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Fixed point results for a new multivalued Geraghty type contraction via $C_{\mathcal{G}}$-simulation functions

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Abstract. The aim of this paper is to introduce the new concept of a multivalued Geraghty type contraction mapping using $C_{\mathcal{G}}$-simulation functions and C-class functions. Additionally, through this type of contraction, we establish fixed point results that generalize several known fixed point results in the literature. As consequences, we arrive at fixed point results endowed with graph. To demonstrate the credibility of our results, we give an example that proves it.

1. Introduction

The concept of fixed point theory is a fundamental tool in solving various mathematical problems. The Banach contraction principle [1] is a crucial component of metric fixed point theory and has been extensively studied by numerous scholars due to its practical applications (see, for example, [2–8, 10–12, 32]). Geraghty [12] introduced a generalization of this principle by utilizing an auxiliary function. Nadler [13] expanded the Banach contraction principle to include multivalued mappings, which opened up new avenues for metric fixed point theory. Subsequently, the notion of Banach contraction was extended through the use of multivalued mappings and the concept of noncompactness measure (refer to [14, 15]) for further details).

The notion of $\mathcal{Z}$-contraction was first introduced by Khojasteh et al.[16] using a set of control functions called simulation functions, and a generalized version of Banach contraction principle was presented. Olgun et al.[17] derived fixed point results for generalized $\mathcal{Z}$-contraction. De-Hierro et al.[18] presented some coincidence point theorems were obtained by extending the class of simulation functions for a pair of mappings. Chandok et al.[19] combined the concepts of simulation functions and C-class functions, resulting in the existence and uniqueness of the point of coincidence, which generalized the results in [16, 17].

On a different note, Samet et al.[4] introduced the concept of α-admissibility and extended the Banach contraction principle. Karapinar [20] generalized the results of Samet et al.[4] and Khojasteh et al.[16] by introducing the notion of α-admissible $\mathcal{Z}$-contraction. Recently, Patel [21] proved some fixed point theorems for multivalued contractions using generalized simulation functions in α-complete metric spaces.

In 2015, Khojasteh et al.[16] provides a class Θ of functions $\mathcal{Z} : (\mathbb{R}_+ \cup \{0\})^2 \to \mathbb{R}$ which satisfies the following conditions:

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($\mathcal{Z}_1$) : $\mathcal{Z}(0,0) = 0$;
($\mathcal{Z}_2$) : $\mathcal{Z}(\varrho, \theta) < \theta - \varrho$ for every $\varrho, \theta > 0$;
($\mathcal{Z}_3$) : If $\{\varrho_n\}, \{\theta_n\}$ are sequences in $\mathbb{R}^+$ such that $\lim_{n\to\infty} \varrho_n = \lim_{n\to\infty} \theta_n > 0$ then

$$\limsup_{n\to\infty} \mathcal{Z}(\varrho_n, \theta_n) < 0$$

called the simulation functions. This function $\mathcal{Z}$ was used to define the concept of $\mathcal{Z}$-contraction, which was then applied to the generalized Banach contraction principle [1] in order to unify various known contractions that involve the combination of $d(Q\nu, Q\mu)$ and $d(\nu, \mu)$. Khojasteh et al.[16] established the following result

Theorem 1.1. *[16] Let (Υ, d) be a complete metric space and $Q : \Upsilon \to \Upsilon$ be a mapping satisfying*

$$\mathcal{Z}(d(Q\nu, Q\mu), d(\nu, \mu)) \geq 0 \text{ for every } \nu, \mu \in \Upsilon, \tag{1}$$

where $\mathcal{Z} \in \Theta$. Then Q possesses a fixed point $\nu \in \Upsilon$ which is a single and for each $\nu_0 \in \Upsilon$, the Picard sequence $\{\nu_n\}$ where $\nu_n = Q\nu_{n-1}$ for all $n \in \mathbb{N}$ converges to ν.

Karapinar [20] established fixed point results in the complete metric spaces by introducing a new contraction condition using an admissible mapping extended in a simulation function. Hakan [22] et al. presented a fixed point theorem by introducing a generalized simulation function on a quasi metric space. Rold'an-L'opez-de-Hierro et al.[18] modified the simulation function by replacing ($\mathcal{Z}_3$) with ($\mathcal{Z}'_3$), where
($\mathcal{Z}'_3$) : if $\{\varrho_n\}, \{\theta_n\}$ are sequences in $\mathbb{R}_+$ such that $\lim_{n\to\infty} \varrho_n = \lim_{n\to\infty} \theta_n > 0$ and $\varrho_n < \theta_n$, then

$$\limsup_{n\to\infty} \mathcal{Z}(\varrho_n, \theta_n) < 0.$$

The simulation function satisfying conditions ($\mathcal{Z}_1$), ($\mathcal{Z}_2$) and ($\mathcal{Z}'_3$) is called simulation function in the sense of Roldán-López-de-Hierro and is denoted by Ω.

Definition 1.2. *[2] A mapping $\mathcal{G} : (\mathbb{R}_+ \cup \{0\})^2 \to \mathbb{R}$ is called a C-class function, if it is continuous and verifies the following assumptions:*

- ***(1)*** *: $\mathcal{G}(\theta, \varrho) \leq \theta$;*
- ***(2)*** *: $\mathcal{G}(\theta, \varrho) = \theta$ means that either $\theta = 0$ or $\varrho = 0$, for every $\theta, \varrho \in \mathbb{R}_+ \cup \{0\}$.*

Definition 1.3. *[10] A mapping $\mathcal{G} : (\mathbb{R}_+ \cup \{0\})^2 \to \mathbb{R}$ has the property **($C_\mathcal{G}$)**, if there is $C_\mathcal{G} \geq 0$ with*

($\mathcal{G}1$) : $\mathcal{G}(\theta, \varrho) > C_\mathcal{G}$ implies that $\theta > \varrho$;
($\mathcal{G}2$) : $\mathcal{G}(\varrho, \varrho) \leq C_\mathcal{G}$, for each $\varrho \in \mathbb{R}_+ \cup \{0\}$.

Definition 1.4. *[10] A mapping $\zeta : (\mathbb{R}_+ \cup \{0\})^2 \to \mathbb{R}$ is called a $C_\mathcal{G}$-simulation function, if it is verifies the following assumptions:*

- ***(1)*** *: $\zeta(\varrho, \theta) < \mathcal{G}(\theta, \varrho)$ for each $\varrho, \theta > 0$, with $\mathcal{G} : (\mathbb{R}_+ \cup \{0\})^2 \to \mathbb{R}$ is a C-class function which has property **($C_\mathcal{G}$)**;*
- ***(2)*** *: if $\{\varrho_n\}, \{\theta_n\}$ two sequences in $\mathbb{R}_+$ with $\lim_{n\to\infty} \varrho_n = \lim_{n\to\infty} \theta_n > 0$, and $\varrho_n < \theta_n$, then $\limsup_{n\to\infty} \zeta(\varrho_n, \theta_n) < C_\mathcal{G}$.*

Definition 1.5. *[5] Let $\beta : \mathbb{R}_+ \cup \{0\} \to (0, 1)$ that verifies the following condition*

$$\text{for any } \{b_m\} \subset \mathbb{R}^+ \text{ and } \lim_{m\to\infty} \beta(b_m) = 1, \text{ implies } \lim_{m\to\infty} b_m = 0^+,$$

such a function is called a Geraghty function.

We deonte the set of Geraghty functions by $\mathcal{F}$.

Definition 1.6. *[12] Let $Q : \Upsilon \to \Upsilon$ be a self-mapping over a metric space (Υ, d). We say that Q is called Geraghty contraction if there is $\beta \in \mathcal{F}$, such that*

$$d(Q\omega, Q\mu) \leq \beta(d(\omega, \mu))d(\omega, \mu), \text{ for avery } \omega, \mu \in \Upsilon.$$

Theorem 1.7. *[12] Let (Υ, d) be a complete metric space and $Q : \Upsilon \to \Upsilon$ is a Geraghty contraction. Then Q possesses a unique fixed point $\omega \in \Upsilon$, and the sequence $\{Q^n\omega\}$ converges to ω.*

Definition 1.8. *[4] Let $Q : \Upsilon \to \Upsilon$ be a map and $\alpha : \Upsilon^2 \to \mathbb{R}$ be a function. Then Q is called α-admissible if $\alpha(\omega, \mu) \geq 1$ implies that $\alpha(Q\omega, Q\mu) \geq 1$.*

Definition 1.9. *[9] An α-admissible map Q is called triangular α-admissible if $\alpha(\omega, \sigma) \geq 1$ and $\alpha(\sigma, \mu) \geq 1$ implies $\alpha(\omega, \mu) \geq 1$, for all $\omega, \mu, \sigma \in \Upsilon$.*

Cho et al.[23] generalized the notion of Geraghty contraction to a new type of contraction, namely α-Geraghty contraction and proved the fixed point results for this type of contraction.

Definition 1.10. *[23] Let (Υ, d) be a metric space and $\alpha : \Upsilon^2 \to \mathbb{R}$ be a function. A map Q: $\Upsilon \to \Upsilon$ is said to be α-Geraghty contractive if there is $\beta \in \mathcal{F}$ such that, for every $\omega, \mu \in \Upsilon$,*

$$\alpha(\omega, \mu)d(Q\mu, Q\omega) \leq \beta(d(\omega, \mu))d(\omega, \mu).$$

Theorem 1.11. *[23] Let (Υ, d) be a complete metric space and $\alpha : \Upsilon \times \Upsilon \to \mathbb{R}$ be a function. Define a map $Q : \Upsilon \to \Upsilon$ satisfyying the following assumptions:*

(1) *Q is continuous and α-Geraghty contraction;*

(2) *Q be a triangular α-admissible;*

(3) *there is $\omega_1 \in \Upsilon$ with $\alpha(\omega_1, Q\omega_1) \geq 1$.*

Then Q possesses a fixed point $\omega \in \Upsilon$, and the sequence $\{Q^n\omega_1\}$ converges to ω.

