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Elliptic Kirchhoff-type system with two convections terms and under Dirichlet boundary conditions

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Abstract. This work discusses the existence of weak solutions for a system of Kirchhoff-type involving variable exponent $(\alpha_1(m), \alpha_2(m))$-Laplacian operators and under the Dirichlet boundary conditions. Under appropriate hypotheses on the nonlinear terms and the Kirchhoff functions, the existence of weak solutions is obtained on the spaces $W_0^{1,\alpha_1(m)}(\mathcal{D}) \times W_0^{1,\alpha_2(m)}(\mathcal{D})$. The proof of the main result is based on a topological degree argument for a class of demicontinuous operators of (S_+)-type.

1. Introduction

The study of Kirchhoff-type systems is an active area of research in nonlinear analysis and mathematical physics, such as the propagation of waves in media with variable density, stationary thermorheological viscous flows, electrorheological fluids and the dynamics of thin elastic plates [42, 43, 35, 37, 32, 33]. The survey of these systems involves the use of various techniques from nonlinear analysis, such as variational methods, critical point theory, and bifurcation theory. We mention that Kirchhoff [24] studied an extension of the D'Alembert wave equation for free vibrations of elastic strings, of the form

$$\rho \frac{\partial^2 v}{\partial t^2} - \Big(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \Big|\frac{\partial v}{\partial m}\Big|^2 dm\Big)\frac{\partial^2 v}{\partial m^2} = 0,$$

given a specific set of physical parameters, including mass density ρ, initial tension ρ_0, area of cross-section h, Young modulus of the material E, and length of the string L, for more information we refer to [16, 20, 8, 19, 34, 3].

In this paper, we aim to investigate the existence of weak solutions for a nonlinear elliptic system

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involving the $(\alpha_1(m), \alpha_2(m))$-Kirchhoff-Laplacian operator of the form

$$\begin{cases} -\mathcal{K}_1\Big(\int_{\mathcal{D}} \mathcal{I}_u^{\alpha_1} dm\Big)\Big(\Delta_{\alpha_1(m)}u - |u|^{\alpha_1(m)-2}u\Big) + \delta_1|u|^{p(m)-2}u = \lambda_1 f_1(m,u,\nabla u) + \mu_1 f_2(m,u) & \text{in } \mathcal{D}, \\ -\mathcal{K}_2\Big(\int_{\mathcal{D}} \mathcal{I}_v^{\alpha_2} dm\Big)\Big(\Delta_{\alpha_2(m)}v - |v|^{\alpha_2(m)-2}v\Big) + \delta_2|v|^{q(m)-2}v = \lambda_2 g_1(m,v,\nabla v) + \mu_2 g_2(m,v) & \text{in } \mathcal{D}, \\ u = v = 0 & \text{on } \partial\mathcal{D}, \end{cases} \tag{1.1}$$

where $\mathcal{I}_u^{\alpha_1}$ and $\mathcal{I}_v^{\alpha_2}$ are given by

$$\mathcal{I}_u^{\alpha_1} := \frac{|\nabla u|^{\alpha_1(m)} + |u|^{\alpha_1(m)}}{\alpha_1(m)} \quad \text{and} \quad \mathcal{I}_v^{\alpha_2} := \frac{|\nabla v|^{\alpha_2(m)} + |v|^{\alpha_2(m)}}{\alpha_2(m)},$$

with $\mathcal{D}$ is a bounded smooth domain in $\mathbb{R}^N (N \geq 2)$ and λ_i, δ_i, μ_i $(i = 1,2)$ are reel parameters, $\mathcal{K}_i$, f_i, $g_i (i = 1,2)$ are functions that test hypotheses that will be defined at a later stage.

The $\alpha(m)$-Laplacian operator $\Delta_{\alpha(m)}u := \operatorname{div}(|\nabla u|^{\alpha(m)-2}\nabla u)$ is a nonlinear partial differential operator that generalizes the classical α-Laplacian operator (see [29, 30, 11, 12, 13, 10]). The α-Laplacian is a well-known operator in mathematical analysis that has many important applications in various fields, including physics, engineering and finance. Motivated by [26, 28, 1, 27, 31, 36, 6, 5], our main object is to establish the existence of weak solutions for a nonlocal elliptic system involving the $(\alpha_1(m), \alpha_2(m))$-Kirchhoff-Laplacian operators depending on six real parameters under Dirichlet boundary condition with convection term, by using another approach based on a topological degree of Berkovits in the framework of Sobolev space with variable exponent.

Let us quickly summarize the contents of the paper. In the Section 2, we briefly review some basic preliminaries on the functional framework and we present several types of generalized (S_+) operators, together with the topological Berkovits degree. At last, in the Section 3, we state our assumptions together with technical lemmas and we also prove the main result.

2. Preliminaries

2.1. Variable exponent Sobolev spaces

In order to start discussing problem (1.1), we need some theories on spaces $W^{1,\alpha(m)}(\mathcal{D})$ which we call generalized Sobolev spaces. Let us shortly recall some basic facts about the setup for generalized Lebesgue-Sobolev spaces, for more information see for instance [22, 25, 40, 14] and [41]. Let

$$C_+(\overline{\mathcal{D}}) = \Big\{\alpha : \alpha \in C(\overline{\mathcal{D}}),\, \alpha(y) > 1 \text{ for every } y \in \overline{\mathcal{D}}\Big\}.$$

For any $\alpha \in C_+(\overline{\mathcal{D}})$, we establish

$$\alpha^+ := \max\Big\{\alpha(y),\ y \in \overline{\mathcal{D}}\Big\}\ ,\ \alpha^- := \min\Big\{\alpha(y),\ y \in \overline{\mathcal{D}}\Big\}.$$

For any $\alpha \in C_+(\overline{\mathcal{D}})$ we define the generalized Lebesgue spaces $L^{\alpha(m)}(\mathcal{D})$ by

$$L^{\alpha(m)}(\mathcal{D}) = \Big\{v : \mathcal{D} \to \mathbb{R} \text{ measurable fanction, } \int_{\mathcal{D}} |v(m)|^{\alpha(m)} dm < +\infty\Big\}.$$

We define the so-called Luxemburg norm, on this space by the formula

$$|v|_{\alpha(m)} = \inf\Big\{k > 0 \ |\ \rho_{\alpha(m)}\Big(\frac{v}{k}\Big) \leq 1\Big\},$$

where, the modular $\rho_{\alpha(m)} : L^{\alpha(m)}(\mathcal{D}) \to \mathbb{R}$ defiend by

$$\rho_{\alpha(m)}(v) = \int_{\mathcal{D}} |v(m)|^{\alpha(m)} dm, \ \forall v \in L^{\alpha(m)}(\mathcal{D}),$$

and verify some useful properties listed below.

Proposition 2.1. *[22, 9] Let (v_k) and $v \in L^{\alpha(m)}(\mathcal{D})$, then*

$$|v|_{\alpha(m)} < 1(resp. = 1; > 1) \ \Leftrightarrow \ \rho_{\alpha(m)}(v) < 1(resp. = 1; > 1) \ , \tag{2.1}$$

$$|v|_{\alpha(m)} > 1 \ \Rightarrow \ |v|^{\alpha^-}_{\alpha(m)} \leq \rho_{\alpha(m)}(v) \leq |v|^{\alpha^+}_{\alpha(m)}, \tag{2.2}$$

$$|v|_{\alpha(m)} < 1 \ \Rightarrow \ |v|^{\alpha^+}_{\alpha(m)} \leq \rho_{\alpha(m)}(v) \leq |v|^{\alpha^-}_{\alpha(m)}, \tag{2.3}$$

$$\lim_{k\to\infty} |v_k - v|_{\alpha(m)} = 0 \ \Leftrightarrow \ \lim_{k\to\infty} \rho_{\alpha(m)}(v_k - v) = 0. \tag{2.4}$$

Remark 2.2. *Notice that, by* (2.2), (2.3), *we can infer the inequalities*

$$|v|_{\alpha(m)} \leq \rho_{\alpha(m)}(v) + 1, \tag{2.5}$$

$$\rho_{\alpha(m)}(v) \leq |v|^{\alpha^-}_{\alpha(m)} + |v|^{\alpha^+}_{\alpha(m)}. \tag{2.6}$$

Proposition 2.3. *[25, 17] $L^{\alpha(m)}(\mathcal{D})$ is a separable and reflexive Banach spaces.*

Proposition 2.4. *[25, 15] Define $L^{\alpha'(m)}(\mathcal{D})$ as the space conjugate of $L^{\alpha(m)}(\mathcal{D})$, where $\frac{1}{\alpha(m)} + \frac{1}{\alpha'(m)} = 1$ for all $m \in \mathcal{D}$. For any $f \in L^{\alpha(m)}(\mathcal{D})$, and $g \in L^{\alpha'(m)}(\mathcal{D})$, we have the Hölder inequality*

$$\Big| \int_{\mathcal{D}} fg \, dm \Big| \leq (\frac{1}{\alpha-} + \frac{1}{\alpha'^-}) |f|_{\alpha(m)} |g|_{\alpha'(m)} \leq 2|f|_{\alpha(m)} |g|_{\alpha'(m)}. \tag{2.7}$$

Remark 2.5. *If $\alpha_1, \alpha_2 \in C_+(\overline{\mathcal{D}})$ with $\alpha_1(m) \leq \alpha_2(m)$ for any $m \in \overline{\mathcal{D}}$, then there exists the continuous embedding $L^{\alpha_2(m)}(\mathcal{D}) \hookrightarrow L^{\alpha_1(m)}(\mathcal{D})$.*

New, we defien the spaces $W^{1,\alpha(m)}(\mathcal{D})$ by

$$W^{1,\alpha(m)}(\mathcal{D}) = \Big\{ v \in L^{\alpha(m)}(\mathcal{D}) \ \Big| \ |\nabla v| \in L^{\alpha(m)}(\mathcal{D}) \Big\},$$

equipped with the norm

$$\|v\|_{\alpha(m)} = |v|_{\alpha(m)} + |\nabla v|_{\alpha(m)}.$$

We indicate by $W^{1,\alpha(m)}_0(\mathcal{D})$ the closure of $C^\infty_0(\mathcal{D})$ respect to the norm of $W^{1,\alpha(m)}(\mathcal{D})$.

