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Novel types of soft compact and connected spaces inspired by soft Q-sets

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Abstract. In this work, we make use of soft Q-sets to introduce the concepts of soft Q-compact, soft Q-Lindelöf and soft Q-connected spaces. We explore the essential properties of these concepts and elucidate the relationships between them with the assist of examples and counterexamples. We also give each one of these concepts a complete description and investigate how they behave under specific kinds of soft mappings. Moreover, we demonstrate the unique characterizations of these concepts which are not satisfied for their counterpart notions existing in the published literature; for example, we prove that every soft Q-subset of soft Q-compact and soft Q-Lindelöf spaces is respectively soft Q-compact and soft Q-Lindelöf as well as we discover the conditions under which the concepts of soft connected and soft Q-connected spaces are equivalent. The role of extended and full soft topologies to obtain some relationships between these concepts and their counterparts via parametric topologies is also discussed.

1. Introduction

Relying on traditional mathematical tools, we find a lot of difficulty in uncertainty problems nowadays. Soft sets and fuzzy sets have been introduced to solve these types of problems and make it easier to model these topics into math problems. The notion of soft sets was first presented by Molodtsov [25] in 1999. He defined them as a mathematical approach to handle uncertainty problems and clarify vague situations. Molodtsov presented in this work many applications in different fields which draws attention to the strengths of soft sets compared to fuzzy sets and probability theory. The definition given by Molodtsov of soft sets has opened the doors to many other mathematicians who are interested in the problem of uncertainties. In their paper [24, 25] Maji et al. studied decision-making problems and tried to present an initial conception of the operations that can be defined between soft sets. These operations have facilitated the task of several researchers in their studies, which makes it possible to define yet other operations suitable for particular cases and specific objectives [6].

The application of soft set theory in topology started with Shabir and Naz in 2011 in their paper [31] where they introduced the definition of soft topological spaces. Their definition is based on a non-empty set of elements (universe) and a fixed set of parameters. After the introduction of this definition of soft topological spaces, several other contributions have been developed respecting the properties of soft sets

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but the basic reference for the definition of soft topology remains that of Shabir and Naz [31]. Studies in soft topological spaces were frequent and in various directions such as Min's paper [26] in which he began to study soft separation axioms where he demonstrated that all soft T_3-spaces are soft T_2-spaces. Again, membership relations and their problems were introduced to studies for giving the definition of new types of separation axioms [19]. Separation axioms still remain an important field of study and we can daily see the development of new works which are interested in this subject such as the work in [32, 33]. Alcantud [2] presented a new way of building soft topologies from bases for topologies. He [3] also investigated the relationship between soft topology and classical topology.

Another direction in the studies of soft topological spaces is the subject of compactness of these spaces which began with Aygünoğlu and Aygün [17]. The diversity in the presentation of the notion of belonging in soft set theory helps Hida [21] to introduce another kind of compactness of soft topological spaces. We can also find some properties of classical topological spaces which become invalid if we try to generalize them to soft topological spaces such as the property says that "compact subsets of Hausdorff spaces are closed"; such these missing properties were discussed in detail in [19]. There are other studies that focus on the problem of compactness such as the paper of Al-Jarrah et al. [7] where they studied soft compact spaces and Lindelöf spaces via soft regular closed sets. Al-shami and Kočinac [13, 14, 23] explored covering properties in soft structures and elucidated their relationships with their classical counterparts. Looking now at the concept of connectedness, this concept was generalized in soft topological spaces in [30, 34]. Also, in the paper [16] of Asaad, we find some characteristics of soft extremally disconnected topological spaces. The idea of maximal soft connected topology has been recently introduced by Al-Ghour and Ameen [5].

The definition of soft mappings given by Kharl and Ahmad [22] uses two crisp mappings, the first between universal sets and the second between sets of parameters. This definition was reformulated by Al-shami [9] using the notion of soft points which helps to simplify the calculation difficulties encountered when using the initial definition of soft mapping. We can still find the notion of continuity of soft mappings and its characterizations in [35, 36]. Other concepts of classical topology and their relationship with soft topology were the goal of a work presented by Al-shami and Kočinac [12].

Given that the definition of topologies is based on the concept of open subsets, we find in the literature several generalizations of the notion of soft open sets in soft topology. These definitions lead to the development of other studies related to separation axioms and the problem of compactness in soft topological spaces. This matter began by Akdag and Ozkan [1] and Chen [18] who established the ideas of soft α-open and soft semi-open sets, respectively. They probed the essential properties of these kind of soft set and applied to define some soft continuity functions. Among the other generalizations of the notion of soft open sets we find soft γ-open sets given in [15] and soft Q-sets presented by Al-Ghour [4]. Al-Ghour gave further applications of soft Q-sets in the Boolean Algebra which is very useful in probability theory and information theory. Another generalization of the notion of open subsets is the notion of somewhat open sets which is used by Al-shami [10] to introduce other types of separation axioms and compactness in soft topological spaces. Al-shami et al. [11] discussed some types of soft mappings with respect to soft somewhere dense sets.

