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Weak solutions for elliptic problems in weighted anisotropic Sobolev space

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Abstract. Using Mountain Pass Theorem, the existence of weak solutions for

$$-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(a(x)|\frac{\partial u}{\partial x_i}|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\right)=\lambda\gamma(x)|u|^{q(x)-2}u-\lambda\delta(x)|u|^{r(x)-2}u,$$

with Dirichlet boundary condition is studied.

1. Introduction

In this paper, we prove the existence of solutions for the weighted anisotropic elliptic problem

$$\begin{cases} -\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(a(x)|\frac{\partial u}{\partial x_i}|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\right)=\lambda\gamma(x)|u|^{q(x)-2}u-\lambda\delta(x)|u|^{r(x)-2}u & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$ $(N\geq 3)$, $\lambda\in\mathbb{R}$, $q,r\in C_+(\overline{\Omega})$ and $\overrightarrow{p}:\Omega\to\mathbb{R}^N$ given by

$$\overrightarrow{p}(x)=(p_1(x),\cdots,p_N(x)),$$

that for each $i\in\{1,\cdots,N\}$, $p_i:\Omega\to\mathbb{R}$ is a continuous function with

$$p_i(x)\geq 2 \quad \textit{for all} \quad x\in\Omega;$$

also, we assume that

(H_0) $\gamma,\delta\in L^\infty(\Omega)$ where $\inf_{x\in\Omega}\gamma(x)>0$ and $\inf_{x\in\Omega}\delta(x)>0$;

(H_1) $1<q^-<q(x)<q^+<\underline{p}<p(x)<\overline{p}<r^-<r(x)<r^+(x)<p_{s_\infty}$

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where for $h \in \{p, q, r\} \subset C_+(\Omega)$, we define

$$h^+ := \sup_{x\in\overline{\Omega}} h(x), \quad h^- := \inf_{x\in\overline{\Omega}} h(x),$$

and

$$h^* := \frac{N}{(\sum_{i=1}^N \frac{1}{h_i^-}) - 1}, \qquad h_+^- := \max\{h_i^- : i = 1, \cdots, N\}.$$

We also set

$$\overline{h} := \max\{h_i^+ : i = 1, \cdots, N\}, \qquad \underline{h} := \min\{h_i^- : i = 1, \cdots, N\}$$

and

$$h_\infty := \max\{h^*, h_+^-\}, \quad h_s := \inf\{\frac{sh_i(x)}{s+1} : i = 1, \cdots, N\}.$$

Many problems in physics and mechanics are modeled by $p(x)$-Laplace operator and are studied by different methods such as variational method [2, 12–16, 18–20, 23], sub-super solutions method [21] and etc. For example, Zhou and Wu [23] studied the problem

$$-div\left(a(x)|\nabla u|^{p(x)-2}\nabla u\right) = \lambda\left(b(x)|u|^{q(x)-2}u - c(x)|u|^{r(x)-2}u\right),$$

with the Dirichlet boundary condition on smooth boundary domain $\Omega \subset \mathbb{R}^N$. They proved the existence of solutions for the above problem by putting the suitable conditions and using the method of variations.
On the other hand, we deal with problems such as elastic mechanics, crystal growth and etc. that in their modeling, the exponent should be able to vary in different directions. The anisotropic Sobolev spaces $W^{1,\overrightarrow{p}(x)}(\Omega)$ with $\overrightarrow{p}(x) = (p_1(x), \cdots, p_N(x))$ is a suitable space for investigating these kinds of problems.
Many authors have examined the problems in this space with the $\overrightarrow{p}$-Laplacian operator

$$\Delta_{\overrightarrow{p}(x)} u = \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(|\frac{\partial u}{\partial x_i}|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\right).$$

For example, Razani et al. [15] have studied results in weighted anisotropic Sobolev spaces. They have proved the existence and approximation results for degenerated anisotropic (p, q)-Laplacian with weights

$$-\sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\left(a(x)|\frac{\partial u}{\partial x_i}|^{p_i-2} + b(x)|\frac{\partial u}{\partial x_i}|^{q_i-2}\right)\frac{\partial u}{\partial x_i}\right) = f(x, u, \nabla u)$$

and a competing anisotropic (p, q)-Laplacian with weights

$$-\sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\left(a(x)|\frac{\partial u}{\partial x_i}|^{p_i-2} - b(x)|\frac{\partial u}{\partial x_i}|^{q_i-2}\right)\frac{\partial u}{\partial x_i}\right) = f(x, u, \nabla u)$$

with Dirichlet boundary condition on a bounded smooth domain in $\mathbb{R}^N, N \geq 3$, where $f : \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ is a Carathõdory function. Their proofs are based on weighted antitropic Sobolev spaces, Nemytskij operators and finite dimensional approximation.
In the next section, we refer to the function spaces, some definitions, theorems and lemmas that we will use to prove the results. And also, we will introduce the weighted antitropic Sobolev spaces as a solution space.