For a non-empty set Υ, if (Υ, d) is a metric space. Let $\mathcal{P}(\Upsilon)$ the power set of Υ and $U, V \in \mathcal{P}(\Upsilon)$, we define

$$\begin{aligned}
\mathcal{N}(\Upsilon) &= \mathcal{P}(\Upsilon) \setminus \{\emptyset\}, \\
CB(\Upsilon) &= \{W \in \mathcal{N}(\Upsilon) : \text{ } W \text{ is closed and bounded }\}, \\
\mathcal{K}(\Upsilon) &= \{W \in \mathcal{N}(\Upsilon) : \text{ } W \text{ is compact }\}, \\
d(\mu, W) &= \inf\{d(\mu, \omega) : \text{ } \mu \in \Upsilon \text{ and } \omega \in W\}, \\
d(U, W) &= \inf\{d(\mu, \omega) : \text{ } \mu \in U \text{ and } \omega \in W\}, \\
\mathcal{H}(U, W) &= \max\{\sup_{\mu \in U} d(\mu, W), \sup_{\omega \in W} d(U, \omega)\}
\end{aligned}$$

Mohammadi et al.[24] introduced the concept of α-admissibility for multivalued mappings as follows

Definition 1.12. *[24] Let Υ be a non empty set, $Q : \Upsilon \to \mathcal{N}(\Upsilon)$ and $\alpha : \Upsilon^2 \to [0, \infty)$ be two maps. Then Q is called an α-admissible whenever for each $\omega \in \Upsilon$ and $\mu \in Q\omega$*

$$\alpha(\omega, \mu) \geq 1 \Rightarrow \alpha(\mu, \sigma) \geq 1, \text{ for all } \sigma \in T\mu.$$

Definition 1.13. *[25] Let (Υ, d) be a metric space and $\alpha : \Upsilon^2 \to [0, \infty)$. We say that the space (Υ, d) is α-complete, if and only if for any Cauchy sequence $\{\omega_n\}$ where $\alpha(\omega_n, \ \omega_{n+1}) \geq 1$ for every $n \in \mathbb{N}$ converges in Υ.*

Definition 1.14. *[26] Let (Υ, d) be a metric space and $\alpha : \Upsilon^2 \to \mathbb{R}_+ \cup \{0\}$ and $\mathcal{Q} : \Upsilon \to \mathcal{K}(\Upsilon)$ be two maps. Then $\mathcal{Q}$ is called α-continuous multivalued mapping on $(\mathcal{K}(\Upsilon), \mathcal{H})$, if for all sequences $\{\omega_n\}$ with $\alpha(\omega_n, \omega_{n+1}) \geq 1$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} \omega_n = \omega \in \Upsilon$, we have $\lim_{n\to\infty} \mathcal{Q}\omega_n = \mathcal{Q}\omega$, that is,*

$$\left.\begin{array}{l}\lim_{n\to\infty} d(\omega_n, \omega) = 0 \\ \textit{and} \\ \alpha(\omega_n, \omega_{n+1}) \geq 1 \textit{ for every } n \in \mathbb{N}\end{array}\right\} \Rightarrow \lim_{n\to\infty} \mathcal{H}(\mathcal{Q}\omega_n, \mathcal{Q}\omega) = 0.$$

Definition 1.15. *[21] Let Υ be a nonempty set, $\mathcal{Q} : \Upsilon \to \mathcal{N}(\Upsilon)$ and $\alpha : \Upsilon^2 \to \mathbb{R}_+ \cup \{0\}$ be two maps. Then $\mathcal{Q}$ is called a triangular α-admissible if $\mathcal{Q}$ is α-admissible which satisfying the following condition*

$$\left.\begin{array}{l}\alpha(\omega, \mu) \geq 1 \\ \textit{and} \\ \alpha(\mu, \sigma) \geq 1\end{array}\right\} \textit{implies that, } \alpha(\omega, \sigma) \geq 1, \ \forall \sigma \in \mathcal{Q}\mu.$$

Lemma 1.16. *[21] Let $\mathcal{Q} : \Upsilon \to \mathcal{N}(\Upsilon)$ be a triangular α-admissible mapping. If there is $\omega_0 \in \Upsilon$ and $\omega_1 \in \mathcal{Q}\omega_0$ such that $\alpha(\omega_0, \omega_1) \geq 1$. Then for a sequence $\{\omega_n\}$, $\omega_{n+1} \in \mathcal{Q}\omega_n$, we get $\alpha(\omega_n, \omega_m) \geq 1$ for every $n, m \in \mathbb{N}$ with $n < m$.*

Lemma 1.17. *[31] Let (Υ, d) be a metric space and $\{\omega_n\}$ be a sequence in Υ with*

$$\lim_{n\to\infty} d(\omega_n, \ \omega_{n+1}) = 0. \tag{2}$$

If $\{\omega_n\}$ is not a Cauchy sequence in Υ, then there exists $\epsilon > 0$ and the sequences $\{m_k\}$, $\{n_k\}$ of positive natural numbers with $n_k > m_k > k$ and

$$\begin{aligned}\lim_{k\to\infty} d(\omega_{m_k}, \omega_{n_k}) &= \lim_{k\to\infty} d(\omega_{m_k}, \omega_{n_k+1}) \\ &= \lim_{k\to\infty} d(\omega_{m_k-1}, \omega_{n_k}) \\ &= \lim_{k\to\infty} d(\omega_{m_k-1}, \omega_{n_k+1}) \\ &= \lim_{k\to\infty} d(\omega_{m_k+1}, \omega_{n_k+1}) = \epsilon.\end{aligned}$$

In 2020, Hussain et al.[33] presented a fixed point theorem for a multivalued Geraghty type contractive mapping via simulation functions along with C-class functions on a metric space.

Definition 1.18. *[33] Let (Υ, d) be a metric space and $\mathcal{Q} : \Upsilon \to \mathcal{K}(\Upsilon)$ be a maps, and let $\alpha : \Upsilon^2 \to \mathbb{R} \cup \{0\}$ be a function. We say that $\mathcal{Q}$ is $\mathcal{Z}_{(\alpha,\mathcal{G})}$-Geraghty multivalued contraction with respect to a $C_{\mathcal{G}}$-simulation function ζ if there exists $\beta \in \mathcal{F}$ such that*

$$\zeta\Big(\alpha(\omega, \mu)\mathcal{H}(\mathcal{Q}\omega, \mathcal{Q}\mu), \ \beta(M(\omega, \mu))M(\omega, \mu)\Big) \geq C_{\mathcal{G}}$$

for every $\omega, \mu \in \Upsilon$ with $\omega \neq \mu$, where

$$M(\omega, \mu) = \max\Big\{d(\omega, \mu), d(\omega, \mathcal{Q}\omega), d(\mu, \mathcal{Q}\mu)\Big\}.$$

Theorem 1.19. *[33] Let (Υ, d) be a metric space and $\mathcal{Q} : \Upsilon \to \mathcal{K}(\Upsilon)$ be a $\mathcal{Z}_{(\alpha,\mathcal{G})}$-Geraghty multivalued contraction satisfying:*

- ***(1)*** *: (Υ, d) is an α-complete metric space;*
- ***(2)*** *: there are $\omega_0 \in \Upsilon$ and $\omega_1 \in \mathcal{Q}\omega_0$ with $\alpha(\omega_0, \ \omega_1) \geq 1$;*
- ***(3)*** *: $\mathcal{Q}$ is triangular α-admissible;*

(4) *: Q is an α-continuous multivalued mapping.*

Then Q possesses a fixed point.

The aim of this document is to demonstrate certain fixed point outcomes for a novel form of multivalued Geraghty contraction using $C_{\mathcal{G}}$-simulation functions featuring C-class functions which is a generalize the result of the previous Theorem. Furthermore, we provide an illustration to exhibit the credibility of our findings with a graph. Additionally, as consequences, we demonstrate that several known fixed point theorems can be easily shown by these main results.