In this paper we continue precedent works about soft Q-sets. This article contains 4 sections in addition to this introduction which organize it as follows. In Section 2, we present some results and definitions needed in our work. Section 3 is devoted to defining the notion of soft Q-compact and soft Q-Lindelöf spaces elucidate the relationships among them. In Section 4, we introduce the concept of soft Q-connected spaces and elaborate its main characterizations. Finally, in the last section we present our conclusion and we give some ideas for future works.

2. Preliminaries

In this part, we give the necessary definitions and results, in the literature, related to our main concept which help us to make this manuscript self contained.

Definition 2.1. ([27]) An ordered pair (f, Σ) is called a soft set over $Z \neq \emptyset$ (known as the universal set) provided that f is a mapping from a nonempty set Σ (known as a set of parameters) to power set 2^Z of Z. Usually, we express a soft set by the following formulation

$$(f, \Sigma) = \{(\sigma, f(\sigma)) : \sigma \in \Sigma \text{ and } f(\sigma) \in 2^Z\}.$$

We call every $f(\sigma)$ a component of (f, Σ).

Henceforth, the notation (f, Σ) will refer a soft set over Z. Also, we sometimes write the abbreviation "S-set" in a place of soft set or soft subset.

Definition 2.2. ([27]) An S-set (f, Σ) is said to be an absolute (resp., a null) S-set if $f(\sigma) = Z$ (resp., $f(\sigma) = \emptyset$) for all $\sigma \in \Sigma$. They are denoted by $\widetilde{Z}$ and ϕ, respectively.

Definition 2.3. ([29]) Let (f, Σ) be an S-set over Z. If there exist $z \in Z$ and $a \in \Sigma$ such that

$$f(\sigma) = \begin{cases} z & : \quad \sigma = a \\ \emptyset & : \quad \forall \sigma \neq a, \end{cases}$$

then we call (f, Σ) a soft point.

Definition 2.4. ([29]) An S-set (f, Σ) over Z defined by

$$f(\sigma) = \begin{cases} Z \\ \emptyset \end{cases}$$

for each $\sigma \in \Sigma$ is said to be a pseudo constant S-set.

If $f(\sigma)$ is finite (resp., countable) for all $\sigma \in \Sigma$, then (f, Σ) is called a finite (resp., countable) S-set; otherwise (f, Σ) is called an infinite (resp., uncountable) S-set.

Definition 2.5. ([20]) We call (f, Σ) an S-subset of (g, Σ) (or (g, Σ) an S-superset of (g, Σ)), denoted by $(f, \Sigma)\widetilde{\subseteq}(g, \Sigma)$ if $f(\sigma) \subseteq g(\sigma)$ for all $\sigma \in \Sigma$.

Definition 2.6. ([6]) If $g(\sigma) = Z - f(\sigma)$ for all $\sigma \in \Sigma$, then we call (g, Σ) a complement of (f, Σ). The complement of (f, Σ) is symbolized by $(f, \Sigma)^c = (f^c, \Sigma)$.

Definition 2.7. Let (f, Σ) and (h, Σ) be S-sets. Then:

(i) $(f, \Sigma)\widetilde{\bigcup}(h, \Sigma) = (k, \Sigma)$, where $k(\sigma) = f(\sigma) \bigcup h(\sigma)$ for each $\sigma \in \Sigma$ [6].

(ii) $(f, \Sigma)\widetilde{\bigcap}(h, \Sigma) = (k, \Sigma)$, where $k(\sigma) = f(\sigma) \bigcap h(\sigma)$, for each $\sigma \in \Sigma$ [25].

Definition 2.8. ([19]) Let (f, Σ) be an S-set and $z \in Z$. Then:

(i) $z \in (f, \Sigma)$ if $z \in f(\sigma)$ for *each* $\sigma \in \Sigma$.

(ii) $z \notin (f, \Sigma)$ if $z \notin f(\sigma)$ for *some* $\sigma \in \Sigma$.

(iii) $z \Subset (f, \Sigma)$ if $z \in f(\sigma)$ for *some* $\sigma \in \Sigma$.

(iv) $z \not\Subset (f, \Sigma)$ if $z \notin f(\sigma)$ for *each* $\sigma \in \Sigma$.