Proposition 2.6. *[38] If the exponent $\alpha(m)$ satisfies the log-Hölder continuity condition, i.e. $\exists \ \tau > 0$ such that for each $a, \ b \in \mathcal{D}, \ a \neq b$ with $|a - b| \leq \dfrac{1}{2}$, we have*

$$|\alpha(a) - \alpha(b)| \leq \frac{\tau}{-\log |a - b|}, \tag{2.8}$$

then there exists a constant $C_{\mathcal{G},\alpha} > 0$ such that

$$|v|_{\alpha(m)} \leq C_{\mathcal{G},\alpha} |\nabla v|_{\alpha(m)}, \ \forall \ v \in W^{1,\alpha(m)}_0(\mathcal{D}). \tag{2.9}$$

We could use the following equivalent norm on $W_0^{1,\alpha(m)}(\mathcal{D})$

$$|v|_{1,\alpha(m)} = |\nabla v|_{\alpha(m)},$$

is equivalent to $\|v\|_{\alpha(m)}$.

Proposition 2.7. *[25, 21] The spaces $W^{1,\alpha(m)}(\mathcal{D})$ and $W_0^{1,\alpha(m)}(\mathcal{D})$ are separable and reflexive Banach spaces.*

Remark 2.8. *The dual space of $W_0^{1,\alpha(m)}(\mathcal{D})$ denoted $W^{-1,\alpha'(m)}(\mathcal{D})$, is equipped with the norm*

$$|v|_{-1,\alpha'(m)} = \inf\left\{|v_0|_{\alpha'(m)} + \sum_{j=1}^{N} |v_j|_{\alpha'(m)}\right\},$$

where the infinimum is performed on all possible separations $v = v_0 - \operatorname{div}\mathcal{L}$ with $v_0 \in L^{\alpha'(m)}(\mathcal{D})$ and $\mathcal{L} = (v_1, \dots, v_N) \in (L^{\alpha'(m)}(\mathcal{D}))^N$.

In the following discussions, we will use the product space

$$\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)} := W_0^{1,\alpha_1(m)}(\mathcal{D}) \times W_0^{1,\alpha_2(m)}(\mathcal{D}),$$

which is equipped with the norm

$$\|(u,v)\|_{\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}} = \max\left\{\|u\|_{\alpha_1(m)}, \|v\|_{\alpha_2(m)}\right\},$$

where $\|\cdot\|_{\alpha_1(m)}$ is the norm of $W_0^{1,\alpha_1(m)}(\mathcal{D})$ and $\|\cdot\|_{\alpha_2(m)}$ is the norm of $W_0^{1,\alpha_2(m)}(\mathcal{D})$.

The space $\left(\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}\right)^*$ is the dual space of $\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}$ corresponding to the Orlicz–Sobolev space $W_0^{-1,\alpha_1'(m)}(\mathcal{D}) \times W_0^{-1,\alpha_2'(m)}(\mathcal{D})$ equipped with the norm

$$\|\cdot\|_{\left(\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}\right)^*} := \max\{\|\cdot\|_{-1,\alpha_1'(m)}, \|\cdot\|_{-1,\alpha_2'(m)}\}.$$

The continuous pairing between the dual spaces $\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}$ and $\left(\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}\right)^*$ given by

$$\langle\cdot,\cdot\rangle_{\alpha_1,\alpha_2} = \langle\cdot,\cdot\rangle_{1,\alpha_1(m)} + \langle\cdot,\cdot\rangle_{1,\alpha_2(m)}.$$

2.2. Topological degree theory

Now, we would like to review a few definitions and fundamental characteristics of Berkovits degree theory for demicontinuous operators in a real reflexive space.
Let $\mathcal{G}$ be a real separable reflexive Banach space and $\mathcal{E}$ be a nonempty subset of $\mathcal{G}$. The symbol $\langle\cdot,\cdot\rangle_{\mathcal{G}}$ means what the usual dual paring between $\mathcal{G}^*$ and $\mathcal{G}$.

Definition 2.9. *Let E be a second real Banach space. A mapping $\mathcal{A} : \mathcal{E} \subset \mathcal{G} \to E$ is*

- *bounded, If it transforms any bounded set into a bounded set.*
- *demicontinuous, if for any $(v_k) \subset \mathcal{E}$, $v_k \to v$ then $\mathcal{A}(v_k) \rightharpoonup \mathcal{A}(v)$.*
- *compact, if $\mathcal{A}$ is continuous and for any $A \subset \mathcal{G}$ bounded we have $\mathcal{A}(A)$ is relatively compact.*

Definition 2.10. *An operator $\mathcal{A} : \mathcal{E} \subset \mathcal{G} \to \mathcal{G}^*$ is called*

- *of type (S_+), if for any $(v_k) \subset \mathcal{E}$ with $v_k \rightharpoonup v$ and $\limsup_{k\to\infty}\langle\mathcal{A}(v_k), v_k - v\rangle \leq 0$, we will have $v_k \to v$.*

- *quasimonotone, if for any sequence* $(v_k) \subset \mathcal{E}$ *with* $v_k \rightharpoonup v$, *we will have* $\limsup_{k\to\infty}\langle \mathcal{A}(v_k),\ v_k - v\rangle \geq 0.$

Definition 2.11. *Let* $\mathrm{Z} : \mathcal{E}_1 \subset \mathcal{G} \to \mathcal{G}^*$ *be a bounded mapping with* $\mathcal{E} \subset \mathcal{E}_1$. *For any operator* $\mathcal{A} : \mathcal{E} \subset \mathcal{G} \to \mathcal{G}$, *we claim that*

- $\mathcal{A}$ *satisfies condition* $(S_+)_{\mathrm{Z}}$, *if for any* $(v_k) \subset \mathcal{E}$ *with* $v_k \rightharpoonup v,\ u_k := \mathrm{Z}(v_k) \rightharpoonup u$ *and* $\limsup_{k\to\infty}\langle \mathcal{A}(v_k),\ u_k - u\rangle \leq 0$, *we have* $v_k \to v$.
- $\mathcal{A}$ *possess the property* $(QM)_{\mathrm{Z}}$, *if for any sequence* $(v_k) \subset \mathcal{E}$ *with* $v_k \rightharpoonup v,\ \ u_k := \mathrm{Z}(v_k) \rightharpoonup u$, *we have* $\limsup_{k\to\infty}\langle \mathcal{A}(v_k), u - u_k\rangle \geq 0$.

In the follow-up, we take into account the following groups of operators:

$$\mathcal{T}_1(\mathcal{E}) := \Big\{\mathcal{A} : \mathcal{E} \to \mathcal{G}^* \mid \mathcal{A} \text{ is bounded, demicontinuous and of type } (S_+)\Big\},$$
$$\mathcal{T}_{\mathrm{Z}}(\mathcal{E}) := \Big\{\mathcal{A} : \mathcal{E} \to \mathcal{G} \mid \mathcal{A} \text{ is demicontinuous, satifies condition } (S_+)_{\mathrm{Z}}\Big\},$$
$$\mathcal{T}_{\mathrm{Z},B}(\mathcal{E}) := \Big\{\mathcal{A} : \mathcal{E} \to \mathcal{G} \mid \mathcal{A} \text{ is bounded, demicontinuous and satifies condition} (S_+)_{\mathrm{Z}}\Big\},$$

for any $\mathcal{E} \subset D(\mathcal{A})$, where $D(\mathcal{A})$ indicates the domain of $\mathcal{A}$, and any $\mathrm{Z} \in \mathcal{T}_1(\mathcal{E})$.
Let Θ be the collection of all bounded open sets in $\mathcal{G}$.

$$\mathcal{T}(\mathcal{G}) := \{\mathcal{A} \in \mathcal{T}_{\mathrm{Z}}(\overline{\theta}) \mid \theta \in \Theta,\ \mathrm{Z} \in \mathcal{T}_1(\overline{\theta})\},$$

where, $\mathrm{Z} \in \mathcal{T}_1(\overline{\theta})$ is known as essential inner map to $\mathcal{A}$.

Lemma 2.12. *[23] Let* $\mathrm{Z} \in \mathcal{T}_1(\overline{\theta})$ *is continuous and* $\mathcal{J} : D(\mathcal{J}) \subset \mathcal{G}^* \to \mathcal{G}$ *is demicontinuous such that* $\mathrm{Z}(\overline{\theta}) \subset D(\mathcal{J})$, *where* θ *is a bounded open set in a real reflexive Banach space* $\mathcal{G}$. *Therefore, the assertions below are correct:*

- *If* $\mathcal{J}$ *is quasimonotone, then* $I + \mathcal{J} \circ \mathrm{Z} \in \mathcal{T}_{\mathrm{Z}}(\overline{\theta})$, *where the identity operator is indicated by* I.
- *If* $\mathcal{J}$ *is of class* (S_+), *then* $\mathcal{J} \circ \mathrm{Z} \in \mathcal{T}_{\mathrm{Z}}(\overline{\theta})$.

Definition 2.13. *Consider that* θ *is bounded open subset of a real reflexive Banach space* $\mathcal{G}$, $\mathrm{Z} \in \mathcal{T}_1(\overline{\theta})$ *is continuous and let* $\mathcal{A}, \mathcal{J} \in \mathcal{T}_{\mathrm{Z}}(\overline{\theta})$. *The affine homotopy* $\mathcal{H} : [0,1] \times \overline{\theta} \to \mathcal{G}$ *defined by*

$$\mathcal{H}(t,v) := (1-t)\mathcal{A}v + t\mathcal{J}v, \quad \text{for} \quad (t,v) \in [0,1] \times \overline{\theta}$$

is called an admissible affine homotopy with the common continuous essential inner map Z.

Remark 2.14. *[23, 18]* The above affine homotopy satisfies condition $(S_+)_{\mathrm{Z}}$.

Next, as in [23] we present the topological degree for the class $\mathcal{T}(\mathcal{G})$.