2. Main results

We begin with the following definition:

Definition 2.1. *Let (Υ, d) be a metric space and $Q : \Upsilon \to \mathcal{K}(\Upsilon)$ be a maps, and let $\alpha : \Upsilon^2 \to \mathbb{R} \cup \{0\}$ be a function. We say that Q is $\mathcal{Z}_{(C_{\mathcal{G}},\alpha)}$-Geraghty multivalued contraction with respect to a $C_{\mathcal{G}}$-simulation function ζ if there exists $\beta \in \mathcal{F}$ such that*

$$\zeta\Big(\alpha(\omega,\mu)\mathcal{H}(Q\omega, Q\mu),\ \beta(M(\omega,\mu))M(\omega,\mu) + L\,N(\omega,\mu)\Big) \geq C_{\mathcal{G}} \tag{3}$$

for every $\omega, \mu \in \Upsilon$ with $\omega \neq \mu$ and $L \geq 0$, where

$$M(\omega,\mu) = \max\Big\{d(\omega,\mu), d(\omega,Q\omega), d(\mu,Q\mu), \frac{d(\omega,Q\mu) + d(Q\omega,\mu)}{2}\Big\},$$
$$N(\omega,\mu) = \min\Big\{d(\omega,Q\omega), d(\mu,Q\mu), d(\omega,Q\mu), d(\mu,Q\omega)\Big\}.$$

Theorem 2.2. *Let (Υ, d) be a metric space and $Q : \Upsilon \to \mathcal{K}(\Upsilon)$ be a $\mathcal{Z}_{(C_{\mathcal{G}},\alpha)}$-Geraghty multivalued contraction satisfying:*

(1) *: (Υ, d) is an α-complete metric space;*

(2) *: there are $\omega_0 \in \Upsilon$ and $\omega_1 \in Q\omega_0$ with $\alpha(\omega_0, \omega_1) \geq 1$;*

(3) *: Q is triangular α-admissible;*

(4) *: Ethier*

(4a) *: Q is an α-continuous multivalued mapping,*
or

(4b) *: If $\{\omega_n\} \subset \Upsilon$ with $\alpha(\omega_n, \omega_{n+1}) \geq 1$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} \omega_n = \omega \in \Upsilon$, then we have $\alpha(\omega_n, \omega) \geq 1$ for every $n \in \mathbb{N}$.*

Then Q possesses a fixed point.

Proof. Let $\omega_0 \in \Upsilon$ and $\omega_1 \in Q\omega_0$ such that $\alpha(\omega_0, \omega_1) \geq 1$. If $\omega_0 = \omega_1$ or $\omega_1 \in Q\omega_1$, then ω_1 is a fixed point of Q and the proof is complete. Assume that $\omega_1 \notin Q\omega_1$. Since Q is $\mathcal{Z}_{(C_{\mathcal{G}},\alpha)}$-Geraghty multivalued contraction therefore taking $\omega = \omega_0$ and $\mu = \omega_1$ in **(3)**, we get

$$\zeta\Big(\alpha(\omega_0,\omega_1)\mathcal{H}(Q\omega_0,\ Q\omega_1),\ \beta(M(\omega_0,\ \omega_1))M(\omega_0,\ \omega_1) + L\,N(\omega_0,\omega_1)\Big) \geq C_{\mathcal{G}}.$$

Also we get that there exists $\omega_2 \in Q\omega_1, \omega_2 \neq \omega_1$ such that

$$\zeta\Big(\alpha(\omega_1,\ \omega_2)\mathcal{H}(Q\omega_1,\ Q\omega_2),\ \beta(M(\omega_1,\ \omega_2))M(\omega_1,\ \omega_2) + L\,N(\omega_1,\omega_2)\Big) \geq C_{\mathcal{G}}$$

and α-admissibility of Q gives $\alpha(\omega_1,\ \omega_2) \geq 1$. Repeating this process, we find that there exists a sequence $\{\omega_n\}$ with initial point ω_0 such that $\omega_{n+1} \in Q\omega_n, \omega_n \neq \omega_{n+1},\ \forall n \geq 0$, we derive

$$\alpha(\omega_n,\ \omega_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{4}$$

By taking $\omega = \omega_n$ and $\mu = \omega_{n+1}$ in **(3)**, we get that

$$\zeta\Big(\alpha(\omega_n,\ \omega_{n+1})\mathcal{H}(Q\omega_n,\ Q\omega_{n+1}),\ \beta(M(\omega_n,\ \omega_{n+1}))M(\omega_n,\ \omega_{n+1}) + L\, N(\omega_n, \omega_{n+1})\Big) \geq C_{\mathcal{G}}.$$

Since Q is $\mathcal{Z}_{(C_{\mathcal{G}},\alpha)}$-Geraghty multivalued contractive, we get

$$\begin{aligned} C_{\mathcal{G}} &\leq \zeta\Big(\alpha(\omega_n,\omega_{n+1})\mathcal{H}(Q\omega_n,Q\omega_{n+1}),\beta(M(\omega_n,\omega_{n+1}))M(\omega_n,\omega_{n+1}) + L\, N(\omega_n,\omega_{n+1})\Big) \\ &< \mathcal{G}\Big(\beta(M(\omega_n,\omega_{n+1}))M(\omega_n,\omega_{n+1}) + L\, N(\omega_n,\omega_{n+1}),\alpha(\omega_n,\omega_{n+1})\mathcal{H}(Q\omega_n,Q\omega_{n+1})\Big) \end{aligned}$$

Using we get that

$$\alpha(\omega_n,\ \omega_{n+1})\mathcal{H}(Q\omega_n,\ Q\omega_{n+1}) < \beta(M(\omega_n,\ \omega_{n+1}))M(\omega_n,\ \omega_{n+1}) + L\, N(\omega_n,\omega_{n+1}). \tag{5}$$

Since Q is compact, therefore

$$d(\omega_{n+1},\ \omega_{n+2}) \leq \mathcal{H}(Q\omega_n,\ Q\omega_{n+1}). \tag{6}$$

Thus, from inequalities **(5)** and **(6)** we have

$$\begin{aligned} d(\omega_{n+1},\ \omega_{n+2}) &\leq \alpha(\omega_n,\ \omega_{n+1})\mathcal{H}(Q\omega_n,\ Q\omega_{n+1}) \\ &\leq \beta(M(\omega_n,\ \omega_{n+1}))M(\omega_n,\ \omega_{n+1}) + L\, N(\omega_n,\omega_{n+1}) \\ &< M(\omega_n,\ \omega_{n+1}) + L\, N(\omega_n,\omega_{n+1}), \end{aligned} \tag{7}$$

On the other hand, we have

$$\begin{aligned} M(\omega_n,\omega_{n+1}) &= \max\Big\{d(\omega_n,\omega_{n+1}),d(\omega_n,Q\omega_n), d(\omega_{n+1},Q\omega_{n+1}), \frac{d(\omega_n,Q\omega_{n+1}) + d(\omega_{n+1},Q\omega_n)}{2}\Big\} \\ &= \max\Big\{d(\omega_n,\omega_{n+1}),d(\omega_n,Q\omega_n), d(\omega_{n+1},Q\omega_{n+1}), \frac{d(\omega_n,Q\omega_{n+1})}{2}\Big\} \end{aligned}$$

Since

$$\begin{aligned} \frac{1}{2}d(\omega_n,Q\omega_{n+1}) &\leq \frac{1}{2}\Big[d(\omega_n,\omega_{n+1}) + d(\omega_{n+1},Q\omega_{n+1})\Big] \\ &\leq \max\Big\{d(\omega_n,\omega_{n+1}), d(\omega_{n+1},Q\omega_{n+1})\Big\} \end{aligned}$$

and

$$(\omega_n,Q\omega_n) \leq d(\omega_n,\omega_{n+1}).$$

Then

$$\begin{aligned} M(\omega_n,\omega_{n+1}) &= \max\Big\{d(\omega_n,\omega_{n+1}), (\omega_n,Q\omega_n) + d(\omega_{n+1},Q\omega_{n+1})\Big\} \\ &= \max\Big\{d(\omega_n,\omega_{n+1}), d(\omega_{n+1},Q\omega_{n+1})\Big\}. \end{aligned}$$

and

$$\begin{aligned} N(\omega_n, \omega_{n+1}) &= \min\left\{d(\omega_n, Q\omega_n), d(\omega_{n+1}, Q\omega_{n+1}), d(\omega_n, Q\omega_{n+1}), d(\omega_{n+1}, Q\omega_n)\right\} \\ &= \min\left\{d(\omega_n, Q\omega_n), d(\omega_{n+1}, Q\omega_{n+1}), d(\omega_n, Q\omega_{n+1}), 0\right\} \\ &= 0. \end{aligned}$$

If $M(\omega_n,\ \omega_{n+1}) = d(\omega_{n+1}, Q\omega_{n+1})$, inequality **(7)** gives

$$\begin{aligned} d(\omega_{n+1},\ \omega_{n+2}) &< d(\omega_{n+1}, Q\omega_{n+1}) \\ &\leq d(\omega_{n+1}, \omega_{n+2}), \end{aligned}$$

a contradiction. Hence $M(\omega_n,\ \omega_{n+1}) = d(\omega_n,\ \omega_{n+1})$, and consequently from **(7)**, we have

$$d(\omega_{n+1},\ \omega_{n+2}) < d(\omega_n,\ \omega_{n+1})\ .$$

Hance for all $n \in \mathbb{N} \cup \{0\}$, we have $d(\omega_n,\ \omega_{n+1}) > d(\omega_{n+1},\ \omega_{n+2})$. Therefore, $d(\omega_n,\ \omega_{n+1})$ is a decreasing sequence of strictly positive real numbers, this implies that there is $\gamma \geq 0$ such that

$$\lim_{n\to\infty} d(\omega_n,\ \omega_{n+1}) = \lim_{n\to\infty} M(\omega_n,\ \omega_{n+1}) = \gamma.$$

Assume that $r > 0$. So by inequality the **(7)**, we obtain

$$\lim_{n\to\infty} \alpha(\omega_n,\ \omega_{n+1})\mathcal{H}(Q\omega_n,\ Q\omega_{n+1}) = \gamma \tag{8}$$

and

$$\lim_{n\to\infty} \beta(d(\omega_n,\ \omega_{n+1}))d(\omega_n,\ \omega_{n+1}) = \gamma. \tag{9}$$

