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Global energy conservation for distributional solutions to incompressible Hall-MHD equations without resistivity

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Abstract. This paper concerns the global energy conservation for distributional solutions to incompressible Hall-MHD equations without resistivity. Motivated by the works of Tan and Wu in [arXiv:2111.13547v2] and Wu in [J. Math. Fluid Mech. 24, 111 (2022)], we establish the energy balance for a distributional solution in whole spaces $\mathbb{R}^d (d \geq 2)$ provided that $u, b \in L^4L^4$ and $\nabla b \in L^{\frac{8}{3}}L^{\frac{8}{3}}$. Moreover, as a corollary, we also obtain the energy conservation criterion for a Leray-Hopf weak solution.

1. Introduction

In this paper, we are concentrated on the global energy conservation of distributional solutions for the incompressible Hall-MHD equations without resistivity. The incompressible Hall-MHD equations without resistivity take the following form:

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - \mu\Delta u + \nabla p = (b \cdot \nabla)b, \\ \partial_t b + (u \cdot \nabla)b + \delta\nabla \times [(\nabla \times b) \times b] = (b \cdot \nabla)u, \\ \operatorname{div} u = \operatorname{div} b = 0, \\ u(x, 0) = u_0(x), b(x, 0) = b_0(x), \end{cases} \tag{1}$$

where $u(t, x) = (u_1(t, x), u_2(t, x), \cdots, u_d(t, x))$ and $b(t, x) = (b_1(t, x), b_2(t, x), \cdots, b_d(t, x))$, $(x, t) \in [0, \infty) \times \mathbb{R}^d$, are the fluid velocity and magnetic field. $\mu \geq 0$ is the viscosity, $\mu = 0$ and $\mu > 0$ correspond to the inviscid and viscous flow respectively, $\delta \geq 0$ is the Hall effect coefficient. We will consider the Cauchy problem for (1), we prescribe the initial data satisfying the compatibility condition $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$ in the distributional sense. From the equations for the magnetic field b, it is easy to see that if one prescribes the divergence condition $\operatorname{div} b_0$ on the initial data b_0, then $\operatorname{div} b = 0$ for later time.

Comparing with the well-known MHD system, the Hall term $\nabla \times [(\nabla \times b) \times b]$ is included due to the Ohm's law, which is believed to be a key issue for understanding magnetic reconnection. Note that the Hall term is quadratic in the magnetic field and involves the second order derivatives. Magnetic reconnection corresponds to a physical process in highly conducting plasmas in which the magnetic topology is

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rearranged and magnetic energy is converted to kinetic energy, thermal energy, and particle acceleration. During this process, the magnetic shear is large, the Hall term becomes dominant. Lighthill[21] started the systematic study of the application of Hall-MHD on plasma, which is followed by [7]. One may refer to [29] for a physical review of the background for Hall-MHD.

If the Hall effect coefficient $\delta = 0$, we obtain the classical incompressible non-resistive MHD system, which entirely different from the standard MHD system. In this case, the study of well-posedness will become more difficult. Even the global existence of Leray-Hopf weak solutions is not known due to the lack of suitable strong convergence in magnetic field b. In spite of the difficulties due to the lack of magnetic diffusion, significant progress has been made in [23, 30], under the assumption that the initial data (u_0, b_0) is close to the equilibrium state $\left(0,(1,0)^T\right)$, the global well-posedness was proven. For more progress in this direction, one may refer to the survey paper [24] and the references therein. In addition, for 2D non-resistive MHD equations (1), Jiu-Niu [18] first proved the local existence and uniqueness of the classical solution in the Sobolev space H^s with $s \geq 3$. Fefferman-McCormick-Robinson-Rodrigo [13] were able to weaken the regularity assumption to $(u_0, b_0) \in H^s$ with $s > \frac{d}{2}$ and obtained local-in-time existence of strong solutions to (1) in $\mathbb{R}^d$, $d = 2, 3$. And then, they made a further improvement by assuming $(u_0, b_0) \in H^{s-1+\varepsilon}\left(\mathbb{R}^d\right) \times H^s\left(\mathbb{R}^d\right), s > \frac{d}{2}, 0 < \varepsilon < 1$ in [14]. Chemin-McCormick-Robinson-Rodrigo [9] presented the local existence of weak solutions to (1) in $\mathbb{R}^d$, $d = 2, 3$ with the initial data $(u_0, b_0) \in B_{2,1}^{\frac{d}{2}-1}\left(\mathbb{R}^d\right) \times B_{2,1}^{\frac{d}{2}}\left(\mathbb{R}^d\right)$ and also proved the corresponding solution is unique in 3D case. Wan in [36] resolved the uniqueness of the solution in the 2D case by using mixed space-time Besov spaces. Recently, Li-Tan-Yin [20] made an important progress by reducing the functional setting to homogeneous Besov space $(u_0, b_0) \in \dot{B}_{p,1}^{d-1}\left(\mathbb{R}^d\right) \times \dot{B}_{p,1}^{\frac{d}{p}}\left(\mathbb{R}^d\right)$ where $p \in [1, 2d]$. Other studies on the regularity of fluid equations can be referred to recent papers [1, 2, 31] and references therein.