2. Preliminaries

Let $\Omega \subset \mathbb{R}^N$ be an open bounded subset. Adjust

$$C_+(\overline{\Omega}) := \left\{ p : \overline{\Omega} \longrightarrow \mathbb{R} \ \text{ measurable} : \ 1 < p^- \leq p^+ < \infty \right\}.$$

For any $p \in C_+(\overline{\Omega})$, we introduce the Lebesgue space with the variable exponent defined by

$$L^{p(x)}(\Omega) := \left\{ u : \Omega \longrightarrow \mathbb{R} \ \text{ measurable and } \ \int_\Omega |u(x)|^{p(x)} dx < \infty \right\},$$

with the Luxemburg norm

$$\|u\|_p := \|u\|_{L^{p(x)}(\Omega)} = \inf \left\{ \sigma > 0 \ : \ \int_\Omega |\frac{u(x)}{\sigma}|^{p(x)} dx \leq 1 \right\}.$$

We recall the following theorem [8, Theorem 2.8].

Theorem 2.1. *Assume that Ω is a bounded and smooth domain in $\mathbb{R}^N$. Let $p_1, p_2 \in C_+(\overline{\Omega})$. Then,*

$$L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$$

if and only if $p_1(x) \leq p_2(x)$ a.e. in Ω.

Proposition 2.2. *[8] The space $L^{p(x)}(\Omega)$ is a separable, uniform convex Banach space and its conjugate space is $L^{p'(x)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have*

$$|\int_\Omega uv dx| \leq \left(\frac{1}{p^-} + \frac{1}{(p^-)'} \right) \|u\|_p \|v\|_{p'} \leq 2 \|u\|_p \|v\|_{p'}.$$

Proposition 2.3. *[6] Set $\rho(u) = \int_\Omega a(x)|u|^{p(x)} dx$, for all $u \in L^{p(x)}(a, \Omega)$. Then,*

(1) $\rho(u) > 1 (= 1; < 1)$ if and only if $\|u\|_{L^{p(x)}(a,\Omega)} > 1 (= 1; < 1)$, respectively;

(2) if $\|u\|_{L^{p(x)}(a,\Omega)} > 1$, then $\|u\|^{p^-}_{L^{p(x)}(a,\Omega)} \leq \rho(u) \leq \|u\|^{p^+}_{L^{p(x)}(a,\Omega)}$;

(3) if $\|u\|_{L^{p(x)}(a,\Omega)} < 1$, then $\|u\|^{p^+}_{L^{p(x)}(a,\Omega)} \leq \rho(u) \leq \|u\|^{p^-}_{L^{p(x)}(a,\Omega)}$.

Here, we introduce the weighted Sobolev space

$$W^{1,p(x)}(a, \Omega) := \left\{ u \in L^{p(x)}(\Omega) : \int_\Omega a(x)|\nabla u|^{p(x)} dx < \infty \right\},$$

endowed with the norm

$$\|u\|_{W^{1,p(x)}(a,\Omega)} := \inf \left\{ \sigma > 0, \int_\Omega \left(a(x)|\frac{\nabla u}{\sigma}|^{p(x)} + |\frac{u}{\sigma}|^{p(x)} \right) dx \leq 1 \right\}$$

Note that $C_0^\infty(\Omega) \subset W^{1,p(x)}(a, \Omega)$ and the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(a, \Omega)$ with respect to the norm $\|.\|_{W^{1,p(x)}(a,\Omega)}$ is the space $W_0^{1,p(x)}(a, \Omega)$.
A reduction in the regularization of classical Sobolev spaces is based on the following condition from [7]: Let a be a measurable positive and a.e. finite function in $\mathbb{R}^N$ satisfying that

(H_1') $a \in L^1_{loc}(\Omega)$ and $a^{\frac{-1}{p(x)-1}} \in L^1_{loc}(\Omega)$;

(H_2') $a^{-s(x)} \in L^1(\Omega)$ *for some* $s(x) \in \left(\max\{\frac{N}{p(x)}, \frac{1}{p(x)-1}\}, +\infty\right)$.

