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A generalized quaternionic sequence with Vietoris' number components

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Abstract. In this investigation, the aim is to determine a generalized quaternionic sequence with Vietoris' number components depending on 2-parameters α and β. Considering specific real values α and β, various types of classical quaternionic sequence with Vietoris' number components can be obtained as real, split, split-semi and so on. The fundamental algebraic structures, several classical expressions, a two and three term recurrence relations are identified, as well as Catalan-like, generating function and Binet-like formulas. Furthermore, a determinantal approach is used to generate the generalized quaternionic sequence with Vietoris' number components.

1. Backgrounds and Motivations

Classical number sequences focus on the study of and applications of integer sequences. But at the same time, we also have rational sequences. Integer sequences are a special case of rational sequences. The Vietoris' sequence of rational numbers can be considered on a set of Appell polynomials several hypercomplex variables in [5]. For more details of Appell polynomials we refer to the reader to [28–30]. The recent studies of the Vietoris' number sequence are [2–5, 7, 8, 13, 27, 32, 33]. The classical works of the Vietoris' number sequence are for combinatorial properties [33] and for some interesting generalizations of combinatorial properties [27]. The inspiration and starting points of this paper are recent works: [7, 8]. In their distinguished paper [8], the authors examined in detail some properties of the Vietoris' sequence whose some of them with the use of the Catalan's sequence properties, and gave special type of matrices to generate this rational sequence.

Now, let us make an overview of the Vietoris' number sequence $\{v_n\}_{n\geq 0}$. The n-th element is of the compact form

$$v_n = \frac{1}{2^n}\binom{n}{\lfloor \frac{n}{2} \rfloor}, \quad n \geq 0,$$

where $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ is the central binomial coefficient, [13]. Here, the notation $\lfloor , \rfloor$ represents the floor function. Even elements of $\{v_n\}_{n\geq 0}$ are given by: $v_{2n} = \frac{1}{2^{2n}}\binom{2n}{n}$, $n \geq 0$, where $v_{2n-1} = v_{2n}$, [32]. The first several

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values of this sequence are (related with the sequence A283208 in [26]):

$$1, \frac{1}{2}, \frac{1}{2}, \frac{3}{8}, \frac{3}{8}, \frac{5}{16}, \frac{5}{16}, \frac{35}{128}, \frac{35}{128}, \frac{63}{256}, \frac{63}{256}, \ldots$$

The two term recurrence relation for $\{v_{2n}\}_{n\geq 0}$ is given by the following identity:

$$v_{2n+2} = d(2n)v_{2n}, \quad n \geq 0, \tag{1}$$

where $d(k) = \frac{k+1}{k+2}, k \geq 0$. Hence one can see v_{2n} in terms of any v_{2k} as follows:

$$v_{2n} = \prod_{l=1}^{n-k} d(2n-2l)v_{2k}, \quad n > k, \tag{2}$$

or in terms of v_0 as

$$v_{2n+2} = \prod_{i=0}^{n} d(2i)v_0 = \frac{(2n+1)!!}{(2n+2)!!}, \tag{3}$$

[7, 8]. The three consecutive term recurrence relation is:

$$v_{2n+2} = \frac{1}{2}v_{2n+1} + \frac{1}{2}d\,(2n)\,v_{2n}, \tag{4}$$

and the three term consecutive with even index recurrence relation is:

$$v_{2n+2} = \frac{1}{2}d\,(2n)\,v_{2n} + \frac{1}{2}d\,(2n)\,d\,(2n-2)\,v_{2n-2}. \tag{5}$$

The generating function is (see [5]):

$$g(x) = \frac{\sqrt{1+x} - \sqrt{1-x}}{x\sqrt{1-x}} = \sum_{s=0}^{\infty} v_s x^s, \quad 0 < |x| < 1. \tag{6}$$

The Binet's like formula is given by:

$$v_{2n} = c_1\,(2n)\,r_1^{2n}\,(2n) + c_2\,(2n)\,r_2^{2n}\,(2n)\,, \tag{7}$$

where

$$r_1\,(2n) = \frac{1}{4}\Big(1 - \sqrt{1+8d(2n)}\Big), \quad r_2\,(2n) = \frac{1}{4}\Big(1 + \sqrt{1+8d(2n)}\Big) \tag{8}$$

and

$$\begin{cases} c_1\,(2n) = \dfrac{r_2^{2n}(2n) - v_2}{r_2^{2n}(2n) - r_1^{2n}(2n)} \displaystyle\prod_{k=1}^{n-1} (2r_1(2k)-1)r_1(2k), \\ c_2\,(2n) = \dfrac{v_2 - r_1^{2n}(2n)}{r_2^{2n}(2n) - r_1^{2n}(2n)} \displaystyle\prod_{k=1}^{n-1} (2r_2(2k)-1)r_2(2k). \end{cases} \tag{9}$$

Some basic properties of r_1 and r_2 are $r_2(0) = \frac{1+\sqrt{5}}{4}$ is half of the golden ratio[1)], $r_1(2n) + r_2(2n) = \frac{1}{2}$ and $r_1(2n)r_2(2n) = -\frac{1}{2}d(2n)$, (see [7]).

[1)]The ratios of sequential Fibonacci numbers approach the golden ratio.