For regular solution, system (1) obeys the following energy balance (i.e., energy identity):

$$\frac{1}{2}\int_{\mathbb{R}^d}\left(|u|^2+|b|^2\right)(t)dx+\mu\int_0^t\int_{\mathbb{R}^d}|\nabla u|^2dxds=\frac{1}{2}\int_{\mathbb{R}^d}\left(|u_0|^2+|b_0|^2\right)dx. \tag{2}$$

However, for Leray-Hopf type weak solutions with less regularity, (2) may be invalid. That is, a weak solution (u, b) to (1), for each $T > 0$, is in the class

$$u\in L^\infty\left(0,T;L^2\left(\mathbb{R}^d\right)\right)\cap L^2\left(0,T;H^1\left(\mathbb{R}^d\right)\right), b\in L^\infty\left(0,T;L^2\left(\mathbb{R}^d\right)\right) \tag{3}$$

and solves (1) in a distributional sense. In addition, such solution (u, b) satisfies the following energy inequality:

$$\frac{1}{2}\int_{\mathbb{R}^d}\left(|u|^2+|b|^2\right)(t)dx+\mu\int_0^t\int_{\mathbb{R}^d}|\nabla u|^2dxds\leq\frac{1}{2}\int_{\mathbb{R}^d}\left(|u_0|^2+|b_0|^2\right)dx. \tag{4}$$

As a matter of fact, it is still an outstanding open question whether there exist global solutions to (1) satisfying the energy identity (2) for arbitrary $(u_0, b_0) \in L^2$.

Before discussing the contents of this paper let us recall some well-established facts regarding the incompressible Navier-Stokes equations (corresponding to the case when $b = 0$ in (1)), in relation with this work. Pioneering works of Leray [26] and Hopf [17] showed the global existence of a weak solution called Leray-Hopf solution with initial data $u_0 \in L^2\left(\mathbb{R}^3\right)$. As usual, a weak solution u satisfies the energy inequality

$$\|u(t)\|_{L^2}^2+2\mu\int_0^t\int_{\mathbb{R}^3}|\nabla u|^2dxdt\leq\|u_0\|_{L^2}^2 \tag{5}$$

for any $t \in (0, T)$. On the other hand, regular solution to the 3D Navier-Stokes equations satisfy the energy equality:

$$\|u(t)\|_{L^2}^2+2\mu\int_0^t\int_{\mathbb{R}^3}|\nabla u|^2dxdt=\|u_0\|_{L^2}^2\,. \tag{6}$$

A natural question that still remains open is whether energy equality, which should be expected from a physical point of view, is valid for weak solutions. Thus, an interesting question is how badly behaved u can keep the energy conservation. Lions [27] and Ladyzhenskaya [25] proved independently that such solutions satisfy the (global) energy equality (6) under the additional assumption $u \in L^4L^4$. Shinbrot [35] generalized the Lions-Ladyzhenskaya condition to

$$u \in L^r(0,T;L^s(\mathbb{R}^d)) \text{ with } \frac{2}{r}+\frac{2}{s} \leq 1, s \geq 4. \tag{7}$$

Yu [37] showed a new proof to the Shinbrot energy conservation criterion. Recently, Tan and Yin [33] proved that suitable weak solution $u \in L^{2,\infty}(0,T;BMO)$ implies energy equality (6). However, due to the well recognized dominant role of the boundary in the generation of turbulence [4], it seems very reasonable to investigate the energy conservation in domains with boundary. Cheskidov, Friedlander and Shvydkoy [10] proved energy equality for $u \in L^3D\left(A^{\frac{5}{12}}\right)$ on a bounded domain, an extension to exterior domains was proved in [12] by Farwig and Taniuchi. Berselli and Chiodaroli [5] established some new criteria, involving the gradient of the velocity. Yu [38] proved the validity of energy equality under the following assumptions

$$u \in L^q([0,T];L^p(\Omega)) \cap L^s(0,T;B^{\alpha,\infty}_s(\Omega))$$

for $\frac{1}{q}+\frac{1}{p} \leq \frac{1}{2}, p \geq 4$ and $\frac{1}{2}+\frac{1}{s} < \alpha < 1, s > 2$. Later, Chen-Liang-Wang-Xu [8] showed that the Shinbrot's condition (7) together with $P \in L^2\left(0,T;L^2(\partial\Omega)\right)$ guaranteed the energy equality. The additional assumptions $u \in L^s(0,T;B^{\alpha,\infty}_s(\Omega))$ in [38] and $P \in L^2\left(0,T;L^2(\partial\Omega)\right)$ in [8] are used to treat with the boundary effects.

When considering distributional solutions (see Definition 1.1), in this case there is not any available regularity on velocity field u, apart the solution being in $L^2_{\text{loc}}\left(\mathbb{R}^3 \times [0,T)\right)$. The interest for distributional solutions dates back to Foias [15], who proved their uniqueness under the solution in Serrin class (i.e, $u \in L^r(0,T;L^s(\Omega))$ with $\frac{2}{r}+\frac{3}{s}=1$, $s>3$). Later, Fabes, Jones and Riviere [11] proved the existence of distributional solutions for the Cauchy problem, while the case of the initial-boundary value problem has been studied mainly starting from the work of Amann [3]. Recently, the possible connection between distributional solutions and the energy equality has been considered by Galdi [16], who proved that if distributional solution in $L^4(0,T;L^4(\mathbb{R}^3))$, and with initial data u_0 in $L^2(\mathbb{R}^3)$, then energy equality (6) holds true. The key observation is the use of the duality argument and the above conditions to improve the regularity of the solution (i.e., $L^\infty\left(0,T;L^2\left(\mathbb{R}^3\right)\right) \cap L^2\left(0,T;H^1\left(\mathbb{R}^3\right)\right)$).