Theorem 2.15. *Let*

$$D = \Big\{(\mathcal{J}, \theta, h) \Big|\ \theta \in \Theta,\ \mathrm{Z} \in \mathcal{T}_1(\overline{\theta}),\ \mathcal{J} \in \mathcal{T}_{\mathrm{Z},B}(\overline{\theta}),\ h \notin \mathcal{J}(\partial\theta)\Big\}.$$

Hence, there exists a unique degree function $d : D \longrightarrow \mathbb{Z}$ *that satisfies the following properties:*

1. (Normalization) *For any* $h \in \mathcal{J}(\theta)$, *we find that*

$$d(I, \theta, h) = 1.$$

2. (Homotopy invariance) *If* $\mathcal{H} : [0,1] \times \overline{\theta} \to \mathcal{G}$ *is a bounded admissible affine homotopy with a common continuous essential inner map and h:* $[0,1] \to \mathcal{G}$ *is a continuous path in* $\mathcal{G}$ *such that* $h(t) \notin \mathcal{H}(t, \partial\theta)$ *for all* $t \in [0,1]$, *then*

$$d(\mathcal{H}(t,\cdot), \theta, h(t)) = C \text{ for all } t \in [0,1].$$

3. (Existence) *If* $d(\mathcal{J}, \theta, h) \neq 0$, *then the equation* $\mathcal{J}v = h$ *has a solution in* θ.

Definition 2.16. *[23] The above degree is defined as follows:*

$$d(\mathcal{J}, \theta, h) := d_B(\mathcal{J}|_{\overline{\theta}_0}, \theta_0, h),$$

where d_B *is the Berkovits degree [2] and* θ_0 *is any open subset of* θ *with* $\mathcal{J}^{-1}(h) \subset \theta_0$ *and* $\mathcal{J}$ *is bounded on* $\overline{\theta}_0$.

3. Hypotheses and Main results

In this section, we will discuss the existence of a weak solution of (1.1). To do so, we give the hypotheses related to our problem. Assumed $\mathcal{D} \subset \mathbb{R}^N (N \geq 2)$ to be a bounded domain with a Lipschitz boundary. In addition, we suppose that $\alpha_1, \alpha_2, \in C_+(\overline{\mathcal{D}})$ satisfying (2.8) and $p, q \in C_+(\overline{\mathcal{D}})$ with $2 \leq p^- \leq p(m) \leq p^+ < \alpha_1^-$ and $2 \leq q^- \leq q(m) \leq q^+ < \alpha_2^-$. For more detais we suppose the following assumptions:
(A_1) $f_1, g_1 : \mathcal{D} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $f_2, g_2 : \mathcal{D} \times \mathbb{R} \to \mathbb{R}$ are four functions satisfying the Carathéodory condition.
(A_2) There exists $C_1, C_2 > 0$ and $\gamma_1, \gamma_2 \in L^{\alpha_1(m)}(\mathcal{D})$ such that

$$|f_1(m, \eta, \varpi)| \leq C_1(\gamma_1(m) + |\eta|^{r_1(m)-1} + |\varpi|^{r_1(m)-1}) \quad \text{and} \quad |f_2(m, \eta)| \leq C_2(\gamma_2(m) + |\eta|^{r_2(m)-1}),$$

for a.e. $m \in \mathcal{D}$ and all $(\eta, \varpi) \in \mathbb{R} \times \mathbb{R}^N$, where $2 \leq r_1^- \leq r_1(m) \leq r_1^+ < \alpha_1^-$ and $2 \leq r_2^- \leq r_2(m) \leq r_2^+ < \alpha_1^-$.
(A_3) There exists $C_1', C_2' > 0$ and $\gamma_1', \gamma_2' \in L^{\alpha_1(m)}(\mathcal{D})$ such that

$$|g_1(m, \eta, \varpi)| \leq C_1'(\gamma_1'(m) + |\eta|^{s_1(m)-1} + |\varpi|^{s_1(m)-1}) \quad \text{and} \quad |g_2(m, \eta)| \leq C_2'(\gamma_2'(m) + |\eta|^{s_2(m)-1}),$$

for a.e. $m \in \mathcal{D}$ and all $(\eta, \varpi) \in \mathbb{R} \times \mathbb{R}^N$, where $2 \leq s_1^- \leq s_1(m) \leq s_1^+ < \alpha_2^-$ and $2 \leq s_2^- \leq s_2(m) \leq s_2^+ < \alpha_2^-$.
(M_0) We suppose that the Kirchhoff functions $\mathcal{K}_i : [0, +\infty) \to [0, +\infty)$ $(i = 1, 2)$ are continuous and increasing functions such that

$$a_1 t^{c_1(m)-1} \leq \mathcal{K}_1(t) \leq a_2 t^{c_1(m)-1},$$
$$b_1 t^{c_2(m)-1} \leq \mathcal{K}_2(t) \leq b_2 t^{c_2(m)-1},$$

where a_i, $b_i (i = 1, 2)$ are real numbers such that $a_1 \leq a_2$, $b_1 \leq b_2$, and $c_1(m)$, $c_2(m) \geq 1$.
In this paper, we will use the definition of the weak solution of (1.1) in the following sense:

Definition 3.1. *We say that* $(u, v) \in \mathcal{W}_0^{\alpha_1(m), \alpha_2(m)}$ *is a weak solution of* (1.1) *if*

$$\mathcal{K}_1\Big(\int_{\mathcal{D}} \mathcal{I}_u^{\alpha_1} \mathrm{d}m\Big) \int_{\mathcal{D}} \Big(|\nabla u|^{\alpha_1(m)-2} \nabla u \nabla \vartheta + |u|^{\alpha_1(m)-2} u \ \vartheta\Big) dm + \mathcal{K}_2\Big(\int_{\mathcal{D}} \mathcal{I}_v^{\alpha_2} \mathrm{d}m\Big) \int_{\mathcal{D}} \Big(|\nabla v|^{\alpha_2(m)-2} \nabla v \nabla \zeta + |v|^{\alpha_2(m)-2} v \ \zeta\Big) dm$$
$$= \int_{\mathcal{D}} \Big(-\delta_1 |u|^{p(m)-2} u + \lambda_1 f_1(m, u, \nabla u) + \mu_1 f_2(m, v)\Big) \vartheta dm + \int_{\mathcal{D}} \Big(-\delta_2 |v|^{q(m)-2} v + \lambda_2 g_1(m, v, \nabla v) + \mu_2 g_2(m, v)\Big) \zeta dm,$$

for each $(\vartheta, \zeta) \in \mathcal{W}_0^{\alpha_1(m), \alpha_2(m)}$.

We now present a lemma and a proposition that will be utilized in demonstrating the main result.

Lemma 3.2. *If* (M_0) *holds, then the operator* $\mathcal{S} : \mathcal{W}_0^{\alpha_1(m), \alpha_2(m)} \to \Big(\mathcal{W}_0^{\alpha_1(m), \alpha_2(m)}\Big)^*$ *defined by*

$$\langle \mathcal{S}(u, v), (\vartheta, \zeta) \rangle$$
$$= \mathcal{K}_1\Big(\int_{\mathcal{D}} \mathcal{I}_u^{\alpha_1} dm\Big) \int_{\mathcal{D}} \Big(|\nabla u|^{\alpha_1(m)-2} \nabla u \nabla \vartheta + |u|^{\alpha_1(m)-2} u \ \vartheta\Big) dm + \mathcal{K}_2\Big(\int_{\mathcal{D}} \mathcal{I}_v^{\alpha_2} dm\Big) \int_{\mathcal{D}} \Big(|\nabla v|^{\alpha_2(m)-2} \nabla v \nabla \zeta + |v|^{\alpha_2(m)-2} v \ \zeta\Big) dm,$$

is continuous, bounded, strictly monotone, coercive, and is of type (S_+).

Proof. Let $(u,v)\in \mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}$, the functional Ψ Defined on $\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}$ by

$$\Psi(u,v):=\widehat{\mathcal{K}_1}\Big(\int_{\mathcal{D}}\mathcal{I}_u^{\alpha_1}dm\Big)+\widehat{\mathcal{K}_2}\Big(\int_{\mathcal{D}}\mathcal{I}_v^{\alpha_2}dm\Big)$$

where $\widehat{\mathcal{K}_i}(s)=\displaystyle\int_0^s\mathcal{K}_i(\tau)\mathrm{d}\tau$, (i=1,2) is continuously Gateaux differentiable whose Gâteaux derivative at the point $(u,v)\in \mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}$ is the functional $\mathcal{S}:=\Psi'(u,v)\in\Big(\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}\Big)^*$, given by

$$\langle\mathcal{S}(u,v),(\vartheta,\zeta)\rangle=\langle\mathcal{S}_{\alpha_1}(u),(\vartheta)\rangle+\langle\mathcal{S}_{\alpha_2}(v),(\zeta)\rangle,$$

where

$$\langle\mathcal{S}_{\alpha_1}(u),\vartheta\rangle=\mathcal{K}_1\Big(\int_{\mathcal{D}}\mathcal{I}_u^{\alpha_1}dm\Big)\int_{\mathcal{D}}\Big(|\nabla u|^{\alpha_1(m)-2}\nabla u\nabla\vartheta+|u|^{\alpha_1(m)-2}u\ \vartheta\Big)dm,$$
$$\langle\mathcal{S}_{\alpha_2}(v),\zeta\rangle=\mathcal{K}_2\Big(\int_{\mathcal{D}}\mathcal{I}_v^{\alpha_2}dm\Big)\int_{\mathcal{D}}\Big(|\nabla v|^{\alpha_2(m)-2}\nabla v\nabla\zeta+|v|^{\alpha_2(m)-2}v\ \zeta\Big)dm.$$