In the literature, a number of authors have studied the properties of hypercomplex numbers with diverse sequences components in many points of view. Some recent studies on quaternions with special sequences are [1, 7, 11, 15–17]. The quaternions have many applications in robotics, computer visualisation, navigation, mechanisms, engineering and many other areas via rotation and orientation. For example, the quaternions can be used for efficient calculations of the position of a rotated aircraft and spacecraft in space. Moreover, the quaternions offers a basic and extensive representations of signals for the simultaneous control of several components. The Fourier transform[2)], which is the integral transform, has been extended to the quaternions. The Fourier transform is particularly relevant to physics, navigation and signal processing. Especially, the special affine Fourier transform (SAFT for short) has a crucial role of the effective representation of quaternion-valued signals and significant uses in a variety of image and processing, edge detection, sampling theory, radar and optical systems, communication and electrical systems, speech recognition, pattern recognition, data compression and analyzing temporary signals. Since, the quaternions are non-commutative, there are three type of SAFT in [19]. The paper [31] works on the short-time SAFT as part of 2-dimensional real quaternion-valued signals, which the readers can find some other presumably novel directions of further researches.

Let us present some preliminary definitions and known results related to generalized quaternions. A generalized quaternion is of the following form:

$$q = q_0 + q_1\mathsf{i} + q_2\mathsf{j} + q_3\mathsf{k},$$

where $q_0, q_1, q_2, q_3 \in \mathbb{R}$ and $\mathsf{i}, \mathsf{j}, \mathsf{k}$ are the non-real quaternionic units, obeying the following multiplication rules:

$$\mathsf{i}^2 = -\alpha, \quad \mathsf{j}^2 = -\beta, \quad \mathsf{k}^2 = -\alpha\beta, \quad \mathsf{ij} = -\mathsf{ji} = \mathsf{k}, \quad \mathsf{jk} = -\mathsf{kj} = \beta\mathsf{i}, \quad \mathsf{ki} = -\mathsf{ik} = \alpha\mathsf{j}, \tag{10}$$

where $\alpha, \beta \in \mathbb{R}$. It can also be written as $q = (q_0, q_1, q_2, q_3)$, where q_0 is the scalar (real) part and denoted by S_q and (q_1, q_2, q_3) is the 3-vector (pure) part and denoted by V_q. The quaternion conjugate is given by $\overline{q} = S_q - V_q$. The addition of two generalized quaternions is defined component-wise, whereas the product of two generalized quaternions is, also a generalized quaternion, calculated by:

$$\begin{aligned} qp = & \ (q_0 + q_1\mathsf{i} + q_2\mathsf{j} + q_3\mathsf{k})(p_0 + p_1\mathsf{i} + p_2\mathsf{j} + p_3\mathsf{k}) \\ = & \ (q_0p_0 - \alpha q_1p_1 - \beta q_2p_2 - \alpha\beta q_3p_3) + (q_0p_1 + q_1p_0 + \beta q_2p_3 - \beta q_3p_2)\mathsf{i} \\ & + (q_0p_2 - \alpha q_1p_3 + q_2p_0 + \alpha q_3p_1)\mathsf{j} + (q_0p_3 + q_1p_2 - q_2p_1 + q_3p_0)\mathsf{k}. \end{aligned}$$

The norm is denoted by $\|,\|$ and defined in a similar way as complex numbers, $\|q\|^2 = q\overline{q} = \overline{q}q$. The inverse of q is defined as $q = \frac{\overline{q}}{\|q\|^2}$, $\|q\|^2 \neq 0$. The generalized quaternions form an associative and a non-commutative algebra of dimension four over $\mathbb{R}$ and include other well-known 4-dimensional algebras as special cases, [9, 10, 12, 14, 18, 20, 22–25]. For $\alpha = \beta = 1$ real, for $\alpha = 1, \beta = -1$ split, for $\alpha = 1, \beta = 0$ semi, for $\alpha = -1, \beta = 0$ split-semi and for $\alpha = \beta = 0$ quasi quaternions are obtained.

If a quaternion algebra is discussed from the viewpoint of Vietoris' sequence, Catarino and Almeida have introduced the real quaternion sequence with Vietoris' numbers, see [7]. They have presented fundamental two and three terms recurrence formulas and examined the determinant of some tridiagonal matrices with the quaternionic Vietoris' sequence, in [7]. Taking all these quite details into account, a natural question to ask is if the paper [7] can be generalized. Our main interest in this paper is to develop the generalized quaternionic sequence with Vietoris' number components to make this generalization. To do this, the article is organized as follows. In Section 2, the Vietoris' generalized quaternionic sequence is examined in-depth and in Section 3 this sequence is generated by an determinantal approach.

2. The Vietoris Generalized Quaternionic Sequence $\{\mathbf{Q}_s\}_{s\geq 0}$

Now, let us introduce some notations that are used in our main results.

[2)] The Fourier transform converts a function of time, to a function of frequency. The affine transform includes translations, rotations, reflections, scalings, shears, etc., as well as their combinations, forming an affine group.

Definition 2.1. *The set of generalized quaternions with Vietoris' number components is denoted by* $\mathsf{Q}_{\alpha\beta}$ *and the mathematical expression is of the form*

$$\mathsf{Q}_s = v_s + v_{s+1}\mathsf{i} + v_{s+2}\mathsf{j} + v_{s+3}\mathsf{k}, \ \ s \geq 0, \tag{11}$$

where v_s *is the s-th element of the Vietoris' number and the quaternionic units satisfy the conditions in equation* (10). *An equivalent definition similar to the Cayley–Dickson form is that*

$$\mathsf{Q}_s = \mathsf{q}_s + \mathsf{q}_{s+2}\mathsf{j}, \tag{12}$$

where $\mathsf{q}_s = v_s + v_{s+1}\mathsf{i}$.