In this paper we investigate the energy conservation of distributional solutions for (1). Kang, Deng and Bie [19] have discussed the energy conservation of weak solutions for the ideal Hall-MHD system. In the following we will prove that for the incompressible Hall-MHD system without resistivity (1), the solution can preserve energy under some suitable regularity condition on velocity field and magnetic field. As is well known, the global energy conservation problem of weak solutions for the 3-D incompressible Navier-Stokes equations is still widely open. To our best knowledge, our conservation results on the incompressible Hall-MHD without resistivity seems to be the first physically interesting example, showing that there exists conservation with Hall effects. From this point of view, we believe that the Hall-MHD system has its own interest and deserves more attention from mathematician. An interesting question is whether or not the energy equality (2) is holds for a Leray-Hopf weak or even distributional solution (i.e., requirement (3) is entirely removed)? Although the equations of magnetic field b are of hyperbolic type, by establishing uniform estimates of velocity u and a lemma introduced by Lions, we shall give an positive answer for this problem provided that $(u,b) \in L^4L^4$ and $\nabla \times b \in L^{\frac{8}{3}}L^{\frac{8}{3}}$. It is worth mentioning that our proof does not rely on Galdi's duality argument in [16], but based on the uniform estimates of velocity u and a lemma introduced by Lions.

In a same fashion with [16], we first define our distributional solutions as follows.

Definition 1.1. *Let* $(u_0,b_0) \in L^2(\mathbb{R}^d)$ *with* $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$, $T>0$. *The function* $(u,b) \in L^2_{\text{loc}}\left(\mathbb{R}^d \times [0,T)\right)$ *is a distributional solution to the non-resistive Hall-MHD system* (1) *if*

1. for any $\Phi \in \mathcal{D}_T$ *and* $\mathcal{D}_T := \left\{\Phi \in C_0^\infty\left(\mathbb{R}^d \times [0,T)\right) : \operatorname{div}\Phi = 0\right\}$, *we have*

$$\int_0^T \int_{\mathbb{R}^d} u \cdot \partial_t \Phi + \mu u \cdot \Delta\Phi + u \otimes u : \nabla \otimes \Phi - b \otimes b : \nabla \otimes \Phi dxdt = -\int_{\mathbb{R}^d} u(x,0) \cdot \Phi(x,0)dx,$$

$$\int_0^T \int_{\mathbb{R}^d} b \cdot \partial_t \Phi + u \otimes b : \nabla \otimes \Phi - \delta[(\nabla \times b) \times b] : \nabla \otimes \Phi - b \otimes u : \nabla \otimes \Phi dxdt$$

$$= -\int_{\mathbb{R}^d} b(x,0) \cdot \Phi(x,0)dx;$$

3. for any $\varphi \in C_0^\infty(\mathbb{R}^d)$, *it holds that*

$$\int_{\mathbb{R}^d} u \cdot \nabla\varphi dx = \int_{\mathbb{R}^d} b \cdot \nabla\varphi dx = 0$$

a.e. $t \in (0,T)$;

We state our main result as follows:

Theorem 1.2. *Let* $\mu, \delta > 0$ *and* $(u,b) \in L^2_{\text{loc}}\left(\mathbb{R}^d \times [0,T)\right)$ *be a distributional solution in the sense of Definition 1.1 to system* (1). *In additional, if*

$$(u,b) \in L^4\left(0,T; L^4(\mathbb{R}^d)\right) \text{ and } \nabla \times b \in L^{\frac{8}{3}}\left(0,T; L^{\frac{8}{3}}(\mathbb{R}^d)\right),$$

then

$$\int_{\mathbb{R}^d} |u(t,x)|^2 + |b(t,x)|^2 dx + 2\mu \int_0^t \int_{\mathbb{R}^d} |\nabla u|^2 dxdt = \int_{\mathbb{R}^d} |u_0|^2 + |b_0|^2 \, dx$$

for any $t \in [0,T]$.

Remark 1.3. *This result extends the well-known Galdi's energy conservation criterion to the non-resistive Hall-MHD system* (1).

Remark 1.4. *It would be very interesting and challenging to improve this result to Shinbrot type exponents and remove the regularity condition on* $\nabla \times b$. *This paper does not answer this question, we hope we can investigate this problem in the near future.*

As a direct corollary to Theorem 1.2, we have

Corollary 1.5. *Let* $u \in L^\infty\left(0,T; L^2\left(\mathbb{R}^d\right)\right) \cap L^2\left(0,T; H^1\left(\mathbb{R}^d\right)\right), b \in L^\infty\left(0,T; L^2\left(\mathbb{R}^d\right)\right)$ *and* (u,b) *be a weak solution to system* (1). *In additional, if*

$$(u,b) \in L^p\left(0,T; L^q(\mathbb{R}^d)\right) \text{ and } \nabla \times b \in L^{\frac{2p}{p-1}}\left(0,T; L^{\frac{2q}{q-1}}(\mathbb{R}^d)\right), \ \frac{2}{p} + \frac{2}{q} \le 1, \ q \ge 4,$$

then

$$\int_{\mathbb{R}^d} |u(t,x)|^2 + |b(t,x)|^2 dx + 2\mu \int_0^t \int_{\mathbb{R}^d} |\nabla u|^2 dxdt = \int_{\mathbb{R}^d} |u_0|^2 + |b_0|^2 \, dx$$

for any $t \in [0,T]$.