Using **(3)** and **(2)** of Definition **1.4**, we get

$$\begin{aligned} C_{\mathcal{G}} &\leq \limsup_{n\to\infty} \limsup_{n\to\infty} \zeta\Big(\alpha(\omega_n, \omega_{n+1})\mathcal{H}(Q\omega_n, Q\omega_{n+1}),\ \beta(M(\omega_n, \omega_{n+1}))M(\omega_n, \omega_{n+1}) + L\ N(\omega_n, \omega_{n+1})\Big) \\ &= \limsup_{n\to\infty} \zeta\Big(\alpha(\omega_n,\ \omega_{n+1})\mathcal{H}(Q\omega_n,\ Q\omega_{n+1}),\ \beta(d(\omega_n,\ \omega_{n+1}))d(\omega_n, \omega_{n+1})\Big) \\ &<\ C_{\mathcal{G}}, \end{aligned}$$

which is absurd, this implies that $\gamma = 0$. Now, we prove that $\{\omega_n\}$ is a Cauchy sequence. Assume on contrary that it is not, hence by Lemma **1.17**, we have

$$\lim_{k\to\infty} d(\omega_{m_k},\ \omega_{n_k}) = \lim_{k\to\infty} d(\omega_{m_k+1},\ \omega_{n_k+1}) = \epsilon \tag{10}$$

and consequently,

$$\lim_{k\to\infty} M(\omega_{m_k},\ \omega_{n_k}) = \epsilon, \tag{11}$$

$$\lim_{k\to\infty} N(\omega_{m_k},\ \omega_{n_k}) = 0. \tag{12}$$

Let $\omega = \omega_{m_k}$, $\mu = \omega_{n_k}$. Since Q is triangular α-admissible, then by using Lemma **1.16**, we have $\alpha(\omega_{m_k},\ \omega_{n_k}) \geq 1$. Then by **(3)**,

$$\begin{aligned} C_{\mathcal{G}} &\leq \zeta\Big(\alpha(\omega_{m_k}, \omega_{n_k})\mathcal{H}(Q\omega_{m_k}, Q\omega_{n_k}), \beta(M(\omega_{m_k}, \omega_{n_k}))M(\omega_{m_k}, \omega_{n_k}) + L\ N(\omega_{m_k}, \omega_{n_k})\Big) \\ &= \zeta\Big(\alpha(\omega_{m_k}, \omega_{n_k})\mathcal{H}(Q\omega_{m_k}, Q\omega_{n_k}), \beta(M(\omega_{m_k}, \omega_{n_k}))M(\omega_{m_k}, \omega_{n_k}) + L\ N(\omega_{m_k}, \omega_{n_k})\Big) \\ &<\ \mathcal{G}\Big(\beta(M(\omega_{m_k}, \omega_{n_k}))M(\omega_{m_k}, \omega_{n_k}), \alpha(\omega_{m_k}, \omega_{n_k})\mathcal{H}(Q\omega_{m_k} + L\ N(\omega_{m_k}, \omega_{n_k}), Q\omega_{n_k})\Big). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} d(\omega_{m_k+1},\ \omega_{n_k+1}) &\leq \alpha(\omega_{m_k},\ \omega_{n_k})\mathcal{H}(Q\omega_{m_k},\ Q\omega_{n_k}) \\ &< \beta(M(\omega_{m_k},\ \omega_{n_k}))M(\omega_{m_k},\ \omega_{n_k}) + L\,N(\omega_{m_k},\ \omega_{n_k}) \\ &< M(\omega_{m_k},\ \omega_{n_k}) + L\,N(\omega_{m_k},\ \omega_{n_k}). \end{aligned} \tag{13}$$

Using **(10)** and **(11)** in **(13)**, we get

$$\lim_{k\to\infty} \alpha(\omega_{m_k},\ \omega_{n_k})\mathcal{H}(Q\omega_{m_k},\ Q\omega_{n_k}) = \epsilon,$$

and

$$\lim_{k\to\infty} \beta(M(\omega_{m_k},\ \omega_{n_k}))M(\omega_{m_k},\ \omega_{n_k}) + L\,N(\omega_{m_k},\ \omega_{n_k}) = \epsilon.$$

Therefore using **(2)** of Definition **1.4** and **(3)**, we get

$$\begin{aligned} C_{\mathcal{G}} &\leq \lim_{k\to\infty} \sup \zeta\Big(\alpha(\omega_{m_k}, \omega_{n_k})\mathcal{H}(Q\omega_{m_k}, Q\omega_{n_k}), \beta(M(\omega_{m_k}, \omega_{n_k}))M(\omega_{m_k}, \omega_{n_k}) + L\,N(\omega_{m_k}, \omega_{n_k})\Big) \\ &= \lim_{k\to\infty} \sup \zeta\Big(\alpha(\omega_{m_k}, \omega_{n_k})\mathcal{H}(Q\omega_{m_k}, Q\omega_{n_k}), \beta(M(\omega_{m_k}, \omega_{n_k}))M(\omega_{m_k}, \omega_{n_k}) + L\,N(\omega_{m_k}, \omega_{n_k})\Big) \\ &< C_{\mathcal{G}}, \end{aligned}$$

which is absurd. This implies that $\{\omega_n\}$ is a Cauchy sequence. By using the α-completeness of (Υ, d), there is $\omega \in \Upsilon$ with $\lim_{n\to\infty} \omega_n = \omega$.

Case 1: If Q is α-continuity multivalued mapping, from **(4a)** we deduce

$$\lim_{n\to\infty} \alpha(\omega_n, \omega)\mathcal{H}(Q\omega_n, Q\omega) = 0. \tag{14}$$

Thus we obtain

$$d(\omega, Q\omega) = \lim_{n\to\infty} d(\omega_{n+1}, Q\omega) \leq \lim_{n\to\infty} \alpha(\omega_n, \omega)\mathcal{H}(Q\omega_n, Q\omega) = 0.$$

Therefore, $\omega \in Q\omega$ and hence Q has a fixed point.

Case 2: If Q is not α-continuous multivalued mapping, from the condition **(4b)**, we can get $\alpha(\omega_n, \omega) \geq 1$ for every $n \in \mathbb{N}$. According to **(3)**, we obtain

$$\begin{aligned} C_{\mathcal{G}} &\leq \zeta\Big(\alpha(\omega_n, \omega)\mathcal{H}(Q\omega_n, Q\omega), \beta(M(\omega_n, \omega))M(\omega_n, \omega) + L\,N(\omega_n, \omega)\Big) \\ &< \mathcal{G}\Big(\beta(M(\omega_n, \omega))M(\omega_n, \omega) + L\,N(\omega_n, \omega), \alpha(\omega_n, \omega)\mathcal{H}(Q\omega_n, Q\omega)\Big) \end{aligned} \tag{15}$$

implies that

$$\begin{aligned} \alpha(\omega_n, \omega)\mathcal{H}(Q\omega_n, Q\omega) &\leq \beta(M(\omega_n, \omega))M(\omega_n, \omega) + L\,N(\omega_n, \omega) \\ &< M(\omega_n, \omega) + L\,N(\omega_n, \omega), \end{aligned} \tag{16}$$

where

$$M(\omega_n, \omega) = \max\Big\{d(\omega_n, \omega), d(\omega_n, Q\omega_n), d(\omega, Q\omega), \frac{d(\omega_n, Q\omega) + d(Q\omega_n, \omega)}{2}\Big\}$$

and

$$N(\omega_n, \omega) = \min\Big\{d(\omega_n, Q\omega_n), d(\omega, Q\omega), d(\omega_n, Q\omega), d(Q\omega_n, \omega)\Big\}.$$

Since $\omega_{n+1} \in Q\omega_n$, then $d(\omega_n, Q\omega_n) \leq d(\omega_n, \omega_{n+1})$ and $d(\omega, Q\omega_n) \leq d(\omega, \omega_{n+1})$, implies that

$$\lim_{n\to\infty} d(\omega_n, \omega_{n+1}) = 0, \text{ implies that } \lim_{n\to\infty} d(\omega_n, Q\omega_n) = 0,$$
$$\lim_{n\to\infty} d(\omega, \omega_{n+1}) = 0, \text{ implies that } \lim_{n\to\infty} d(\omega, Q\omega_n) = 0.$$

So, we can obtain

$$\begin{aligned} &\lim_{n\to\infty} M(\omega_n, u) = d(\omega, Q\omega),\\ &\lim_{n\to\infty} N(\omega_n, \omega) = 0. \end{aligned} \tag{17}$$

Suppose that $d(\omega, Q\omega) > 0$. Since we know that $d(\omega_{n+1}, Q\omega) \leq \mathcal{H}(Q\omega_n, Q\omega)$ and $\alpha(\omega_n, \omega) \geq 1$, therefore

$$d(\omega_{n+1}, Q\omega) \leq \alpha(\omega_n, \omega)\mathcal{H}(Q\omega_n, Q\omega). \tag{18}$$

Then, from **(18)** and **(16)** we deduce

$$\begin{aligned} d(\omega_{n+1}, Q\omega) &\leq \alpha(\omega_n, u)\mathcal{H}(Q\omega_n, Q\omega)\\ &\leq \beta\Big(M(\omega_n, \omega)\Big)M(\omega_n, \omega) + L\, N(\omega_n, \omega)\\ &< M(\omega_n, \omega) + L\, N(\omega_n, \omega). \end{aligned} \tag{19}$$