So, $\mathcal{S}$ is continuons, bounded and since $\mathcal{S}_{\alpha_1}$ and $\mathcal{S}_{\alpha_2}$ are strictly monotone (see [7, Theorem 2.1]), then $\mathcal{S}$ is strictly monotone.
Let us show that $\mathcal{S}$ is coercive. By using (2.5) and (2.6), we have

$$\begin{aligned}&\frac{\langle\mathcal{S}(u,v),(u,v)\rangle}{\|(u,v)\|}\\
&=\frac{\mathcal{K}_1\Big(\int_{\mathcal{D}}\mathcal{I}_u^{\alpha_1}dm\Big)\int_{\mathcal{D}}\Big(|\nabla u|^{\alpha_1(m)}+|u|^{\alpha_1(m)}\Big)}{\|(u,v)\|}dm+\frac{\mathcal{K}_2\Big(\int_{\mathcal{D}}\mathcal{I}_v^{\alpha_2}dm\Big)\int_{\mathcal{D}}\Big(|\nabla v|^{\alpha_2(m)}+|v|^{\alpha_2(m)}\Big)}{\|(u,v)\|}dm\\
&\geq\frac{\frac{a_1}{(\alpha_1^+)^{c_1^--1}}\min\Big(\|u\|_{\alpha_1(m)}^{\alpha_1^-(c_1^--1)},\|u\|_{\alpha_1(m)}^{\alpha_1+(c_1^--1)}\Big)\min\Big(\|u\|_{\alpha_1(m)}^{\alpha_1^-},\|u\|_{\alpha_1(m)}^{\alpha_1^+}\Big)}{\|(u,v)\|}\\
&+\frac{\frac{b_1}{(\alpha_2^+)^{c_2^--1}}\min\Big(\|v\|_{\alpha_2(m)}^{\alpha_2^-(c_2^--1)},|v\|_{\alpha_2(m)}^{\alpha_2^+(c_2^--1)}\Big)\min\Big(\|v\|_{\alpha_2(m)}^{\alpha_2^-},\|v\|_{\alpha_2(m)}^{\alpha_2^+}\Big)}{\|(u,v)\|}.\end{aligned}$$

This shows that, $\lim_{\|(u,v)\|\to+\infty}\frac{\langle\mathcal{S}(u,v),(u,v)\rangle}{\|(u,v)\|}=+\infty$, then $\mathcal{S}$ is coercive on $\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}$.
Now, we will show that the operator $\mathcal{S}$ is of type (S_+).
Let $(u_k,v_k)\subset\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}$ with $(u_k,v_k)\rightharpoonup(u,v)$ and $\overline{\lim\limits_{k\longrightarrow\infty}}\langle\mathcal{S}(u_k,v_k)-\mathcal{S}(u,v),(u_k-u,v_k-v)\rangle\leqslant0$, we show that $(u_k,v_k)\to(u,v)$.
If $(u_k,v_k)\rightharpoonup(u,v)$ and $\overline{\lim}_{k\longrightarrow\infty}\langle\mathcal{S}(u_k,v_k)-\mathcal{S}(u,v),(u_k-u,v_k-v)\rangle\leqslant0$, then

$$\lim_{k\to\infty}\langle\mathcal{S}(u_k,v_k)-\mathcal{S}(u,v),(u_k-u,v_k-v)\rangle=0.$$

Therefore,

$$\langle\mathcal{S}_{\alpha_1}(u_k)-\mathcal{S}_{\alpha_1}(u),u_k-u\rangle+\langle\mathcal{S}_{\alpha_2}(v_k)-\mathcal{S}_{\alpha_2}(v),v_k-v\rangle\to0.$$

Since $\mathcal{S}_{\alpha_1}$ and $\mathcal{S}_{\alpha_2}$ are monotone, then

$$\langle\mathcal{S}_{\alpha_1}(u_k)-\mathcal{S}_{\alpha_1}(u),u_k-u\rangle\to0,\quad\langle\mathcal{S}_{\alpha_2}(v_k)-\mathcal{S}_{\alpha_2}(v),v_k-v\rangle\to0.\tag{3.1}$$

Arguing as in [7], we obtain

$$\begin{aligned}\langle\mathcal{S}_{\alpha_1}(u_k)-\mathcal{S}_{\alpha_1}(u),u_k-u\rangle&\geq\mathcal{K}_1\Big(\int_{\mathcal{D}}\mathcal{I}_u^{\alpha_1}dm\Big)\Big(\int_{\mathcal{D}}\frac{1}{2}\Big(|\nabla u_k|^{\alpha_1(m)-2}-|\nabla u|^{\alpha_1(m)-2}\Big)\Big(|\nabla u_k|^2-|\nabla u|^2\Big)dm\Big)\\
&+\mathcal{K}_1\Big(\int_{\mathcal{D}}\mathcal{I}_v^{\alpha_2}dm\Big)\Big(\int_{\mathcal{D}}\frac{1}{2}\Big(|u_k|^{\alpha_1(m)-2}-|u|^{\alpha_1(m)-2}\Big)\Big(|u_k|^2-|u|^2\Big)dm\Big)\\
&\geq0.\end{aligned}$$

With this information and (3.1), $\nabla u_k(m) \to \nabla u(m)$ and $u_k(m) \to u(m)$ for a.e. $m \in \mathcal{D}$. Using Fatou's lemma, we get

$$\varliminf_{k\to\infty} \int_{\mathcal{D}} \frac{\left(|\nabla u_k|^{\alpha_1(m)} + |u_k|^{\alpha_1(m)}\right)}{\alpha_1(m)} dm \geq \int_{\mathcal{D}} \frac{\left(|\nabla u|^{\alpha_1(m)} + |u|^{\alpha_1(m)}\right)}{\alpha_1(m)} dm. \tag{3.2}$$

In the same way, we can obtain

$$\varliminf_{k\to\infty} \int_{\mathcal{D}} \frac{\left(|\nabla v_k|^{\alpha_2(m)} + |v_k|^{\alpha_2(m)}\right)}{\alpha_2(m)} dm \geq \int_{\mathcal{D}} \frac{\left(|\nabla v|^{\alpha_2(m)} + |v|^{\alpha_2(m)}\right)}{\alpha_2(m)} dm. \tag{3.3}$$

However, we also have

$$\begin{aligned}\lim_{k\to\infty} \langle \mathcal{S}_{\alpha_1}(u_k), u_k - u\rangle + \langle \mathcal{S}_{\alpha_2}(v_k), v_k - v\rangle &= \lim_{k\to\infty} \langle \mathcal{S}(u_k, v_k), (u_k, v_k) - (u, v)\rangle \\ &= \lim_{k\to\infty} \langle \mathcal{S}(u_k, v_k) - \mathcal{S}(u, v), (u_k, v_k) - (u, v)\rangle = 0.\end{aligned}$$

We can see using Young's inequality that,

$$\begin{aligned}
&\langle \mathcal{S}_{\alpha_1}(u_k), u_k - u\rangle + \langle \mathcal{S}_{\alpha_2}(v_k), v_k - v\rangle \\
&= \mathcal{K}_1\Big(\int_{\mathcal{D}} \frac{|\nabla u_k|^{\alpha_1(m)} + |u_k|^{\alpha_1(m)}}{\alpha_1(m)} dm\Big)\Big(\int_{\mathcal{D}} \left(|\nabla u_k|^{\alpha_1(m)} + |u_k|^{\alpha_1(m)}\right) dm - \int_{\mathcal{D}} \left(|\nabla u_k|^{\alpha_1(m)-2}\nabla u_k \cdot \nabla u + |u_k|^{\alpha_1(m)-2} u_k u\right) dm\Big) \\
&+ \mathcal{K}_2\Big(\int_{\mathcal{D}} \frac{|\nabla v_k|^{\alpha_2(m)} + |v_k|^{\alpha_2(m)}}{\alpha_2(m)} dm\Big)\Big(\int_{\mathcal{D}} (|\nabla v_k|^{\alpha_2(m)} + |v_k|^{\alpha_2(m)}) dm - \int_{\mathcal{D}} \left(|\nabla v_k|^{\alpha_2(m)-2}\nabla v_k . \nabla v + |v_k|^{\alpha_2(m)-2} v_k v\right) dm\Big) \\
&\geq \mathcal{K}_1\Big(\int_{\mathcal{D}} \frac{|\nabla u_k|^{\alpha_1(m)} + |u_k|^{\alpha_1(m)}}{\alpha_1(m)} dm\Big)\Big(\int_{\mathcal{D}} \frac{|\nabla u_k|^{\alpha_1(m)} + |u_k|^{\alpha_1(m)}}{\alpha_1(m)} dm - \int_{\mathcal{D}} \frac{|\nabla u|^{\alpha_1(m)} + |u|^{\alpha_1(m)}}{\alpha_1(m)} dm\Big) \\
&+ \mathcal{K}_2\Big(\int_{\mathcal{D}} \frac{|\nabla v_k|^{\alpha_2(m)} + |v_k|^{\alpha_2(m)}}{\alpha_2(m)} dm\Big)\Big(\int_{\mathcal{D}} \frac{|\nabla v_k|^{\alpha_2(m)} + |v_k|^{\alpha_2(m)}}{\alpha_2(m)} dm - \int_{\mathcal{D}} \frac{|\nabla v|^{\alpha_2(m)} + |v|^{\alpha_2(m)}}{\alpha_2(m)} dm\Big) \\
&\geq \frac{a_1}{\left(\alpha_1^+\right)^{c_1^- - 1}}\Big(\int_{\mathcal{D}} \left(|\nabla u_k|^{\alpha_1(m)} + |u_k|^{\alpha_1(m)}\right) dm\Big)^{c_1^- - 1}\Big(\int_{\mathcal{D}} \frac{|\nabla u_k|^{\alpha_1(m)} + |u_k|^{\alpha_1(m)}}{\alpha_1(m)} dm - \int_{\mathcal{D}} \frac{|\nabla u|^{\alpha_1(m)} + |u|^{\alpha_1(m)}}{\alpha_1(m)} dm\Big) \\
&+ \frac{b_1}{\left(\alpha_2^+\right)^{c_2^- - 1}}\Big(\int_{\mathcal{D}} |\nabla v_k|^{\alpha_2(m)} + |v_k|^{\alpha_2(m)} dm\Big)^{c_2^- - 1}\Big(\int_{\mathcal{D}} \frac{|\nabla v_k|^{\alpha_2(m)} + |v_k|^{\alpha_2(m)}}{\alpha_2(m)} dm - \int_{\mathcal{D}} \frac{|\nabla v|^{\alpha_2(m)} + |v|^{\alpha_2(m)}}{\alpha_2(m)} dm\Big).
\end{aligned}$$

Combining (3.2) and (3.3) we get

$$\lim_{k\to\infty} \int_{\mathcal{D}} \frac{\left(|\nabla u_k|^{\alpha_1(m)} + |u_k|^{\alpha_1(m)}\right)}{\alpha_1(m)} dm = \int_{\mathcal{D}} \frac{\left(|\nabla u|^{\alpha_1(m)} + |u|^{\alpha_1(m)}\right)}{\alpha_1(m)} dm,$$

and

$$\lim_{k\to\infty} \int_{\mathcal{D}} \frac{\left(|\nabla v_k|^{\alpha_2(m)} + |v_k|^{\alpha_2(m)}\right)}{\alpha_2(m)} dm = \int_{\mathcal{D}} \frac{\left(|\nabla v|^{\alpha_2(m)} + |v|^{\alpha_2(m)}\right)}{\alpha_2(m)} dm.$$

