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Framed clad helices in Euclidean 3-space

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Abstract. In this study, we introduce framed clad helices, which are a generalization of clad (i.e. C-slant or 2-slant) helices in Euclidean 3-space. They also include framed helices and framed slant helices. After, we give a characterization of the framed clad helices using the alternative adapted frame, which is more useful than the adapted frame of a framed curve. Moreover, we prove the existence of framed spherical images of any framed curve using alternative adapted frames. Additionally, we obtain interesting results regarding the relationship between a framed clad helix and its framed spherical images. Finally, we support the concept with some nice graphics.

1. Introduction

Honda and Takahashi introduced the concept of framed curves in Euclidean space as smooth curves with singular points [14]. A framed curve is a curve with a moving frame, which generalizes not only regular curves with the linear independent condition [13] but also regular curves with the Bishop frame [5]. Additionally, framed curves may have singular points, and they are also a generalization of Legendre curves [8, 31]. Similar to curvature functions of a regular curve, Honda et al. [14] also defined framed curvature functions which are well-defined even at singular points. Also, Fukunaga and Takahashi [12] studied existence conditions of framed curves. Moreover, Wang et al. [34] defined an adapted frame which is an alternative to the moving frame of a framed curve in Euclidean space, and elements of the adapted frame are called a generalized tangent vector, generalized principal normal vector, and generalized binormal vector, respectively. Afterward, as a generalized version of regular curves, the concept of special framed curves were studied in Euclidean space, In this sense, framed helix [15], framed slant helix [28], framed rectifying curve [34, 36], framed normal curve [37], framed Bertrand and framed Mannheim curves [16], and also the references [7, 18, 35] noteworthy studies that contribute to the theory of framed curves. Moreover, Legendre curves are a special case of framed curves. Therefore, references [6, 8–11, 17, 22–24, 32, 38] are other notable studies that contribute to the field of framed curves, specifically in the category of frontal or front curves.

In the theory of curves in differential geometry, if a vector field associated with the Frenet frame of the curve makes a constant angle with a fixed direction along the curve, then this curve is a type of helix curve or helical curve. Overall, the helical curve has many applications in various fields of mathematics and other sciences, and its unique shape makes it a useful tool for solving a wide range of problems (ex. the motion of a particle traveling in a circular path with a constant velocity in Physics, the design of mechanical components, such as screws, bolts, and springs in Engineering, structures of certain chiral and crystal in

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Chemistry, the structure of certain molecules, such as DNA and proteins in Biology, etc.). In the classical sense, a helix, which resembles a spiral bow or a spiral staircase in Euclidean 3-space, was first defined by Lancret in 1802 using the tangent vector of the curve. It was characterized by Saint-Venant in 1845 as a curve whose the ratio of its curvatures (τ/κ) is a constant. A helix is also called a general helix or cylindrical helix. Especially, if its curvatures are non-zero constants then it is called a circular helix. By using the principal normal vector of the curve, the slant helix, which is the first generalization of the helix, is introduced by Izumiya and Takeuchi in 2004. By the same authors, the slant helix is characterized by a constant function σ which corresponds to the geodesic curvature of the spherical image of the unit principal normal of the curve is a constant [19]. Scofield introduced a new helical curve whose Darboux vector field (centrode) revolves around a fixed line in space with constant angle and constant speed. It is called a curve of constant precession [29]. Kula and Yaylı proved that the tangent, principal normal, and binormal images of a slant helix are spherical helices, a circle, and a spherical helix, respectively. Also, they show that a curve of constant precession is a slant helix [20]. By considering similar idea to slant helix, Takahashi and Takeuchi in 2014 defined clad helix and g-clad helix if the spherical image of the unit principal normal is a spherical helix and a spherical slant helix, respectively. They give characterizations of the clad helix and g-clad helix by constant functions φ and ψ, respectively [30]. However, as a consecutive slant helix, Ali in 2012 defined a k-slant helix by using sequentially k-vector field. He examined the axis and characterization for k-slant helices with respect to the Frenet apparatus [1]. We remark that 0-type, 1-type, 2-type, and 3-type slant helix corresponds to helix, slant helix, clad helix, and g-clad helix, respectively. Moreover, Uzunoglu et al. in 2016 defined an alternative frame that is generated by the Frenet frame. They defined the C-slant helix by using the vector field C and gave its axis with respect to the alternative frame and its characterization [33]. However, we point out that the C-slant helix corresponds to clad helix (or 2-slant helix) and the vector field C corresponds to the co-Darboux vector field (co-centrode) with respect to the Frenet frame of a regular space curve. Also, about the type of helix curves, other remarkable articles are [2, 4, 21, 26, 27].

In this paper, as a generalization of concepts of clad helix[30] (C-slant helix[33] or 2-slant helix [1]), we introduce framed clad helix (or co-Darboux helix) in Euclidean 3-space by using the adapted frame in [34]. Also, we obtain that family of framed clad helix includes family of framed helices [15] and framed slant helices [28] in Euclidean 3-space. First of all, we give framed Darboux (centrode) vector and framed co-Darboux (co-centrode) vector with respect to the adapted frame [34]. As motivated by [33], we defined a new alternative adapted frame apparatus which is generated by the adapted frame apparatus of a framed curve. After, by applying sequentially procedures, we give the construction method of k-alternative adapted frame apparatus of a framed curve, and according to these adapted frames, we defined unit type-k framed Darboux and unit type-k framed co-Darboux vector fields. Next, we prove the existence of framed spherical images of any framed curve by using k-alternative adapted frames and examine the adapted frame apparatus of framed spherical images. In the last section, we introduce framed clad helix (co-Darboux helix) whose unit framed co-Darboux vector makes a constant angle with a fixed unit vector. After, we give explicitly the axis of a framed clad helix with respect to its alternative adapted frame apparatus. Moreover, we investigate exciting results between the framed clad helix and its generalized tangent, principal normal, binormal, unit framed Darboux, unit framed co-Darboux images, respectively. Finally, we supported the concept with some nice graphics.

2. Preliminary

Let $\mathbb{R}^3$ denote the Euclidean 3-space, that is, the 3-dimensional real vector space endowed with the standard inner product $\langle x, y\rangle = \sum_{i=1}^{3} x_i y_i$, for all $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in \mathbb{R}^3$. The norm of a vector $x \in \mathbb{R}^3$ is defined by $\|x\| = \sqrt{\langle x, x\rangle}$. Also, the cross product of vectors x and y is given by $x \wedge y = (x_2y_3 - x_3y_2, -x_1y_3 + x_3y_1, x_1y_2 - x_2y_1)$.

2.1. Framed Curves in Euclidean 3-Space

Let $\gamma : I \to \mathbb{R}^3$ be a space curve. If $\dot{\gamma}(t_0) = \frac{d\gamma}{dt}(t_0) = 0$ at $t_0 \in I$ then t_0 is called a singular point of γ. It is easy to see that the Frenet frame of any space curve is not well-defined at any singular of the curve. Now, let's give the following concepts about framed curves which is a regular or singular space curve in $\mathbb{R}^3$ (see [12, 14, 15, 34] for more detail and background).

Let us take the set $\Delta_2 = \left\{\boldsymbol{u} = (\boldsymbol{u}_1, \boldsymbol{u}_2) \in \mathbb{S}^2 \times \mathbb{S}^2 \,|\langle \boldsymbol{u}_1, \boldsymbol{u}_2\rangle = 0\right\}$ as a 3-dimensional manifold.

Definition 2.1. *$\left(\gamma, \mu_1, \mu_2\right) : I \to \mathbb{R}^3 \times \Delta_2 \subset \mathbb{R}^3 \times \mathbb{S}^2 \times \mathbb{S}^2$ is called a framed curve, if $\left\langle \dot{\gamma}(t), \mu_i(t)\right\rangle = 0$ for all $t \in I$. $\gamma : I \to \mathbb{R}^3$ is also called a framed curve (or framed base curve) if there exists $\mu = \left(\mu_1, \mu_2\right) : I \to \Delta_2$ such that $\left(\gamma, \mu_1, \mu_2\right)$ is a framed curve [14].*

Now, unlike the Frenet frame, a well-defined moving frame in singular points can be constructed along the framed curve γ which may have singular points. Let $\left(\gamma, \mu_1, \mu_2\right)$ be a framed curve and $\vartheta : I \to \mathbb{S}^2$ be a regular spherical curve such that $\vartheta(t) = \mu_1(t) \wedge \mu_2(t)$ for all $t \in I$. Hence, $\left\{\mu_1, \mu_2, \vartheta\right\}$ is an orthonormal frame which is a moving frame along the framed curve γ in $\mathbb{R}^3$. Then, the Frenet-Serret type formula are given by

$$\dot{\mu}_1(t) = \mathfrak{l}(t)\mu_2(t) + \mathfrak{m}(t)\vartheta(t), \quad \dot{\mu}_2(t) = -\mathfrak{l}(t)\mu_1(t) + \mathfrak{n}(t)\vartheta(t), \quad \dot{\vartheta}(t) = -\mathfrak{m}(t)\mu_1(t) - \mathfrak{n}(t)\mu_2(t),$$

and there exists a smooth function $\mathfrak{a} : I \to \mathbb{R}$ such that

$$\dot{\gamma}(t) = \mathfrak{a}(t)\vartheta(t). \tag{1}$$

Here, the quadruple smooth functions $(\mathfrak{l}, \mathfrak{m}, \mathfrak{n}, \mathfrak{a}) = \left(\left\langle \dot{\mu}_1, \mu_2\right\rangle, \left\langle \dot{\mu}_1, \vartheta\right\rangle, \left\langle \dot{\mu}_2, \vartheta\right\rangle, \left\langle \dot{\gamma}, \vartheta\right\rangle\right)$ are called the curvature of the framed curve γ.