For $p, s \in C_+(\Omega)$, denote

$$p_s(x) := \frac{p(x)s(x)}{1+s(x)} < p(x),$$

where $s(x)$ is given in (H_2'). Also we set

$$p_s^\star(x) := \begin{cases} \dfrac{p(x)s(x)N}{(s(x)+1)N - p(x)s(x)} & \text{for } N > p_s(x), \\ \infty & \text{for } N \leq p_s(x). \end{cases}$$

for almost all $x \in \Omega$. Next, we recall the following Proposition according to [7, Theorem 2.11].

Proposition 2.4. *Let* $p, s \in C_+(\Omega)$ *and* (H_1') *and* (H_2') *be satisfied. Then, we have the following compact embedding*

$$W^{1,p(x)}(a,\Omega) \hookrightarrow W^{1,p_s(x)}(\Omega) \hookrightarrow\hookrightarrow L^{r(x)}(\Omega),$$

provided that

$$r \in C_+(\overline{\Omega}), \quad 1 \leq r(x) < p_s^\star(x) \text{ for all } x \in \Omega.$$

For all $u \in W_0^{1,p(x)}(a,\Omega)$,

$$\|u\|_{W_0^{1,p(x)}(a,\Omega)} := \inf\left\{\sigma > 0, \int_\Omega a(x)|\frac{\nabla u}{\sigma}|^{p(x)}dx \leq 1\right\}$$

is an equivalent norm on $W_0^{1,p(x)}(a,\Omega)$ for which $W_0^{1,p(x)}(a,\Omega)$ becomes a uniformly convex Banach space.

In the following, we consider the vectorial function $\overrightarrow{p} : \overline{\Omega} \to \mathbb{R}^N$ as follows

$$\overrightarrow{p} := \overrightarrow{p}(x) = (p_1(x), \cdots, p_N(x)),$$

where

$$p_i \in C_+(\overline{\Omega}) \quad \text{for all} \quad i \in \{1, \cdots, N\}.$$

The anisotropic variable exponent Sobolev space defined by

$$\begin{cases} W^{1,\overrightarrow{p}}(\Omega) = \left\{u \in W^{1,1}(\Omega) : \dfrac{\partial u}{\partial x_i} \in L^{p_i(x)}(\Omega) \text{ for } i = 1, \cdots, N\right\}, \\ W_0^{1,\overrightarrow{p}}(\Omega) = W^{1,\overrightarrow{p}}(\Omega) \cap W_0^{1,1}(\Omega), \end{cases}$$

The anisotropic variable exponent Sobolev space $W_0^{1,\overrightarrow{p}}(\Omega)$ can also be defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,\overrightarrow{p}}(\Omega)$ with respect to the norm

$$\|u\|_{W_0^{1,\overrightarrow{p}}(\Omega)} := \sum_{i=1}^N \|\frac{\partial u}{\partial x_i}\|_{p_i}.$$

These spaces are separable and reflexive Banach spaces [4, 10].
The suitable space for the solutions to our problem is the weighted anisotropic variable exponent Sobolev space, i.e.,

$$W^{1,\overrightarrow{p}}(a,\Omega) := \left\{u \in W^{1,1}(a,\Omega) : \frac{\partial u}{\partial x_i} \in L^{p_i(x)}(a,\Omega) \text{ for } i = 1, \cdots, N\right\}. \tag{2}$$

We remind that $W_0^{1,\overrightarrow{p}}(a,\Omega)$ can be defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,\overrightarrow{p}}(a,\Omega)$ with respect to the norm

$$\begin{aligned}\|u\| = \|u\|_{W_0^{1,\overrightarrow{p}}(a,\Omega)} &:= \sum_{i=1}^{N}\left(\int_\Omega a(x)|\frac{\partial u}{\partial x_i}|^{p_i}dx\right)^{\frac{1}{p_i}}\\ &= \sum_{i=1}^{N}\|u\|_{W_0^{1,p_i(x)}(a,\Omega)}.\end{aligned}$$

Theorem 2.5. *[10] Suppose, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary. If, for all $i = 1,\cdots,N$,*

$$\sum_{i=1}^{N}\frac{1}{p_i^-} > 1.$$

(1) For any $\alpha \in C(\overline{\Omega})$ verifying

$$1 < \alpha(x) < p_\infty \quad \textit{for all} \quad x \in \overline{\Omega},$$

the embedding

$$W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow L^{\alpha(x)}(\Omega),$$

is continuous and compact.

(2) Assume that $\underline{p} > N$, then the embedding

$$W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow C^0(\overline{\Omega})$$

is compact.