Then

$$\lim_{n\to\infty} \int_{\mathcal{D}} \left(|\nabla u_k|^{\alpha_1(m)} + |u_k|^{\alpha_1(m)}\right) dm = \int_{\mathcal{D}} \left(|\nabla u|^{\alpha_1(m)} + |u|^{\alpha_1(m)}\right) dm,$$

and

$$\lim_{n\to\infty} \int_{\mathcal{D}} \left(|\nabla v_k|^{\alpha_2(m)} + |v_k|^{\alpha_2(m)}\right) dm = \int_{\mathcal{D}} \left(|\nabla v|^{\alpha_2(m)} + |v|^{\alpha_2(m)}\right) dm.$$

Using a technique comparable to that of [22], we find that

$$\lim_{n\to\infty}\int_{\mathcal{D}}\Big(|\nabla u_k-\nabla u|^{\alpha_1(m)}+|u_k-u|^{\alpha_1(m)}\Big)dm=0,$$

and

$$\lim_{n\to\infty}\int_{\mathcal{D}}\Big(|\nabla v_k-\nabla v|^{\alpha_2(m)}+|v_k-v|^{\alpha_2(m)}\Big)dm=0.$$

Therefore, $(u_k,v_k)\to(u,v)$, i.e $\mathcal{S}$ is of type (S_+).

Proposition 3.3. *Under the assumptions $(A_1)-(A_3)$, the operator $C:\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}\to\Big(\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}\Big)^*$ defined by*

$$\begin{aligned}\langle C(u,v),(\vartheta,\zeta)\rangle=-\Big(\int_{\mathcal{D}}&(-\delta_1|u|^{p(m)-2}u+\lambda_1f_1(m,u,\nabla u)+\mu_1f_2(m,u))\vartheta\\&+(-\delta_2|v|^{q(m)-2}v+\lambda_2g_1(m,v,\nabla v)+\mu_2g_2(m,v))\zeta dm\Big),\end{aligned}$$

is compact.

Proof. In order to prove this lemma, we proceed in four steps.
Step 1 : Let $\Upsilon_{f_1,g_1}:\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}\to L^{\alpha_1'(m)}(\mathcal{D})\times L^{\alpha_2'(m)}(\mathcal{D})$ be an operator defined by

$$\Upsilon_{f_1,g_1}(u,v)=(-\lambda_1f_1(m,u,\nabla u),-\lambda_2g_1(m,v,\nabla v))=(\Upsilon_{f_1}(u),\Upsilon_{g_2}(v))$$

In this step, we prove that the operator Υ_{f_1,g_1} is bounded and continuous.
First, let $(u,v)\in\mathcal{W}_0^{\alpha_1(m),\alpha_1(m)}$, we have

$$\begin{aligned}|\Upsilon_{f_1,g_1}(u,v)|_{L^{\alpha_1'(m)}\times L^{\alpha_2'(m)}}&=\max\Big(\Big|-\lambda_1f_1(m,u,\nabla u)\Big|_{\alpha_1'(m)},\Big|-\lambda_2g_1(m,v,\nabla v)\Big|_{\alpha_2'(m)}\Big)\\&\le\Big|-\lambda_1f_1(m,u,\nabla u)\Big|_{\alpha_1'(m)}+\Big|-\lambda_2g_1(m,v,\nabla v)\Big|_{\alpha_2'(m)}.\end{aligned}$$

With (2.5), (2.6), (A_2) and (A_3), we get

$$\begin{aligned}|\Upsilon_{f_1,g_1}(u,v)|_{L^{\alpha_1'(m)}\times L^{\alpha_2'(m)}}&\le\int_{\mathcal{D}}\Big|-\lambda_1f_1(mu,\nabla u)\Big|^{\alpha_1'(m)}dm+\int_{\mathcal{D}}\Big|-\lambda_2g_1(m,v,\nabla v)\Big|^{\alpha_2'(m)}dm+2\\&\le C\Big(|\lambda_1|^{\alpha_1'^-}+|\lambda_1|^{\alpha_1'^+}\Big)\Big(\rho_{\alpha_1'(m)}(\gamma_1)+\rho_{e_1(m)}(u)+\rho_{e_1(m)}(\nabla u)\Big)\\&\quad+C\Big(|\lambda_2|^{\alpha_2'^-}+|\lambda_2|^{\alpha_2'^+}\Big)\Big(\rho_{\alpha_2'(m)}(\gamma_1')+\rho_{z_1(m)}(v)+\rho_{z_1(m)}(\nabla v)\Big)+2\\&\le C\Big(|\gamma_1|^{\alpha_1'^+}_{\alpha_1(m)}+|u|^{e_1^+}_{e_1(m)}+|u|^{e_1^-}_{e_1(m)}+|\nabla u|^{e_1^+}_{e_1(m)}+|\nabla u|^{e_1^-}_{e_1(m)}\Big)\\&\quad+C\Big(|\gamma_1'|^{\alpha_2'^+}_{\alpha_2(m)}+|v|^{z_1^+}_{z_1(m)}+|v|^{z_1^-}_{z_1(m)}+|\nabla v|^{z_1^+}_{z_1(m)}+|\nabla v|^{z_1^-}_{z_1(m)}\Big)+2,\end{aligned}$$

where $e_1(m)=(r_1(m)-1)\alpha_1'(m)<\alpha_1(m)$, $z_1(m)=(s_1(m)-1)\alpha_2'(m)<\alpha_2(m)$. Using $L^{\alpha_1(m)}\hookrightarrow L^{e_1(m)}$, $L^{\alpha_2(m)}\hookrightarrow L^{z_1(m)}$ and (2.9), we find

$$\begin{aligned}|\Upsilon_{f_1,g_1}(u,v)|_{L^{\alpha_1'(m)}\times L^{\alpha_2'(m)}}&\le C\left(|\gamma_1|^{\alpha_1'^+}_{\alpha_1(m)}+|u|^{e_1^+}_{1,\alpha_1(m)}+|u|^{e_1^-}_{1,\alpha_1(m)}\right)+C\Big(|\gamma_1'|^{\alpha_2'^+}_{\alpha_2(m)}+|v|^{z_1^+}_{1,\alpha_2(m)}+|v|^{z_1^-}_{1,\alpha_2(m)}\Big)+2\\&\le C_{\max}\Big(|\gamma_1|^{\alpha_1'^+}_{\alpha_1(m)}+|u|^{e_1^+}_{1,\alpha_1(m)}+|u|^{e_1^-}_{1,\alpha_1(m)}+|\gamma_1'|^{\alpha_2'^+}_{\alpha_2(m)}+|v|^{z_1^+}_{1,\alpha_2(m)}+|v|^{z_1^-}_{1,\alpha_2(m)}\Big)+2,\end{aligned}$$

then, Υ_{f_1,g_1} is bounded on $\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}(\mathcal{D})$.
Let us also show that Υ_{f_1,g_1} is continuous. Let $(u_n,v_n)\to(u,v)$ in $\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}(\mathcal{D})$, therefore, there exists a subsequence (u_k,v_k) of (u_n,v_n) and (ϱ_1,ϱ_2) in $L^{\alpha_1(m)}(\mathcal{D})\times L^{\alpha_2(m)}(\mathcal{D})$, (ϱ_3,ϱ_4) in $(L^{\alpha_1(m)}(\mathcal{D})\times L^{\alpha_2(m)}(\mathcal{D}))^N$ with

$$u_k(m)\to u(m),v_k(m)\to v(m)\text{ and }\nabla u_k(m)\to\nabla u(m),\nabla v_k(m)\to\nabla v(m),\tag{3.4}$$

$$|u_k(m)| \leq \varrho_1(m), |v_k(m)| \leq \varrho_2(m) \text{ and } |\nabla u_k(m)| \leq |\varrho_3(m)|, |\nabla v_k(m)| \leq |\varrho_4(m)|. \tag{3.5}$$

So, using (A_1) and (3.4), we obtain, as $k \longrightarrow \infty$

$$\Big(f_1(m, u_k(m), \nabla u_k(m)), g_1(m, v_k(m), \nabla v_k(m))\Big) \longrightarrow (f_1(m, u(m), \nabla u(m)), g_1(m, v(m), \nabla v(m))) \tag{3.6}$$

a.e. $m \in \mathcal{D}$.
Furthermore, according to (A_2), (A_3), and (3.5) we get

$$|f_1(m, u_k(m), \nabla u_k(m))| \leq C_1(\gamma_1(m) + |\varrho_1(m)|^{r_1(m)-1} + |\varrho_3(m)|^{r_1(m)-1})),$$

$$|g_1(m, v_k(m), \nabla v_k(m))| \leq C_1'(\gamma_1'(m) + |\varrho_2(m)|^{s_1(m)-1} + |\varrho_4(m)|^{s_1(m)-1}).$$

Since

$$\gamma_1 + |\varrho_1|^{r_1(m)-1} + |\varrho_3(m)|^{r_1(m)-1} \in L^{\alpha_1(m)}(\mathcal{D}),$$

$$\gamma_2 + |\varrho_2|^{s_1(m)-1} + |\varrho_4(m)|^{s_1(m)-1} \in L^{\alpha_2(m)}(\mathcal{D}),$$

and

$$\rho_{\alpha_1(m)}(\Upsilon_{f_1}(u_k) - \Upsilon_{f_1}(u)) = \int_{\mathcal{D}} |\lambda_1 f_1(m, u_k(m), \nabla u_k(m)) - \lambda_1 f_1(m, u(m), \nabla u(m))|^{\alpha_1(m)} dm,$$

$$\rho_{\alpha_2(m)}(\Upsilon_{g_1}(v_k) - \Upsilon_{g_1}(v)) = \int_{\mathcal{D}} |\lambda_2 g_1(m, v_k(m), \nabla v_k(m)) - \lambda_2 g_1(m, v(m), \nabla v(m))|^{\alpha_2(m)} dm.$$