Remark 2.2. *It is clear that if $t_0 \in I$ is a singular point of γ then $\mathfrak{a}(t_0) = 0$. Moreover, since we suppose that ϑ is a regular spherical curve (i.e. $\dot{\vartheta}(t) \neq 0$) then $(\mathfrak{m}(t), \mathfrak{n}(t)) \neq (0, 0)$ for all $t \in I$.*

As similar to Bishop frame [5] of regular curves, Wang et al. [34] give the following adapted frame which is an alternative to the moving frame of the framed curve:

Let $\left(\eta_1, \eta_2\right) \in \Delta_2$ and $\theta : I \to \mathbb{R}$ be a smooth function such that

$$\begin{pmatrix} \eta_1(t) \\ \eta_2(t) \end{pmatrix} = \begin{pmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{pmatrix} \begin{pmatrix} \mu_1(t) \\ \mu_2(t) \end{pmatrix}.$$

It is easy to see that $\left(\gamma, \eta_1, \eta_2\right) : I \to \mathbb{R}^3 \times \Delta_2$ is also a framed curve and $\vartheta = \mu_1 \wedge \mu_2 = \eta_1 \wedge \eta_2$. Now, we assume that $\mathfrak{m}(t) = -\mathfrak{p}(t)\cos\theta(t)$ and $\mathfrak{n}(t) = \mathfrak{p}(t)\sin\theta(t)$ such that $\mathfrak{m}(t)\sin\theta(t) + \mathfrak{n}(t)\cos\theta(t) = 0$, then we have an adapted frame $\left\{\vartheta, \eta_1, \eta_2\right\}$ along the framed curve γ and the following Frenet-Serret type formula

$$\dot{\vartheta}(t) = \mathfrak{p}(t)\eta_1(t), \quad \dot{\eta}_1(t) = -\mathfrak{p}(t)\vartheta(t) + \mathfrak{q}(t)\eta_2(t), \quad \dot{\eta}_2(t) = -\mathfrak{q}(t)\eta_1(t), \tag{2}$$

where

$$\begin{cases} \mathfrak{p} = \left\langle \dot{\vartheta}, \eta_1\right\rangle = \left\|\dot{\vartheta}\right\| = \sqrt{\mathfrak{m}^2 + \mathfrak{n}^2} > 0, \\ \mathfrak{q} = \left\langle \dot{\eta}_1, \eta_2\right\rangle = \mathfrak{l} - \dot{\theta} = \mathfrak{l} + \left(\dfrac{\mathfrak{m}^2}{\mathfrak{m}^2 + \mathfrak{n}^2}\right)\left(\dfrac{\mathfrak{n}}{\mathfrak{m}}\right)^{\bullet}, \\ \mathfrak{a} = \left\langle \dot{\gamma}, \vartheta\right\rangle. \end{cases} \tag{3}$$

The triple smooth functions $(\mathfrak{p}, \mathfrak{q}, \mathfrak{a})$ are called framed curvature with respect to adapted frame $\left\{\vartheta, \eta_1, \eta_2\right\}$ along the framed curve γ. Moreover, the vectors $\vartheta(t), \eta_1(t), \eta_2(t)$ are called the generalized tangent vector, the generalized principle normal vector, and the generalized binormal vector of the framed curve, respectively.

Proposition 2.3. *Let $\left(\gamma, \eta_1, \eta_2\right) : I \to \mathbb{R}^3 \times \Delta_2$ be a framed curve with framed curvature $(\mathfrak{p}, \mathfrak{q}, \mathfrak{a})$. If the framed curve γ is a regular curve with curvature κ and torsion τ, then we have $\kappa = \frac{\mathfrak{p}}{|\mathfrak{a}|}$ and $\tau = \frac{\mathfrak{q}}{\mathfrak{a}}$ [34].*

Now, we give the following generalized versions of well-known definitions and characterizations for some regular space curves.

Definition 2.4. *Let $\left(\gamma, \eta_1, \eta_2\right)$ be a framed curve in $\mathbb{R}^3 \times \Delta_2$. Then, γ is called a framed planar curve if it lies on a plane in $\mathbb{R}^3$ [14].*

By using Proposition 3.3 in [14] with equations (3), we give the following characterization of framed planar curves with respect to the adapted curvature.

Theorem 2.5. *Let $\left(\gamma, \eta_1, \eta_2\right)$ be a framed curve in $\mathbb{R}^3 \times \Delta_2$ with framed curvature $(\mathfrak{p}, \mathfrak{q}, \mathfrak{a})$. Then, γ is a framed planar curve if and only if $\mathfrak{q} = 0$.*

Definition 2.6. *Let $\left(\gamma, \eta_1, \eta_2\right)$ be a framed curve in $\mathbb{R}^3 \times \Delta_2$. Then, γ is called a framed spherical curve if it lies on a sphere with a radius r in $\mathbb{R}^3$ [34].*

We give the Theorem 2.7 by using Proposition 2 and Corollary 1 in [34] with equations (3).

Theorem 2.7. *Let $\left(\gamma, \eta_1, \eta_2\right)$ be a framed curve in $\mathbb{R}^3 \times \Delta_2$ with framed curvature $(\mathfrak{p}, \mathfrak{q} = 0, \mathfrak{a})$. Then, γ is a framed spherical curve which is a circle in $\mathbb{R}^3$ if and only if $\mathfrak{q} = 0$ and $\frac{\mathfrak{p}}{|\mathfrak{a}|}$ is a constant such that $\mathfrak{a} \neq 0$.*

Definition 2.8. *Let $\left(\gamma, \eta_1, \eta_2\right)$ be a framed curve in $\mathbb{R}^3 \times \Delta_2$ with framed curvature $(\mathfrak{p}, \mathfrak{q}, \mathfrak{a})$. Then, the framed harmonic curvature of γ is given by $\mathfrak{h} = \frac{\mathfrak{q}}{\mathfrak{p}}$ [28].*

Definition 2.9. *Let $\left(\gamma, \eta_1, \eta_2\right)$ be a framed curve in $\mathbb{R}^3 \times \Delta_2$ with adapted frame $\left\{\boldsymbol{\vartheta}, \eta_1, \eta_2\right\}$. Then, γ is called a framed helix if its generalized tangent vector $\boldsymbol{\nu}$ makes a constant angle with a fixed unit vector $\boldsymbol{\zeta}$ [15, 34].*

Theorem 2.10. *Let $\left(\gamma, \eta_1, \eta_2\right)$ be a framed curve in $\mathbb{R}^3 \times \Delta_2$ with framed curvature $(\mathfrak{p}, \mathfrak{q}, \mathfrak{a})$. Then, γ is a framed helix if and only if $\mathfrak{h} = \cot\phi$ such that ϕ is a constant angle [34].*

Definition 2.11. *Let $\left(\gamma, \eta_1, \eta_2\right)$ be a framed curve in $\mathbb{R}^3 \times \Delta_2$ with adapted frame $\left\{\boldsymbol{\vartheta}, \eta_1, \eta_2\right\}$. Then, γ is called a framed slant helix if its generalized principal normal vector η_1 makes a constant angle with a fixed unit vector $\boldsymbol{\zeta}$. That is, $\left\langle \eta_1, \zeta \right\rangle = \cos\phi$ where $\phi \neq \pi/2$ is a constant angle [28].*

Theorem 2.12. *Let $\left(\gamma, \eta_1, \eta_2\right)$ be a framed curve in $\mathbb{R}^3 \times \Delta_2$ with framed curvature $(\mathfrak{p}, \mathfrak{q}, \mathfrak{a})$. Then, γ is a framed slant helix if and only if*

$$\sigma_1 = \frac{\mathfrak{p}^2\left(\frac{\mathfrak{q}}{\mathfrak{p}}\right)^{\bullet}}{\left(\mathfrak{p}^2 + \mathfrak{q}^2\right)^{3/2}} = \frac{\dot{\mathfrak{h}}}{\mathfrak{p}\left(1 + \mathfrak{h}^2\right)^{3/2}} \tag{4}$$

is a constant function [28].