The first five Vietoris' generalized quaternionic numbers are

$$\begin{array}{llllll} \mathsf{Q}_0 = & 1 + \frac{1}{2}\mathsf{i} + \frac{1}{2}\mathsf{j} + \frac{3}{8}\mathsf{k}, & \mathsf{Q}_1 = & \frac{1}{2} + \frac{1}{2}\mathsf{i} + \frac{3}{8}\mathsf{j} + \frac{3}{8}\mathsf{k}, & \mathsf{Q}_2 = & \frac{1}{2} + \frac{3}{8}\mathsf{i} + \frac{3}{8}\mathsf{j} + \frac{5}{16}\mathsf{k}, \\ \mathsf{Q}_3 = & \frac{3}{8} + \frac{3}{8}\mathsf{i} + \frac{5}{16}\mathsf{j} + \frac{5}{16}\mathsf{k}, & \mathsf{Q}_4 = & \frac{3}{8} + \frac{5}{16}\mathsf{i} + \frac{5}{16}\mathsf{j} + \frac{35}{128}\mathsf{k}, & \mathsf{Q}_5 = & \frac{5}{16} + \frac{5}{16}\mathsf{i} + \frac{35}{128}\mathsf{j} + \frac{35}{128}\mathsf{k}. \end{array}$$

Since $v_{2n-1} = v_{2n}$, the even and odd indexed numbers of $\{\mathsf{Q}_s\}_{s\geq 0}$ can be examined in detail. For $s = 2n$, we have

$$\mathsf{Q}_{2n} = \ \ \mathsf{q}_{2n} + \mathsf{q}_{2n+2}\mathsf{j} = v_{2n} + v_{2n+2}\,(\mathsf{i} + \mathsf{j}) + v_{2n+4}\mathsf{k}. \tag{13}$$

Applying equation (1), equation (13) also can be rewritten as:

$$\mathsf{Q}_{2n} \ \ = \ \ v_{2n}\,(1 + d(2n)\mathsf{i} + d(2n)\mathsf{j} + d(2n)d(2n+2)\mathsf{k}) = v_{2n}\mathsf{X}(2n), \tag{14}$$

where $\mathsf{X}(2n) = 1 + d(2n)\mathsf{i} + d(2n)\mathsf{j} + d(2n)d(2n+2)\mathsf{k}$. Additionally, for $s = 2n + 1$, we obtain

$$\mathsf{Q}_{2n+1} = \ \ \mathsf{q}_{2n+1} + \mathsf{q}_{2n+3}\mathsf{j} = 2\mathsf{q}_1\,(v_{2n+2} + v_{2n+4}\mathsf{j}) = v_{2n+2}\,(1 + \mathsf{i}) + v_{2n+4}\,(\mathsf{j} + \mathsf{k}) \tag{15}$$

and it is rewritten as:

$$\mathsf{Q}_{2n+1} \ \ = \ \ v_{2n}d(2n)\,(1 + \mathsf{i} + d(2n+2)\,(\mathsf{j} + \mathsf{k})) = v_{2n+2}\mathsf{Y}(2n+2), \tag{16}$$

where $\mathsf{Y}(2n+2) = 1 + \mathsf{i} + d(2n+2)\mathsf{j} + d(2n+2)\mathsf{k}$.

Lemma 2.2. *Let* $\{\mathsf{Q}_s\}_{s\geq 0}$ *be the Vietoris' generalized quaternionic sequence. Then,*

$$\mathsf{Q}_{2n+1} \ \ = \frac{(2n+1)!!}{(2n+2)!!}\mathsf{Y}(2n+2), \tag{17}$$

and

$$\mathsf{Q}_{2n+2} \ \ = \frac{(2n+1)!!}{(2n+2)!!}\mathsf{X}(2n+2). \tag{18}$$

Proof. It follows directly from equation (3), (14) and (16). □

Remark 2.3. *Using Lemma 2.2, all statements can also be recalculated as this form.*

By using operations over generalized quaternion algebra, basic arithmetic operations can be given in a standard way.

Theorem 2.4. *Let* $\{\mathsf{Q}_s\}_{s\geq 0}$ *be the Vietoris' generalized quaternionic sequence. Then we have:*

$$\lim_{n\to\infty} \frac{\mathsf{Q}_{2n-1}}{v_{2n}} = \lim_{n\to\infty} \frac{\mathsf{Q}_{2n}}{v_{2n}} = \lim_{n\to\infty} \frac{\mathsf{Q}_{2n+1}}{v_{2n}} = \lim_{n\to\infty} \frac{\mathsf{Q}_{2n+2}}{v_{2n}} = 1 + \mathsf{i} + \mathsf{j} + \mathsf{k}. \tag{19}$$

Proof. Taking equation (1) and the relation $d(k) = \frac{k+1}{k+2}$ into account, the proof is straightforward. □

Proposition 2.5. *Let* $\{\mathsf{Q}_s\}_{s\geq 0}$ *be the Vietoris' generalized quaternionic sequence. Then the followings hold:*

(i) $\mathsf{Q}_{2n} + \mathsf{Q}_{2n+1} = v_{2n}\left[1 + d(2n) + 2d(2n)\mathsf{i} + d(2n)(1 + d(2n+2))\mathsf{j} + 2d(2n)d(2n+2)\mathsf{k}\right]$,

