \mathbb{N})$ such that $s_n \in S_n$ for each $n \in \mathbb{N}$ and there is n' such that $s_n \in W$ for each $n > n'$.

Let us recall that a subset A of X *converges* to x if A is infinite, $x \notin A$, and for each neighborhood U of x, $A \setminus U$ is finite. We write $x = \lim A$ if A converges to x. Consider the following collections:

• $\Gamma_x = \{A \subseteq X : x = \lim A\}$;

• $w\Gamma_x$ = the family of all subsets of X admitting a partition $\mathcal{S} = \{S_n : n \in \mathbb{N}\}$ such that for every n the set S_n is finite and $\mathcal{S}$ weakly converges to x.

Theorem 3.4. *The following conditions are equivalent for a space X:*

1. $C_p(X)$ *satisfies* $S_{fin}(\Gamma_{\mathbf{0}}, w\Gamma_{\mathbf{0}})$;
2. X *satisfies* $S_{fin}(\Gamma_F, \Omega^{gr})$;
3. X *satisfies* $U_{fin}(\Gamma_F, \Gamma)$;
4. X *satisfies* $U_{fin}(\mathcal{O}^{\omega}_{cz}, \Gamma)$;
5. X *is projectively Hurewicz.*

Proof. (3) ⇔ (2). By Theorem 3.4 in [10], the equality $U_{fin}(\mathcal{O}, \Gamma) = S_{fin}(\Gamma, \Omega^{gr})$ is true in the class of metric separable spaces. Let X satisfies $U_{fin}(\Gamma_F, \Gamma)$. By Theorem 5.4 in [9] and Theorem 3.1, $U_{fin}(\Gamma_F, \Gamma) = U_{fin}(\mathcal{O}^{\omega}_{cz}, \Gamma)$, i.e., X is projectively Hurewicz.

Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of countable γ_F-shrinkable covers of X. For every $n \in \mathbb{N}$ and every $U \in \mathcal{U}_n$ fix a continuous function $f_U : X \to \mathbb{R}$ such that $U = f_U^{-1}[\mathbb{R} \setminus \{0\}]$. Put $h = \prod\{f_U : U \in \mathcal{U}_n, n \in \mathbb{N}\}$. Then h is a continuous mapping from X onto $h(X) \subseteq \mathbb{R}^{\omega}$, thus $h(X)$ satisfies $U_{fin}(\mathcal{O}, \Gamma) = S_{fin}(\Gamma, \Omega^{gr})$. Let $h(\mathcal{U}_n) = \{h(U) : U \in \mathcal{U}_n\}$. Since $(h(\mathcal{U}_n) : n \in \mathbb{N})$ be a sequence of γ-covers of $h(X)$ we get (2). Since a continuous metrizable image of a space satisfying the property $S_{fin}(\Gamma_F, \Omega^{gr})$ is a space with this property and $S_{fin}(\Gamma_F, \Omega^{gr}) = S_{fin}(\Gamma, \Omega^{gr})$ for metrizable spaces, the implication (2) ⇒ (3) is proved similarly.

(4) ⇔ (5). By Theorem 3.1.

(5) ⇒ (3). By Theorem 5.4 in [9] (or Theorem 4.1 in [8]).

(3) ⇒ (4). Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of countable cozero covers of X. Enumerate $\mathcal{U}_n = \{U^n_m : m \in \mathbb{N}\}$.

For $n, m \in \mathbb{N}$, fix a continuous function $f_{n,m} : X \to [0, 1]$ that witnesses U^n_m being cozero, i.e. $f^{-1}(0, 1] = U^n_m$. For every $n, m, i \in \mathbb{N}$, let us define

$W^n_{m,i} = f^{-1}_{n,m}(\frac{1}{i+1}, 1]$ and $H^n_{m,i} = f^{-1}_{n,m}[\frac{1}{i+1}, 1]$.

Clearly, the set $W^n_{m,i}$ is cozero and $H^n_{m,i}$ is a zero-set. Note that

$W^n_{m,i} \subseteq H^n_{m,i} \subseteq W^n_{m,i+1} \subseteq U^n_m$ and $U^n_m = \bigcup\limits_{i=1}^{\infty} W^n_{m,i}$.