2.2. *Construction Method for Alternative Frames of Framed Curves*

Firstly, we define the framed Darboux vector (or framed centrode) and the framed co-Darboux vector (or framed co-centrode) with respect to the adapted frame $\left\{\boldsymbol{\vartheta}, \eta_1, \eta_2\right\}$ of framed curve γ, respectively.

Definition 2.13. *Let $\left(\gamma, \eta_1, \eta_2\right)$ be a framed curve in $\mathbb{R}^3 \times \Delta_2$ with adapted frame apparatus $\left\{\boldsymbol{\vartheta}, \eta_1, \eta_2, (\mathfrak{p}, \mathfrak{q}, \mathfrak{a})\right\}$. Then, the framed Darboux (type-1 Darboux) vector of the framed curve γ is defined by $\boldsymbol{\Omega}(t) = \mathfrak{q}(t)\boldsymbol{\vartheta}(t) + \mathfrak{p}(t)\eta_2(t)$ which is satisfying the following equations:*

$$\dot{\boldsymbol{\vartheta}}(t) = \boldsymbol{\Omega}(t) \wedge \boldsymbol{\vartheta}(t), \quad \dot{\eta}_1(t) = \boldsymbol{\Omega}(t) \wedge \eta_1(t), \quad \dot{\eta}_2(t) = \boldsymbol{\Omega}(t) \wedge \eta_2(t).$$

Moreover, we call that

$$\mathbf{\Omega}_1(t) = \frac{\mathfrak{q}(t)\boldsymbol{\vartheta}(t) + \mathfrak{p}(t)\boldsymbol{\eta}_2(t)}{\sqrt{\mathfrak{p}^2(t) + \mathfrak{q}^2(t)}} \tag{5}$$

is the unit framed Darboux vector of γ.

Definition 2.14. *Let $(\gamma, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ be a framed curve in $\mathbb{R}^3 \times \Delta_2$ with adapted frame apparatus $\{\boldsymbol{\vartheta}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, (\mathfrak{p}, \mathfrak{q}, \mathfrak{a})\}$. Then, the framed co-Darboux (type-1 co-Darboux) vector of the framed curve γ is defined by $\widehat{\mathbf{\Omega}}(t) = -\mathfrak{p}(t)\boldsymbol{\vartheta}(t) + \mathfrak{q}(t)\boldsymbol{\eta}_2(t)$. Moreover, we call that*

$$\widehat{\mathbf{\Omega}}_1(t) = \frac{-\mathfrak{p}(t)\boldsymbol{\vartheta}(t) + \mathfrak{q}(t)\boldsymbol{\eta}_2(t)}{\sqrt{\mathfrak{p}^2(t) + \mathfrak{q}^2(t)}} \tag{6}$$

is the unit framed co-Darboux vector of γ. Also, it is easy to see that $\widehat{\mathbf{\Omega}}_1 = \frac{\dot{\boldsymbol{\eta}}_1}{\|\dot{\boldsymbol{\eta}}_1\|}$.

Now, since $\mathbf{\Omega}_1 = \boldsymbol{\eta}_1 \wedge \widehat{\mathbf{\Omega}}_1$ by Definition 2.13 and Definition 2.14, we get a new orthonormal frame $\{\boldsymbol{\eta}_1, \widehat{\mathbf{\Omega}}_1, \mathbf{\Omega}_1\}$ which is called an alternative (1-alternative) adapted frame along the framed curve γ. By using (2) and (3), we obtain the following equations:

$$\dot{\boldsymbol{\eta}}_1 = \mathfrak{f}_1\widehat{\mathbf{\Omega}}_1, \quad \dot{\widehat{\mathbf{\Omega}}}_1 = -\mathfrak{f}_1\boldsymbol{\eta}_1 + \mathfrak{g}_1\mathbf{\Omega}_1, \quad \dot{\mathbf{\Omega}}_1 = -\mathfrak{g}_1\widehat{\mathbf{\Omega}}_1, \tag{7}$$

where

$$\begin{cases} \mathfrak{f}_1 = \left\langle \dot{\boldsymbol{\eta}}_1, \widehat{\mathbf{\Omega}}_1 \right\rangle = \sqrt{\mathfrak{p}^2 + \mathfrak{q}^2} = \mathfrak{p}\sqrt{1 + \mathfrak{h}^2} > 0, \\ \mathfrak{g}_1 = \left\langle \dot{\widehat{\mathbf{\Omega}}}_1, \mathbf{\Omega}_1 \right\rangle = \dfrac{\mathfrak{p}^2\left(\frac{\mathfrak{q}}{\mathfrak{p}}\right)^{\bullet}}{\mathfrak{p}^2 + \mathfrak{q}^2} = \dfrac{\dot{\mathfrak{h}}}{1 + \mathfrak{h}^2} = \sigma_1\mathfrak{f}_1, \\ \mathfrak{a} = \left\langle \dot{\gamma}, \boldsymbol{\vartheta} \right\rangle, \end{cases} \tag{8}$$

and σ_1 is given by (4). The triple smooth functions $(\mathfrak{f}_1, \mathfrak{g}_1, \mathfrak{a})$ are called the framed curvature with respect to the alternative adapted frame $\{\boldsymbol{\eta}_1, \widehat{\mathbf{\Omega}}_1, \mathbf{\Omega}_1\}$ along the framed curve γ. Thus, we call that $\{\boldsymbol{\eta}_1, \widehat{\mathbf{\Omega}}_1, \mathbf{\Omega}_1, (\mathfrak{f}_1, \mathfrak{g}_1, \mathfrak{a})\}$ is an alternative adapted frame apparatus of the framed curve γ.

Now, let's construct an another alternative adapted frame which is generated by $\{\boldsymbol{\eta}_1, \widehat{\mathbf{\Omega}}_1, \mathbf{\Omega}_1\}$ along the framed curve γ. The following vectors:

$$\mathbf{\Omega}_2(t) = \frac{\mathfrak{g}_1(t)\boldsymbol{\eta}_1(t) + \mathfrak{f}_1(t)\mathbf{\Omega}_1(t)}{\sqrt{\mathfrak{f}_1{}^2(t) + \mathfrak{g}_1{}^2(t)}},$$

$$\widehat{\mathbf{\Omega}}_2(t) = \frac{\dot{\widehat{\mathbf{\Omega}}}_1(t)}{\left\|\dot{\widehat{\mathbf{\Omega}}}_1(t)\right\|} = \frac{-\mathfrak{f}_1(t)\boldsymbol{\eta}_1(t) + \mathfrak{g}_1(t)\mathbf{\Omega}_1(t)}{\sqrt{\mathfrak{f}_1{}^2(t) + \mathfrak{g}_1{}^2(t)}},$$

are called the unit framed type-2 Darboux and the unit framed type-2 co-Darboux vectors of the framed curve γ with respect to the 1-alternative adapted frame $\{\boldsymbol{\eta}_1, \widehat{\mathbf{\Omega}}_1, \mathbf{\Omega}_1\}$, respectively. Then, we construct a new orthonormal frame $\left\{\widehat{\mathbf{\Omega}}_1, \widehat{\mathbf{\Omega}}_2 = \frac{\dot{\widehat{\mathbf{\Omega}}}_1}{\|\dot{\widehat{\mathbf{\Omega}}}_1\|}, \mathbf{\Omega}_2 = \widehat{\mathbf{\Omega}}_1 \wedge \widehat{\mathbf{\Omega}}_2\right\}$ which is called a 2-alternative adapted frame along the framed curve γ. Similarly to the previous method, by using (7) and (8), we obtain the following equations:

$$\dot{\widehat{\mathbf{\Omega}}}_1 = \mathfrak{f}_2\widehat{\mathbf{\Omega}}_2, \quad \dot{\widehat{\mathbf{\Omega}}}_2 = -\mathfrak{f}_2\widehat{\mathbf{\Omega}}_1 + \mathfrak{g}_2\mathbf{\Omega}_2, \quad \dot{\mathbf{\Omega}}_2 = -\mathfrak{g}_2\widehat{\mathbf{\Omega}}_2, \tag{9}$$

where

$$\begin{cases} \mathfrak{f}_2 = \left\langle \dot{\widehat{\boldsymbol{\Omega}}}_1, \widehat{\boldsymbol{\Omega}}_2 \right\rangle = \sqrt{\mathfrak{f}_1{}^2 + \mathfrak{g}_1{}^2} = \mathfrak{f}_1 \sqrt{1+\sigma_1{}^2} > 0, \\ \mathfrak{g}_2 = \left\langle \dot{\widehat{\boldsymbol{\Omega}}}_2, \boldsymbol{\Omega}_2 \right\rangle = \dfrac{\mathfrak{f}_1{}^2 \left(\frac{\mathfrak{g}_1}{\mathfrak{f}_1}\right)^{\bullet}}{\mathfrak{f}_1{}^2 + \mathfrak{g}_1{}^2} = \dfrac{\dot{\sigma}_1}{1+\sigma_1{}^2} = \sigma_2 \mathfrak{f}_2, \\ \mathfrak{a} = \left\langle \dot{\gamma}, \vartheta \right\rangle, \end{cases} \tag{10}$$

such that $\sigma_2 = \frac{\mathfrak{f}_1{}^2\left(\frac{\mathfrak{g}_1}{\mathfrak{f}_1}\right)^{\bullet}}{\mathfrak{f}_2{}^3} = \frac{\dot{\sigma}_1}{\mathfrak{f}_1(1+\sigma_1{}^2)^{\frac{3}{2}}}$. Thus, we call that $\left\{\widehat{\boldsymbol{\Omega}}_1, \widehat{\boldsymbol{\Omega}}_2, \boldsymbol{\Omega}_2, (\mathfrak{f}_2, \mathfrak{g}_2, \mathfrak{a})\right\}$ is a type-2 alternative adapted frame apparatus of the framed curve γ.