Definition 2.9. ([9]) A soft mapping Ω_π from the class of all soft points over Z with Σ to the class of all soft points over X with Γ generated by the crisp mappings $\Omega : Z \to X$ and $\pi : \Sigma \to \Gamma$ is defined as follows

$$\Omega_\pi(P^z_\sigma) = P^{\Omega(z)}_{\pi(\sigma)} \text{ for each } P^z_\sigma \in P(Z_\Sigma).$$

In addition, $\Omega_{\pi}^{-1}(P_b^x) = \widetilde{\bigcup}_{\substack{\sigma\in\pi^{-1}(\varrho)\\ z\in\Omega^{-1}(x)}} P_{\sigma}^{z}$ for each $P_{\varrho}^{x} \in P(X_\Gamma)$.

Definition 2.10. ([31]) A soft topology θ on the universal set Z with a set of parameters Σ is a subfamily of $P(Z_\Sigma)$ satisfying that $\widetilde{Z}$ and ϕ are members of θ and θ is closed under finite soft intersections and arbitrary soft unions.

The triplet (Z, θ, Σ) is termed a soft topological space (ST-space). An S-set belongs to θ is named soft open and its complement is named soft closed.

Proposition 2.11. ([31]) *Let (Z, θ, Σ) be an ST-space. Then for any $\sigma \in \Sigma$, the family $\theta_\sigma = \{f(\sigma) : (f, \Sigma) \in \theta\}$ forms a topology on Z. From now on, we will name this topology a parametric topology.*

Definition 2.12. ([13]) Let (f, Σ) be an S-subset of an ST-space (Z, θ, Σ). Then $(cl(f), \Sigma)$ is defined by $cl(f)(\sigma) = cl(f(\sigma))$, where $cl(f(\sigma))$ is the closure of $f(\sigma)$ in (Z, θ_σ).

Definition 2.13. ([17, 29]) Let (Z, θ, Σ) be an ST-space. Then θ is called

(i) an enriched ST provided if every pseudo constant S-subset of $\widetilde{Z}$ is a member of θ.

(ii) an extended ST if $(f, \Sigma) \in \theta \iff f(\sigma) \in \theta_\sigma$ for all $\sigma \in \Sigma$.

In [13], we could find the proof of the identity between enriched ST and extended ST. In this work we use the terminology "extended ST" for this type of soft topology. In this case we say that (Z, θ, Σ) is an extended ST-space. In the theorem below, we present the behaviour of these concepts between classical topological spaces and STSs.

Theorem 2.14. *([13]) An ST-space (Z, θ, Σ) is extended iff $(int(f), \Sigma) = int(f, \Sigma)$ and $(cl(f), \Sigma) = cl(f, \Sigma)$ for any S-subset (f, Σ).*

Definition 2.15. ([28]) An ST-space (Z, θ, Σ) is named full if and only if all components of any non-null soft open set are nonempty.

Definition 2.16. ([17]) If any soft open cover of an ST-space (Z, θ, Σ) has a finite (resp., countable) subcover, then (Z, θ, Σ) is named soft compact (resp., soft Lindelöf).

Definition 2.17. ([8]) A family $C = \{(f_\rho, \Sigma) : \rho \in I\}$ possesses the property of finite (resp., countable) intersection if the members of any finite (resp., countable) family of C has a non-null soft intersection.

Definition 2.18. ([30]) If the null and absolute S-sets are the only soft clopen subsets of an ST-space (Z, θ, Σ), then (Z, θ, Σ) is said to be soft connected.

Definition 2.19. We call an S-subset (f, Σ) of (Z, θ, Σ)

(i) a soft Q-set [4] if $int(cl(f, \Sigma) = cl(int(f, \Sigma)$.

(ii) a soft sw-open set [28] if $int(f, \Sigma) \neq \Phi$ or $(f, \Sigma) = \Phi$.

Theorem 2.20. ([4]) *An S-set (f, Σ) is a soft Q-set iff (f^c, Σ) is a soft Q-set.*

Definition 2.21. ([28]) If any soft sw-open cover of an ST-space (Z, θ, Σ) has a finite (resp., countable) subcover, then (Z, θ, Σ) is named soft sw-compact (resp., soft sw-Lindelöf).

3. Soft Q-compact and soft Q-Lindelöf spaces

In this section, novel kinds of soft compact and soft Lindelöf spaces are displayed and studied. Their fundamental features are explored and some examples are initiated to illustrate the relationships between them. The property that reports "soft j-closed subset of soft j-compact and soft j-Lindelöf spaces is respectively soft j-compact and soft j-Lindelöf ($j \in \{\alpha, semi, pre, b\}$)" is updated herein for the concepts of soft Q-compactness and soft Q-Lindelöfness. Finally, we scrutinize the relation between these concepts via soft and crisp topological structures and provide some specific relations under a condition of extended or full soft topologies.