So,

$$\frac{d(\omega_{n+1}, Q\omega)}{d(\omega, Q\omega)} \leq \beta\Big(M(\omega_n, \omega)\Big)\frac{M(\omega_n, \omega)}{d(\omega, Q\omega)} + L\frac{N(\omega_n, \omega)}{d(\omega, Q\omega)} < \frac{M(\omega_n, \omega) + L\, N(\omega_n, \omega)}{d(\omega, Q\omega)}. \tag{20}$$

By using **(17)**, **(20)** and letting $n \to \infty$, we get

$$\lim_{n\to\infty} \beta\Big(M(\omega_n, \omega)\Big) = 1.$$

Since β is a Geraghty function, then

$$\lim_{n\to\infty} M(\omega_n, \omega) = 0.$$

which a contraduction. Therefore, $d(\omega, Q\omega) = 0$, that is, $\omega \in Q\omega$. The proof is finished. □

Theorem 2.3. *Let (Υ, d) be a metric space and $Q : \Upsilon \to \mathcal{K}(\Upsilon)$ be a maps, such that*

$$\zeta\Big(\alpha(\omega, \mu)\mathcal{H}(Q\omega, Q\mu), \beta(M_d(\omega, \mu))M_d(\omega, \mu)\Big) \geq C_{\mathcal{G}}$$

for every $\omega, \mu \in \Upsilon$ with $\omega \neq \mu$, $L \geq 0$ and

$$\begin{aligned} M_d(\omega, \mu) &= \max\Big\{d(\omega, \mu), \frac{d(\omega, Q\omega) + d(\mu, Q\mu)}{2}, \frac{d(\omega, Q\mu) + d(Q\omega, \mu)}{2}\Big\},\\ N(\omega, \mu) &= \min\Big\{d(\omega, Q\omega), d(\mu, Q\mu), d(\omega, Q\mu), d(\mu, Q\omega)\Big\}. \end{aligned}$$

Also assume that

(1) *: (Υ, d) is an α-complete metric space;*

(2) *: There are $\omega_0 \in \Upsilon$ and $\omega_1 \in Q\omega_0$ with $\alpha(\omega_0, \omega_1) \geq 1$;*

(3) *: Q is triangular α-admissible;*

(4) *: Ethier*

(4a) *: Q is an α-continuous multivalued mapping,*
or

(4b) *: If $\{\omega_n\} \subset \Upsilon$ with $\alpha(\omega_n, \omega_{n+1}) \geq 1,\ n \in \mathbb{N}$ and $\lim_{n\to\infty} \omega_n = \omega \in \Upsilon$, then we get $\alpha(\omega_n, \omega) \geq 1$ for every $n \in \mathbb{N}$.*

Then Q possesses a fixed point.

Proof. We have

$$\begin{aligned} M_d(\omega, \mu) &= \max\Big\{d(\omega, \mu), \frac{d(\omega, Q\omega) + d(\mu, Q\mu)}{2}, \frac{d(\omega, Q\mu) + d(Q\omega, \mu)}{2}\Big\} \\ &\leq \max\Big\{d(\omega, \mu), d(\omega, Q\omega), d(\mu, Q\mu), \frac{d(Q\omega, \mu) + d(\omega, Q\mu)}{2}\Big\} \\ &\leq M(\omega, \mu). \end{aligned}$$

Therefore, Q is a $\mathcal{Z}_{(C_\mathcal{G},\alpha)}$-Geraghty multivalued contraction. As in the proof of Theorem **2.2**, we conclude that Q possesses a unique fixed point. □

Theorem 2.4. *Let (Υ, d) be a metric space and $Q : \Upsilon \to \mathcal{K}(\Upsilon)$ be a maps, such that*

$$\alpha(\omega, \mu) \geq 1 \ \Rightarrow\ \zeta\Big(\mathcal{H}(Q\omega, Q\mu),\ \beta(E(\omega, \mu))E(\omega, \mu) + L\, N(\omega, \mu)\Big) \geq C_\mathcal{G} \tag{21}$$

for every $\omega, \mu \in \Upsilon$ with $\omega \neq \mu$, where $L \geq 0$ and

$$\begin{aligned} &E(\omega, \mu) = d(\omega, \mu) + |d(\omega, Q\omega) - d(\mu, Q\mu)|, \\ &N(\omega, \mu) = \min\Big\{d(\omega, Q\omega), d(\mu, Q\mu), d(\omega, Q\mu), d(\mu, Q\omega)\Big\}. \end{aligned}$$

Also assume that

(1) *: (Υ, d) is an α-complete metric space;*

(2) *: There are $\omega_0 \in \Upsilon$ and $\omega_1 \in Q\omega_0$ such that $\alpha(\omega_0,\ \omega_1) \geq 1$;*

(3) *: Q is triangular α-admissible;*

(4) *: If $\{\omega_n\} \subset \Upsilon$ with $\lim_{n\to\infty} d(\omega_n, \omega_{n+1}) = r \in \mathbb{R}_+$, then $\lim_{n\to\infty} d(\omega_n, Q\omega_n) = r$;*

(5) *: Ethier*

(5a) *: Q is an α-continuous multivalued mapping,*
or

(5b) *: If $\{\omega_n\} \subset \Upsilon$ with $\alpha(\omega_n, \omega_{n+1}) \geq 1,\ n \in \mathbb{N}$ and $\lim_{n\to\infty} \omega_n = \omega \in \Upsilon$, then we get $\alpha(\omega_n, \omega) \geq 1$ for every $n \in \mathbb{N}$.*

Then Q possesses a fixed point.

Proof. Let $\omega_0 \in \Upsilon$ and $\omega_1 \in Q\omega_0$ such that $\alpha(\omega_0,\ \omega_1) \geq 1$. If $\omega_0 = \omega_1$ or $\omega_1 \in Q\omega_1$, then ω_1 is a fixed point of Q and the proof is complete. Assume that $\omega_1 \notin Q\omega_1$. From **(21)**, and taking $\omega = \omega_0$ and $\mu = \omega_1$, we get

$$\zeta\Big(\mathcal{H}(Q\omega_0,\ Q\omega_1),\ \beta(E(\omega_0,\ \omega_1))E(\omega_0,\ \omega_1) + L\, N(\omega_0, \omega_1)\Big) \geq C_\mathcal{G}.$$

Also we get that there exists $\omega_2 \in \mathcal{Q}\omega_1, \omega_2 \neq \omega_1$ such that

$$\zeta\Big(\mathcal{H}(\mathcal{Q}\omega_1,\ \mathcal{Q}\omega_2),\ \beta(E(\omega_1,\ \omega_2))E(\omega_1,\ \omega_2) + L\ N(\omega_1, \omega_2)\Big) \geq C_{\mathcal{G}}$$

and α-admissibility of $\mathcal{Q}$ gives $\alpha(\omega_1,\ \omega_2) \geq 1$. Repeating this process, we find that there exists a sequence $\{\omega_n\}$ with initial point ω_0 such that $\omega_{n+1} \in \mathcal{Q}\omega_n, \omega_n \neq \omega_{n+1} \forall n \geq 0$, we derive

$$\alpha(\omega_n,\ \omega_{n+1}) \geq 1 \text{ for every } n \in \mathbb{N} \cup \{0\}. \tag{22}$$

We want to conclude that $d(\omega_n, \omega_{n+1})$ is decreasing. Assume that

$$d(\omega_n, \omega_{n+1}) < d(\omega_{n+1}, \omega_{n+2}). \tag{23}$$

From **(22)** and **(21)**, we find that

$$\begin{aligned} C_{\mathcal{G}} &\leq \zeta\Big(\mathcal{H}(\mathcal{Q}\omega_n, \mathcal{Q}\omega_{n+1}), \beta(E(\omega_n, \omega_{n+1}))E(\omega_n, \omega_{n+1}) + L\ N(\omega_n, \omega_{n+1})\Big) \\ &< \mathcal{G}\Big(\beta(E(\omega_n, \omega_{n+1}))E(\omega_n, \omega_{n+1}) + L\ N(\omega_n, \omega_{n+1}), \mathcal{H}(\mathcal{Q}\omega_n, \mathcal{Q}\omega_{n+1})\Big) \end{aligned}$$

we get that

$$\mathcal{H}(\mathcal{Q}\omega_n,\ \mathcal{Q}\omega_{n+1}) < \beta(E(\omega_n,\ \omega_{n+1}))E(\omega_n,\ \omega_{n+1}) + L\ N(\omega_n, \omega_{n+1}). \tag{24}$$

Since $\mathcal{Q}$ is compact, therefore

$$d(\omega_{n+1},\ \omega_{n+2}) \leq \mathcal{H}(\mathcal{Q}\omega_n,\ \mathcal{Q}\omega_{n+1}). \tag{25}$$

Thus, from inequalities **(24)** and **(25)** we have

$$\begin{aligned} d(\omega_{n+1},\ \omega_{n+2}) &\leq \mathcal{H}(\mathcal{Q}\omega_n, \mathcal{Q}\omega_{n+1}) \\ &< \beta(E(\omega_n, \omega_{n+1}))E(\omega_n, \omega_{n+1}) + L\ N(\omega_n, \omega_{n+1}) \\ &< E(\omega_n, \omega_{n+1}) + L\ N(\omega_n, \omega_{n+1}), \end{aligned} \tag{26}$$