Also, it is shown in [10, Theorem 1] that $W_0^{1,\overrightarrow{p}}(\Omega)$ is continuously embedded in $W_0^{1,\overrightarrow{p^-}}(\Omega)$. The following condition gives interesting results of embedding in $W_0^{1,\overrightarrow{p}}(a,\Omega)$.

(H_2) $a^{-s(x)} \in L^1(\Omega)$ *for some* $s(x) \in \left(\max\{\sum_{i=1}^{N}\frac{1}{p_i(x)}, \frac{1}{\underline{p}-1}\}, +\infty\right)$.

Notice that one can prove the following embedding result.

Proposition 2.6. *Assume (H_2) is hold. There are continuous embeddings*

$$W_0^{1,\overrightarrow{p}}(a,\Omega) \hookrightarrow W_0^{1,\overrightarrow{p_s}}(\Omega) \hookrightarrow L^\alpha(\Omega). \tag{3}$$

Also, the embedding

$$W_0^{1,\overrightarrow{p_s}}(\Omega) \hookrightarrow L^\alpha(\Omega),$$

is compact for $\alpha < p_{s_\infty}$. Furthermore, $W_0^{1,\overrightarrow{p}}(\Omega)$ is a uniformly convex Banach space.

Now, we recall the definition of Palais-Smale condition.

Definition 2.7. *Let ϕ and ψ be two continuously Gâteaux-differentiable functionals defined on a real Banach space X and fix $r \in \mathbb{R}$. The functional $I = \phi - \psi$ is said to confirm the Palais-Smale condition (in short $(PS)^{[r]}$), if any sequence $\{u_n\}_{n\in N}$ in X such that*

- *$\{I(u_n)\}$ is bounded;*
- $\lim_{n\to\infty} \|I'(u_n)\|_{X^\star} = 0$;
- *$\phi(u_n) < r$ for each $n \in N$;*

has a convergent subsequence.

If $r = \infty$, we say that the functional $I = \phi - \psi$ verify the Palais-Smale condition or in short (PS).
In this paper, we intend to prove the existence of a nontrivial weak solution for problem (1) and also show that this problem has infinitely many solutions. Due to do this, one can define the functional

$$\Phi : W_0^{1,\overrightarrow{p}}(a, \Omega) \to \mathbb{R},$$

by

$$\Phi(u) := \sum_{i=1}^{N} \int_\Omega \frac{a(x)}{p_i(x)} |\frac{\partial u}{\partial x_i}|^{p_i(x)} dx - \lambda \int_\Omega \frac{\gamma(x)}{q(x)} |u|^{q(x)} dx + \lambda \int_\Omega \frac{\delta(x)}{r(x)} |u|^{r(x)} dx$$

for all $u \in W_0^{1,\overrightarrow{p}}(a, \Omega)$. Notice that

$$\Phi(u)'v = \sum_{i=1}^{N} \int_\Omega a(x) |\frac{\partial u}{\partial x_i}|^{p_i(x)-2} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx$$
$$- \lambda \int_\Omega \gamma(x)|u|^{q(x)-2} uv dx - \lambda \int_\Omega \delta(x)|u|^{r(x)-2} uv dx,$$

for all $v \in W_0^{1,\overrightarrow{p}}(a, \Omega)$.

Definition 2.8. *It is called $u \in W_0^{1,\overrightarrow{p}}(a, \Omega)$ is a weak solution of problem (1) if it verifies*

$$\sum_{i=1}^{N} \int_\Omega a(x) |\frac{\partial u}{\partial x_i}|^{p_i(x)-2} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx =$$
$$\lambda \int_\Omega \gamma(x)|u|^{q(x)-2} uv dx - \lambda \int_\Omega \delta(x)|u|^{r(x)-2} uv dx$$

for all $v \in W_0^{1,\overrightarrow{p}}(a, \Omega)$.

It is clear that the critical points of Φ are weak solutions of the Problem (1).
Now, we can state the main results in this paper.

Theorem 2.9. *Suppose $(H_0), (H_1)$ and (H_2) hold.*

(A) if $\lambda > 0$, then problem (1) has a nontrivial solution which is a minimizer of the associated integral functional of Φ.

(B) if $\lambda < 0$, then problem (1) has a sequence of solutions $\{\pm u_n\}$ such that

$$\Phi(\pm u_n) \to +\infty \quad as \quad n \to +\infty.$$

In the next section we prove the existence of a weak solution of (1), when $\lambda > 0$ and the existence of a sequence of solutions when $\lambda < 0$.