Then, by Lebesgue's theorem and (2.4), we have

$$\Upsilon_{f_1}(u_k) \to \Upsilon_{f_1}(u) \text{ in } L^{\alpha_1(m)}(\mathcal{D}) \quad \text{and} \quad \Upsilon_{g_1}(v_k) \to \Upsilon_{g_1}(v) \text{ in } L^{\alpha_2(m)}(\mathcal{D}).$$

Then Υ_{f_1,g_1} is continuous.
Step 2 : We define the operator $\Phi : \mathcal{W}_0^{\alpha_1(m),\alpha_2(m)} \to L^{\alpha_1'(m)}(\mathcal{D}) \times L^{\alpha_2'(m)}(\mathcal{D})$ by

$$\Phi(u, v) = (\delta_1 |u|^{p(m)-2} u, \delta_2 |v|^{q(m)-2} v).$$

We will prove that Φ is bounded and continuous.
It is clear that Φ is continuous. Next we show that Φ is bounded.
Let $(u, v) \in \mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}$, using (2.5) and (2.6), we obtain

$$\begin{aligned}
|\Phi(u,v)|_{L^{\alpha_1'(m)} \times L^{\alpha_2'(m)}} &\leq \int_{\mathcal{D}} |\delta_1 |u|^{p(m)-2} u|^{\alpha_1'(m)} dm + \int_{\mathcal{D}} |\delta_2 |v|^{q(m)-2} v|^{\alpha_2'(m)} dm + 2 \\
&\leq \Big(|\delta_1|^{\alpha_1'^-} + |\delta_1|^{\alpha_1'^+}\Big) \rho_{l_1(m)}(u) + \Big(|\delta_2|^{\alpha_2'^-} + |\delta_2|^{\alpha_2'^+}\Big) \rho_{l_2(m)}(v) + 2 \\
&\leq \Big(|\delta_1|^{\alpha_1'^-} + |\delta_1|^{\alpha_1'^+}\Big)\Big(|u|_{l_1(m)}^{l_1^-} + |u|_{l_1(m)}^{l_1^+}\Big) + \Big(|\delta_2|^{\alpha_2'^-} + |\delta_2|^{\alpha_2'^+}\Big)\Big(|v|_{l_2(m)}^{l_2^-} + |v|_{l_2(m)}^{l_2^+}\Big) + 2,
\end{aligned}$$

where $l_1(m) = (p(m) - 1)\alpha_1'(m)$ and $l_2(m) = (q(m) - 1)\alpha_2'(m)$.
Hence, we deduce from $L^{\alpha_1(m)} \hookrightarrow L^{l_1(m)}$, $L^{\alpha_2(m)} \hookrightarrow L^{l_2(m)}$ and (2.9) that

$$|\Phi(u,v)|_{L^{\alpha_1'(m)} \times L^{\alpha_2'(m)}} \leq C\Big(|u|_{1,\alpha_1(m)}^{l_1^-} + |u|_{1,\alpha_1(m)}^{l_1^+}\Big) + C\Big(|v|_{1,\alpha_2(m)}^{l_2^-} + |v|_{1,\alpha_2(m)}^{l_2^+}\Big) + 2.$$

Then Φ is bounded on $\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}$.
Step 3: Let $\Upsilon_{f_2,g_2} : \mathcal{W}_0^{\alpha_1(m),\alpha_2(m)} \to L^{\alpha_1'(m)}(\mathcal{D}) \times L^{\alpha_2'(m)}(\mathcal{D})$ be an operator defined by

$$\Upsilon_{f_2,g_2}(u, v) = (-\mu_1 f_2(m, u), -\mu_2 g_2(m, v)) = (\Upsilon_{f_2}(u), \Upsilon_{g_2}(v)).$$

In this step, we prove that the operator Υ_{f_2,g_2} is bounded and continuous.
First, let $(u,v) \in \mathcal{W}_0^{\alpha_1(m),\alpha_1(m)}$, we have

$$|\Upsilon_{f_2,g_2}(u,v)|_{L^{\alpha_1'(m)}\times L^{\alpha_2'(m)}} = \max\left(\Big|-\mu_1 f_2(m,u)\Big|_{\alpha_1'(m)}, \Big|-\mu_2 g_2(m,v)\Big|_{\alpha_2'(m)}\right)$$
$$\leq \Big|-\mu_1 f_2(m,u)\Big|_{\alpha_1'(m)} + \Big|-\mu_2 g_2(m,v)\Big|_{\alpha_2'(m)}.$$

With (2.5), (2.6), (A_2) and (A_3), we get

$$\begin{aligned} |\Upsilon_{f_2,g_2}(u,v)|_{L^{\alpha_1'(m)}\times L^{\alpha_2'(m)}} &\leq \int_{\mathcal{D}} \Big|-\mu_2 f_2(m,v)\Big|^{\alpha_1'(m)} dm + \int_{\mathcal{D}} \Big|-\mu_1 g_2(m,v)\Big|^{\alpha_2'(m)} dm + 2 \\ &\leq C\Big(|\mu_1|^{\alpha_1'^-} + |\mu_1|^{\alpha_1'^+}\Big)\Big(\rho_{\alpha_1'(m)}(\gamma_2) + \rho_{e_2(m)}(u)\Big) + C\Big(|\mu_2|^{\alpha_2'^-} + |\mu_2|^{\alpha_2'^+}\Big)\Big(\rho_{\alpha_2'(m)}(\gamma_2') + \rho_{z_2(m)}(v)\Big) + 2 \\ &\leq C\Big(|\gamma_2|^{\alpha_1'^+}_{\alpha_1(m)} + |u|^{e_2^+}_{e_2(m)} + |u|^{e_2^-}_{e_2(m)}\Big) + C\Big(|\gamma_2'|^{\alpha_2'^+}_{\alpha_2(m)} + |v|^{z_2^+}_{z_2(m)} + |v|^{z_2^-}_{z_2(m)}\Big) + 2, \end{aligned}$$

where $e_2(m) = (r_2(m)-1)\alpha_1'(m) < \alpha_1(m)$, $z_2(m) = (s_2(m)-1)\alpha_2'(m) < \alpha_2(m)$. Using $L^{\alpha_1(m)} \hookrightarrow L^{e_2(m)}$, $L^{\alpha_2(m)} \hookrightarrow L^{z_2(m)}$ and (2.9), we find

$$\begin{aligned} |\Upsilon_{f_2,g_2}(u,v)|_{L^{\alpha_1'(m)}\times L^{\alpha_2'(m)}} &\leq C\left(|\gamma_1|^{\alpha_1'^+}_{\alpha_1(m)} + |u|^{e^+}_{1,\alpha_1(m)} + |u|^{e^-}_{1,\alpha_1(m)}\right) + C\Big(|\gamma_2|^{\alpha_2'^+}_{\alpha_2(m)} + |v|^{z^+}_{1,\alpha_2(m)} + |v|^{z^-}_{1,\alpha_2(m)}\Big) + 2 \\ &\leq C_{\max}\Big(|\gamma_2|^{\alpha_1'^+}_{\alpha_1(m)} + |u|^{e^+}_{1,\alpha_1(m)} + |u|^{e^-}_{1,\alpha_1(m)} + |\gamma_2'|^{\alpha_2'^+}_{\alpha_2(m)} + |v|^{z_2^+}_{1,\alpha_2(m)} + |v|^{z_2^-}_{1,\alpha_2(m)}\Big) + 2, \end{aligned}$$

according to this, Υ_{f_2,g_2} is bounded on $\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}(\mathcal{D})$.
Let us also show that Υ_{f_2,g_2} is continuous. Let $(u_n,v_n) \to (u,v)$ in $\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}(\mathcal{D})$, therefore, there exists a subsequence (u_k,v_k) of (u_n,v_n) and (ω_1,ω_2) in $L^{\alpha_1(m)}(\mathcal{D}) \times L^{\alpha_2(m)}(\mathcal{D})$, with

$$u_k(m) \to u(m) \text{ and } v_k(m) \to v(m), \tag{3.7}$$

$$|u_k(m)| \leq \omega_1(m) \text{ and } |v_k(m)| \leq \omega_2(m). \tag{3.8}$$

So, as a (A_1) and (3.7) , we obtain, as $k \longrightarrow \infty$

$$\Big(f_2(m,u_k(m)), g_2(m,v_k(m))\Big) \longrightarrow (f_2(m,u(m)), g_2(m,v(m)))$$

a.e. $m \in \mathcal{D}$.
Furthermore, according to (A_2), (A_3) and (3.8) we get

$$|f_2(m,u_k(m))| \leq C_2(\gamma_2(m) + |\omega_1(m)|^{r_2(m)-1}),$$

$$|g_2(m,v_k(m))| \leq C_2'(\gamma_2'(m) + |\omega_2(m)|^{s_2(m)-1}).$$

Since

$$\gamma_2 + |\omega_1|^{r_2(m)-1} \in L^{\alpha_1(m)}(\mathcal{D}), \quad \gamma_2' + |\omega_2|^{s_2(m)-1} \in L^{\alpha_2(m)}(\mathcal{D}),$$

and

$$\rho_{\alpha_1(m)}(\Upsilon_{f_2}(u_k) - \Upsilon_{f_2}(u)) = \int_{\mathcal{D}} |\mu_1 f_2(m,u_k(m)) - \mu_1 f_2(m,u(m))|^{\alpha_1(m)} dm,$$

$$\rho_{\alpha_2(m)}(\Upsilon_{g_2}(v_k) - \Upsilon_{g_2}(v)) = \int_{\mathcal{D}} |\mu_2 g_2(m,v_k(m)) - \mu_2 g_2(m,v(m))|^{\alpha_2(m)} dm.$$

Then, by Lebesgue's theorem and (2.4), we have

$$\Upsilon_{f_2}(u_k) \to \Upsilon_{f_2}(u) \text{ in } L^{\alpha_1(m)}(\mathcal{D}) \quad \text{and} \quad \Upsilon_{g_2}(v_k) \to \Upsilon_{g_2}(v) \text{ in } L^{\alpha_2(m)}(\mathcal{D}).$$

Then Υ_{f_2,g_2} is continuous.