Now, as a generalization, we can give the definition of (k+1)-alternative adapted frame, which is generated by the k-alternative adapted frame of the framed curve γ. Let's give the following vectors:

$$\boldsymbol{\Omega}_{k+1}(t) = \frac{\mathfrak{g}_k(t)\widehat{\boldsymbol{\Omega}}_{k-1}(t) + \mathfrak{f}_k(t)\boldsymbol{\Omega}_k(t)}{\sqrt{\mathfrak{f}_k{}^2(t) + \mathfrak{g}_k{}^2(t)}},$$

$$\widehat{\boldsymbol{\Omega}}_{k+1}(t) = \frac{\dot{\widehat{\boldsymbol{\Omega}}}_k(t)}{\left\|\dot{\widehat{\boldsymbol{\Omega}}}_k(t)\right\|} = \frac{-\mathfrak{f}_k(t)\widehat{\boldsymbol{\Omega}}_{k-1}(t) + \mathfrak{g}_k(t)\boldsymbol{\Omega}_k(t)}{\sqrt{\mathfrak{f}_k{}^2(t) + \mathfrak{g}_k{}^2(t)}},$$

are called the unit framed type-(k+1) Darboux and the unit framed type-(k+1) co-Darboux vectors of the framed curve γ with respect to the k-alternative adapted frame $\left\{\widehat{\boldsymbol{\Omega}}_{k-1}, \widehat{\boldsymbol{\Omega}}_k, \boldsymbol{\Omega}_k\right\}$ for $k \in \mathbb{N}$, respectively. Then,

$$\left\{\widehat{\boldsymbol{\Omega}}_k, \widehat{\boldsymbol{\Omega}}_{k+1} = \frac{\dot{\widehat{\boldsymbol{\Omega}}}_k}{\left\|\dot{\widehat{\boldsymbol{\Omega}}}_k\right\|}, \boldsymbol{\Omega}_{k+1} = \widehat{\boldsymbol{\Omega}}_k \wedge \widehat{\boldsymbol{\Omega}}_{k+1}\right\}$$

is called a (k+1)-alternative adapted frame along the framed curve γ. Similarly, we have

$$\dot{\widehat{\boldsymbol{\Omega}}}_k = \mathfrak{f}_{k+1}\widehat{\boldsymbol{\Omega}}_{k+1}, \quad \dot{\widehat{\boldsymbol{\Omega}}}_{k+1} = -\mathfrak{f}_{k+1}\widehat{\boldsymbol{\Omega}}_k + \mathfrak{g}_{k+1}\boldsymbol{\Omega}_{k+1}, \quad \dot{\boldsymbol{\Omega}}_{k+1} = -\mathfrak{g}_{k+1}\widehat{\boldsymbol{\Omega}}_{k+1}, \tag{11}$$

where

$$\begin{cases} \mathfrak{f}_{k+1} = \left\langle \dot{\widehat{\boldsymbol{\Omega}}}_k, \widehat{\boldsymbol{\Omega}}_{k+1} \right\rangle = \sqrt{\mathfrak{f}_k{}^2 + \mathfrak{g}_k{}^2} = \mathfrak{f}_k \sqrt{1+\sigma_k{}^2} > 0, \\ \mathfrak{g}_{k+1} = \left\langle \dot{\widehat{\boldsymbol{\Omega}}}_{k+1}, \boldsymbol{\Omega}_{k+1} \right\rangle = \dfrac{\mathfrak{f}_k{}^2 \left(\frac{\mathfrak{g}_k}{\mathfrak{f}_k}\right)^{\bullet}}{\mathfrak{f}_k{}^2 + \mathfrak{g}_k{}^2} = \dfrac{\dot{\sigma}_k}{1+\sigma_k{}^2} = \sigma_{k+1} \mathfrak{f}_{k+1}, \\ \mathfrak{a} = \left\langle \dot{\gamma}, \vartheta \right\rangle, \end{cases} \tag{12}$$

such that

$$\sigma_{k+1} = \frac{\mathfrak{f}_k{}^2\left(\frac{\mathfrak{g}_k}{\mathfrak{f}_k}\right)^{\bullet}}{\mathfrak{f}_{k+1}{}^3} = \frac{\dot{\sigma}_k}{\mathfrak{f}_k(1+\sigma_k{}^2)^{\frac{3}{2}}}.$$

Thus, we call that $\left\{\widehat{\boldsymbol{\Omega}}_k, \widehat{\boldsymbol{\Omega}}_{k+1}, \boldsymbol{\Omega}_{k+1}, (\mathfrak{f}_{k+1}, \mathfrak{g}_{k+1}, \mathfrak{a})\right\}$ is a (k+1)-alternative adapted frame apparatus of the framed curve γ.

Remark 2.15. *In particular, we use the following notations for the 0-alternative adapted frame, which corresponds to the adapted frame of the framed curve γ:*

$$\left\{\widehat{\boldsymbol{\Omega}}_{-1} = \vartheta, \widehat{\boldsymbol{\Omega}}_0 = \boldsymbol{\eta}_1, \boldsymbol{\Omega}_0 = \boldsymbol{\eta}_2\right\}, \quad \mathfrak{f}_0 = \mathfrak{p}, \ \mathfrak{g}_0 = \mathfrak{q}, \ \sigma_0 = \mathfrak{h}. \tag{13}$$

2.3. Framed Spherical Indicatrices of Framed Curves

Now, we must give the following proposition before giving definitions of some framed spherical indicatrices of any framed curve γ in $\mathbb{R}^3$.

Proposition 2.16. $\left(\mathbf{\Omega}_k, \widehat{\mathbf{\Omega}}_{k+1}, \mathbf{\Omega}_{k+1}\right)$, $\left(\widehat{\mathbf{\Omega}}_k, \widehat{\mathbf{\Omega}}_{k+2}, \mathbf{\Omega}_{k+2}\right)$ *are framed spherical curves in* $\mathbb{S}^2 \times \Delta_2$ *for* $k \in \mathbb{N}$.

Proof. By using the (k+2)-alternative adapted frame $\left\{\widehat{\mathbf{\Omega}}_{k+1}, \widehat{\mathbf{\Omega}}_{k+2}, \mathbf{\Omega}_{k+2}\right\}$ of the framed curve γ, it is easy to see that $\left(\mathbf{\Omega}_k, \widehat{\mathbf{\Omega}}_{k+1}, \mathbf{\Omega}_{k+1}\right)$ and $\left(\widehat{\mathbf{\Omega}}_k, \widehat{\mathbf{\Omega}}_{k+2}, \mathbf{\Omega}_{k+2}\right)$ satisfy the framed curve conditions (Definition 2.1). Also, since $\mathbf{\Omega}_k$ and $\widehat{\mathbf{\Omega}}_k$ are in $\mathbb{S}^2$, the proof is clear. □

Now, we give the following definition of framed spherical indicatrices of the framed curve γ by using notations (13) and Proposition 2.16.