For $k \in \mathbb{N}$, write $W^n_k = \bigcup\{W^n_{m,i+1} : i, m \leq k\}$ and let $\mathcal{W}_n = \{W^n_1, W^n_2, ...\}$. Observe that $\mathcal{W}_n \in \Gamma_F$ because $H^n_k = \bigcup\{H^n_{m,i} : i, m \leq k\}$ is a zero-set contained in W^n_k. Moreover the family $\{H^n_k : k \in \mathbb{N}\}$ is a γ-cover of X since one readily checks that the family $\{\bigcup\{W^n_{m,i} : i, m \leq k\} : k \in \mathbb{N}\}$ is a γ-cover and $\bigcup\{W^n_{m,i} : i, m \leq k\} \subseteq H^n_k$. Now apply the property $U_{fin}(\Gamma_F, \Gamma)$ to the sequence $(\mathcal{W}_n : n \in \mathbb{N})$ together with the fact that $\mathcal{W}_n$ is a finer cover that $\mathcal{U}_n$ for all n.

(1) ⇒ (2). Let $\{\mathcal{V}_i : i \in \mathbb{N}\} \in [\Gamma_F]^{\omega}$. Note that we assume that all γ_F-shrinkable covers are countable.

Since $\mathcal{V}_i = \{V_{i,j} : j \in \mathbb{N}\} \in \Gamma_F$, there is $\{F_{i,j} : j \in \mathbb{N}\} \in \Gamma$ such that $F_{i,j}$ is a zero-set in X and $F_{i,j} \subset V_{i,j} \in \mathcal{V}_i$ for each $j \in \mathbb{N}$. Let $T_i = \{f_{i,j} \in C_p(X) : f_{i,j}(F_{i,j}) = 0$ and $f_{i,j}(X \setminus V_{i,j}) = 1$ for each $i, j \in \mathbb{N}\}$. Since $\{F_{i,j} : j \in \mathbb{N}\}$ is a γ-cover, we have $\lim\limits_{j\to\infty} T_i = \mathbf{0}$ for each $i \in \mathbb{N}$. By (1), there are finite subsets T'_i of T_i and a partition of the set $\bigcup T'_i$ into finite parts such that for each neighborhood $O = [K, (-\epsilon, \epsilon)]$ of the function $\mathbf{0}$ where K is a finite subset of X and $\epsilon > 0$, and all but finitely many parts $\mathcal{P}$ of the partition, there is a function $g \in \mathcal{P}$ with $g \in O$.

Let $\mathcal{P} = \{\{g_{l,1}, ..., g_{l,k_l}\} : l \in \mathbb{N}\}$. Since $g_{l,m} = f_{i_s,j_s}$ for some $i_s, j_s \in \mathbb{N}$, we can consider $Q = \{V_{l,m} : V_{l,m} = V_{i_s,j_s}, f_{i_s,j_s}(X \setminus V_{i_s,j_s}) = 1, f_{i_s,j_s} = g_{l,m}, l \in \mathbb{N}\}$. Then Q has a partition $\mathcal{Q} = \{\{V_{l,1}, ..., V_{l,k_l}\} : l \in \mathbb{N}\}$ and, for any finite subset K of X all but finitely many parts $\mathcal{Q}$ of the partition, there is $V_{l,k}$ with $K \subseteq V_{l,k}$. Thus, $Q \in \Omega^{gr}$.

(2) $\Rightarrow$ (1). Let $T_i \in \Gamma_{\mathbf{0}}$ for each $i \in \mathbb{N}$. By passing to a countable infinite subset, we can without loss of generality assume that each T_i is countable. Enumerate $T_i = \{f_{i,j} \in C_p(X) : j \in \mathbb{N}\}$.

For i, j define $V_{i,j} = f_{i,j}^{-1}((-\frac{1}{i}, \frac{1}{i}))$ (we can without loss of generality assume that each $V_{i,j}$ is non-empty), and let $\mathcal{V}_i = \{V_{i,j} : j \in \mathbb{N}\}$.