(ii) $\mathsf{Q}_{2n} - \mathsf{Q}_{2n+1} = v_{2n}\left[1 - d(2n) + d(2n)(1 - d(2n+2))\mathsf{j}\right]$,

(iii) $\mathsf{Q}_{2n} + \mathsf{Q}_{2n-1} = v_{2n}\left[2 + (d(2n)+1)\mathsf{i} + 2d(2n)\mathsf{j} + d(2n)(d(2n+2)+1)\mathsf{k}\right]$,

(iv) $\mathsf{Q}_{2n} - \mathsf{Q}_{2n-1} = v_{2n}\left[(d(2n)-1)\mathsf{i} + d(2n)(d(2n+2)-1)\mathsf{k}\right]$,

(v) $\mathsf{Q}_{2n+1} + \mathsf{Q}_{2n-1} = v_{2n}\left[(d(2n)+1)(1+\mathsf{i}) + d(2n)(d(2n+2)+1)(\mathsf{j}+\mathsf{k})\right]$,

(vi) $\mathsf{Q}_{2n+1} - \mathsf{Q}_{2n-1} = v_{2n}\left[(d(2n)-1)(1+\mathsf{i}) + d(2n)(d(2n+2)-1)(\mathsf{j}+\mathsf{k})\right]$,

(vii) $\mathsf{Q}_{2n} + \mathsf{Q}_{2n+2} = v_{2n}\left[(1 + d(2n)) + d(2n)(d(2n+2)+1)(\mathsf{i}+\mathsf{j}) + d(2n)d(2n+2)(d(2n+4)+1)\mathsf{k}\right]$,

(viii) $\mathsf{Q}_{2n+2} - \mathsf{Q}_{2n} = v_{2n}\left[(d(2n)-1) + d(2n)(d(2n+2)-1)(\mathsf{i}+\mathsf{j}) + d(2n)d(2n+2)(d(2n+4)-1)\mathsf{k}\right]$.

Proof. To prove items, we use equations (1), (14) and (16). □

Theorem 2.6. *Let* $\{\mathsf{Q}_s\}_{s\geq 0}$ *be the Vietoris' generalized quaternionic sequence. Then, the following relations are satisfied:*

(i) $\mathsf{Q}_{2n} - \mathsf{Q}_{2n+1}\mathsf{i} - \mathsf{Q}_{2n+2}\mathsf{j} - \mathsf{Q}_{2n+3}\mathsf{k} = v_{2n} + \alpha v_{2n+2} + \beta v_{2n+4} + \alpha\beta v_{2n+6}$,

(ii) $\mathsf{Q}_{2n+1} - \mathsf{Q}_{2n+2}\mathsf{i} - \mathsf{Q}_{2n+3}\mathsf{j} - \mathsf{Q}_{2n+4}\mathsf{k} = v_{2n+2} + \alpha v_{2n+4} + \beta v_{2n+6} + \alpha\beta v_{2n+8}$,

(iii) $\|\mathsf{Q}_{2n}\|^2 = v_{2n}^2\left[1 + d^2(2n)(\alpha + \beta + \alpha\beta d^2(2n+2))\right]$,

(iv) $\|\mathsf{Q}_{2n+1}\|^2 = v_{2n+2}^2(1+\alpha)(1+\beta d^2(2n+2))$,

(v) $\mathsf{Q}_s + \overline{\mathsf{Q}}_s = 2v_s$,

(vi) $\mathsf{Q}_s^2 + \|\mathsf{Q}_s\|^2 = 2v_s\mathsf{Q}_s$.

Proof. (ii) From equations (10) and (14), and considering definition of norm, we obtain:

$$\begin{aligned}
\|\mathsf{Q}_{2n}\|^2 &= v_{2n}^2\mathsf{X}_{2n}\overline{\mathsf{X}}_{2n}\\
&= v_{2n}^2\left(1 + d(2n)\mathsf{i} + d(2n)\mathsf{j} + d(2n)d(2n+2)\mathsf{k}\right)\left(1 - d(2n)\mathsf{i} - d(2n)\mathsf{j} - d(2n)d(2n+2)\mathsf{k}\right)\\
&= v_{2n}^2\left[1 + d^2(2n)(\alpha + \beta + \alpha\beta d^2(2n+2))\right].
\end{aligned}$$

The other items can be shown easily and left to the readers. □

Proposition 2.7. *Let* $\{\mathsf{Q}_s\}_{s\geq 0}$ *be the Vietoris' generalized quaternionic sequence. Then, the following properties can be given:*

(i) $\mathsf{Q}_{2n}\mathsf{Q}_{2m} - \overline{\mathsf{Q}}_{2n}\overline{\mathsf{Q}}_{2m} = 2v_{2n}v_{2m}\left(V_{\mathsf{X}(2n)} + V_{\mathsf{X}(2m)}\right)$,

(ii) $\mathsf{Q}_{2n}\mathsf{Q}_{2m+1} - \overline{\mathsf{Q}}_{2n}\overline{\mathsf{Q}}_{2m+1} = 2v_{2n}v_{2m+2}\left(V_{\mathsf{X}(2n)} + V_{\mathsf{Y}(2m+2)}\right)$,

(iii) $\mathsf{Q}_{2n+1}\mathsf{Q}_{2m} - \overline{\mathsf{Q}}_{2n+1}\overline{\mathsf{Q}}_{2m} = 2v_{2n+2}v_{2m}\left(V_{\mathsf{Y}(2n+2)} + V_{\mathsf{X}(2m)}\right)$,