3. Weak solutions

Throughout the article, the letters $c, c_i, i = 1, 2, \cdots$ show positive constants that are possible change from one line to another.
Note that, by using (H_0) and Proposition 2.3, we have

$$\int_\Omega \frac{\gamma(x)}{q(x)}|u|^{q(x)} \leq \frac{\|\gamma\|_\infty}{q^-}\int_\Omega |u|^{q(x)} \leq \frac{\|\gamma\|_\infty}{q^-}\left[\|u\|_q^{q^+} + \|u\|_q^{q^-}\right],$$

and from Proposition 2.6,

$$\int_\Omega \frac{\gamma(x)}{q(x)}|u|^{q(x)} \leq \frac{\|\gamma\|_\infty}{q^-}\left[c_1\|u\|^{q^+} + c_2\|u\|^{q^-}\right].$$

Proof. First, we examine part (A) of Theorem 2.9 and prove the existence of a nontrivial solution for problem (1). For this purpose, we show that if $\lambda > 0$ then the functional Φ is coercive. Let $\|u\| > 1$, according to the definition of the functional Φ, we have

$$\begin{aligned}\Phi(u) &= \sum_{i=1}^N \int_\Omega \frac{a(x)}{p_i(x)}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)}dx - \lambda\int_\Omega \frac{\gamma(x)}{q(x)}|u|^{q(x)}dx + \lambda\int_\Omega \frac{\delta(x)}{r(x)}|u|^{r(x)}dx \\ &\geq \frac{1}{\overline{p}N^{\underline{p}-1}}\|u\|^{\underline{p}} - \lambda\frac{c\|\gamma\|_\infty}{q^-}\|u\|^{q^+},\end{aligned} \tag{4}$$

because $\underline{p} > q^+$, so Φ is coercive and has a minimizer which is a solution for problem (1). Now, we show that the minimizer is nonzero. Indeed, for $t > 0$ small enough and $v_0 \in W_0^{1,\overrightarrow{p}}(a, \Omega)$

$$\begin{aligned}&\Phi(tv_0) = \\ &\sum_{i=1}^N \int_\Omega \frac{a(x)}{p_i(x)}t^{p_i(x)}\left|\frac{\partial v_0}{\partial x_i}\right|^{p_i(x)}dx - \lambda\int_\Omega \frac{\gamma(x)}{q(x)}t^{q(x)}|v_0|^{q(x)}dx + \lambda\int_\Omega \frac{\delta(x)}{r(x)}t^{r(x)}|v_0|^{r(x)}dx \\ &\leq \frac{t^{\underline{p}}}{\underline{p}}\sum_{i=1}^N \int_\Omega a(x)\left|\frac{\partial v_0}{\partial x_i}\right|^{p_i(x)}dx - \lambda\frac{t^{q^+}\|\gamma\|_\infty}{q^+}\int_\Omega |v_0|^{q(x)}dx + \lambda\frac{t^{r^-}\|\delta\|_\infty}{r^-}\int_\Omega |v_0|^{r(x)}dx \\ &\leq \frac{t^{\underline{p}}}{\underline{p}}\sum_{i=1}^N \int_\Omega a(x)\left|\frac{\partial v_0}{\partial x_i}\right|^{p_i(x)}dx - \lambda\frac{t^{q^+}\|\gamma\|_\infty}{q^+}\int_\Omega |v_0|^{q(x)}dx + \lambda\frac{t^{\underline{p}}\|\delta\|_\infty}{r^-}\int_\Omega |v_0|^{r(x)}dx \\ &\leq c_3t^{\underline{p}} - c_4t^{q^+} \\ &< 0.\end{aligned}$$

Since $q^+ < \underline{p}$, the last inequality is obtained. Now, we prove part (B) of Theorem 2.9. Set

$$X := W_0^{1,\overrightarrow{p}}(a, \Omega),$$

this is a reflexive and separable Banach space. We remind that if X be a reflexive and separable Banach space, then there exist $\{\varrho_j\} \subset X$ and $\{\varrho_j^*\} \subset X^*$ such that,

$$X = \overline{span\{\varrho_j : j = 1, 2, 3, \cdots\}}, \quad X^* = \overline{\mathrm{span}\{\varrho_j^* : j = 1, 2, 3, \cdots\}}$$

and

$$\langle \varrho_j^*, \varrho_i\rangle = \begin{cases} 1 & j = i, \\ 0 & j \neq i.\end{cases}$$

Where $\langle ., . \rangle$ denotes the duality product between X and X^*. For convenience, we write

$$X_j = span\{\varrho_j\}, \quad Y_k = \oplus_{j=1}^{k} X_j \text{ and } Z_k = \oplus_{j=k}^{\infty} X_j. \tag{5}$$

The following lemma is our main tool for proving the infinitely many solutions to problem (1).