Definition 2.17. *Let* $\left(\gamma, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2\right)$ *be a framed curve in* $\mathbb{R}^3 \times \Delta_2$ *with adapted frame apparatus* $\left\{\boldsymbol{\vartheta}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, (\mathfrak{p}, \mathfrak{q}, \mathfrak{a})\right\}$ *and unit vectors* $\mathbf{\Omega}_1, \widehat{\mathbf{\Omega}}_1$ *be given by* (5),(6), *respectively. Then, the following framed spherical curves:*

- $\left(\boldsymbol{\vartheta}, \widehat{\mathbf{\Omega}}_1, \mathbf{\Omega}_1\right), \left(\boldsymbol{\eta}_1, \widehat{\mathbf{\Omega}}_2, \mathbf{\Omega}_2\right), \left(\boldsymbol{\eta}_2, \widehat{\mathbf{\Omega}}_1, \mathbf{\Omega}_1\right)$ *in* $\mathbb{S}^2 \times \Delta_2$ *are called geneneralized tangent (or framed* $\boldsymbol{\vartheta}-$*) indicatrix, generalized principal normal (or framed* $\boldsymbol{\eta}_1-$*) indicatrix, generalized binormal (or framed* $\boldsymbol{\eta}_2-$*)indicatrix of framed curve* γ*, respectively.*
- $\left(\mathbf{\Omega}_1, \widehat{\mathbf{\Omega}}_2, \mathbf{\Omega}_2\right)$ *and* $\left(\widehat{\mathbf{\Omega}}_1, \widehat{\mathbf{\Omega}}_3, \mathbf{\Omega}_3\right)$ *in* $\mathbb{S}^2 \times \Delta_2$ *are called framed Darboux (or framed* $\mathbf{\Omega}_1-$*) indicatrix and framed co-Darboux (or framed* $\widehat{\mathbf{\Omega}}_1-$*) indicatrix of framed curve* γ*, respectively.*

Proposition 2.18. *Let* $\left(\boldsymbol{\vartheta}, \widehat{\mathbf{\Omega}}_1, \mathbf{\Omega}_1\right)$ *be framed* $\boldsymbol{\vartheta}-$*indicatrix of framed curve* γ*. Then, the adapted frame apparatus of framed* $\boldsymbol{\vartheta}-$*indicatrix is given by* $\left\{\boldsymbol{\eta}_1, \widehat{\mathbf{\Omega}}_1, \mathbf{\Omega}_1, (\mathfrak{f}_1, \mathfrak{g}_1, \mathfrak{p})\right\}$.

Proof. Let framed $\boldsymbol{\vartheta}-$indicatrix of γ be given by a framed spherical curve $\boldsymbol{\alpha} = \boldsymbol{\vartheta}$ in $\mathbb{S}^2$ with the adapted frame apparatus $\left\{\boldsymbol{\vartheta}_\alpha, \boldsymbol{\eta}_{1\alpha}, \boldsymbol{\eta}_{2\alpha}, (\mathfrak{p}_\alpha, \mathfrak{q}_\alpha, \mathfrak{a}_\alpha)\right\}$. If we start to taking the derivative of $\boldsymbol{\alpha} = \boldsymbol{\vartheta}$ and use equations (1) and (2), then we have

$$\mathfrak{a}_\alpha \boldsymbol{\vartheta}_\alpha = \mathfrak{p}\boldsymbol{\eta}_1.$$

From this equation, we get

$$\boldsymbol{\vartheta}_\alpha = \boldsymbol{\eta}_1, \tag{14}$$

such that $\mathfrak{a}_\alpha = \mathfrak{p} > 0$. By taking the derivative of the last equation and using equations (2),(3) and (6), we have

$$\mathfrak{p}_\alpha \boldsymbol{\eta}_{1\alpha} = \mathfrak{f}_1 \widehat{\mathbf{\Omega}}_1.$$

From the last equation, we get

$$\boldsymbol{\eta}_{1\alpha} = \widehat{\mathbf{\Omega}}_1, \tag{15}$$

such that $\mathfrak{p}_\alpha = \mathfrak{f}_1$. Later, since $\boldsymbol{\eta}_{2\alpha} = \boldsymbol{\vartheta}_\alpha \wedge \boldsymbol{\eta}_{1\alpha}$ with respect to the adapted frame of $\boldsymbol{\alpha}$ and $\mathbf{\Omega}_1 = \boldsymbol{\eta}_1 \wedge \widehat{\mathbf{\Omega}}_1$ with respect to the 1-alternative adapted frame of γ, we get

$$\boldsymbol{\eta}_{2\alpha} = \mathbf{\Omega}_1, \tag{16}$$

by using equations (14) and (15). Finally, since $\mathfrak{q}_\alpha = \left\langle \dot{\boldsymbol{\eta}}_{1\alpha}, \boldsymbol{\eta}_{2\alpha} \right\rangle$ by using equations (3) with respect to the adapted frame of $\boldsymbol{\alpha}$ and $\mathfrak{g}_1 = \left\langle \dot{\widehat{\mathbf{\Omega}}}_1, \widehat{\mathbf{\Omega}}_1 \right\rangle$ with respect to the 1-alternative adapted frame of γ, it leads to $\mathfrak{q}_\alpha = \mathfrak{g}_1$ by using equations (15) and (16). □

Proposition 2.19. *Let $\left(\boldsymbol{\eta}_1, \widehat{\boldsymbol{\Omega}}_2, \boldsymbol{\Omega}_2\right)$ be framed $\boldsymbol{\eta}_1$−indicatrix of framed curve $\boldsymbol{\gamma}$. Then, the adapted frame apparatus of framed $\boldsymbol{\eta}_1$−indicatrix is given by $\left\{\widehat{\boldsymbol{\Omega}}_1, \widehat{\boldsymbol{\Omega}}_2, \boldsymbol{\Omega}_2, (\mathfrak{f}_2, \mathfrak{g}_2, \mathfrak{f}_1)\right\}$.*

Proof. Let framed $\boldsymbol{\eta}_1$−indicatrix of $\boldsymbol{\gamma}$ be given by a framed spherical curve $\boldsymbol{\beta} = \boldsymbol{\eta}_1$ in $\mathbb{S}^2$ with the adapted frame apparatus $\left\{\boldsymbol{\vartheta}_\beta, \boldsymbol{\eta}_{1\beta}, \boldsymbol{\eta}_{2\beta}, \left(\mathfrak{p}_\beta, \mathfrak{q}_\beta, \mathfrak{a}_\beta\right)\right\}$. Then, the proof is clear by using the technique similar to the proof of Proposition 2.18 with equations (7),(8),(9) and (10). □

Proposition 2.20. *Let $\left(\boldsymbol{\eta}_2, \pm\widehat{\boldsymbol{\Omega}}_1, \boldsymbol{\Omega}_1\right)$ be framed $\boldsymbol{\eta}_2$−indicatrix of framed curve $\boldsymbol{\gamma}$. Then, the adapted frame apparatus of framed $\boldsymbol{\eta}_2$−indicatrix is given by $\left\{\pm\boldsymbol{\eta}_1, \pm\widehat{\boldsymbol{\Omega}}_1, \boldsymbol{\Omega}_1, (\mathfrak{f}_1, \pm\mathfrak{g}_1, \mp\mathfrak{q})\right\}$.*

Proof. Let framed $\boldsymbol{\eta}_2$−indicatrix of $\boldsymbol{\gamma}$ be given by a framed spherical curve $\boldsymbol{\xi} = \boldsymbol{\eta}_2$ in $\mathbb{S}^2$ with the adapted frame apparatus $\left\{\boldsymbol{\vartheta}_\xi, \boldsymbol{\eta}_{1\xi}, \boldsymbol{\eta}_{2\xi}, (\mathfrak{p}_\xi, \mathfrak{q}_\xi, \mathfrak{a}_\xi)\right\}$. Then, the proof is clear by using the technique similar to the proof of Proposition 2.18 with equations (7) and (8). □

Proposition 2.21. *Let $\left(\boldsymbol{\Omega}_1, \pm\widehat{\boldsymbol{\Omega}}_2, \boldsymbol{\Omega}_2\right)$ be framed $\boldsymbol{\Omega}_1$−indicatrix of framed curve $\boldsymbol{\gamma}$. Then, the adapted frame apparatus of framed $\boldsymbol{\Omega}_1$−indicatrix is given by $\left\{\pm\widehat{\boldsymbol{\Omega}}_1, \pm\widehat{\boldsymbol{\Omega}}_2, \boldsymbol{\Omega}_2, (\mathfrak{f}_2, \pm\mathfrak{g}_2, \mp\mathfrak{g}_1)\right\}$.*

Proof. Let framed $\boldsymbol{\Omega}_1$−indicatrix of $\boldsymbol{\gamma}$ be given by a framed spherical curve $\boldsymbol{\omega} = \boldsymbol{\Omega}_1$ in $\mathbb{S}^2$ with the adapted frame apparatus $\left\{\boldsymbol{\vartheta}_\omega, \boldsymbol{\eta}_{1\omega}, \boldsymbol{\eta}_{2\omega}, (\mathfrak{p}_\omega, \mathfrak{q}_\omega, \mathfrak{a}_\omega)\right\}$. Then, the proof is clear by using the technique similar to the proof of Proposition 2.18 with equations (7),(8),(9) and (10). □

Proposition 2.22. *Let $\left(\widehat{\boldsymbol{\Omega}}_1, \widehat{\boldsymbol{\Omega}}_3, \boldsymbol{\Omega}_3\right)$ be framed $\widehat{\boldsymbol{\Omega}}_1$−indicatrix of framed curve $\boldsymbol{\gamma}$. Then, the adapted frame apparatus of framed $\widehat{\boldsymbol{\Omega}}_1$−indicatrix is given by $\left\{\pm\widehat{\boldsymbol{\Omega}}_2, \pm\widehat{\boldsymbol{\Omega}}_3, \boldsymbol{\Omega}_3, (\mathfrak{f}_3, \mathfrak{g}_3, \mathfrak{f}_2)\right\}$.*