Remark 1.6. *When Hall effccts coefficient* $\delta = 0$, *problem* (1) *reduces to d-D MHD equations without resistivity, whose global energy equality of distributional solutions has been established by the author in his PhD thesis. But, here, the results take into account the impact of the Hall effect on energy conservation.*

2. Proof of Theorem 1.2

Let $\eta : \mathbb{R}^d \to \mathbb{R}$ be a standard mollifier, i.e. $\eta(x) = Ce^{\frac{1}{|x|^2-1}}$ for $|x| < 1$ and $\eta(x) = 0$ for $|x| \geqslant 1$, where constant $C > 0$ selected such that $\int_{\mathbb{R}^d} \eta(x)\mathrm{d}x = 1$. For any $\varepsilon > 0$, we define the rescaled mollifier $\eta_\varepsilon(x) = \varepsilon^{-d}\eta\left(\frac{x}{\varepsilon}\right)$. For any function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, its mollified version is defined as

$$f^\varepsilon(x) = (f * \eta_\varepsilon)(x) = \int_{\mathbb{R}^d} \eta_\varepsilon(x-y)f(y)\mathrm{d}y.$$

If $f \in W^{1,p}(\mathbb{R}^d)$, the following local approximation is well known

$$f^\varepsilon(x) \to f \quad \text{in} \quad W^{1,p}_{loc}(\mathbb{R}^d) \quad \forall p \in [1,\infty).$$

The crucial ingredient to prove Theorem 1.2 is the following crucial lemma proved by Lions [28]

Lemma 2.1. *Let ∂ be a partial derivative in one direction. Let $f, \partial f \in L^p\left(\mathbb{R}^+ \times \mathbb{R}^d\right)$, $g \in L^q\left(\mathbb{R}^+ \times \mathbb{R}^d\right)$ with $1 \le p, q \le \infty$, and $\frac{1}{p} + \frac{1}{q} \le 1$. Then, we have*

$$\|\partial(fg) * \eta_\varepsilon - \partial\left(f(g * \eta_\varepsilon)\right)\|_{L^r(\mathbb{R}^+\times\Omega)} \le C\|\partial f\|_{L^p\left(\mathbb{R}^+\times\mathbb{R}^d\right)}\|g\|_{L^q\left(\mathbb{R}^+\times\mathbb{R}^d\right)}$$

for some constant $C > 0$ independent of ε, f and g, and with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. In addition,

$$\partial(fg) * \eta_\varepsilon - \partial\left(f(g * \eta_\varepsilon)\right) \to 0 \quad \text{in } L^r\left(\mathbb{R}^+ \times \mathbb{R}^d\right)$$

as $\varepsilon \to 0$, if $r < \infty$.

Lemma 2.2. *Let $(u_0, b_0) \in L^2(\mathbb{R}^d)$ with $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$ and let (u, b) be a distributional solution in the sense of Definition 1.1 to system* (1) *and satisfies*

$$(u, b) \in L^4\left(0, T; L^4(\mathbb{R}^d)\right),$$

then we have

$$\sup_{t\ge0} \|u^\varepsilon(\cdot,t)\|^2_{L^2} + \frac{3\mu}{2}\int_0^t \int_{\mathbb{R}^d} |\nabla u^\varepsilon|^2 dx d\tau \le C, \quad \forall\, t \in [0,T],$$

in particular, it holds that

$$\sup_{t\ge0} \|u(\cdot,t)\|^2_{L^2} + \frac{3\mu}{2}\int_0^t \int_{\mathbb{R}^d} |\nabla u|^2 dx d\tau \le C,$$

where C is a constant depending only on viscosity μ, $\|u_0\|_{L^2}$ and $\int_0^T \|u\|^4_{L^4} + \|b\|^4_{L^4} dt$.

Proof of Lemma 2.2. By the definition of distributional solutions, we obtain that following identity

$$\int_{\mathbb{R}^d} u \cdot \partial_t \Phi^\varepsilon + \mu u \cdot \Delta\Phi^\varepsilon + u \otimes u : \nabla \otimes \Phi^\varepsilon - b \otimes b : \nabla \otimes \Phi^\varepsilon dx = \frac{d}{dt}\int_{\mathbb{R}^d} u(x,t) \cdot \Phi^\varepsilon(x,t) dx,$$

for all $\Phi^\varepsilon \in \mathcal{D}_T$. Which in turn gives

$$\int_{\mathbb{R}^d} u^\varepsilon \cdot \partial_t \Phi + \mu u^\varepsilon \cdot \Delta\Phi + (u \otimes u)^\varepsilon : \nabla \otimes \Phi - (b \otimes b)^\varepsilon : \nabla \otimes \Phi dx dt = \frac{d}{dt}\int_{\mathbb{R}^d} u^\varepsilon(x,t) \cdot \Phi(x,t) dx.$$

Now, choosing $\Phi = u^\varepsilon$ in above identity, integrate by parts to find