On the other hand, we have

$$E(\omega_n, \omega_{n+1}) = d(\omega_n, \omega_{n+1}) + |d(\omega_n, \mathcal{Q}\omega_n) - d(\omega_{n+1}, \mathcal{Q}\omega_{n+1})|$$

and

$$\begin{aligned} N(\omega_n, \omega_{n+1}) &= \min\Big\{d(\omega_n, \mathcal{Q}\omega_n), d(\omega_{n+1}, \mathcal{Q}\omega_{n+1}), d(\omega_n, \mathcal{Q}\omega_{n+1}), d(\omega_{n+1}, \mathcal{Q}\omega_n)\Big\} \\ &= \min\Big\{d(\omega_n, \mathcal{Q}\omega_n), d(\omega_{n+1}, \mathcal{Q}\omega_{n+1}), d(\omega_n, \mathcal{Q}\omega_{n+1}), 0\Big\} \\ &= 0. \end{aligned} \tag{27}$$

By using **(23)**, we have

$$\begin{aligned} E(\omega_n, \omega_{n+1}) &= d(\omega_n, \omega_{n+1}) - d(\omega_n, \mathcal{Q}\omega_n) + d(\omega_{n+1}, \mathcal{Q}\omega_{n+1}) \\ &\leq d(\omega_{n+1}, \omega_{n+2}). \end{aligned} \tag{28}$$

Hence, from **(28)** and inequality **(26)** turns into

$$\begin{aligned} d(\omega_{n+1},\ \omega_{n+2}) &\leq \mathcal{H}(\mathcal{Q}\omega_n, \mathcal{Q}\omega_{n+1}) && (29) \\ &< \beta(E(\omega_n, \omega_{n+1}))E(\omega_n, \omega_{n+1}) + L\ N(\omega_n, \omega_{n+1}) \\ &< E(\omega_n, \omega_{n+1}) + L\ N(\omega_n, \omega_{n+1}) \\ &\leq d(\omega_{n+1}, \omega_{n+2}), && (30) \end{aligned}$$

a contradiction. Consequently, we conclude that $\{d(\omega_n, \omega_{n+1})\}$ is a decreasing sequence . Therefore, $d(\omega_n, \omega_{n+1})$ is a decreasing sequence of positive real numbers, hence there is $\gamma \geq 0$ such that

$$\lim_{n\to\infty} d(\omega_n,\ \omega_{n+1}) = \gamma.$$

According to assumption **(4)**, we have $\lim\limits_{n\to\infty} d(Q\omega_n, \omega_n) = \gamma$. So,

$$\lim_{n\to\infty} E(\omega_n,\ \omega_{n+1}) = \gamma.$$

Assume that $\gamma > 0$. So by inequality **(26)** and **(27)**, we obtain

$$\lim_{n\to\infty} \beta(E(\omega_n,\ \omega_{n+1})) = 1. \tag{31}$$

Since β is a Geraghty function, then

$$\lim_{n\to\infty} E(\omega_n, \omega_{n+1}) = 0,$$

a contradiction and hence $\gamma = 0$.

Now, we prove that $\{\omega_n\}$ is a Cauchy sequence. Assume on contrary that it is not, then by Lemma **1.17**, we have

$$\lim_{k\to\infty} d(\omega_{m_k}, \omega_{n_k}) = \lim_{k\to\infty} d(\omega_{m_k+1},\ \omega_{n_k+1}) = \epsilon \tag{32}$$

and consequently,

$$\lim_{k\to\infty} E(\omega_{m_k}, \omega_{n_k}) = \epsilon. \tag{33}$$

Let $\omega = \omega_{m_k}, \mu = \omega_{n_k}$. Since Q is triangular α-admissible, then by using Lemma **1.16**, we can obtain $\alpha(\omega_{m_k},\ \omega_{n_k}) \geq 1$. Then by **(21)**,

$$\begin{aligned}
C_{\mathcal{G}} &\leq \zeta\Big(\mathcal{H}(Q\omega_{m_k}, Q\omega_{n_k}), \beta(E(\omega_{m_k}, \omega_{n_k}))E(\omega_{m_k}, \omega_{n_k}) + L\ N(\omega_{m_k}, \omega_{n_k})\Big) \\
&= \zeta\Big(\mathcal{H}(Q\omega_{m_k}, Q\omega_{n_k}), \beta(E(\omega_{m_k}, \omega_{n_k}))E(\omega_{m_k}, \omega_{n_k}) + L\ N(\omega_{m_k}, \omega_{n_k})\Big) \\
&< \ \mathcal{G}\Big(\beta(E(\omega_{m_k}, \omega_{n_k}))E(\omega_{m_k}, \omega_{n_k}) + L\ N(\omega_{m_k}, \omega_{n_k}), \mathcal{H}(Q\omega_{m_k}, Q\omega_{n_k})\Big).
\end{aligned}$$

So,

$$\begin{aligned}
\mathcal{H}(Q\omega_{m_k}, Q\omega_{n_k}) &\leq \mathcal{H}(Q\omega_{m_k}, Q\omega_{n_k}) \\
&< \beta(E(\omega_{m_k}, \omega_{n_k}))E(\omega_{m_k}, \omega_{n_k}) + L\ N(\omega_{m_k}, \omega_{n_k}) \\
&< E(\omega_{m_k}, \omega_{n_k}) + L\ N(\omega_{m_k}, \omega_{n_k}).
\end{aligned} \tag{34}$$

Using **(34)**, **(32)** and **(33)**, we get

$$\lim_{k\to\infty} \mathcal{H}(Q\omega_{m_k},\ Q\omega_{n_k}) = \epsilon,$$

and

$$\lim_{k\to\infty} \beta(E(\omega_{m_k},\ \omega_{n_k}))E(\omega_{m_k},\ \omega_{n_k}) = \epsilon.$$

Therefore using **(21)** and **(2)** of Definition **1.4**, we get

$$\begin{aligned}
C_{\mathcal{G}} &\leq \lim_{k\to\infty} \sup \zeta\Big(\mathcal{H}(Q\omega_{m_k}, Q\omega_{n_k}), \beta(E(\omega_{m_k}, \omega_{n_k}))E(\omega_{m_k}, \omega_{n_k}) + L\ N(\omega_{m_k}, \omega_{n_k})\Big) \\
&= \lim_{k\to\infty} \sup \zeta\Big(\mathcal{H}(Q\omega_{m_k}, Q\omega_{n_k}), \beta(E(\omega_{m_k}, \omega_{n_k}))E(\omega_{m_k}, \omega_{n_k}) + L\ N(\omega_{m_k}, \omega_{n_k})\Big) \\
&< \ C_{\mathcal{G}},
\end{aligned}$$

a contradiction. This implires that $\{\omega_n\}$ is a Cauchy sequence. By using the α-completeness of (Υ, d), there is $\omega \in \Upsilon$ with $\omega_n \to \omega$ as $n \to \infty$.

Case 1: If Q is α-continuous multivalued mapping, from **(5a)** we deduce

$$\lim_{n\to\infty} \mathcal{H}(Q\omega_n, Q\omega) = 0.$$

Thus we obtain

$$d(\omega, Q\omega) = \lim_{n\to\infty} d(\omega_{n+1}, Q\omega) \le \lim_{n\to\infty} \mathcal{H}(Q\omega_n, Q\omega) = 0.$$

Therefore, $\omega \in Q\omega$ and hence Q has a fixed point.

Case 2: If Q is not α-continuous multivalued mapping, From the condition **(5b)**, we get $\alpha(\omega_n, \omega) \ge 1$ for every $n \in \mathbb{N}$. According to **(21)**, we have

$$\begin{aligned} C_{\mathcal{G}} &\le \zeta\Big(\mathcal{H}(Q\omega_n, Q\omega), \beta(E(\omega_n,\omega))E(\omega_n,\omega) + L\,N(\omega_n,\omega)\Big) \\ &< \mathcal{G}\Big(\beta(E(\omega_n,\omega))E(\omega_n,\omega) + L\,N(\omega_n,\omega), \mathcal{H}(Q\omega_n, Q\omega)\Big) \end{aligned} \tag{35}$$

Using we can get

$$\begin{aligned} \mathcal{H}(Q\omega_n, Q\omega) &\le \beta(E(\omega_n,\omega))E(\omega_n,\omega) + L\,N(\omega_n,\omega) \\ &< E(\omega_n,\omega) + L\,N(\omega_n,\omega), \end{aligned} \tag{36}$$

where

$$E(\omega_n,\omega) = d(\omega_n,\omega) + |d(\omega_n, Q\omega_n) - d(\omega, Q\omega)|$$

and

$$\begin{aligned} N(\omega_n,\omega) &= \min\Big\{d(\omega_n, Q\omega_n), d(\omega, Q\omega), d(\omega_n, Q\omega), d(Q\omega_n, \omega)\Big\} \\ &= \min\Big\{d(\omega_n, \omega_{n+1}), d(\omega, Q\omega), d(\omega_n, Q\omega), d(\omega_{n+1}, \omega)\Big\}. \end{aligned}$$

Since $\lim_{n\to\infty} d(\omega_n, \omega_{n+1}) = 0$ implies that $\lim_{n\to\infty} d(\omega_n, Q\omega_n) = 0$. Then, we can obtain

$$\lim_{n\to\infty} E(\omega_n,\omega) = d(\omega, Q\omega) \text{ and } \lim_{n\to\infty} N(\omega_n,\omega) = 0. \tag{37}$$

Since we know that $d(\omega_{n+1}, Q\omega) \le \mathcal{H}(Q\omega_n, Q\omega)$ and $\alpha(\omega_n,\omega) \ge 1$, therefore from **(36)** we deduce

$$\begin{aligned} d(\omega_{n+1}, Q\omega) &\le \mathcal{H}(Q\omega_n, Q\omega) \\ &\le \beta(E(\omega_n,\omega))E(\omega_n,\omega) + L\,N(\omega_n,\omega) \\ &< E(\omega_n,\omega) + L\,N(\omega_n,\omega). \end{aligned} \tag{38}$$