Proof. Let framed $\widehat{\boldsymbol{\Omega}}_1$−indicatrix of $\boldsymbol{\gamma}$ be given by a framed spherical curve $\hat{\boldsymbol{\omega}} = \widehat{\boldsymbol{\Omega}}_1$ in $\mathbb{S}^2$ with the adapted frame apparatus $\left\{\boldsymbol{\vartheta}_{\hat\omega}, \boldsymbol{\eta}_{1\hat\omega}, \boldsymbol{\eta}_{2\hat\omega}, (\mathfrak{p}_{\hat\omega}, \mathfrak{q}_{\hat\omega}, \mathfrak{a}_{\hat\omega})\right\}$. Then, the proof is clear by using the technique similar to the proof of Proposition 2.18 with equations (9), (10), (11) and (12). □

3. Framed Clad Helix and Its Spherical Indicatrices

In this section, as a generalization of the framed helix and framed slant helix, we introduce framed clad helix (or co-Darboux helix) in $\mathbb{R}^3$. After, we give a characterization of framed clad helix and give explicitly its axis with respect to its alternative adapted frame. Moreover, we obtain interesting results for framed spherical indicatrices of the framed clad helix.

Definition 3.1. *Let $\left(\boldsymbol{\gamma}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2\right)$ be a framed curve in $\mathbb{R}^3 \times \Delta_2$. Then $\boldsymbol{\gamma}$ is called a framed clad helix (or co-Darboux helix) with the axis $\boldsymbol{u}$ if its unit framed co-Darboux vector $\widehat{\boldsymbol{\Omega}}_1$ makes a constant angle $\phi \neq \frac{\pi}{2}$ with a fixed unit vector $\boldsymbol{u}$. That is,*

$$\left\langle \widehat{\boldsymbol{\Omega}}_1, \boldsymbol{u} \right\rangle = \cos\phi, \quad \phi = constant \neq \frac{\pi}{2} \tag{17}$$

along the framed curve $\boldsymbol{\gamma}$.

Theorem 3.2. *Let $\left(\boldsymbol{\gamma}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2\right)$ be a framed curve in $\mathbb{R}^3 \times \Delta_2$ with alternative adapted frame apparatus $\left\{\boldsymbol{\eta}_1, \widehat{\boldsymbol{\Omega}}_1, \boldsymbol{\Omega}_1, (\mathfrak{f}_1, \mathfrak{g}_1, \mathfrak{a})\right\}$. If $\boldsymbol{\gamma}$ is a framed clad helix then its axis $\boldsymbol{u}$ is given by*

$$\boldsymbol{u} = \left(\frac{\mathfrak{f}_1 \sigma_1 \left(1 + {\sigma_1}^2\right)}{\dot{\sigma}_1} \boldsymbol{\eta}_1 + \widehat{\boldsymbol{\Omega}}_1 + \frac{\mathfrak{f}_1 \left(1 + {\sigma_1}^2\right)}{\dot{\sigma}_1} \boldsymbol{\Omega}_1 \right) \cos\phi, \tag{18}$$

where σ_1 is given by (4).

Proof. Suppose that γ is a framed clad helix with axis $\boldsymbol{u}$. Then, we have $\left\langle \widehat{\boldsymbol{\Omega}}_1, \boldsymbol{u} \right\rangle = \cos\phi$. After successive differentiation of this equation and by using equations (7) and (17), we get the following equations

$$\begin{cases} -\mathfrak{f}_1 \left\langle \boldsymbol{\eta}_1, \boldsymbol{u} \right\rangle + \mathfrak{g}_1 \left\langle \boldsymbol{\Omega}_1, \boldsymbol{u} \right\rangle = 0, \\ -\dot{\mathfrak{f}}_1 \left\langle \boldsymbol{\eta}_1, \boldsymbol{u} \right\rangle + \dot{\mathfrak{g}}_1 \left\langle \boldsymbol{\Omega}_1, \boldsymbol{u} \right\rangle = \left(\mathfrak{f}_1{}^2 + \mathfrak{g}_1{}^2 \right) \cos\phi, \end{cases} \tag{19}$$

respectively. After the solution of system (19) and by using equations (10), we have

$$\begin{aligned} \left\langle \boldsymbol{\eta}_1, \boldsymbol{u} \right\rangle &= \frac{\mathfrak{g}_1 \left(\mathfrak{f}_1{}^2 + \mathfrak{g}_1{}^2 \right)}{\mathfrak{f}_1{}^2 \left(\frac{\mathfrak{g}_1}{\mathfrak{f}_1} \right)^{\bullet}} \cos\phi = \frac{\mathfrak{f}_1 \sigma_1 \left(1 + \sigma_1{}^2 \right)}{\dot{\sigma}_1} \cos\phi, \\ \left\langle \boldsymbol{\Omega}_1, \boldsymbol{u} \right\rangle &= \frac{\mathfrak{f}_1{}^2 + \mathfrak{g}_1{}^2}{\mathfrak{f}_1 \left(\frac{\mathfrak{g}_1}{\mathfrak{f}_1} \right)^{\bullet}} \cos\phi = \frac{\mathfrak{f}_1 \left(1 + \sigma_1{}^2 \right)}{\dot{\sigma}_1} \cos\phi, \end{aligned} \tag{20}$$

where σ_1 is given by (4). Hence, it is clear that the axis $\boldsymbol{u}$ has the form of (18) from equations (17) and (20). □

Theorem 3.3. *Let $\left(\gamma, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2\right)$ be a framed curve in $\mathbb{R}^3 \times \Delta_2$ with alternative adapted frame apparatus $\left\{ \boldsymbol{\eta}_1, \widehat{\boldsymbol{\Omega}}_1, \boldsymbol{\Omega}_1, (\mathfrak{f}_1, \mathfrak{g}_1, \mathfrak{a}) \right\}$. Then, γ is a framed clad helix if and only if*

$$\sigma_2 = \frac{\dot{\sigma}_1}{\mathfrak{f}_1 (1 + \sigma_1{}^2)^{\frac{3}{2}}} \tag{21}$$

is a constant function where σ_1 is given by (4).

Proof. Suppose that γ is a framed clad helix with axis $\boldsymbol{u}$. Then, its axis $\boldsymbol{u}$ has the form of (18) with respect to the alternative adapted frame of γ. Also, since $\boldsymbol{u}$ is a fixed unit vector, we get

$$\sigma_2 = \frac{\dot{\sigma}_1}{\mathfrak{f}_1 (1 + \sigma_1{}^2)^{\frac{3}{2}}} = \pm \cot\phi, \tag{22}$$

where ϕ is the constant angle between framed co-Darboux vector $\widehat{\boldsymbol{\Omega}}_1$ and $\boldsymbol{u}$. Thus, it is clear that σ_2 is a constant function which is given by (22).

On the other hand, suppose that σ_2 is the constant function that is given by (22) and $\boldsymbol{u}$ is a vector field that has the form of (18) where ϕ is a differentiable function. After differentiating (18) by using equations (7), we get

$$-\mathfrak{f}_1 \left\langle \boldsymbol{\eta}_1, \boldsymbol{u} \right\rangle + \mathfrak{g}_1 \left\langle \boldsymbol{\Omega}_1, \boldsymbol{u} \right\rangle = -\dot{\phi} \sin\phi, \tag{23}$$

Now, if we take into account the equations $\left\langle \boldsymbol{\eta}_1, \boldsymbol{u} \right\rangle$, and $\left\langle \boldsymbol{\Omega}_1, \boldsymbol{u} \right\rangle$ which have the form of (20) in equation (23), then $\dot{\phi} \sin\phi = 0$. That is, ϕ is a constant function. Thus, it leads to that the framed curve γ is a framed clad helix. □

Corollary 3.4. *Let $\left(\gamma, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2\right)$ be a framed curve in $\mathbb{R}^3 \times \Delta_2$ with adapted frame apparatus $\left\{ \boldsymbol{\vartheta}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, (\mathfrak{p}, \mathfrak{q}, \mathfrak{a}) \right\}$. Then γ is a framed clad helix if and only if the following statements are hold:*

(i) *framed $\boldsymbol{\vartheta}$–indicatrix of γ is a framed spherical slant helix in $\mathbb{S}^2 \times \Delta_2$,*

(ii) *framed $\boldsymbol{\eta}_1$–indicatrix of γ is a framed spherical helix in $\mathbb{S}^2 \times \Delta_2$,*

(iii) *framed $\boldsymbol{\eta}_2$–indicatrix of γ is a framed spherical slant helix in $\mathbb{S}^2 \times \Delta_2$,*

(iv) *framed $\boldsymbol{\Omega}_1$-indicatrix of γ is a framed spherical helix in $\mathbb{S}^2 \times \Delta_2$,* *(v) framed* $\widehat{\mathbf{\Omega}}_1$*−indicatrix of* γ *is a framed circle in* $\mathbb{S}^2 \times \Delta_2$.