Lemma 3.1. *[22] X is a Banach space, $\Phi \in C^1(X, \mathbb{R})$ is an even functional, the subspaces Y_k and Z_k are defined as (5). If, for each $k = 1, 2, 3, \cdots$, there exists $\rho_k > d_k > 0$ such that*

(1) $\max_{u \in Y_k, \|u\| = \rho_k} \Phi(u) \leq 0$;

(2) $\inf_{u \in Z_k, \|u\| = d_k} \Phi(u) \to \infty \quad as \quad k \to \infty$;

(3) The functional Φ satisfies the (P.S.) condition;

Then, Φ has an unbounded sequence of critical values.

Using Lemma 3.1, we prove that if $\lambda < 0$ then problem (1) has infinitely many solutions. Assume, $t > 1$ and $\nu \in Y_k$ with $\|\nu\| = 1$. Adjust

$$\ell_k = \inf_{\nu \in Y_k, \|\nu\| = 1} \int_\Omega \frac{\delta(x)}{r(x)} |\nu|^{r(x)} dx.$$

From definition Φ, we obtain

$$\begin{aligned}
&\Phi(t\nu) \\
&= \sum_{i=1}^{N} \int_\Omega \frac{a(x)}{p_i(x)} t^{p_i(x)} |\frac{\partial \nu}{\partial x_i}|^{p_i(x)} dx - \lambda \int_\Omega \frac{\gamma(x)}{q(x)} t^{q(x)} |\nu|^{q(x)} dx + \lambda \int_\Omega \frac{\delta(x)}{r(x)} t^{r(x)} |\nu|^{r(x)} dx \\
&\leq \frac{t^{\overline{p}}}{\underline{p}} \sum_{i=1}^{N} \int_\Omega a(x) |\frac{\partial \nu}{\partial x_i}|^{p_i(x)} dx - \lambda \frac{\|\gamma\|_\infty}{q^-} t^{q^+} \int_\Omega |\nu|^{q(x)} dx + \lambda t^{r^-} \int_\Omega \frac{\delta(x)}{r(x)} |\nu|^{r(x)} dx \\
&\leq \frac{t^{\overline{p}}}{\underline{p} N^{\overline{p}-1}} \|\nu\|^{\overline{p}} + c_5 \frac{\|\gamma\|_\infty}{q^-} t^{q^+} \|\nu\|^{q^+} + \lambda t^{r^-} \ell_k \\
&\leq c_6 t^{\overline{p}} + c_7 t^{q^+} - c_8 t^{r^-} \ell_k,
\end{aligned}$$

according to (H_1), $r^- > \overline{p} > q^+$, so for a $t_0 \in [1, \infty[$, $\Phi(t_0 \nu) < 0$. As a result, there exists large $\rho_k > 0$ so that

$$\max_{u \in Y_k, \|u\| = \rho_k} \Phi(u) < 0.$$

Therefore, (1) of Lemma 3.1 is satisfied. Now, we show that (2) also holds. For this purpose, adjust

$$\beta_k = \sup_{\nu \in Z_k, \|\nu\| \leq 1} \int_\Omega \frac{\delta(x)}{r(x)} |\nu|^{r(x)} dx.$$

According to the definition of Z_k, we see that

$$0 \leq \beta_{k+1} \leq \beta_k \quad and \quad \beta_k \to 0 \quad as\ k \to \infty,$$

so, there exists $\{u_k\}$ such that,

$$|\beta_k - \int_\Omega \frac{\delta(x)}{r(x)} |u_k(x)|^{r(x)} dx| < \frac{1}{\kappa}$$

for all $k \geq 1$. Considering that $\{u_k\}$ is bounded in $W_0^{1,\overrightarrow{p}(x)}(\Omega)$, we can conclude that,

$$u_k \to u_0 \quad in \quad W_0^{1,\overrightarrow{p}(x)}(\Omega),$$

therefore,

$$\varrho_j^*(u_k) \to 0 \quad as \quad k \to \infty,$$

on the other hand, we have

$$\varrho_j^*(u_0) = 0 \quad for\ all \quad j \geq 1,$$

thus, $u_0 = 0$. Now, consider $u \in W_0^{1,\overrightarrow{p}(x)}(\Omega)$ with $\|u\| > 1$. According to the definition $\Phi(u)$, we have