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} |u^\varepsilon|^2 dx + \mu \int_{\mathbb{R}^3} |\nabla u^\varepsilon|^2 dx &= \int_{\mathbb{R}^d} (u \otimes u)^\varepsilon \cdot \nabla u^\varepsilon dx - \int_\Omega (b \otimes b)^\varepsilon \cdot \nabla u^\varepsilon dx \\ &= I_1 + I_2. \end{aligned} \tag{8}$$

For I_1, applying the Hölder inequality, we have

$$\begin{aligned} I_1 &\le \left| \int_{\mathbb{R}^d} (u \otimes u)^\varepsilon \cdot \nabla u^\varepsilon dx \right| \le C \|(u \otimes u)^\varepsilon\|_{L^2} \|\nabla u^\varepsilon\|_{L^2} \\ &\le C \|(u \otimes u)\|_{L^2} \|\nabla u^\varepsilon\|_{L^2} \le C \|u\|_{L^4}^2 \|\nabla u^\varepsilon\|_{L^2} \\ &\le C(\mu) \|u\|_{L^4}^4 + \frac{\mu}{4} \|\nabla u^\varepsilon\|_{L^2}^2. \end{aligned} \tag{9}$$

Similarily, for I_2, we obtain

$$\begin{aligned} I_2 &\le \left| \int_{\mathbb{R}^d} (b \otimes b)^\varepsilon \cdot \nabla u^\varepsilon dx \right| \le C \|b\|_{L^4}^2 \|\nabla u^\varepsilon\|_{L^2} \\ &\le C(\mu) \|b\|_{L^4}^4 + \frac{\mu}{4} \|\nabla u^\varepsilon\|_{L^2}^2. \end{aligned} \tag{10}$$

Putting the above estimates (9)-(10) into (8), one concludes that

$$\frac{d}{dt} \int_{\mathbb{R}^d} |u^\varepsilon|^2 dx + \frac{3\mu}{2} \int_{\mathbb{R}^d} |\nabla u^\varepsilon|^2 dx \le C(\mu) \left(\|u\|_{L^4}^4 + \|b\|_{L^4}^4 \right), \tag{11}$$

and it follows that

$$\sup_{t \ge 0} \|u^\varepsilon(\cdot, t)\|_{L^2}^2 + \frac{3\mu}{2} \int_0^t \int_{\mathbb{R}^d} |\nabla u^\varepsilon|^2 dx d\tau \le \|u_0\|_{L^2}^2 + C \int_0^t \left(\|u\|_{L^4}^4 + \|b\|_{L^4}^4 \right) d\tau \le C \tag{12}$$

for all $t \in [0, T]$, where C is a constant depending only on viscosity μ, u_0 and $\int_0^T \|u\|_{L^4}^4 + \|b\|_{L^4}^4 dt$. Let $\varepsilon \to 0$ in (12), one has

$$\sup_{t \ge 0} \|u(\cdot, t)\|_{L^2}^2 + \frac{3\mu}{2} \int_0^t \int_{\mathbb{R}^d} |\nabla u|^2 dx d\tau \le C. \tag{13}$$

This concludes the proof of Lemma 2.2.

Proof of Theorem 1.2. With Lemma 2.1 and Lemma 2.2 in hand, we are ready to prove our main result. First, we define two new functions $\Xi = u^\varepsilon$, $\Sigma = b^\varepsilon$, and note that, we have

$$\operatorname{div} u^\varepsilon = 0, \ \operatorname{div} b^\varepsilon = 0.$$

Using Ξ^ε and Σ^ε to test $(1)_1$ and $(1)_2$, respectively, one has

$$\begin{cases} \int_{\mathbb{R}^d} \Xi^\varepsilon \left(\partial_t u + u \cdot \nabla u - \mu \Delta u + \nabla p - b \cdot \nabla b \right) dx = 0, \\ \int_{\mathbb{R}^d} \Sigma^\varepsilon \left(\partial_t b + (u \cdot \nabla) b + \delta \nabla \times [(\nabla \times b) \times b] - (b \cdot \nabla) u \right) dx = 0. \end{cases} \tag{14}$$

Naturally,

$$\begin{cases} \int_{\mathbb{R}^d} u^\varepsilon \left(\partial_t u + (u \cdot \nabla) u - \mu \Delta u + \nabla p - (b \cdot \nabla) b \right)^\varepsilon dx = 0, \\ \int_{\mathbb{R}^d} b^\varepsilon \left(\partial_t b + (u \cdot \nabla) b + \delta \nabla \times [(\nabla \times b) \times b] - (b \cdot \nabla) u \right)^\varepsilon dx = 0. \end{cases} \tag{15}$$

This yields

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} \left(|u^\varepsilon|^2 + |b^\varepsilon|^2 \right) dx + \mu \int_{\mathbb{R}^d} |\nabla u^\varepsilon|^2 dx \\ &= - \int_{\mathbb{R}^d} \operatorname{div}(u \otimes u)^\varepsilon \cdot u^\varepsilon dx + \int_{\mathbb{R}^d} \operatorname{div}(b \otimes b)^\varepsilon \cdot u^\varepsilon dx - \int_{\mathbb{R}^d} (u \cdot \nabla b)^\varepsilon \cdot b^\varepsilon dx \\ &\quad + \int_{\mathbb{R}^d} (b \cdot \nabla u)^\varepsilon \cdot b^\varepsilon dx - \delta \int_{\mathbb{R}^d} [(\nabla \times b) \times b]^\varepsilon \cdot (\nabla \times b)^\varepsilon dx. \end{aligned} \tag{16}$$