Definition 3.1. Let Γ be a family Q-subsets of (Z, θ, Σ). Then, we say that Γ is a soft Q-cover of $\widetilde{Z}$ if $\widetilde{Z}$ is a subset of Γ.

Definition 3.2. Let (Z, θ, Σ) be an ST-space. Then, we say that (Z, θ, Σ) is soft Q-compact (resp., soft Q-Lindelöf) if every soft Q-cover of (Z, θ, Σ) has a finite (resp., countable) sub-cover of $\widetilde{Z}$.

Now, we give two examples of soft topological spaces, the first one is soft Q-compact and the second one is not soft Q-Lindelöf.

Example 3.3. In the real numbers set $\mathbb{R}$ we take the soft topology θ defined by

$$\theta = \{(f, \Sigma) \widetilde{\subseteq} \widetilde{\mathbb{R}} : 1 \notin (f, \Sigma)\} \cup \{\widetilde{\mathbb{R}}\}$$

where Σ is a finite set of parameters.

Let (g, Σ) be any non-null proper S-subset of $\widetilde{\mathbb{R}}$. Then we have two cases:

(i) $1 \in (g, \Sigma)$. So $cl(int(g, \Sigma)) = (g, \Sigma)$, whereas $1 \notin int(cl(g, \Sigma))$. Thus $cl(int(g, \Sigma)) \neq int(cl(g, \Sigma))$.

(ii) Or $1 \notin (g, \Sigma)$. So $1 \in cl(int(g, \Sigma))$, whereas $1 \notin int(cl(g, \Sigma))$. Thus $cl(int(g, \Sigma)) \neq int(cl(g, \Sigma))$.

This implies (g, Σ) is not a soft Q-set. This automatically means that the only soft Q-subsets of $\widetilde{\mathbb{R}}$ are the null and absolute S-sets. Hence, $(\mathbb{R}, \theta, \Sigma)$ is a soft Q-compact space.

Example 3.4. In the real numbers set $\mathbb{R}$ we take the soft topology θ defined by $\theta = \{(f, \Sigma) \widetilde{\subseteq} \widetilde{\mathbb{R}} : (f^c, \Sigma)$ is finite$\} \cup \{\phi\}$ where Σ is a finite set of parameters.

An S-subset of $(\mathbb{R}, \theta, \Sigma)$ which is finite is clearly a soft Q-set. Then, $(\mathbb{R}, \theta, \Sigma)$ is not a soft Q-Lindelöf space.

Proposition 3.5. *(i) soft Q-compactness implies soft Q-Lindelöfness.*

(ii) The classes of soft Q-compact and soft Q-Lindelöf subsets are respectively closed under finite and countable unions.

Proof. Obvious. □

Proposition 3.6. *Every soft Q-subset of a soft Q-compact (resp., soft Q-Lindelöf) ST-space (Z, θ, Σ) is soft Q-compact (resp., soft Q-Lindelöf).*

Proof. Assume that $C = \{(f_\rho, \Sigma) : \rho \in I\}$ is a soft Q-cover of (g, Σ) which is a soft Q-subset of (Z, θ, Σ). By Theorem 2.20, (g^c, Σ) is also a soft Q-set, so $\{(f_\rho, \Sigma) : \rho \in I\} \widetilde{\bigcup} (g^c, \Sigma)$ is a soft Q-cover of (Z, θ, Σ) which is soft Q-compact. So, $\widetilde{Z} = \widetilde{\bigcup}_{\rho=1}^{n} (f_\rho, \Sigma) \widetilde{\bigcup} (g^c, \Sigma)$. Thus, $(g, \Sigma) \widetilde{\subseteq} \widetilde{\bigcup}_{\rho=1}^{n} (f_\rho, \Sigma)$, so we can see that (g, Σ) is soft Q-compact. We can proceed similarly to prove the case soft Q-Lindelöf. □

Corollary 3.7. *If (f, Σ) is soft Q-Lindelöf (resp., soft Q-compact) and (g, Σ) is a soft Q-set, then the soft intersection of them is soft Q-Lindelöf (resp., soft Q-compact).*

Theorem 3.8. *An ST-space* (Z, θ, Σ) *is soft Q-Lindelöf (resp., soft Q-compact) iff* $\widetilde{\bigcap}_{\rho\in I}(f_\rho, \Sigma) \neq \phi$ *for any class of soft Q-sets has a countable (resp., finite) intersection property.*

Proof. We provide the proof in case of soft Q-compactness.