$$\begin{aligned}
\Phi(u) &= \sum_{i=1}^{N} \int_\Omega \frac{a(x)}{p_i(x)} |\frac{\partial u}{\partial x_i}|^{p_i(x)} dx - \lambda \int_\Omega \frac{\gamma(x)}{q(x)} |u|^{q(x)} dx + \lambda \int_\Omega \frac{\delta(x)}{r(x)} |u|^{r(x)} dx \\
&\geq \frac{1}{\overline{p}} \sum_{i=1}^{N} \int_\Omega a(x) |\frac{\partial u}{\partial x_i}|^{p_i(x)} dx - \lambda \frac{\gamma_0}{q^+} \int_\Omega |u|^{q(x)} dx + \lambda \frac{\|\delta\|_\infty}{r^-} \int_\Omega |u|^{r(x)} dx \\
&\geq \frac{1}{\overline{p} N^{\underline{p}-1}} \|u\|^{\underline{p}} - \lambda \int_\Omega \frac{\gamma(x)}{q(x)} |u|^{q(x)} dx - C\beta_k \|u\|^{r^+} \\
&\geq \frac{1}{\overline{p} N^{\underline{p}-1}} \|u\|^{\underline{p}} - C\beta_k \|u\|^{r^+}, \qquad (6)
\end{aligned}$$

note that $\lambda < 0$, so the last inequality holds. Put

$$\|u\| = \mu_k := \left(\frac{1}{2\overline{p} N^{\underline{p}-1} C\beta_k} \right)^{\frac{1}{r^+ - \underline{p}}},$$

so,

$$\|u\| \to \infty \quad as \quad \beta_k \to 0.$$

Therefore, we obtain from (6)

$$\inf_{u \in Z_k, \|u\| = \mu_k} \Phi(u) \geq \frac{1}{2\overline{p} N^{\underline{p}-1}} \mu_k^{\underline{p}} \to +\infty \quad as \quad k \to \infty.$$

Now, it suffices to show that the functional Φ verifies the Palais-Smale condition on $W_0^{1,\overrightarrow{p}}(a,\Omega)$. Suppose, the sequence $\{u_n\}$ satisfies in the Palais-Smale condition i.e.,

$$\Phi(u_n) \to c \quad and \quad \Phi'(u_n) \to 0.$$

So, for $\|u_n\|$ large enough, we obtain

$$\begin{aligned}
r^- c + 1 &\geq r^- \Phi(u_n) - \Phi'(u_n) u_n \\
&= \sum_{i=1}^{N} \int_\Omega [\frac{r^-}{p_i(x)} - 1] a(x) |\frac{\partial u_n}{\partial x_i}|^{p_i(x)} dx - \lambda \int_\Omega [\frac{r^-}{q(x)} - 1] \gamma(x) |u_n|^{q(x)} dx \\
&+ \lambda \int_\Omega [\frac{r^-}{r(x)} - 1] \delta(x) |u_n|^{r(x)} dx \geq [\frac{r^-}{\overline{p}} - 1] \sum_{i=1}^{N} \int_\Omega a(x) |\frac{\partial u_n}{\partial x_i}|^{p_i(x)} dx \\
&- \lambda \int_\Omega [\frac{r^-}{q(x)} - 1] \gamma(x) |u_n|^{q(x)} dx + \lambda \int_\Omega [\frac{r^-}{r(x)} - 1] \delta(x) |u_n|^{r(x)} dx \\
&\geq \frac{1}{N^{\underline{p}-1}} [\frac{r^-}{\overline{p}} - 1] \|u_n\|^{\underline{p}}, \qquad (7)
\end{aligned}$$

the last inequality holds according to the condition (H_1). From (7), we conclude that $\{u_n\}$ is a bounded sequence in $W_0^{1,\overrightarrow{p}}(a,\Omega)$. We consider the subsequence still denoted by $\{u_n\}$. Suppose

$$\sum_{i=1}^{N}\int_{\Omega}\frac{a(x)}{p_i(x)}|\frac{\partial u_n}{\partial x_i}|^{p_i(x)}dx \longrightarrow \xi \quad as \quad n\to\infty, \tag{8}$$

on the other hand, there exists $u_0 \in W_0^{1,\overrightarrow{p}}(a,\Omega)$ such that

$$u_n \rightharpoonup u_0 \quad in \quad W_0^{1,\overrightarrow{p}}(a,\Omega),$$
$$u_n(x) \to u_0(x) \quad a.e. \quad x\in\Omega,$$

and

$$u_n \to u_0 \quad in \quad L^{q(x)}(\Omega),$$
$$u_n \to u_0 \quad in \quad L^{r(x)}(\Omega).$$