Note that $V_{i,j}$ is a cozero-set in X for each $i, j \in \mathbb{N}$.

Thus we have a mapping $\Phi : \bigcup \mathcal{V}_i \to \bigcup T_i$ such that $\Phi(V_{i,j}) = f_{i,j}$ for $i, j \in \mathbb{N}$.

Since $\lim\limits_{j\to\infty} T_i = \mathbf{0}$, for any finite subset F of X and $\epsilon > 0$ (we can assume that $\epsilon < \frac{1}{i}$), there is $j' \in \mathbb{N}$ such that $f_{i,j} \in [F, (-\epsilon, \epsilon)]$ for each $j > j'$. Thus, $F \subset V_{i,j}$ for each $j > j'$. Thus, $\mathcal{V}_i \in \Gamma_{cz}$.

For i, j define $F_{i,j} = f_{i,j}^{-1}([-\frac{1}{i+1}, \frac{1}{i+1}])$, and let $\mathcal{F}_i = \{F_{i,j} : j \in \mathbb{N}\}$.

Then $F_{i,j} \subset V_{i,j}$ for each $j \in \mathbb{N}$ and $\mathcal{F}_i \in \Gamma$. Note also that $F_{i,j}$ is a zero-set and $V_{i,j}$ is a cozero-set in X for each $j \in \mathbb{N}$. It follows that $\mathcal{V}_i \in \Gamma_F$.

By (2), there are finite subsets $D_i \subset \mathcal{V}_i$ for each $i \in \mathbb{N}$ such that $\bigcup D_i$ is a cozero ω-groupable cover of the space X.

Let $\mathcal{P} = \{\mathcal{P}_k : k \in \mathbb{N}\}$ be a partition of the cover $\bigcup D_i$ into finite parts such that for each finite set $F \subset X$ and all but finitely many parts $\{\mathcal{P}_k : k \in \mathbb{N}\}$ of the partition, there is a set $V_{i(k),j(k)} \in \mathcal{P}_k$ with $F \subset V_{i(k),j(k)}$.

For each k define $S_k = \{f_V : \Phi(V) = f_V, V \in \mathcal{P}_k\}$. The family $\mathcal{S} = \{S_k : k \in \mathbb{N}\}$ is a partition of $\bigcup\{f_{i,j} : V_{i,j} \in D_i, i \in \mathbb{N}\}$. Then, for each finite set $F \subset X$ and $\epsilon > 0$, and all but finitely many parts of the partition $\mathcal{S}$, there is a function $f_{i(k),j(k)} \in \mathcal{S}_k$ with $f_{i(k),j(k)} \in [F, (-\epsilon, \epsilon)]$. Thus, $\bigcup\{f_{i,j} : f_{i,j} \in T_i, V_{i,j} \in D_i, i \in \mathbb{N}\} \in w\Gamma_{\mathbf{0}}$ and $C_p(X)$ satisfies $S_{fin}(\Gamma_{\mathbf{0}}, w\Gamma_{\mathbf{0}})$. $\square$

Note that the property $S_{fin}(\Gamma_x, w\Gamma_x)$ is a *topological* property. Thus, if $C_p(X)$ is homeomorphic to $C_p(Y)$ and $C_p(X)$ satisfies $S_{fin}(\Gamma_{\mathbf{0}}, w\Gamma_{\mathbf{0}})$, then $C_p(Y)$ satisfies $S_{fin}(\Gamma_g, w\Gamma_g)$ for each $g \in C_p(Y)$.

Theorem 3.5. *Suppose that $C_p(X)$ and $C_p(Y)$ are homeomorphic. Then X has the projectively Hurewicz property if and only if Y has the projectively Hurewicz property.*

Problem 3.6. *Let $\mathcal{P} \in \{Menger, Rothberger, Scheepers, S_1(\Gamma, \mathcal{O})\}$. Will the projectively $\mathcal{P}$ property be t-invariant?*

Conjecture 3.7. *The projectively Scheepers Diagram is t-invariant, i.e., each projectively selection property in the Scheepers Diagram is t-invariant.*

If the conjecture is true, then, applying Velichko's result, the Scheepers Diagram is l-invariant.

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