Clearly,

$$\begin{aligned}
&\int_{\mathbb{R}^d}\left(|u^\varepsilon|^2+|b^\varepsilon|^2\right)dx-\int_{\mathbb{R}^d}\left(|u_0^\varepsilon|^2+|b_0^\varepsilon|^2\right)dxdt+2\mu\int_0^t\int_{\mathbb{R}^d}|\nabla u^\varepsilon|^2dxd\tau\\
=&-2\int_0^t\int_{\mathbb{R}^d}\operatorname{div}(u\otimes u)^\varepsilon\cdot u^\varepsilon dxd\tau+2\int_0^t\int_{\mathbb{R}^d}\operatorname{div}(b\otimes b)^\varepsilon\cdot u^\varepsilon dxd\tau-2\int_0^t\int_{\mathbb{R}^d}(u\cdot\nabla b)^\varepsilon\cdot b^\varepsilon dxd\tau\\
&+2\int_0^t\int_{\mathbb{R}^d}(b\cdot\nabla u)^\varepsilon\cdot b^\varepsilon dxd\tau-2\delta\int_0^t\int_{\mathbb{R}^d}[(\nabla\times b)\times b]^\varepsilon\cdot(\nabla\times b)^\varepsilon dxd\tau\\
=&J_1+J_2+J_3+J_4+J_5.
\end{aligned}\tag{17}$$

Notice that

$$-2\int_0^t\int_{\mathbb{R}^d}\operatorname{div}(u^\varepsilon\otimes u^\varepsilon)\cdot u^\varepsilon dxd\tau=0,$$

thus by using the Höder equality and Lemma 2.2, one has

$$\begin{aligned}
J_1=&-2\int_0^t\int_{\mathbb{R}^d}\operatorname{div}(u\otimes u)^\varepsilon\cdot u^\varepsilon-\operatorname{div}(u^\varepsilon\otimes u^\varepsilon)\cdot u^\varepsilon dxd\tau\\
=&2\int_0^t\int_{\mathbb{R}^d}[(u\otimes u)^\varepsilon-(u^\varepsilon\otimes u^\varepsilon)]\cdot\nabla u^\varepsilon dxd\tau\\
\le&2\int_0^t\int_{\mathbb{R}^d}|(u\otimes u)^\varepsilon-u^\varepsilon\otimes u^\varepsilon|\,|\nabla u^\varepsilon|\,dxd\tau\\
\le&2\int_0^t\int_{\mathbb{R}^d}\left(|(u\otimes u)^\varepsilon-u\otimes u|+|u\otimes u-u\otimes u^\varepsilon|+|u\otimes u^\varepsilon-u^\varepsilon\otimes u^\varepsilon|\right)|\nabla u^\varepsilon|\,dxd\tau\\
\le&C\,\|(u\otimes u)^\varepsilon-u\otimes u\|_{L^2(0,T;L^2(\mathbb{R}^d))}\,\|\nabla u^\varepsilon\|_{L^2(0,T;L^2(\mathbb{R}^d))}\\
&+C\,\|u-u^\varepsilon\|_{L^4(0,T;L^4(\mathbb{R}^d))}\,\|u\|_{L^4(0,T;L^4(\mathbb{R}^d))}\|\nabla u^\varepsilon\|_{L^2(0,T;L^2(\mathbb{R}^d))}\\
&+C\,\|u-u^\varepsilon\|_{L^4(0,T;L^4(\mathbb{R}^d))}\,\|u^\varepsilon\|_{L^4(0,T;L^4(\mathbb{R}^d))}\|\nabla u^\varepsilon\|_{L^2(0,T;L^2(\mathbb{R}^d))}\\
&\to 0,\quad as\quad \epsilon\to 0.
\end{aligned}\tag{18}$$

We next rewrite

$$\begin{aligned}
(b\otimes b)_{ij}^\varepsilon&=[(b\otimes b)_{ij}^\varepsilon-(b\otimes b^\varepsilon)_{ij}]+[(b\otimes b^\varepsilon)_{ij}-(b^\varepsilon\otimes b^\varepsilon)_{ij}]+(b^\varepsilon\otimes b^\varepsilon)_{ij}\\
&=[(b_ib_j)^\varepsilon-b_ib_j^\varepsilon]+[b_ib_j^\varepsilon-b_i^\varepsilon b_j^\varepsilon]+b_i^\varepsilon b_j^\varepsilon,
\end{aligned}$$

For the term J_2, we claim that

$$\begin{aligned}
J_2&=2\int_0^t\int_{\mathbb{R}^d}\operatorname{div}(b\otimes b)^\varepsilon\cdot u^\varepsilon dxdt=-2\int_0^t\int_{\mathbb{R}^d}(b\otimes b)^\varepsilon\cdot\nabla u^\varepsilon dxd\tau\\
&=-2\int_0^t\int_{\mathbb{R}^d}[(b_ib_j)^\varepsilon-b_ib_j^\varepsilon]+[b_ib_j^\varepsilon-b_i^\varepsilon b_j^\varepsilon]\partial_ju_i^\varepsilon dxd\tau+2\int_0^t\int_{\mathbb{R}^d}b_j^\varepsilon\partial_jb_i^\varepsilon u_i^\varepsilon dxd\tau.
\end{aligned}\tag{19}$$