Necessity: Let $C = \{(f_\rho, \Sigma) : \rho \in I\}$ be a class of soft Q-subsets of a soft Q-compact space (Z, θ, Σ). Suppose that $\widetilde{\bigcap}_{\rho\in I}(f_\rho, \Sigma) = \phi$. Then, $\widetilde{Z} = \widetilde{\bigcup}_{\rho\in I}(f^c_\rho, \Sigma)$. By assumption, $\widetilde{Z} = \widetilde{\bigcup}^n_{\rho=1}(f^c_\rho, \Sigma)$. This means that $\phi = (\widetilde{\bigcup}^n_{\rho=1}(H^c_\rho, \Sigma))^c = \widetilde{\bigcap}^n_{\rho=1}(f_\rho, \Sigma)$. Hence, C has has a finite intersection property, as required.

Sufficiency: Consider $C = \{(f_\rho, \Sigma) : \rho \in I\}$ as a soft Q-cover of (Z, θ, Σ). Then $\phi = \widetilde{\bigcap}_{\rho\in I}(f^c_\rho, \Sigma)$. According to the property of finite intersection of C, we obtain $\phi = \widetilde{\bigcap}^n_{\rho=1}(f^c_\rho, \Sigma)$. Thus, $\widetilde{Z} = \widetilde{\bigcup}^n_{\rho=1}(f_\rho, \Sigma)$, which proves that (Z, θ, Σ) is soft Q-compact. □

Lemma 3.9. *The property of being a soft Q-set is a soft topological property.*

Proof. It follows from the fact that we have $\Omega_\pi(int(cl(g, \Sigma))) = int(cl(\Omega_\pi(g, \Sigma)))$ for every subset provided that Ω_π is a soft homeomorphism mapping. □

Proposition 3.10. *The soft homeomorphism image of a soft Q-compact (resp., soft Q-Lindelöf) set is soft Q-compact (resp., soft Q-Lindelöf).*

Proof. Consider $\Omega_\pi : (Z, \theta_E, \Sigma) \to (X, \theta_X, \Sigma)$ is a soft homeomorphism mapping and let (g, Σ) be a soft Q-Lindelöf subset of (Z, θ_E, Σ). Let us consider $C = \{(f_\rho, \Sigma) : \rho \in I\}$ is a soft Q-cover of $\Omega_\pi(g, \Sigma)$. Obviously, $(g, \Sigma)\widetilde{\subseteq}\widetilde{\bigcup}_{\rho\in I}\Omega^{-1}_\pi(f_\rho, \Sigma)$. It follows from Lemma 3.9 that $\Omega^{-1}_\pi(f_\rho, \Sigma)$ is a soft Q-set for each $\rho \in I$. According to soft Q-Lindelöfness of (g, Σ), there is a countable set δ such that $(g, \Sigma)\widetilde{\subseteq}\widetilde{\bigcup}_{\rho\in\delta}\Omega^{-1}_\pi(f_\rho, \Sigma)$. Now, we obtain $\Omega_\pi(g, \Sigma)\widetilde{\subseteq}\widetilde{\bigcup}_{\rho\in\delta}\Omega_\pi(\Omega^{-1}_\pi(f_\rho, \Sigma))\widetilde{\subseteq}(f_\rho, \Sigma)$, which means that $\Omega_\pi(g, \Sigma)$ is soft Q-Lindelöf. It can be prove the case of soft Q-compact in a similar way. □

In what follows, we study the properties of Q-compactness and Q-Lindelöfness respecting the relation between soft topology and its parametric topologies. First, the parametric topologies inherit the properties of soft Q-compactness and Q-Lindelöfness from its soft topologies. The other types of compactness and Lindelöfness which use soft b-open sets, soft pre-open, soft semi-open and soft α-open do not keep these properties.

Theorem 3.11. *A subset* (f, Σ) *of an extended ST-space* (Z, θ, Σ) *is soft Q-set if and only if all components are Q-sets.*

Proof. Follows from Theorem 2.14. □

Theorem 3.12. *Let* (Z, θ_σ) *be a parametric topological space inspired by an extended soft Q-compact (resp., extended soft Q-Lindelöf) space* (Z, θ, Σ)*. Then* (Z, θ_σ) *is Q-compact (resp., Q-Lindelöf) for each* $\sigma \in \Sigma$*.*