Suppose that $d(\omega,\ Q\omega) > 0$, then from**(37)**, **(38)** and Letting $n \to \infty$, we get

$$\lim_{n\to\infty} \beta(E(\omega_n,\omega)) = 1,$$

Since β is a Gerahty function, then

$$\lim_{n\to\infty} E(\omega_n,\omega) = 0,$$

a contradiction. This implies that $d(\omega, Q\omega) = 0$, and thus, $\omega \in Q\omega$. the proof is finished. □

Example 2.5. *Let $\Upsilon = [-3, 3]$, $d(\omega,\mu) = |\omega - \mu|$ and a function*

$$\alpha : \Upsilon^2 \to \mathbb{R}_+ \cup \{0\}$$
$$(\omega,\mu) \mapsto \alpha(\omega,\mu) = \begin{cases} 1 & \text{if } (\omega,\mu) \in [0,1] \\ 0 & \text{otherwise.} \end{cases}$$

and $Q : \Upsilon \to \mathcal{K}(\Upsilon)$ be given by

$$Q\omega = \begin{cases} \{0\} & \text{if } \omega \in (-10, 0)\ , \\ \left[0, \frac{2+\omega}{4}\right] & \text{if } \omega \in [0,1]\,, \\ \left[\frac{\omega-3}{2}, \frac{\omega-1}{2}\right] & \text{if } \omega \in (1,2]\,, \\ \{2\} & \text{if } \omega \in (2,3)\ , \end{cases}$$

Then the space (Υ, d) is α-complete and Q is α-continuous. Moreover, we have the multivalued mapping Q is a triangular α-admissible, on the other hand if $\alpha(\omega,\mu) \geq 1$, then for any $\omega, \mu \in [0,1]$ we can get

$$Q\omega = [0, \frac{2+\omega}{4}]$$
$$Q\mu = [0, \frac{2+\mu}{4}],$$

implies that

$$\alpha(a,b) \geq 1 \textit{ for all } a \in Q\omega \textit{ and } b \in Q\mu.$$

Thus, Q is α-admissible. In addition, if $\alpha(\omega,\mu) \geq 1$, we deduce $\omega, \mu \in [0,1]$. So,

$$\omega \in [0,1]$$
$$Q\mu = [0, \frac{2+\mu}{4}].$$

Let $\sigma \in Q\mu$. Then we can obtain $\alpha(\mu,\sigma) \geq 1$.

*Finally, For $\omega \in [0,1]$ and $\sigma \in [0, \frac{2+\omega}{4}]$ we have $\alpha(\omega,\sigma) \geq 1$. So the mapping Q is a triangular α-admissible. If we take $\omega_0 = 1$ then the condition **(2)** of Theorem **2.2** is satisfied. Next, Consider the following mappings*

$$\mathcal{G}(\theta,\varrho) = \theta - \varrho;$$
$$\zeta(\varrho,\theta) = \frac{3}{4}\varrho - \theta;$$
$$\beta(t) = \frac{1}{t+1},$$

for every $t, \theta, \varrho \in \mathbb{R}_+ \cup \{0\}$, it is therefore clear that β, $\mathcal{G}$ and ζ are functions respectively Geraghty function, C-class function and $C_{\mathcal{G}}$-simulation function.

Let $\omega, \mu \in [0,1], \omega \neq \mu$ then $\alpha(\omega,\mu) = 1$, we are going to assess the values of $\mathcal{H}(Q\omega, Q\mu)$ and

$$M(\omega,\mu) = \max\Big\{d(\omega,\mu), d(\omega,Q\omega), d(\mu,Q\mu), \frac{d(\omega,Q\mu) + d(Q\omega,\mu)}{2}\Big\},$$
$$N(\omega,\mu) = \min\Big\{d(\omega,Q\omega), d(\mu,Q\mu), d(\omega,Q\mu), d(\mu,Q\omega)\Big\}.$$

And let $L \geq 0$. So,

$$\begin{aligned}\zeta\Big(\alpha(\omega,\mu)\mathcal{H}(Q\omega,Q\mu),\beta(M(\omega,\mu))M(\omega,\mu)+L\,N(\omega,\mu)\Big) &= \frac{3}{4}\Big[\beta(M(\omega,\mu))M(\omega,\mu)+L\,N(\omega,\mu))\Big]\\ &\quad-\alpha(\omega,\mu)\mathcal{H}(Q\omega,Q\mu)\\ &= \frac{8}{9}\Big[\frac{M(\omega,\mu)}{M(\omega,\mu)+1}+L\,N(\omega,\mu))\Big]\\ &\quad-\mathcal{H}(Q\omega,Q\mu) \end{aligned}\tag{39}$$

and

$$\begin{aligned}\mathcal{G}\Big(\beta(M(\omega,\mu))M(\omega,\mu)+L\,N(\omega,\mu),\alpha(\omega,\mu)\mathcal{H}(Q\omega,Q\mu)\Big) &= \beta(M(\omega,\mu))M(\omega,\mu)+L\,N(\omega,\mu))\\ &\quad-\alpha(\omega,\mu)\mathcal{H}(Q\omega,Q\mu)\\ &= \frac{M(\omega,\mu)}{M(\omega,\mu)+1}+L\,N(\omega,\mu))\\ &\quad-\mathcal{H}(Q\omega,Q\mu)\end{aligned}\tag{40}$$

*Accoding to **(40)** and **(40)**, we conclude*

$$\begin{aligned}0 <\ &\zeta\Big(\alpha(\omega,\mu)\mathcal{H}(Q\omega,Q\mu),\beta(M(\omega,\mu))M(\omega,\mu)+L\,N(\omega,\mu)\Big)\\ <\ &\mathcal{G}\Big(\beta(M(\omega,\mu))M(\omega,\mu)+L\,N(\omega,\mu),\alpha(\omega,\mu)\mathcal{H}(Q\omega,Q\mu),\Big)\end{aligned}\tag{41}$$

*Then, by using the inequality **(41)** and the Definition **2.1**, it is therefore clear that the mapping Q is an $\mathcal{Z}_{(\alpha,C_{\mathcal{G}},)}$-Geraghty multivalued contraction with $C_{\mathcal{G}} = 0$. Thus the assumptions of Theorem **2.2** are verified. So Q possesses a fixed points in Υ.*

3. Consequences

The concept of $\mathbf{G}_R$-contraction on a metric space with a graph $\mathbf{G}_R$ was introduced by Jachymski [28] in 2008, along with a fixed point theorem that extends the results of Ran and Reurings [29]. Jachymski's results have since been expanded to include multivalued mappings in [30, 31]. In this paper, we explore the consequences of our primary findings on graphs. To do so, we require several essential notions. We consider a metric space (Υ, d) and let $\Delta = \{(\omega,\omega) \ : \ \omega \in \Upsilon\}$. We define a graph $\mathbf{G}_R$ as having vertices $\mathcal{V}(\mathbf{G}_R)$ equal to Υ and edges $\mathbb{E}(\mathbf{G}_R)$, such that

$$(\omega,\mu),(\mu,\omega) \in \mathbb{E}(\mathbf{G}_R), \text{ implies that } \omega = \mu.$$

If the edges have an associated direction, we call $\mathbf{G}_R$ a directed graph. Now, with the pair $\Big(\mathcal{V}(\mathbf{G}_R), \mathbb{E}(\mathbf{G}_R)\Big)$, we can get the graph $\mathbf{G}_R$. We may treat $\mathbf{G}_R$ as a weighted graph by assigning the distance between its vertices to each edge.

Definition 3.1. *[21] Let a non-empty set Υ equipped with a graph $\mathbf{G}_R$, and let Q be a multivalued mapping from Υ to $\mathcal{N}(\Upsilon)$. If Q satisfies the condition that for each $\omega \in \Upsilon$ and $\mu \in Q\omega$ with $(\omega,\mu),(\mu,\sigma) \in \mathbb{E}(\mathbf{G}_R)$, we get $(\omega,\sigma) \in \mathbb{E}(\mathbf{G}_R)$ for all $\sigma \in Q\mu$, then we say that Q is a triangular edge preserving mapping.*

Definition 3.2. *[21] Let $\mathbf{G}_R$ be a graph on the metric space (Υ, d), we say that Υ is $\mathbb{E}(\mathbf{G}R)$-complete if any Cauchy sequence $\{\omega_n\}$ in Υ with $(\omega_n,\omega_{n+1}) \in \mathbb{E}(\mathbf{G}_R)$ for all $n \in \mathbb{N}$ converges in Υ.*

Definition 3.3. *[21] Let $\mathbf{G}_R$ is a graph on the metric space (Υ, d). A mapping $Q : \Upsilon \to \mathcal{K}(\Upsilon)$ is an $\mathbb{E}(\mathbf{G}_R)$-continuous on $(\mathcal{K}(\Upsilon), \mathcal{H})$, if for a given point $\omega \in \Upsilon$ and a sequence $\{\omega_n\}$ with*

$$\left.\begin{array}{l}\lim\limits_{n\to\infty} d(\omega_n,\omega) = 0\\ \textit{and}\\ (\omega_n,\omega_{n+1}) \in \mathbb{E}(\mathbf{G}_R), \quad \forall n \in \mathbb{N}\end{array}\right\} \textit{implies that, } \lim_{n\to\infty}\mathcal{H}(Q\omega_n, Q\omega) = 0.$$

Definition 3.4. *Let $\mathbf{G}_R$ is a graph on the metric space (Υ, d). A mapping $\mathcal{Q} : \Upsilon \to \mathcal{K}(\Upsilon)$ is an $\mathbb{E}(\mathbf{G}_R)$-$\mathcal{Z}_{\mathcal{G}}$-Geraghty multivalued contraction, if there exist a $C_{\mathcal{G}}$-simulation function ζ and $\alpha : \Upsilon^2 \to \mathbb{R}_+ \cup \{0\}$ with*

$$(\omega, \mu) \in \mathbb{E}(\mathbf{G}_R), \text{ implies that } \quad \zeta(\alpha(\omega, \mu)\mathcal{H}(\mathcal{Q}\omega, \mathcal{Q}\mu), \beta(d(\omega, \mu))d(\omega, \mu)) \geq C_{\mathcal{G}},$$

for every $\omega, \mu \in \Upsilon$.