Step 4:

Let $I^* : L^{\alpha_1(m)}(\mathcal{D}) \times L^{\alpha_2(m)}(\mathcal{D}) \to \left(\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}(\mathcal{D})\right)^*$ be the adjoint operator for the embedding

$$I : \mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}(\mathcal{D}) \to L^{\alpha_1(m)}(\mathcal{D}) \times L^{\alpha_2(m)}(\mathcal{D}).$$

We then define

$$I^* \circ \Upsilon_{f_1,g_1} : \mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}(\mathcal{D}) \to \left(\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}(\mathcal{D})\right)^*,$$

$$I^* \circ \Phi : \mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}(\mathcal{D}) \to \left(\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}(\mathcal{D})\right)^*,$$

and

$$I^* \circ \Upsilon_{f_2,g_2} : \mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}(\mathcal{D}) \to \left(\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}(\mathcal{D})\right)^*.$$

Since I the embedding is compact, it is known that the adjoint operator I^* is also compact. Therefore, the operators $I^* \circ \Upsilon_{f_1,g_1}$, $I^* \circ \Phi$ and $I^* \circ \Upsilon_{f_2,g_2}$ are compact, that means $\mathcal{C} = I^* \circ \Upsilon_{f_1,g_1} + I^* \circ \Phi + I^* \circ \Upsilon_{f_2,g_2}$ is compact.

We now provide our main result.

Theorem 3.4. *Under the assumptions* $(A_1) - (A_3)$ *and* (M_0), *the system* (1.1) *has a weak solutions* (u, v) *in* $\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}$.

Proof. First, $(u, v) \in \mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}(\mathcal{D})$ is a weak solution of (1.1) if and only if

$$\mathcal{S}(u, v) = -\mathcal{C}(u, v), \tag{3.9}$$

the operators $\mathcal{S} : \mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}(\mathcal{D}) \longrightarrow \left(\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}(\mathcal{D})\right)^*$ and $\mathcal{C} : \mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}(\mathcal{D}) \longrightarrow \left(W_0^{\alpha_1(m)}(\mathcal{D}) \times W_0^{\alpha_2(m)}(\mathcal{D})\right)^*$ are defined in Lemmas 3.2 and 3.3 respectively.

As a result of characteristics of the operator $\mathcal{S}$ presented in Lemma 3.2 also considering the Minty-Browder Theorem (see [39, Theorem 26 A], the inverse operator

$$\mathcal{U} := \mathcal{S}^{-1} : \left(\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}(\mathcal{D})\right)^* \to \mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}(\mathcal{D}),$$

is bounded, continuous and of class (S_+).

Secondly, the operator $\mathcal{C}$ is bounded, continuous, and quasimonotone (see Proposition 3.3). Due to this, equation (3.9) is equivalent to

$$(u, v) = \mathcal{U}(\vartheta, \zeta) \text{ and } (\vartheta, \zeta) + (\mathcal{C} \circ \mathcal{U})(\vartheta, \zeta) = 0 \text{ and } (\vartheta, \zeta) \in \left(\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}(\mathcal{D})\right)^*. \tag{3.10}$$

Use of the topological degree theory in subsection 2.2 To solve (3.10). For this, create

$$\mathcal{M} := \left\{(\vartheta, \zeta) \in \left(\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}(\mathcal{D})\right)^* : \exists\, t \in [0, 1] \text{ such that } (\vartheta, \zeta) + t(\mathcal{C} \circ \mathcal{U})(\vartheta, \zeta) = 0\right\}.$$

Afterward, we prove that $\mathcal{M}$ is bounded in $\left(\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}(\mathcal{D})\right)^*$.

Let $(\vartheta, \zeta) \in \mathcal{M}$ and put $(u, v) := \mathcal{U}(\vartheta, \zeta)$, then

$\|\mathcal{U}(\vartheta, \zeta)\|_{\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}} = \|(u, v)\|_{\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}} = \max\{\|u\|_{\alpha_1(m)}, \|v\|_{\alpha_2(m)}\}$. Taking into account that $\|\cdot\|_{\alpha_1(m)} = |\nabla \cdot|_{\alpha_1(m)}$.

Case 1: If $\|\mathcal{U}(\vartheta,\zeta)\|_{\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}} = \|u\|_{\alpha_1(m)} = |\nabla u|_{\alpha_1(m)}$.
Case 1.1: If $|\nabla u|_{\alpha_1(m)} \leq 1$ then $\|\mathcal{U}(\vartheta,\zeta)\|_{\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}}$ is bounded.
Case 1.2: If $|\nabla u|_{\alpha_1(m)} > 1$, then by (2.2), (A_2), (2.1), (2.6) and the Young inequality, we get

$$\begin{aligned}
\|\mathcal{U}(\vartheta,\zeta)\|^{\alpha_1^-} &= |\nabla u|^{\alpha_1-}_{\alpha_1(m)}\\
&\leq \rho_{\alpha_1(m)}(\nabla u)\\
&\leq \langle \mathcal{S}u, u\rangle\\
&= \langle \vartheta, \mathcal{U}\vartheta\rangle\\
&= -t\langle (C\circ\mathcal{U})\vartheta, \mathcal{U}\vartheta\rangle\\
&= t\int_{\mathcal{D}}\Big(-\delta_1|u|^{p(m)-2}u + \lambda_1 f_1(x,u,\nabla u) + \mu_1 f_2(x,u)\Big)u\,dm\\
&\leq C_{\max}\Big(\rho_{p(m)}(u) + \int_{\mathcal{D}}|\gamma_1(m)u(m)|dm + \rho_{r_1(m)}(u) + \int_{\mathcal{D}}|\nabla u|^{r_1(m)-1}|u|dm + \int_{\mathcal{D}}|\gamma_2(m)u(m)|dm + \rho_{r_2(m)}(u)\Big)\\
&\leq C_{\max}\Big(|u|^{p^-}_{p(m)} + |u|^{p^+}_{p(m)} + |\gamma_1|_{\alpha_1(m)}|u|_{\alpha_1(m)} + |u|^{r_1^+}_{r_1(m)} + |u|^{r_1^-}_{r_1(m)} + |u|^{r_2^+}_{r_2(m)} + |u|^{r_2^-}_{r_2(m)}\\
&+ \frac{1}{r_1'^-}\rho_{r_1(m)}(\nabla u) + \frac{1}{r_1-}\rho_{r_1(m)}(u) + |\gamma_2|_{\alpha_1(m)}|u|_{\alpha_1(m)}\Big)\\
&\leq C_{\max}\Big(|u|^{p^-}_{p(m)} + |u|^{p^-}_{p(m)} + |u|_{\alpha_1(m)} + |u|^{r_2^+}_{r_2(m)} + |u|^{r_2^-}_{r_2(m)} + |u|^{r_1^+}_{r_1(m)} + |u|^{r_1^-}_{r_1(m)} + |\nabla u|^{r_1^+}_{r_1(m)}\Big).
\end{aligned}$$

By (2.9), $L^{\alpha_1(m)} \hookrightarrow L^{p(m)}$, $L^{\alpha_1(m)} \hookrightarrow L^{r_1(m)}$ and $L^{\alpha_1(m)} \hookrightarrow L^{r_2(m)}$, we obtain

$$\|\mathcal{U}(\vartheta,\zeta)\|^{\alpha_1^-}_{\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}} \leq C\Big(|\mathcal{U}(\vartheta)|^{p^+}_{1,\alpha_1(m)} + |\mathcal{U}(\vartheta)|_{1,\alpha_1(m)} + |\mathcal{U}(\vartheta)|^{r_2^+}_{1,\alpha_1(m)} + |\mathcal{U}(\vartheta)|^{r_1^+}_{1,\alpha_1(m)}\Big),$$

then $\|\mathcal{U}(\vartheta,\zeta)\|_{\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}}$ is bounded.
Case 2: If $\|\mathcal{U}(\vartheta,\zeta)\|_{\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}} = \|v\|_{\alpha_2(m)} = |\nabla v|_{\alpha_2(m)}$.
Case 2.1: If $|\nabla v|_{\alpha_2(m)} \leq 1$, then $\|\mathcal{U}(\vartheta,\zeta)\|_{\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}}$ is bounded,
Case 2.2: If $|\nabla v|_{\alpha_2(m)} > 1$, then in the same way by (2.2), (A_3), (2.1), (2.6) and the Young inequality, we find that

$$\|\mathcal{U}(\vartheta,\zeta)\|^{\alpha_2^-}_{\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}} \leq C\Big(|\mathcal{U}(\zeta)|^{q^+}_{1,\alpha_2(m)} + |\mathcal{U}(\zeta)|_{1,\alpha_2(m)} + +|\mathcal{U}(\zeta)|^{s_2^+}_{1,\alpha_2(m)} + |\mathcal{U}(\zeta)|^{s_1^+}_{1,\alpha_2(m)}\Big).$$

So, we infer that $\Big\{\mathcal{U}(\vartheta,\zeta)|(\vartheta,\zeta)\in\mathcal{M}\Big\}$ is bounded.
Since the operator C is bounded, and by (3.10), there exists $R > 0$ such that

$$\|(\vartheta,\zeta)\|_{(\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}(\mathcal{D}))^*} < R \ \text{ for all } \ (\vartheta,\zeta)\in\mathcal{M}.$$

Therefore

$$(\vartheta,\zeta) + t(C\circ\mathcal{U})(\vartheta,\zeta) \neq 0 \ \text{ for all } \ (\vartheta,\zeta)\in\partial\mathcal{M}_R(0) \ \text{ and all } \ t\in[0,1],$$

where $\mathcal{M}_R(0)$ is the ball of center 0 and radius R in $\Big(\mathcal{W}^{\alpha_1(m),\alpha_2(m)}(\mathcal{D})\Big)^*$.
Moreover, from Lemma 2.12 we have also

$$I + C\circ\mathcal{U} \in \mathcal{T}_{\mathcal{U}}(\overline{\mathcal{M}_R(0)}) \ \text{ and } \ I = \mathcal{S}\circ\mathcal{U} \in \mathcal{T}_{\mathcal{U}}(\overline{\mathcal{M}_R(0)}).$$

Next, we define the homotopy $\mathcal{H} : [0,1]\times\overline{\mathcal{M}_R(0)} \to \Big(\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}(\mathcal{D})\Big)^*$ by

$$\mathcal{H}(t,(\vartheta,\zeta)) := (\vartheta,\zeta) + t(C\circ\mathcal{U})(\vartheta,\zeta).$$

By Theorem 2.15(homotopy invariance), we conclude that the value of $d(\mathcal{H}(t,\cdot), \mathcal{M}_R(0), 0)$ is constant for all $t \in [0,1]$.