So, we conclude that

$$\lim_{n\to\infty}\int_{\Omega}\gamma(x)\left(|u_n|^{q(x)-2}u_n - |u_0|^{q(x)-2}u_0\right)(u_n-u_0) = 0,$$
$$\lim_{n\to\infty}\int_{\Omega}\delta(x)\left(|u_n|^{r(x)-2}u_n - |u_0|^{r(x)-2}u_0\right)(u_n-u_0) = 0.$$

As a result, we have

$$\lim_{n\to\infty}\sum_{i=1}^{N}\int_{\Omega}a(x)\left(|\frac{\partial u_n}{\partial x_i}|^{p_i(x)-2}\frac{\partial u_n}{\partial x_i} - |\frac{\partial u_0}{\partial x_i}|^{p_i(x)-2}\frac{\partial u_0}{\partial x_i}\right)\left(\frac{\partial u_n}{\partial x_i}-\frac{\partial u_0}{\partial x_i}\right)dx = 0. \tag{9}$$

Using the above discussion, we show that $\{u_n\}$ has a strongly convergent subsequence in the following cases:

(1) $\xi = 0$.
From (8), we obtain

$$\sum_{i=1}^{N}\int_{\Omega}a(x)|\frac{\partial u_n}{\partial x_i}|^{p_i(x)}dx \to 0 \quad as \quad n\to\infty,$$

so, we conclude that $\{u_n\}$ is strongly convergent to 0 in $W_0^{1,\overrightarrow{p}}(a,\Omega)$.

(2) $\xi > 0$.
We recall the following inequality from [9] which states for every $\zeta,\eta\in\mathbb{R}^N$,

$$2^{\theta}\left(|\zeta|^{\theta-2}\zeta - |\eta|^{\theta-2}\eta\right)\cdot(\zeta-\eta) \geq |\zeta-\eta|^{\theta} \quad \text{for all} \quad \theta > 2, \tag{10}$$

where $\zeta\cdot\eta$ represents the inner product in $\mathbb{R}^N$.
From (10), we conclude that there exists $\varepsilon' > 0$ such that

$$\begin{aligned}\int_{\Omega}a(x)&\left(|\frac{\partial u_n}{\partial x_i}|^{p_i(x)-2}\frac{\partial u_n}{\partial x_i} - |\frac{\partial u_0}{\partial x_i}|^{p_i(x)-2}\frac{\partial u_0}{\partial x_i}\right)\left(\frac{\partial u_n}{\partial x_i}-\frac{\partial u_0}{\partial x_i}\right)dx\\ &\geq \varepsilon_i\int_{\Omega}a(x)|\frac{\partial u_n}{\partial x_i}-\frac{\partial u_0}{\partial x_i}|^{p_i(x)}dx\\ &\geq \varepsilon'\int_{\Omega}a(x)|\frac{\partial u_n}{\partial x_i}-\frac{\partial u_0}{\partial x_i}|^{p_i(x)}dx\end{aligned}$$

for $i \in \{1, \cdots, N\}$, where

$$\varepsilon' = \min_{i=1,\cdots,N} \varepsilon_i.$$

Which implies,

$$\lim_{n\to\infty} \sum_{i=1}^{N} \int_\Omega a(x) \left(|\frac{\partial u_n}{\partial x_i}|^{p_i(x)-2} \frac{\partial u_n}{\partial x_i} - |\frac{\partial u_0}{\partial x_i}|^{p_i(x)-2} \frac{\partial u_0}{\partial x_i} \right) \left(\frac{\partial u_n}{\partial x_i} - \frac{\partial u_0}{\partial x_i} \right) dx$$
$$\geq \varepsilon' \sum_{i=1}^{N} \int_\Omega a(x) |\frac{\partial u_n}{\partial x_i} - \frac{\partial u_0}{\partial x_i}|^{p_i(x)}.$$

And from (9), we obtain

$$\sum_{i=1}^{N} \int_\Omega a(x) |\frac{\partial u_n}{\partial x_i} - \frac{\partial u_0}{\partial x_i}|^{p_i(x)} \to 0 \quad as \quad n \to \infty,$$

that's mean,

$$u_n \to u_0 \quad in \quad W_0^{1,\overrightarrow{p}}(a, \Omega),$$

hence, u_n converges strongly to u_0 in $W_0^{1,\overrightarrow{p}}(a, \Omega)$.

So, according to Lemma 3.1, Φ takes an unbounded sequence of critical values. As a result, part (B) of Theorem 2.9 is also proved. □

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