Corollary 3.5. *Let $\mathbf{G}_R$ is a graph on the metric space (Υ, d), and $\mathcal{Q} : \Upsilon \to \mathcal{K}(\Upsilon)$ be $\mathbb{E}(\mathbf{G}_R)$-$\mathcal{Z}_{\mathcal{G}}$-Geraghty multivalued contraction. Assume that the following assumptions hold*

(1) *: (Υ, d) is an $\mathbb{E}(\mathbf{G}_R)$-complete metric space;*

(2) *: there are $\omega_0 \in \Upsilon$ and $\omega_1 \in \mathcal{Q}\omega_0$ with $\alpha(\omega_0, \omega_1) \in \mathbb{E}(\mathbf{G}_R)$;*

(3) *: $\mathcal{Q}$ is triangular edge preserving;*

(4) *: Ethier*

- ***(4a)*** *: $\mathcal{Q}$ is an $\mathbb{E}(\mathbf{G}_R)$-continuous multivalued mapping, or*
- ***(4b)*** *: If $\{\omega_n\} \subset \Upsilon$ with $\alpha(\omega_n, \omega_{n+1}) \in \mathbb{E}(\mathbf{G}_R)$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} \omega_n = \omega \in \Upsilon$, then we get $\alpha(\omega_n, \omega) \in \mathbb{E}(\mathbf{G}_R)$ for every $n \in \mathbb{N}$.*

Then $\mathcal{Q}$ possesses a fixed point.

Proof. We define

$$\alpha : \Upsilon^2 \to \mathbb{R}_+ \cup \{0\}$$
$$(\omega, \mu) \mapsto \alpha(\omega, \mu) = \begin{cases} 1 & \text{if } (\omega, \mu) \in \mathbb{E}(\mathbf{G}_R) \\ 0 & \text{otherwise.} \end{cases}$$

And by taking $M(\omega, \mu) = d(\omega, \mu)$. Then all the assumptions for the Theorem **2.2** hold. Thus $\mathcal{Q}$ possesses a fixed point in Υ. □

Corollary 3.6. *[33] Let (Υ, d) be a metric space and $\mathcal{Q} : \Upsilon \to \mathcal{K}(\Upsilon)$ be a maps, with*

$$\zeta\Big(\alpha(\omega, \mu)\mathcal{H}(\mathcal{Q}\omega, \mathcal{Q}\mu),\ \beta(\mathcal{D}(\omega, \mu))\mathcal{D}(\omega, \mu)\Big) \geq C_{\mathcal{G}}$$

for every $\omega, \mu \in \Upsilon$ with $\omega \neq \mu$, $L \geq 0$ and

$$\mathcal{D}(\omega, \mu) = \max\Big\{d(\omega, \mu), d(\omega, \mathcal{Q}\omega), d(\mu, \mathcal{Q}\mu)\Big\}.$$

Also assume that

(1) *: (Υ, d) is an α-complete metric space;*

(2) *: There are $\omega_0 \in \Upsilon$ and $\omega_1 \in \mathcal{Q}\omega_0$ with $\alpha(\omega_0,\ \omega_1) \geq 1$;*

(3) *: $\mathcal{Q}$ is triangular α-admissible;*

(4) *: Ethier*

- ***(4a)*** *: $\mathcal{Q}$ is an α-continuous multivalued mapping, or*
- ***(4b)*** *: If $\{\omega_n\} \subset \Upsilon$ with $\alpha(\omega_n, \omega_{n+1}) \geq 1,\ n \in \mathbb{N}$ and $\lim_{n\to\infty} \omega_n = \omega \in \Upsilon$, then we get $\alpha(\omega_n, \omega) \geq 1$ for every $n \in \mathbb{N}$.*

Then Q possesses a fixed point.

Proof. We have

$$\begin{aligned}\mathcal{D}(\omega,\mu) &= \max\left\{d(\omega,\mu), d(\omega,Q\omega), d(\mu,Q\mu)\right\}\\ &\leq \max\left\{d(\omega,\mu), d(\omega,Q\omega), d(\mu,Q\mu), \frac{d(Q\omega,\mu)+d(\omega,Q\mu)}{2}\right\}\\ &= M(\omega,\mu).\end{aligned}$$

Then, from Theorem **2.2**, just taking $L = 0$. □

Corollary 3.7. *Let (Υ, d) be a metric space and $Q : \Upsilon \to \mathcal{K}(\Upsilon)$ be a maps, with*

$$\zeta\Big(\alpha(\omega,\mu)\mathcal{H}(Q\omega,Q\mu),\ \beta(\mathcal{D}_m(\omega,\mu))\mathcal{D}_m(\omega,\mu)\Big) \geq C_{\mathcal{G}}$$

for every $\omega, \mu \in \Upsilon$ with $\omega \neq \mu$, $L \geq 0$ and

$$\mathcal{D}_m(\omega,\mu) = \max\left\{d(\omega,\mu), \frac{1}{2}\Big(d(\omega,Q\omega)+d(\mu,Q\mu)\Big)\right\}.$$

Also assume that

(1) *: (Υ, d) is an α-complete metric space;*

(2) *: There are $\omega_0 \in \Upsilon$ and $\omega_1 \in Q\omega_0$ with $\alpha(\omega_0,\ \omega_1) \geq 1$;*

(3) *: Q is triangular α-admissible;*

(4) *: Ethier*

(4a) *: Q is an α-continuous multivalued mapping,*
or

(4b) *: If $\{\omega_n\} \subset \Upsilon$ with $\alpha(\omega_n, \omega_{n+1}) \geq 1,\ n \in \mathbb{N}$ and $\lim_{n\to\infty} \omega_n = \omega \in \Upsilon$, then we get $\alpha(\omega_n, \omega) \geq 1$ for every $n \in \mathbb{N}$.*

Then Q possesses a fixed point.

Proof. From Theorem **2.3**, just taking $L = 0$ and we have

$$\begin{aligned}\mathcal{D}_m(\omega,\mu) &= \max\left\{d(\omega,\mu), \frac{d(\omega,Q\omega), d(\mu,Q\mu)}{2}\right\}\\ &\leq \max\left\{d(\omega,\mu), \frac{d(\omega,Q\omega), d(\mu,Q\mu)}{2}, \frac{d(Q\omega,\mu)+d(\omega,Q\mu)}{2}\right\}\\ &= M_d(\omega,\mu).\end{aligned}$$

□

By using the Theorem **2.4** and $L = 0$, we can get the following result

Corollary 3.8. *Let (Υ, d) be a metric space and $Q : \Upsilon \to \mathcal{K}(\Upsilon)$ be a maps, such that*

$$\alpha(\omega,\mu) \geq 1 \ \Rightarrow\ \zeta\Big(\mathcal{H}(Q\omega,Q\mu),\ \beta(E(\omega,\mu))E(\omega,\mu)\Big) \geq C_{\mathcal{G}}$$

for every $\omega, \mu \in \Upsilon$ with $\omega \neq \mu$, where $L \geq 0$ and

$$E(\omega,\mu) = d(\omega,\mu) + |d(\omega,Q\omega) - d(\mu,Q\mu)|.$$

Also assume that

(1) *:* (Υ, d) *is an* α*-complete metric space;*

(2) *: There exist* $\omega_0 \in \Upsilon$ *and* $\omega_1 \in \mathcal{Q}\omega_0$ *such that* $\alpha(\omega_0,\ \omega_1) \geq 1$*;*

(3) *:* $\mathcal{Q}$ *is triangular* α*-admissible;*

(4) *: If* $\{\omega_n\} \subset \Upsilon$ *with* $\lim_{n\to\infty} d(\omega_n, \omega_{n+1}) = r \in \mathbb{R}_+$*, then* $\lim_{n\to\infty} d(\omega_n, \mathcal{Q}\omega_n) = r$*;*

(5) *: Ethier*

- ***(5a)*** *:* $\mathcal{Q}$ *is an* α*-continuous multivalued mapping,*
  or
- ***(5b)*** *: If* $\{\omega_n\} \subset \Upsilon$ *with* $\alpha(\omega_n, \omega_{n+1}) \geq 1,\ n \in \mathbb{N}$ *and* $\lim_{n\to\infty} \omega_n = \omega \in \Upsilon$*, then we get* $\alpha(\omega_n, \omega) \geq 1$ *for every* $n \in \mathbb{N}$*.*

Then $\mathcal{Q}$ *possesses a fixed point.*

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Conflict of interest

The authors declare that they have no conflict of interest.

Data Availability

The data used to support the findings of this study are included in the references within the article.

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