Similarly, we have

$$J_3=-2\int_0^t\int_{\mathbb{R}^d}(u\cdot\nabla b)^\varepsilon\cdot b^\varepsilon dxd\tau=-2\int_0^t\int_{\mathbb{R}^d}[\partial_j(u_jb_i)^\varepsilon-\partial_j(u_jb_i^\varepsilon)]\cdot b_i^\varepsilon dxd\tau.\tag{20}$$

Analogously, we have also the following identity:

$$(b\cdot\nabla u)_i^\varepsilon=[\partial_j(u_ib_j)^\varepsilon-\partial_j(u_ib_j^\varepsilon)]+[\partial_j(u_ib_j^\varepsilon)-\partial_j(u_i^\varepsilon b_j^\varepsilon)]+b_j^\varepsilon\partial_ju_i^\varepsilon.$$

Thus, for the term J_4, we find that

$$
\begin{aligned}
J_4 &= 2\int_0^t \int_{\mathbb{R}^d} (b\cdot\nabla u)^\varepsilon \cdot b^\varepsilon dxd\tau \\
&= 2\int_0^t \int_{\mathbb{R}^d} [\partial_j(u_ib_j)^\varepsilon - \partial_j(u_ib_j^\varepsilon)] + [\partial_j(u_ib_j^\varepsilon) - \partial_j(u_i^\varepsilon b_j^\varepsilon)]b_i^\varepsilon dxd\tau + 2\int_0^t \int_{\mathbb{R}^d} b_j^\varepsilon \partial_j u_i^\varepsilon b_i^\varepsilon dxd\tau \\
&= 2\int_0^t \int_{\mathbb{R}^d} [\partial_j(u_ib_j)^\varepsilon - \partial_j(u_ib_j^\varepsilon)] + [\partial_j(u_ib_j^\varepsilon) - \partial_j(u_i^\varepsilon b_j^\varepsilon)]b_i^\varepsilon dxd\tau - 2\int_0^t \int_{\mathbb{R}^d} b_j^\varepsilon \partial_j b_i^\varepsilon u_i^\varepsilon dxd\tau.
\end{aligned} \tag{21}
$$

Combining (19)-(21), it yields that

$$
\begin{aligned}
J_2+J_3+J_4 =& -2\int_0^t \int_{\mathbb{R}^d} [(b_ib_j)^\varepsilon - b_ib_j^\varepsilon] + [b_ib_j^\varepsilon - b_i^\varepsilon b_j^\varepsilon]\partial_j u_i^\varepsilon dxd\tau \\
& -2\int_0^t \int_{\mathbb{R}^d} [\partial_j(u_jb_i)^\varepsilon - \partial_j(u_jb_i^\varepsilon)]\cdot b_i^\varepsilon dxd\tau \\
& +2\int_0^t \int_{\mathbb{R}^d} [\partial_j(u_ib_j)^\varepsilon - \partial_j(u_ib_j^\varepsilon)] + [\partial_j(u_ib_j^\varepsilon) - \partial_j(u_i^\varepsilon b_j^\varepsilon)]b_i^\varepsilon dxd\tau \\
=:& K_1^\varepsilon + K_2^\varepsilon + K_3^\varepsilon.
\end{aligned} \tag{22}
$$

For K_1^ε, by using Lemmas 2.2 again, we obtain that

$$
\begin{aligned}
K_1^\varepsilon \le& \left| -2\int_0^t \int_{\mathbb{R}^d} [(b_ib_j)^\varepsilon - b_ib_j^\varepsilon] + [b_ib_j^\varepsilon - b_i^\varepsilon b_j^\varepsilon]\partial_j u_i^\varepsilon dxd\tau \right| \\
\le& 2\int_0^t \int_{\mathbb{R}^d} \left(\left|(b_ib_j)^\varepsilon - b_ib_j\right| + \left|b_j - b_j^\varepsilon\right| |b_i| \right) \left|\partial_j u_i^\varepsilon\right| dxd\tau \\
& + 2\int_0^t \int_{\mathbb{R}^d} \left|b_i - b_i^\varepsilon\right| \left|b_j^\varepsilon\right| \left|\partial_j u_i^\varepsilon\right| dxd\tau \\
\le& C\left\|(b_ib_j)^\varepsilon - (b_ib_j)\right\|_{L^2(0,T;L^2(\mathbb{R}^d))} \|\partial_j u_i^\varepsilon\|_{L^2(0,T;L^2(\mathbb{R}^d))} \\
& + C\left\|b_i^\varepsilon\right\|_{L^4(0,T;L^4(\mathbb{R}^d))} \|b_j^\varepsilon - b_j\|_{L^4(0,T;L^4(\mathbb{R}^d))} \|\partial_j u_i^\varepsilon\|_{L^2(0,T;L^2(\mathbb{R}^d))} \\
& + C\left\|b_j^\varepsilon\right\|_{L^4(0,T;L^4(\mathbb{R}^d))} \|b_i^\varepsilon - b_i\|_{L^4(0,T;L^4(\mathbb{R}^d))} \|\partial_j u_i^\varepsilon\|_{L^2(0,T;L^2(\mathbb{R}^d))} \\
& \to 0, \ as\ \varepsilon \to 0.
\end{aligned} \tag{23}
$$