Since $\mathcal{H}(1,\cdot) = I + C \circ \mathcal{U}$ and $\mathcal{H}(0,\cdot) = I$, then

$$d(I + C \circ \mathcal{U}, \mathcal{M}_R(0), 0) = d(I, \mathcal{M}_R(0),\ 0). \tag{3.11}$$

Applying also the normalization properties of the degree d as in Theorem 2.15, we have

$$d(I, \mathcal{M}_R(0),\ 0) = 1. \tag{3.12}$$

Combining (3.11) and (3.12) we get

$$d(I + C \circ \mathcal{U}, \mathcal{M}_R(0), 0) = d(I, \mathcal{M}_R(0),\ 0) = 1 \neq 0.$$

Since $d(I + C \circ \mathcal{U}, \mathcal{M}_R(0), 0) \neq 0$, then by the existence property of the degree d stated in Theorem 2.15, we find $(\vartheta, \zeta) \in \mathcal{M}_R(0)$ such as

$$(\vartheta, \zeta) + (C \circ \mathcal{U})(\vartheta, \zeta) = 0.$$

At last, we deduce that $(u, v) = \mathcal{U}(\vartheta, \zeta)$ is a weak solution of (1.1). The proof is completed.

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References

[1] Allalou C., El Ouaarabi M., Melliani S.: Existence and uniqueness results for a class of $p(x)$-Kirchhoff-type problems with convection term and Neumann boundary data. Journal of Elliptic and Parabolic Equations. **8**(1), 617–633 (2022).

[2] Berkovits J.: Extension of the Leray–Schauder degree for abstract Hammerstein type mappings. J Differ. Equ. **234**, 289–310 (2007).

[3] Bouaam H., El Ouaarabi M., Allalou C.,Melliani S.: Variable exponent q(m)-Kirchhoff-type problems with nonlocal terms and logarithmic nonlinearity on compact Riemannian manifolds. Bull. Malays. Math. Sci. Soc. **46**, 97 (2023).

[4] Chen Y., Levine S., Rao M.: Variable exponent, linear growth functionals in image restoration. SIAM J. Appl. Math. **66**, 1383–1406 (2006).

[5] Chung N. T.: Multiple solutions for a class of $p(x)$-Kirchhoff type problems with Neumann boundary conditions. Advances in Pure and Applied Mathematics. **4**(2), 165–177 (2013).

[6] Dai G.: Existence of solutions for nonlocal elliptic systems with nonstandard growth conditions. Electronic Journal of Differential Equations. **2011**(137), 1–13 (2011).

[7] Dai G., Ruyun M.: Solutions for a $p(x)$-Kirchhoff type equation with Neumann boundary data. Nonlinear Analysis. Real World Applications. **12**(5), 2666–2680 (2011).

[8] Eddine N.C., Ragusa M.A.: Generalized critical Kirchhoff-type potential systems with Neumann boundary conditions. Appl. Anal. **101**, 3958–3988 (2022).

[9] El Hammar H., El Ouaarabi M., Melliani S., Allalou C.: Variable exponent $p(\cdot)$-Kirchhoff type problem with convection in variable exponent Sobolev spaces. Boletim da Sociedade Paranaense de Matemática. **41**, 1–13 (2023).

[10] El Ouaarabi M., Allalou C., Melliani S.: Existence result for Neumann problems with p(x)-Laplacian-like operators in generalized Sobolev spaces. Rend. Circ. Mat. Palermo, II. Ser **72**, 1337–1350 (2023).

[11] El Ouaarabi M., Allalou C., Melliani S.: Existence of weak solution for a class of p(x)-Laplacian problems depending on three real parameters with Dirichlet condition. Bol. Soc. Mat. Mex. **28**, 31 (2022).

[12] El Ouaarabi M., Allalou C., Melliani S.: On a class of p(x)-Laplacian-like Dirichlet problem depending on three real parameters. Arab. J. Math. **11**(2), 227–239 (2022).

[13] El Ouaarabi M., Allalou C., Melliani S.: Weak solution of a Neumann boundary value problem with p(x)-Laplacian-like operator. Analysis. **42**(4), 271–280 (2022).

[14] El Ouaarabi M., Allalou C.,Melliani, S.: Weak solutions for double phase problem driven by the $(p(x), q(x))$-Laplacian operator under Dirichlet boundary conditions. Boletim da Sociedade Paranaense de Matemática. **41**, 1–14 (2023).

[15] El Ouaarabi M., Allalou C., Melliani S.: Existence of weak solutions for p(x)-Laplacian-like problem with p(x)-Laplacian operator under Neumann boundary condition. São Paulo J. Math. Sci. (2022). https://doi.org/10.1007/s40863-022-00321-z.

[16] El Ouaarabi M., Allalou C., Melliani S.: Existence of weak solutions to a p(x)-Kirchhoff type problems involving the p(x)-Laplacian-like operators. Nonlinear Studies. **30**(1), 333–345 (2023).

[17] El Ouaarabi M., Allalou C., Melliani S.: On a class of nonlinear degenerate elliptic equations in weighted Sobolev spaces. Georgian Mathematical Journal. **30**(1), 81–94 (2023).
[18] El Ouaarabi M., Allalou C., Melliani S.: p(x)-Laplacian-like Neumann problems in variable-exponent Sobolev spaces via topological degree methods. Filomat. **36**(17), 5973–5984 (2022).
[19] El Ouaarabi M., Allalou C., Melliani S.: Existence result for a Neumann boundary value problem governed by a class of p(x)-Laplacian-like equation. Asymptotic Analysis. **132**(1-2), 245–259 (2023).
[20] El Ouaarabi M., Allalou C., Melliani S.: Neumann Problem Involving The p(x)-Kirchhoff-Laplacian-Like Operator in Variable Exponent Sobolev Space. Asia Pac. J. Math. **9**, 18 (2022).
[21] El Ouaarabi, M.; Abbassi, A.; Allalou, C.: (2022). Existence and uniqueness of weak solution in weighted Sobolev spaces for a class of nonlinear degenerate elliptic problems with measure data. Int. J. Nonlinear Anal. Appl. **13**(1), 2635–2653.
[22] Fan X.L., Zhao D.: On the Spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$. J Math Anal Appl. **263**, 424–446 (2001).
[23] Kim I.S., Hong S.J.: A topological degree for operators of generalized (S_+) type. Fixed Point Theory and Appl. **1**, 1–16 (2015).
[24] Kirchhoff G.: Mechanik, Teubner, Leipzig, (1883).
[25] Kováčik O., Rákosník J.: On spaces $L^{p(x)}$ and $W^{1,p(x)}$. Czechoslovak Math. J. **41**(4), 592–618 (1991).
[26] Li C., Tang C. L.: Three solutions for a class of quasilinear elliptic systems involving the (p, q)-Laplacian. Nonlinear Analysis: Theory, Methods and Applications. **69**(10), 3322–3329 (2008).
[27] Luc N.H., Kumar D., Long L.D., Van H.T.K.: Final value problem for parabolic equation with fractional Laplacian and Kirchhoff's term. Journal of Function Spaces. **2021**, 7238678 (2021).
[28] Massar M., Talbi M., Tsouli N.: Multiple solutions for nonlocal system of $(p(x), q(x))$-Kirchhoff type. Applied Mathematics and Computation. **242**. 216–226 (2014).
[29] Mihăilescu M., Rădulescu V., Stancu-Dumitru D.: A Caffarelli–Kohn–Nirenberg-type inequality with variable exponent and applications to PDEs. Complex Variables and Elliptic Equations. **56**(7-9), 659–669 (2011).
[30] Ragusa M.A., Razani A., Safari F.: Existence of radial solutions for a $p(x)$-Laplacian Dirichlet problem. Advances in Difference Equations. **2021**(1), 1–14 (2021).
[31] Ragusa M.A.: Linear growth coefficients in quasilinear equations. Nonlinear Differ. Equ. Appl. **13** (2007), 605–617; Nonlinear Differ. Equ. Appl. **15**, 277–277 (Erratum) (2008).
[32] Ragusa M.A., Tachikawa A.: On continuity of minimizers for certain quadratic growth functionals. Journal of the Mathematical Society of Japan. **57**(3), 691–700 (2005).
[33] Ragusa M.A., Tachikawa A.: Regularity of Minimizers of some Variational Integrals with Discontinuity. Zeitschrift für Analysis und ihre Anwendungen. **27**(4), 469–482 (2008).
[34] Ragusa M.A.: On weak solutions of ultraparabolic equations. Nonlinear Anal. **47**(1), 503–511 (2001)
[35] Rúzicka M.: Electrorheological fuids: modeling and mathematical theory. Springer Science & Business Media. (2000).
[36] Rezvani A., Alimohammady M., Agheli B.: Multiplicity of solutions for Kirchhoff type problem involving eigenvalue. Filomat. **36** (11), 3861–3874 (2022).
[37] Růžička M.: Flow of shear dependent electrorheological fluids. Comptes Rendus de l'Académie des Sciences-Series I-Mathematics. **329**(5), 393–398 (1999).
[38] Samko S.G.: Density of $C_0^\infty(\mathbb{R}^N)$ in the generalized Sobolev spaces $W^{x,p(x)}(\mathbb{R}^N)$. Doklady Mathematics. **60**(3), 382–385 (1999).
[39] Zeidler E.: Nonlinear Functional Analysis and its Applications II/B. Springer-Verlag. New York, (1990).
[40] Zhao D., Qiang W.J., Fan, X.L.: On generalizerd Orlicz spaces $L^{p(x)}(\Omega)$- J. Gansu Sci. **9**(2), 1–7 (1996).
[41] Zhao D.,Fan X.L.: On the Nemytsky operators from $L^{p_1(x)}(\Omega)$ to $L^{p_2(x)}(\Omega)$. J. Lanzhou Uni. **34**(1) 1–5 (1998).
[42] Zhikov V.V.E.: Averaging of functionals of the calculus of variations and elasticity theory. Mathematics of the USSR-Izvestiya. **29**(1), 33–66 (1987).
[43] Zhikov V.V.E.: Averaging of functionals of the calculus of variations and elasticity theory. Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya. **50**(4), 675–710 (1986).