Proof. Let $\{H_\rho : \rho \in I\}$ be a Q-cover of (Z, θ_σ). Then, for all $\rho \in I$ there is a soft Q-subset (f_ρ, Σ) of θ such that $f_\rho(\sigma) = H_\rho$ and $f_\rho(\alpha) = Z$ for each $\alpha \neq \sigma$. Now, the family $\{(f_\rho, \Sigma) : \rho \in I\}$ forms a soft Q-cover of (Z, θ, Σ). By hypothesis of soft Q-compactness of (Z, θ, Σ), we obtain $\widetilde{Z} = \bigcup\limits_{\rho=1}^{n}(f_\rho, \Sigma)$. This implies that $Z = \bigcup\limits_{\rho=1}^{n} f_\rho(\sigma) = \bigcup\limits_{\rho=1}^{n} H_\rho$. Hence, (Z, θ_σ) is Q-compact, as required. Following similar argument, one prove the case between parentheses. □

The converse of Theorem 3.12 is not true as shown in the following example.

Example 3.13. Let $Z = \{x, y\}$ be the universal set, $(Z, \theta, \mathbb{R})$ be the ST-space with the set of real numbers $\mathbb{R}$ as set of parameters and θ be the discrete soft topology. It is clear that all parametric topological spaces inspired by $(Z, \theta, \mathbb{R})$ are Q-compact. But the ST-space $(Z, \theta, \mathbb{R})$ is not soft Q-Lindelöf.

Theorem 3.14. *Let (Z, θ, Σ) be a full ST-space such that Σ is a finite (resp., countable) set of parameters. Then, the following statements are equivalent:*

1. *every parametric topological spaces (Z, θ_σ) inspired by (Z, θ, Σ) is Q-compact (resp., Q-Lindelöf).*
2. *(Z, θ, Σ) is soft Q-compact (resp., soft Q-Lindelöf).*

Proof. We present the proof for the property compact (the case Lindelöf could be seen similarly).

[1. ⇒ 2.] Let $C = \{(f_\rho, \Sigma) : \rho \in I\}$ be a soft Q-cover of (Z, θ, Σ) and $|\Sigma| = m$. Then $Z = \bigcup\limits_{\rho\in I} f_\rho(\sigma)$ for all $\sigma \in \Sigma$. Using Theorem 3.11, we can see that $f_\rho(\sigma)$ is a Q-set for each $\sigma \in \Sigma$. Since (Z, θ_σ) is Q-compact for all $\sigma \in \Sigma$, then we have $Z = \bigcup\limits_{\rho=1}^{n_1} f_\rho(\sigma_1)$, $Z = \bigcup\limits_{\rho=n_1+1}^{n_2} f_\rho(\sigma_2)$,..., $Z = \bigcup\limits_{\rho=n_{m-1}+1}^{n_m} f_\rho(\sigma_m)$. This implies that $\widetilde{Z} = \widetilde{\bigcup}_{\rho=1}^{n_m}(f_\rho, \Sigma)$. Hence, (Z, θ, Σ) is soft Q-compact.

[2. ⇒ 1.] Follows from Theorem 3.12. □

On one a hand, Example 3.13 shows that the evident proposition for classical topologies saying "every countable (resp., finite) topological space is Q-Lindelöf (resp., Q-compact)" can not be generalized to STs. Also, the property reports that "An extended ST-space (Z, θ, Σ) is soft sw-Lindelöf (resp., soft sw-compact) if and only if the universal set Z and set of parameters Σ are countable (resp., finite)." is not valid for this types of compactness and Lindelöfness as illustrated by the next example.

Example 3.15. Any indiscrete soft topology defined over an infinite set with a finite set of parameters is soft Q-compact.

Definition 3.16. Let (Z, θ, Σ) be an ST-space. Then (Z, θ, Σ) is called an almost soft Q-compact (resp., almost soft Q-Lindelöf) if for each soft Q-cover there exists a finite (resp., countable) subfamily which covers $\widetilde{Z}$ by the closure of whose members. Equivalently, if $C = \{(f_\rho, \Sigma) : \rho \in I\}$ is a soft Q-cover, then there exists $\delta \subseteq I$ such that δ is finite (resp., countable) and $\widetilde{Z} = \widetilde{\bigcup}_{\rho\in\delta} sqcl(f_\rho, \Sigma)$.

Definition 3.17. Let (Z, θ, Σ) be an ST-space. Then (Z, θ, Σ) is called weakly soft Q-compact (resp., weakly soft Q-Lindelöf) if for each soft Q-cover there exists a finite (resp., countable) subfamily which covers $\widetilde{Z}$ by its soft Q-closure. Equivalently, if $C = \{(f_\rho, \Sigma) : \rho \in I\}$ is a soft Q-cover, then there exists $\delta \subseteq I$ such that δ is finite (resp., countable) and $\widetilde{Z} = sqcl(\widetilde{\bigcup}_{\rho\in\delta}(f_\rho, \Sigma))$.