By Lions commutator estimates (Lemma 2.1) and the uniform estimates of velocity u (Lemma 9), then these lead to the fact that

$$
\begin{aligned}
K_2^\varepsilon \le& 2\int_0^t \int_{\mathbb{R}^d} \left|\partial_j(u_ib_j)^\varepsilon - \partial_j(u_ib_j^\varepsilon)\right| |b_i^\varepsilon| dxd\tau \\
\le& C\left\|\partial_j\left(u_ib_j\right)^\varepsilon - \partial_j(u_ib_j^\varepsilon)\right\|_{L^{\frac{4}{3}}\left(0,T;L^{\frac{4}{3}}(\mathbb{R}^d)\right)} \left\|b_i^\varepsilon\right\|_{L^4(0,T;L^4(\mathbb{R}^d))} \\
\to& 0, \quad as \quad \epsilon \to 0.
\end{aligned} \tag{24}
$$

Finally, for the term K_3^ε, one see that

$$\begin{aligned}K_3^\varepsilon \leq & 2\left|\int_0^t \int_{\mathbb{R}^d} [\partial_j(u_i b_j)^\varepsilon - \partial_j(u_i b_j^\varepsilon)] + [\partial_j(u_i b_j^\varepsilon) - \partial_j(u_i^\varepsilon b_j^\varepsilon)] b_i^\varepsilon dx d\tau\right| \\ \leq & C\left\|\partial_j\left(u_i b_j\right)^\varepsilon - \partial_j(u_i b_j^\varepsilon)\right\|_{L^{\frac{4}{3}}\left(0,T;L^{\frac{4}{3}}(\mathbb{R}^d)\right)} \left\|b_i^\varepsilon\right\|_{L^4\left(0,T;L^4(\mathbb{R}^d)\right)} \\ & + C\left\|\nabla u - \nabla u^\varepsilon\right\|_{L^2\left(0,T;L^2(\mathbb{R}^d)\right)} \left\||b^\varepsilon|^2\right\|_{L^2\left(0,T;L^2(\mathbb{R}^d)\right)} \\ \to & 0, \quad as \quad \epsilon \to 0.\end{aligned} \tag{25}$$

So, it follows that

$$J_2 + J_3 + J_4 \to 0, \quad as \quad \epsilon \to 0. \tag{26}$$

For the term J_5, we claim that

$$\begin{aligned}&-2\delta \int_0^t \int_{\mathbb{R}^d} [(\nabla \times b) \times b]^\varepsilon \cdot (\nabla \times b)^\varepsilon dx d\tau \to -2\delta \int_0^t \int_{\mathbb{R}^d} [(\nabla \times b) \times b] \cdot (\nabla \times b) dx d\tau = 0 \\ &as \quad \epsilon \to 0.\end{aligned} \tag{27}$$

Indeed,

$$\begin{aligned}&2\delta \int_0^t \int_{\mathbb{R}^d} [(\nabla \times b) \times b]^\varepsilon \cdot (\nabla \times b)^\varepsilon - [(\nabla \times b) \times b] \cdot (\nabla \times b) dx d\tau \\ =& 2\delta \int_0^t \int_{\mathbb{R}^d} [(\nabla \times b) \times b]^\varepsilon \cdot (\nabla \times b)^\varepsilon - [(\nabla \times b) \times b] \cdot (\nabla \times b)^\varepsilon dx d\tau \\ &+ 2\delta \int_0^t \int_{\mathbb{R}^d} [(\nabla \times b) \times b] \cdot (\nabla \times b)^\varepsilon - [(\nabla \times b) \times b] \cdot (\nabla \times b) dx d\tau \\ \leq& 2\delta \left\|[(\nabla \times b) \times b]^\varepsilon - [(\nabla \times b) \times b]\right\|_{L^{\frac{8}{5}}\left(0,T;L^{\frac{8}{5}}(\mathbb{R}^d)\right)} \left\|(\nabla \times b)^\varepsilon\right\|_{L^{\frac{8}{3}}\left(0,T;L^{\frac{8}{3}}(\mathbb{R}^d)\right)} \\ &+ 2\delta \left\|[(\nabla \times b) \times b]\right\|_{L^{\frac{8}{5}}\left(0,T;L^{\frac{8}{5}}(\mathbb{R}^d)\right)} \left\|(\nabla \times b)^\varepsilon - (\nabla \times b)\right\|_{L^{\frac{8}{3}}\left(0,T;L^{\frac{8}{3}}(\mathbb{R}^d)\right)} \\ &\to 0, \ as \ \varepsilon \to 0,\end{aligned} \tag{28}$$

where used the fact that

$$\begin{aligned}\|(\nabla \times b) \times b\|_{L^{\frac{8}{5}}\left(0,T;L^{\frac{8}{5}}(\mathbb{R}^d)\right)} &\leq \|\nabla \times b\|_{L^{\frac{8}{3}}\left(0,T;L^{\frac{8}{3}}(\mathbb{R}^d)\right)} \|b\|_{L^4\left(0,T;L^4(\mathbb{R}^d)\right)} \\ &\leq C.\end{aligned}$$

Letting ε goes to zero in (17), and using the facts (18), (26) and (27), what we have proved is that in the limit

$$\int_{\mathbb{R}^d} \left(|u|^2 + |b|^2\right) dx + 2\mu \int_0^t \int_{\mathbb{R}^d} |\nabla u|^2 dx dt = \int_{\mathbb{R}^d} \left(|u_0|^2 + |b_0|^2\right) dx dt. \tag{29}$$

This completes the proof of Theorem 1.2.

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