(iv) $\mathsf{Q}_{2n+1}\mathsf{Q}_{2m+1} - \overline{\mathsf{Q}}_{2n+1}\overline{\mathsf{Q}}_{2m+1} = 2v_{2n+2}v_{2m+2}\left(V_{\mathsf{Y}(2n+2)} + V_{\mathsf{Y}(2m+2)}\right)$,

(v) $\mathsf{Q}_{2n}\overline{\mathsf{Q}}_{2m} - \overline{\mathsf{Q}}_{2n}\mathsf{Q}_{2m} = 2v_{2n}v_{2m}\left(V_{\mathsf{X}(2n)} - V_{\mathsf{X}(2m)}\right)$,

(vi) $\mathsf{Q}_{2n}\overline{\mathsf{Q}}_{2m+1} - \overline{\mathsf{Q}}_{2n}\mathsf{Q}_{2m+1} = 2v_{2n}v_{2m+2}\left(V_{\mathsf{X}(2n)} - V_{\mathsf{Y}(2m+2)}\right)$,

(vii) $\mathsf{Q}_{2n+1}\overline{\mathsf{Q}}_{2m} - \overline{\mathsf{Q}}_{2n+1}\mathsf{Q}_{2m} = 2v_{2n+2}v_{2m}\left(V_{\mathsf{Y}(2n+2)} - V_{\mathsf{X}(2m)}\right)$,

(viii) $\mathsf{Q}_{2n+1}\overline{\mathsf{Q}}_{2m+1} - \overline{\mathsf{Q}}_{2n+1}\mathsf{Q}_{2m+1} = 2v_{2n+2}v_{2m+2}\left(V_{\mathsf{Y}(2n+2)} - V_{\mathsf{Y}(2m+2)}\right)$.

Proof. (ii) Considering equations (14) and (16), we have:

$$\begin{aligned}\mathsf{Q}_{2n}\mathsf{Q}_{2m+1} - \overline{\mathsf{Q}}_{2n}\overline{\mathsf{Q}}_{2m+1} &= v_{2n}\mathsf{X}(2n)v_{2m+2}\mathsf{Y}(2m+2) - v_{2n}\overline{\mathsf{X}(2n)}v_{2m+2}\overline{\mathsf{Y}(2m+2)} \\ &= 2v_{2n}v_{2m+2}\,(1 + d(2n))\mathsf{i} + (d(2n) + d(2m+2))\mathsf{j} \\ &\quad +(d(2n)d(2n+2) + d(2m+2))\mathsf{k}) \\ &= 2v_{2n}v_{2m+2}\left(V_{\mathsf{X}(2n)} + V_{\mathsf{Y}(2m+2)}\right).\end{aligned}$$

The other items can be proved similarly. □

Proposition 2.8. *Let $\{\mathsf{Q}_s\}_{s\geq 0}$ be the Vietoris' generalized quaternionic sequence. Let us write another useful conjugation of $\mathsf{Q}_s = \mathsf{q}_s + \mathsf{q}_{s+2}\mathsf{j} \in \mathbb{Q}_{\alpha\beta}$ as $\mathsf{Q}_s^* = \mathsf{q}_s^* - \mathsf{q}_{s+2}^*\mathsf{j}$, where $\mathsf{q}_s^* = v_s - v_{s+1}\mathsf{i}$. By considering this form, we can write the following:*

(i) $\mathsf{Q}_{2n} + \mathsf{Q}^*_{2n} = 2v_{2n}\,(1 + d(2n)d(2n+2)\mathsf{k})$,

(ii) $\mathsf{Q}_{2n} - \mathsf{Q}^*_{2n} = 2v_{2n}d(2n)\,(\mathsf{i} + \mathsf{j})$,

(iii) $\mathsf{Q}_{2n+1} + \mathsf{Q}^*_{2n+1} = 2v_{2n}d(2n)\,(1 + d(2n+2)\mathsf{k})$,

(iv) $\mathsf{Q}_{2n+1} - \mathsf{Q}^*_{2n+1} = 2v_{2n}d(2n)\,(\mathsf{i} + d(2n+2)\mathsf{j})$,

(v) $\mathsf{Q}_{2n} + \mathsf{Q}^*_{2n+1} = v_{2n}\,[1 + d(2n) + d(2n)(1 - d(2n+2))\mathsf{j} + 2d(2n)d(2n+2)\mathsf{k}]$,

(vi) $\mathsf{Q}_{2n} - \mathsf{Q}^*_{2n+1} = v_{2n}\,[1 - d(2n) + 2d(2n)\mathsf{i} + d(2n)(1 + d(2n+2))\mathsf{j}]$,

(vii) $\mathsf{Q}_{2n} + \mathsf{Q}^*_{2n-1} = v_{2n}\,[2 + (d(2n) - 1)\mathsf{i} + d(2n)(1 + d(2n+2))\mathsf{k}]$,

(viii) $\mathsf{Q}_{2n} - \mathsf{Q}^*_{2n-1} = v_{2n}\,[(d(2n) + 1)\mathsf{i} + 2d(2n)\mathsf{j} + d(2n)(d(2n+2) - 1)\mathsf{k}]$,

(ix) $\mathsf{Q}_{2n+1} + \mathsf{Q}^*_{2n-1} = v_{2n}\,[1 + d(2n) + (d(2n) - 1)\mathsf{i} + d(2n)(d(2n+2) - 1)\mathsf{j} + d(2n)(d(2n+2) + 1)\mathsf{k}]$,

(x) $\mathsf{Q}_{2n+1} - \mathsf{Q}^*_{2n-1} = v_{2n}\,[d(2n) - 1 + (d(2n) + 1)\mathsf{i} + d(2n)(d(2n+2) + 1)\mathsf{j} + d(2n)(d(2n+2) - 1)\mathsf{k}]$.