Proof. Suppose that $\left(\gamma, \eta_1, \eta_2\right)$ is a framed clad helix in $\mathbb{R}^3 \times \Delta_2$ with adapted frame apparatus $\left\{\vartheta, \eta_1, \eta_2, (\mathfrak{p}, \mathfrak{q}, \mathfrak{a})\right\}$. Then, there exists a constant function σ_2, which is given by (21). Now, we give the parts of proof.

Proof of **(*i*)**: By Proposition 2.18, framed ϑ−indicatrix of γ is given by a framed spherical curve $\boldsymbol{\alpha} = \vartheta$ in $\mathbb{S}^2$ with framed curvature $(\mathfrak{p}_\alpha, \mathfrak{q}_\alpha, \mathfrak{a}_\alpha) = (\mathfrak{f}_1, \mathfrak{g}_1, \mathfrak{p})$. Then, by using equations (8), framed harmonic curvature of $\boldsymbol{\alpha}$ is

$$\mathfrak{h}_\alpha = \frac{\mathfrak{q}_\alpha}{\mathfrak{p}_\alpha} = \frac{\mathfrak{g}_1}{\mathfrak{f}_1} = \sigma_1.$$

Moreover, we see that

$$\sigma_{1\alpha} = \frac{\dot{\mathfrak{h}}_\alpha}{\mathfrak{p}_\alpha\left(1 + \mathfrak{h}_\alpha{}^2\right)^{\frac{3}{2}}} = \frac{\dot{\sigma}_1}{\mathfrak{f}_1(1 + \sigma_1{}^2)^{\frac{3}{2}}} = \sigma_2 = constant.$$

by using (4) and (21). Thus, the desired result is obtained by Theorem 2.12 and Theorem 3.3.

Proof of **(*ii*)**: By Proposition 2.19, framed η_1−indicatrix of γ is given by a framed spherical curve $\boldsymbol{\beta} = \eta_1$ in $\mathbb{S}^2$ with framed curvature $\left(\mathfrak{p}_\beta, \mathfrak{q}_\beta, \mathfrak{a}_\beta\right) = (\mathfrak{f}_2, \mathfrak{g}_2, \mathfrak{f}_1)$. Then, by using equations (10), framed harmonic curvature of $\boldsymbol{\beta}$ is

$$\mathfrak{h}_\beta = \frac{\mathfrak{q}_\beta}{\mathfrak{p}_\beta} = \frac{\mathfrak{g}_2}{\mathfrak{f}_2} = \sigma_2 = constant.$$

Thus, the proof is clear by Theorem 2.10.

Proof of **(*iii*)**: By Proposition 2.20, framed η_2−indicatrix of γ is given by a framed spherical curve $\boldsymbol{\xi} = \eta_2$ in $\mathbb{S}^2$ with framed curvature $(\mathfrak{p}_\xi, \mathfrak{q}_\xi, \mathfrak{a}_\xi) = (\mathfrak{f}_1, \varepsilon\mathfrak{g}_1, -\varepsilon\mathfrak{q})$ such that $\varepsilon = \pm 1$. As similar of the Proof of (**i**), we get

$$\mathfrak{h}_\xi = \frac{\mathfrak{q}_\xi}{\mathfrak{p}_\xi} = \frac{\varepsilon\mathfrak{g}_1}{\mathfrak{f}_1} = \varepsilon\sigma_1, \quad \sigma_{1\xi} = \frac{\dot{\mathfrak{h}}_\xi}{\mathfrak{p}_\xi\left(1 + \mathfrak{h}_\xi{}^2\right)^{\frac{3}{2}}} = \frac{\varepsilon\dot{\sigma}_1}{\mathfrak{f}_1(1 + \sigma_1{}^2)^{\frac{3}{2}}} = \varepsilon\sigma_2 = constant.$$

Thus, it leads to the desired result.

Proof of **(*iv*)**: By Proposition 2.21, framed $\mathbf{\Omega}_1$-indicatrix of γ is given by a framed spherical curve $\boldsymbol{\omega} = \mathbf{\Omega}_1$ in $\mathbb{S}^2$ with framed curvature $(\mathfrak{p}_\omega, \mathfrak{q}_\omega, \mathfrak{a}_\omega) = (\mathfrak{f}_2, \varepsilon\mathfrak{g}_2, -\varepsilon\mathfrak{g}_1)$. As similar of the Proof of (**ii**), we obtain

$$\mathfrak{h}_\omega = \frac{\mathfrak{q}_\omega}{\mathfrak{p}_\omega} = \frac{\varepsilon\mathfrak{g}_2}{\mathfrak{f}_2} = \varepsilon\sigma_2 = constant.$$

Thus, the proof is clear.

Proof of **(*v*)**: By Proposition 2.22, framed $\widehat{\mathbf{\Omega}}_1$−indicatrix of γ is given by a framed spherical curve $\hat{\boldsymbol{\omega}} = \widehat{\mathbf{\Omega}}_1$ in $\mathbb{S}^2$ with framed curvature $(\mathfrak{p}_{\hat{\omega}}, \mathfrak{q}_{\hat{\omega}}, \mathfrak{a}_{\hat{\omega}}) = (\mathfrak{f}_3, \mathfrak{g}_3, \mathfrak{f}_2)$. Then, by using equations (11) and (12) for $k = 3$ with respect to the 3-alternative adapted frame of γ, we get

$$\mathfrak{h}_{\hat{\omega}} = \frac{\mathfrak{q}_{\hat{\omega}}}{\mathfrak{p}_{\hat{\omega}}} = \frac{\mathfrak{g}_3}{\mathfrak{f}_3} = \sigma_3 = \frac{\dot{\sigma}_2}{\mathfrak{f}_2(1 + \sigma_2{}^2)^{\frac{3}{2}}} = 0.$$

Moreover, since $\mathfrak{p}_{\hat{\omega}} > 0$ with respect to the adapted frame of framed spherical curve $\hat{\boldsymbol{\omega}}$ and $\mathfrak{h}_{\hat{\omega}} = 0$ by the last equation, it must be $\mathfrak{q}_{\hat{\omega}} = 0$. As the last step, since $\mathfrak{a}_{\hat{\omega}} = \mathfrak{f}_2 > 0$, we see that the ratio

$$\frac{\mathfrak{p}_{\hat{\omega}}}{\mathfrak{a}_{\hat{\omega}}} = \frac{\mathfrak{f}_3}{\mathfrak{f}_2} = \frac{\mathfrak{f}_2\sqrt{1 + \sigma_2{}^2}}{\mathfrak{f}_2} = \sqrt{1 + \sigma_2{}^2} = constant.$$

by using equations (11). As a result, the desired result is obtained by Theorem 2.7. □

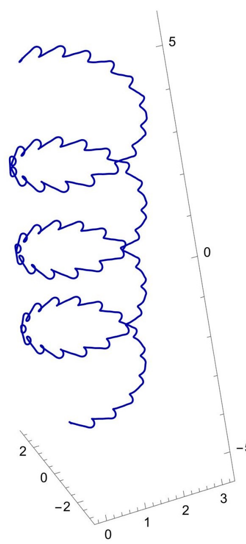
Remark 3.5. *As a regular curve in the classical sense, Ali obtained the explicit parametric representation of a 2-slant helix (i.e. clad helix) and calculated its curvatures in Euclidean 3-space [3]. In special case, we remark that when the framed curvature* (23) *corresponds to the curvature equations (37) of 2-slant helix given in [3] when the curve is unit speed regular (i.e.* $\mathfrak{a}(t) = 1$*). But the parametric representation contained a nested integral operator, which is not appear to be easy calculable. However, we have two options for plotting the graphics of regular clad helices. One is to use its parametrization with a numerical integration method, while the other is to use its curvatures for the numerical solution method of the Frenet differential equation system. Based on this ideas, as a regular or singular curves, we obtain congruent graphics of framed clad helices and their framed spherical indicatrices with given framed curvature (see Example 3.6 and Example 3.7).*

According to the Existence and Uniqueness Theorems of framed curves in [14], if the framed curvature of a framed curve is given, then we can draw a congruent graphic to the framed curve by applying the numerical solution method to Frenet-type differential equation system (1),(2),(3) with the initial conditions in Mathematica. Hence, let's conclude this section with an example.