The following result shows the uniqueness behaviours of the covering properties produced by soft Q-sets.

Theorem 3.18. *The next concepts are equivalent:*

(i) *(Z, θ, Σ) is soft Q-Lindelöf (resp., soft Q-compact).*

(ii) *(Z, θ, Σ) is almost soft Q-Lindelöf (resp., almost soft Q-compact).*

(iii) *(Z, θ, Σ) is weakly soft Q-Lindelöf (resp., weakly soft Q-compact).*

Proof. It is a direct result of Theorem 2.20. □

4. Soft Q-connected spaces

The concept of soft Q-connectedness is introduced and its main properties and characterizations are established in this section. We show the conditions under which the concepts of soft Q-connected and soft connected spaces are equivalent. Some illustrative counterexamples are provided.

Definition 4.1. An ST-space (Z, θ, Σ) is called soft Q-disconnected if it contain a non-null proper soft Q-set. Otherwise, we call (Z, θ, Σ) a soft Q-connected space. That is, (Z, θ, Σ) is soft Q-connected provided that the only soft Q-sets are the absolute and null soft sets.

In the above definition, we do not follow the known technique of defining soft connectedness since the existence of a non-null proper soft Q-subset implies that we obtain two non-null soft Q-subsets such that their soft intersection is the null soft set and soft union is the absolute soft set; see, Theorem 2.20.

Note that the ST-space given in Example 3.3 is soft Q-disconnected, whereas the ST-space given in Example 3.4 is soft Q-connected.

Proposition 4.2. *If an ST-space (Z, θ, Σ) contains a non-null proper soft closed subset with null soft interior, then (Z, θ, Σ) is soft Q-disconnected.*

Proof. Let (f, Σ) be a non-null soft closed subset with null soft interior. Then $int(cl(f, \Sigma)) = int(f, \Sigma) = \phi$, which means that $cl(int(f, \Sigma)) = \phi$ as well. Hence, $int(cl(f, \Sigma)) = cl(int(f, \Sigma))$, as required. □

Example below shows that the converse of the proposition above is incorrect in general.

Example 4.3. Let $\theta = \{\phi, \widetilde{Z}, (f, \Sigma), (f^c, \Sigma)\}$ be a soft topology on $Z = \{x, y\}$, where $\Sigma = \{\sigma_1, \sigma_2\}$. It is clear that (Z, θ, Σ) is soft Q-disconnected. But the soft interior of every non-null soft closed subset is non-null.

Theorem 4.4. *Every soft disconnected space (Z, θ, Σ) is soft Q-disconnected.*

Proof. By soft disconnectedness of (Z, θ, Σ), θ contains a soft clopen subset, say, (h, Σ). Obviously, $int(cl(h, \Sigma)) = cl(int(h, \Sigma))$. Hence, (Z, θ, Σ) is soft Q-disconnected. □

The converse of the above proposition fails as example below elucidates.

Example 4.5. Let $\theta = \{\phi, (f, \Sigma)\widetilde{\subseteq}\widetilde{\mathbb{N}} : 1 \in (f, \Sigma)\}$ be a soft topology on the set of natural numbers $\mathbb{N}$, where $\Sigma = \{\sigma_1, \sigma_2\}$. It is clear that the only soft clopen subsets of (Z, θ, Σ) are the null and absolute soft sets, so (Z, θ, Σ) is soft connected. On the other hand, $(h, \Sigma) = \{(\sigma_1, \{1, 2\}), (\sigma_2, \{1, 2\})\}$ is a soft Q-subset of (Z, θ, Σ) because $int(cl(h, \Sigma)) = cl(int(h, \Sigma)) = \widetilde{\mathbb{N}}$. Hence, (Z, θ, Σ) is soft Q-disconnected.

Lemma 4.6. *An ST-space (Z, θ, Σ) in which every soft subset is soft pre-open iff every soft open is soft closed.*

Proof. Suppose that (f, Σ) is a soft open set. Then $(f^c, \Sigma) = cl(f^c, \Sigma)$ which is soft pre-open, so that $cl(f^c, \Sigma)\widetilde{\subseteq}int(cl(f^c, \Sigma)) = int(f^c, \Sigma)$. This implies that $int(f^c, \Sigma) = (f^c, \Sigma)$. Thus, (f^c, Σ) is soft open, and (f, Σ) is soft closed. Conversely, let (f, Σ) be any soft subset. Then $(cl(f, \Sigma))^c$ is soft open. By assumption, it is also soft closed. Therefore, $(cl(f, \Sigma))^c = cl[(cl(f, \Sigma))^c] = (int(cl(f, \Sigma)))^c$. So that $(f, \Sigma)\widetilde{\subseteq}cl(f, \Sigma) = int(cl(f, \Sigma))$. Hence, (f, Σ) is soft pre-open. □

Recall that we call (Z, θ, Σ) soft clopen if every soft open is soft closed.