Proof. (vi) It is easy to see:

$$\begin{aligned}\mathsf{Q}_{2n} - \mathsf{Q}^*_{2n+1} &= v_{2n}\,(1 + d(2n)\mathsf{i} + d(2n)\mathsf{j} + d(2n)d(2n+2)\mathsf{k}) - v_{2n}d(2n)\,(1 - \mathsf{i} - d(2n+2)\mathsf{j} + d(2n+2)\mathsf{k}) \\ &= v_{2n}\,[1 - d(2n) + 2d(2n)\mathsf{i} + d(2n)(1 + d(2n+2))\mathsf{j}].\end{aligned}$$

Similarly, the proofs of the other parts are simple calculations so we can omit it. □

2.1. The recurrence relations

In the next four theorems, two term recurrence relations are introduced.

Theorem 2.9. *Let $\{\mathsf{Q}_s\}_{s\geq 0}$ be the Vietoris' generalized quaternionic sequence and $\mathsf{Y}(2n+2)\overline{\mathsf{Y}}(2n+2) \neq 0$. Then,*

$$\mathsf{Q}_{2n+2} = \mathsf{Q}_{2n+1}\Phi_R(2n+2) = \Phi_L(2n+2)\mathsf{Q}_{2n+1}, \tag{20}$$

where

$$\begin{cases} \Phi_R(2n+2) = \dfrac{\Phi_0 + \Phi_1\mathsf{i} + \alpha\Phi_2\mathsf{j} + \Phi_3\mathsf{k}}{(1+\alpha)(1+\beta d^2(2n+2))}, \\ \Phi_L(2n+2) = \dfrac{\Phi_0 + \Phi_4\mathsf{i} - \alpha\Phi_2\mathsf{j} + \Phi_2\mathsf{k}}{(1+\alpha)(1+\beta d^2(2n+2))}, \end{cases}$$

with

$$\begin{cases} \Phi_0 = & 1 + \alpha d(2n+2) + \beta d^2(2n+2)(1 + \alpha d(2n+4)), \\ \Phi_1 = & -1 + d(2n+2) + \beta d^2(2n+2)(1 - d(2n+4)), \\ \Phi_2 = & d(2n+2)(d(2n+4) - d(2n+2)), \\ \Phi_3 = & d(2n+2)(-2 + d(2n+2) + d(2n+4)), \\ \Phi_4 = & -1 + d(2n+2) + \beta d^2(2n+2)(d(2n+4) - 1). \end{cases}$$

Proof. We sketch the steps and leave the details to the reader. From equations (14) and (16), we get:

$$\mathsf{Q}_{2n+2} = \quad v_{2n+2}\mathsf{X}(2n+2) = v_{2n+2}\mathsf{Y}(2n+2)\tfrac{\overline{\mathsf{Y}}(2n+2)}{\mathsf{Y}(2n+2)\overline{\mathsf{Y}}(2n+2)}\mathsf{X}(2n+2) = \mathsf{Q}_{2n+1}\Phi_R(2n+2),$$

where $\Phi_R(2n+2) = \dfrac{\overline{\mathsf{Y}}(2n+2)\mathsf{X}_{2n+2}}{\mathsf{Y}(2n+2)\overline{\mathsf{Y}}(2n+2)} = \dfrac{\Phi_0 + \Phi_1\mathsf{i} + \alpha\Phi_2\mathsf{j} + \Phi_3\mathsf{k}}{(1+\alpha)(1+\beta d^2(2n+2))}\cdot$ A similar proof can be used to verify the other case. □

Theorem 2.10. *Let* $\{\mathsf{Q}_s\}_{s\geq 0}$ *be the Vietoris' generalized quaternionic sequence and* $\mathsf{X}(2n)\overline{\mathsf{X}}(2n) \neq 0$. *Then,*

$$\mathsf{Q}_{2n+1} \quad = \mathsf{Q}_{2n}\Psi_R(2n) = \Psi_L(2n)\mathsf{Q}_{2n}, \tag{21}$$

where

$$\begin{cases} \Psi_R(2n) & = d(2n)\dfrac{\Psi_0 + \Psi_1\mathsf{i} + \Psi_2\mathsf{j} + \Psi_3\mathsf{k}}{1 + d^2(2n)(\alpha + \beta + \alpha\beta d^2(2n+2))}, \\ \Psi_L(2n) & = d(2n)\dfrac{\Psi_0 + \Psi_4\mathsf{i} + \Psi_2\mathsf{j} + \Psi_2\mathsf{k}}{1 + d^2(2n)(\alpha + \beta + \alpha\beta d^2(2n+2))}, \end{cases}$$

with

$$\begin{cases} \Psi_0 = & 1 + \alpha d(2n) + \beta d(2n)d(2n+2)(1 + \alpha d(2n+2)), \\ \Psi_1 = & 1 - d(2n) + \beta d(2n)d(2n+2)(-1 + d(2n+2)), \\ \Psi_2 = & d(2n+2) - d(2n), \\ \Psi_3 = & d(2n) + d(2n+2) - 2d(2n)d(2n+2), \\ \Psi_4 = & 1 - d(2n) + \beta d(2n)d(2n+2)(1 - d(2n+2)). \end{cases}$$