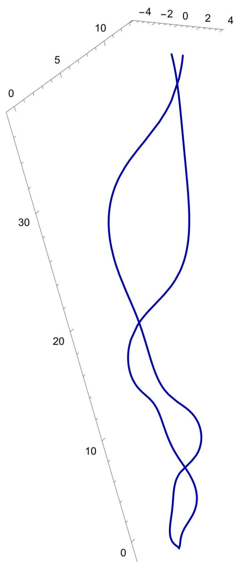
Example 3.6. *Let γ be a framed curve with framed curvature*

$$(\mathfrak{p}(t), \mathfrak{q}(t), \mathfrak{a}(t)) = \left(\frac{\lambda}{\mu}\cos(\lambda t)\cos\left(\frac{1}{\mu}\cos(\lambda t)\right), \frac{-\lambda}{\mu}\cos(\lambda t)\sin\left(\frac{1}{\mu}\cos(\lambda t)\right), \mathfrak{a}(t)\right),$$

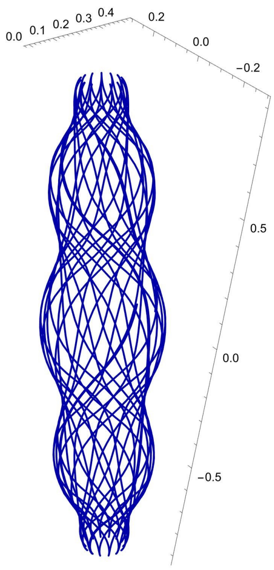
where λ, μ are nonzero constants and $\mathfrak{a}$ is a smooth function. Then, by using Theorem 3.3, we see that γ is a framed clad helix such that $\sigma_2 = \mu$ (see Figure 1).



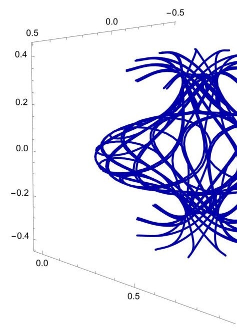
(a) $\lambda = 3$, $\mu = \frac{1}{4}$, $\mathfrak{a}(t) = 1$



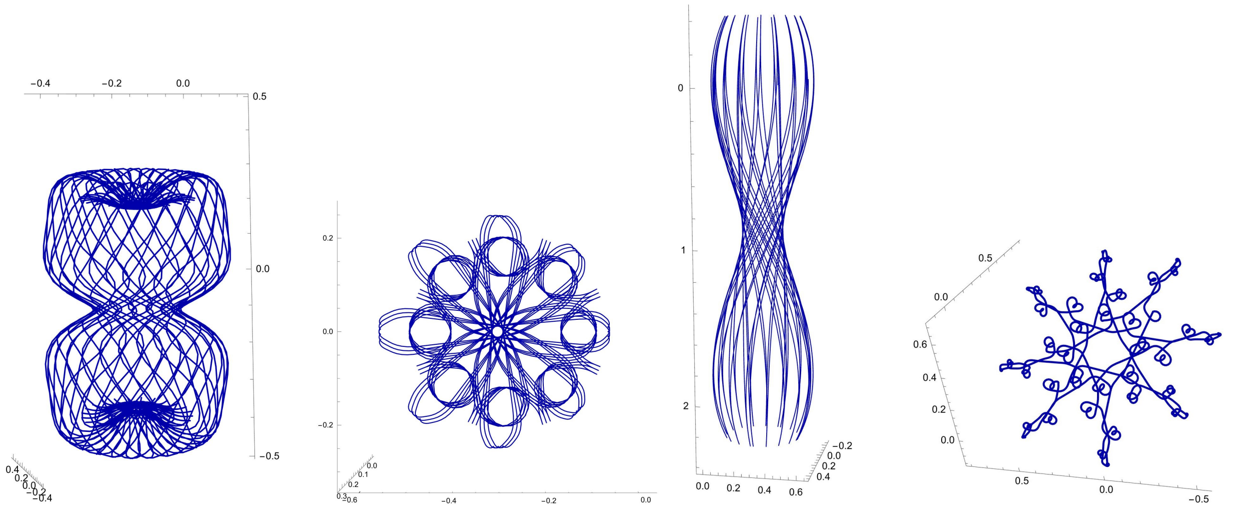
(b) $\lambda = 2$, $\mu = 1$, $\mathfrak{a}(t) = t$



(c) $\lambda = 5$, $\mu = 1$, $\mathfrak{a}(t) = \cos t$



(d) $\lambda = 1$, $\mu = \frac{1}{6}$, $\mathfrak{a}(t) = \cos t$



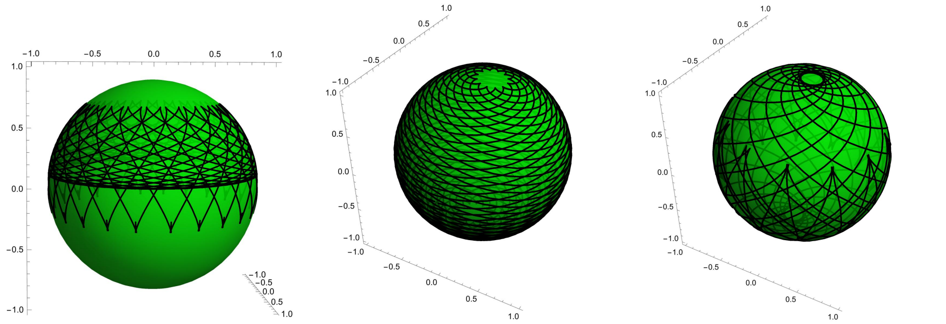
(e) $\lambda = 1$, $\mu = \frac{1}{12}$, $\mathfrak{a}(t) = \cos t$

(f) $\lambda = 3$, $\mu = \frac{1}{6}$, $\mathfrak{a}(t) = \sin 2t$

(g) $\lambda = 2$, $\mu = 2$, $\mathfrak{a}(t) = \cos t + \sin t$

(h) $\lambda = 5$, $\mu = \frac{1}{10}$, $\mathfrak{a}(t) = \cos t + \sin t$

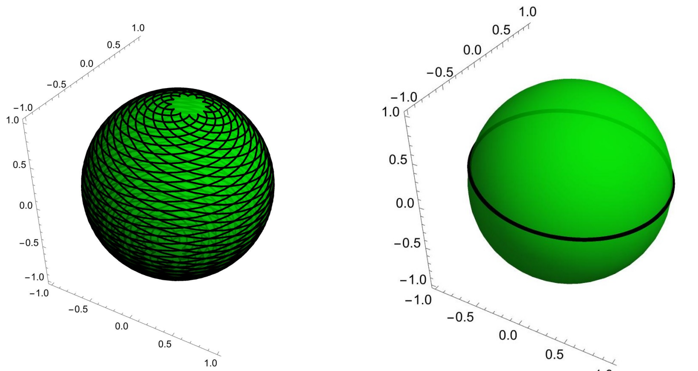
Figure 1: Framed clad helices are generated by some constants λ, μ and some smooth functions $\mathfrak{a}$.



(a) Framed ϑ−indicatrix of γ which is a framed slant helix such that $\sigma_{1\alpha} = \frac{1}{6}$.

(b) Framed η_1−indicatrix of γ which is a framed helix such that $\mathfrak{h}_\beta = \frac{1}{6}$.

(c) Framed η_2−indicatrix of γ which is a framed slant helix such that $\sigma_{1\xi} = \frac{1}{6}$.



(d) Framed $\mathbf{\Omega}_1$-indicatrix of γ which is a framed helix such that $\mathfrak{h}_\omega = \frac{1}{6}$.

(e) Framed $\widehat{\mathbf{\Omega}}_1$-indicatrix of γ which is a framed circle with the radius $\frac{6}{\sqrt{37}}$ such that $\frac{\mathfrak{p}_{\hat{\omega}}}{\mathfrak{a}_{\hat{\omega}}} = \frac{\sqrt{37}}{6}$ and $\mathfrak{q}_{\hat{\omega}} = 0$.

Figure 2: Some framed spherical images of the framed clad helix γ given in Figure 1f.

Finally, as an application of Corollary 3.4, we give an example of some framed spherical indicatrices of a framed clad helix.

Example 3.7. *Let γ be a framed clad helix with the framed curvature*

$$(\mathfrak{p}(t), \mathfrak{q}(t), \mathfrak{a}(t)) = (18\cos(3t)\cos(6\cos(3t)), -18\cos(3t)\sin(6\cos(3t)), \sin(2t)),$$

such that $\sigma_2 = \frac{1}{6}$ (see Figure 1f). Then, by using Propositions 2.18-2.22, we obtain framed spherical indicatrices of the framed clad helix γ (see Figure 2).

4. Conclusion

In this study, we introduce the concept of framed clad helix by generalizing the concept of clad helix [30] (or 2-slant helix [1]) in the theory of regular curves. Moreover, the family of framed clad helices, which may have singular points, encompasses the families of the framed helix and framed slant helix given in [15] and [28], respectively. In future studies, as an extension of concepts of helix, slant helix, and clad helix, we will introduce a family of framed *k*-slant helices by using the unit framed type-(k+1) co-Darboux vector in Euclidean 3-space.

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Conflict of interest

The author declares that there is no conflict of interest.

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