Theorem 4.7. *Let all soft subsets of a soft clopen ST-space (Z, θ, Σ) are not soft dense. Then (Z, θ, Σ) is soft Q-disconnected iff it is soft disconnected.*

Proof. Necessity: Since (Z, θ, Σ) is soft Q-disconnected, there exists a non-null proper soft Q-subset with the property $int(cl(f, \Sigma)) = cl(int(f, \Sigma))$. Putting $(g, \Sigma) = int(cl(f, \Sigma)) = cl(int(f, \Sigma))$, so (g, Σ) is a soft clopen subset. Since (Z, θ, Σ) is soft clopen, (f, Σ) is soft pre-open, so $int(cl(f, \Sigma)) \neq \phi$. By assumption, $cl(int(f, \Sigma)) \neq \widetilde{Z}$. Thus, (g, Σ) is a non-null proper soft clopen subset of (Z, θ, Σ). Hence, (Z, θ, Σ) is soft disconnected.

Sufficiency: It is obtained from Theorem 4.4. □

Proposition 4.8. *Let Ω_π be a soft homeomorphism mapping of a soft Q-disconnected space (Z, θ_Z, Σ) onto (X, θ_X, Σ). Then $\Omega_\pi(\widetilde{Z})$ soft Q-disconnected.*

Proof. Suppose that $\Omega_\pi(\widetilde{Z}) = \widetilde{X}$ is soft Q-disconnected. Then, θ_X contains a non-null proper soft Q-subset (f, Σ). By Lemma 3.9, we get $\Omega_\pi^{-1}(f, \Sigma)$ a non-null proper soft Q-subset in θ_Z. Hence, (Z, θ_Z, Σ) is soft Q-disconnected. □

In the following part of this section, we present the relationships between soft topologies and their parametric topologies respecting this type of soft Q-connectedness.

First, we show by the next example that all parametric topological spaces of a soft Q-disconnected space (Z, θ, Σ) need not be Q-disconnected even if (Z, θ, Σ) is extended.

Example 4.9. Let $\theta = \{\phi, \widetilde{Z}, (f, \Sigma), (f^c, \Sigma)\}$ be a soft topology on $Z = \{x, y\}$ with $\Sigma = \{\sigma_1, \sigma_2\}$, where $(f, \Sigma) = \{(\sigma_1, Z), (\sigma_2, \emptyset)\}$. It is obvious that (Z, θ, Σ) is soft Q-disconnected. Moreover, it is soft disconnected. On the other hand, the parametric topologies inspired by this soft topology are indiscrete, so they are Q-connected.

Theorem 4.10. *Let an ST-space (Z, θ, Σ) be extended such that there exists a parametric topological space (Z, θ_σ) is Q-disconnected, (Z, θ, Σ) is soft Q-disconnected.*

Proof. Let (Z, θ_σ) be Q-disconnected. Then there exists a nonempty proper Q-subset H with the property $int(cl(H)) = cl(int(H))$. Define a soft set (g, Σ) as follows: $g(\sigma) = H$ and $g(\sigma') = X$ for all $\sigma' \neq \sigma$. Since θ is extended, (g, Σ) is a non-null proper soft Q-subset. Hence, (Z, θ, Σ) is soft Q-disconnected. □

5. Concluding remarks

Soft topology is a fruitful branch to study topological properties and produces various kinds for each classical topological concept. For example, it was defined pp-soft T_i, pt-soft T_i, tp-soft T_i and tt-soft T_i as counterparts of T_i-spaces. This variety represents an advantage of soft topology and enhances its importance. In the current paper, we first have presented the concepts of soft Q-compact and soft Q-Lindelöf spaces using soft Q-sets. Then we have introduced the concepts of soft Q-connected and soft Q-disconnected spaces. In general, we have studied their basic properties showed the relationships between them. Also, we have discussed their characterizations and illustrated their unique characterizations which are not satisfied for their counterparts. Moreover, we have elucidated the relationships between these concepts and their counterparts via parametric topologies and demonstrated the role of extended and full soft topologies as a guarantee to satisfy some relations.

In future work, we are going to investigate more soft topological ideas inspired by soft Q-sets such as soft paracompact and soft Menger spaces. Also, we will try to apply the current concepts to information systems as done in [8]. Moreover, we will study the behaviour and characterizations of the notions presented herein in the content of supra soft topologies and infra soft topologies.

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