Proof. Considering equations (14) and (16), we obtain

$$\mathsf{Q}_{2n+1} = \quad v_{2n+2}\mathsf{Y}(2n+2) = d(2n)\mathsf{Y}(2n+2)\tfrac{\overline{\mathsf{X}}(2n)}{\mathsf{X}(2n)\overline{\mathsf{X}}(2n)}\mathsf{X}(2n)v_{2n} = \Psi_L(2n)\mathsf{Q}_{2n},$$

where $\Psi_L(2n) = d(2n)\dfrac{\Psi_0 + \Psi_4\mathsf{i} + \Psi_2\mathsf{j} + \Psi_2\mathsf{k}}{1 + d^2(2n)(\alpha + \beta + \alpha\beta d^2(2n+2))}\cdot$ The other case can be proved by using similar arguments. □

With the help of equations (14) and (16), the following two theorems can be proved similarly.

Theorem 2.11. *Let* $\{\mathsf{Q}_s\}_{s\geq 0}$ *be the Vietoris' generalized quaternionic sequence and* $\mathsf{X}(2n)\overline{\mathsf{X}}(2n) \neq 0$. *Then,*

$$\mathsf{Q}_{2n+2} \quad = \mathsf{Q}_{2n}\Theta_R(2n) = \Theta_L(2n)\mathsf{Q}_{2n}, \tag{22}$$

where

$$\begin{cases} \Theta_R(2n) & = d(2n)\dfrac{\Theta_0 + \Theta_1\mathsf{i} + \Theta_2\mathsf{j} + \Theta_3\mathsf{k}}{1 + d^2(2n)(\alpha + \beta + \alpha\beta d^2(2n+2))}, \\ \Theta_L(2n) & = d(2n)\dfrac{\Theta_0 + \Theta_4\mathsf{i} + \Theta_5\mathsf{j} + \Theta_3\mathsf{k}}{1 + d^2(2n)(\alpha + \beta + \alpha\beta d^2(2n+2))}, \end{cases}$$

with

$$\begin{cases} \Theta_0 = & 1 + d(2n)d(2n+2)(\alpha + \beta + \alpha\beta d(2n+2)d(2n+4)), \\ \Theta_1 = & -d(2n) + d(2n+2) + \beta d(2n)d(2n+2)(d(2n+2) - d(2n+4)), \\ \Theta_2 = & -d(2n) + d(2n+2) + \alpha d(2n)d(2n+2)(d(2n+4) - d(2n+2)), \\ \Theta_3 = & -d(2n+2)(d(2n) - d(2n+4)), \\ \Theta_4 = & -d(2n) + d(2n+2) - \beta d(2n)d(2n+2)(d(2n+2) - d(2n+4)), \\ \Theta_5 = & -d(2n) + d(2n+2) - \alpha d(2n)d(2n+2)(d(2n+4) - d(2n+2)). \end{cases}$$

Theorem 2.12. *Let* $\{\mathsf{Q}_s\}_{s\geq 0}$ *be the Vietoris' generalized quaternionic sequence and* $\mathsf{Y}(2n)\overline{\mathsf{Y}}(2n) \neq 0$. *Then,*

$$\mathsf{Q}_{2n+1} = \mathsf{Q}_{2n-1}\Omega_R(2n) = \Omega_L(2n)\mathsf{Q}_{2n-1}, \tag{23}$$

where

$$\begin{cases} \Omega_R(2n) & = d(2n)\dfrac{\Omega_0 + \Omega_1 \mathsf{j}}{(1+\alpha)(1+\beta d^2(2n))}, \\ \Omega_L(2n) & = d(2n)\dfrac{\Omega_0 + \Omega_2 \mathsf{j} + \Omega_3 \mathsf{k}}{(1+\alpha)(1+\beta d^2(2n))}, \end{cases}$$

with

$$\begin{cases} \Omega_0 = & (1+\alpha)(1+\beta d(2n)d(2n+2)), \\ \Omega_1 = & (1+\alpha)(-d(2n) + d(2n+2)), \\ \Omega_2 = & (1-\alpha)(-d(2n) + d(2n+2)), \\ \Omega_3 = & -2(d(2n) - d(2n+2)). \end{cases}$$

In the sequel, we always assume Φ_R, Ψ_R, Θ_R and Ω_R are the functions in Theorems 2.9–2.12. These functions are the key concepts of this paper.

Theorem 2.13. *Let* $\{\mathsf{Q}_s\}_{s\geq 0}$ *be the Vietoris' generalized quaternionic sequence. Then, a three consecutive term recurrence relation is*

$$\mathsf{Q}_{s+1} = \mathsf{Q}_s\Lambda_1(s) + \mathsf{Q}_{s-1}\Lambda_0(s-1),$$

where

$$\Lambda_1(s) = \begin{cases} \frac{1}{2}\Psi_R(s), & s = 2n \\ \frac{1}{2}\Phi_R(s+1), & s = 2n+1 \end{cases} \quad \text{and} \quad \Lambda_0(s-1) = \begin{cases} \frac{1}{2}\Omega_R(s), & s = 2n \\ \frac{1}{2}\Theta_R(s-1) & s = 2n+1. \end{cases}$$

Proof. The proof is a simple calculation by using Theorems 2.9–2.12 and a relation $\mathsf{Q}_{s+1} = \frac{1}{2}\mathsf{Q}_{s+1} + \frac{1}{2}\mathsf{Q}_{s+1}$. So we can omit it. □

Theorem 2.14. *Let* $\{\mathsf{Q}_s\}_{s\geq 0}$ *be the Vietoris' generalized quaternionic sequence. Then, a three consecutive term with even and odd indexes recurrence relation is*

$$\mathsf{Q}_{s+2} = \mathsf{Q}_s\Gamma_1(s) + \mathsf{Q}_{s-2}\Gamma_0(s-2),$$

where

$$\Gamma_1(s) = \begin{cases} \frac{1}{2}\Theta_R(s), & s = 2n \\ \frac{1}{2}\Omega_R(s+1), & s = 2n+1 \end{cases} \quad \text{and} \quad \Gamma_0(s-2) = \begin{cases} \Theta_R(s-2)\Gamma_1(s), & s = 2n \\ \Omega_R(s-1)\Gamma_1(s) & s = 2n+1. \end{cases}$$

Proof. Let $s = 2n$, then we have $\mathsf{Q}_{2n+2} = v_{2n+2}\mathsf{X}(2n+2)$. Using equation (5), we have

$$\begin{aligned}
\mathsf{Q}_{2n+2} &= \left(\tfrac{1}{2}d(2n)v_{2n} + \tfrac{1}{2}d(2n)d(2n-2)v_{2n-2}\right)\mathsf{X}(2n+2)\\
&= \tfrac{1}{2}v_{2n}\mathsf{X}(2n)d(2n)\tfrac{\overline{\mathsf{X}}(2n)}{\mathsf{X}(2n)\overline{\mathsf{X}}(2n)}\mathsf{X}(2n+2)\\
&\quad+\tfrac{1}{2}v_{2n-2}\mathsf{X}(2n-2)d(2n)d(2n-2)\tfrac{\overline{\mathsf{X}}(2n-2)}{\mathsf{X}(2n-2)\overline{\mathsf{X}}(2n-2)}\mathsf{X}(2n+2)\\
&= \tfrac{1}{2}\mathsf{Q}_{2n}d(2n)\tfrac{\overline{\mathsf{X}}(2n)}{\mathsf{X}(2n)\overline{\mathsf{X}}(2n)}\mathsf{X}(2n+2)\\
&\quad+\tfrac{1}{2}\mathsf{Q}_{2n-2}\left[d(2n-2)\tfrac{\overline{\mathsf{X}}(2n-2)}{\mathsf{X}(2n-2)\overline{\mathsf{X}}(2n-2)}\mathsf{X}(2n)\right]\left[d(2n)\tfrac{\overline{\mathsf{X}}(2n)}{\mathsf{X}(2n)\overline{\mathsf{X}}(2n)}\mathsf{X}(2n+2)\right]\\
&= \tfrac{1}{2}\mathsf{Q}_{2n}\Theta_R(2n) + \tfrac{1}{2}\mathsf{Q}_{2n-2}\Theta_R(2n-2)\Theta_R(2n)\\
&= \mathsf{Q}_{2n}\Gamma_1(2n) + \mathsf{Q}_{2n-2}\Gamma_0(2n-2).
\end{aligned}$$

Let $s = 2n+1$, then we have $\mathsf{Q}_{2n+3} = v_{2n+4}\mathsf{Y}(2n+4)$. From equation (5), we get

$$\begin{aligned}
\mathsf{Q}_{2n+3} &= \left(\tfrac{1}{2}d(2n+2)\right)v_{2n+2} + \tfrac{1}{2}d(2n+2)d(2n)v_{2n})\mathsf{Y}(2n+4)\\
&= \tfrac{1}{2}v_{2n+2}\mathsf{Y}(2n+2)d(2n+2)\tfrac{\overline{\mathsf{Y}}(2n+2)}{\mathsf{Y}(2n+2)\overline{\mathsf{Y}}(2n+2)}\mathsf{Y}(2n+4)\\
&\quad+\tfrac{1}{2}v_{2n}\mathsf{Y}(2n)d(2n+2)d(2n)\tfrac{\overline{\mathsf{Y}}(2n)}{\mathsf{Y}(2n)\overline{\mathsf{Y}}(2n)}\mathsf{Y}(2n+4)\\
&= \tfrac{1}{2}\mathsf{Q}_{2n+1}d(2n+2)\tfrac{\overline{\mathsf{Y}}(2n+2)}{\mathsf{Y}(2n+2)\overline{\mathsf{Y}}(2n+2)}\mathsf{Y}(2n+4)\\
&\quad+\tfrac{1}{2}\mathsf{Q}_{2n-1}\left[d(2n)\tfrac{\overline{\mathsf{Y}}(2n)}{\mathsf{Y}(2n)\overline{\mathsf{Y}}(2n)}\mathsf{Y}(2n+2)\right]\left[d(2n+2)\tfrac{\overline{\mathsf{Y}}(2n+2)}{\mathsf{Y}(2n+2)\overline{\mathsf{Y}}(2n+2)}\mathsf{Y}(2n+4)\right]\\
&= \tfrac{1}{2}\mathsf{Q}_{2n+1}\Omega_R(2n+2) + \tfrac{1}{2}\mathsf{Q}_{2n-1}\Omega_R(2n)\Omega_R(2n+2)\\
&= \mathsf{Q}_{2n+1}\Gamma_1(2n+1) + \mathsf{Q}_{2n-1}\Gamma_0(2n-1).
\end{aligned}$$

□

Theorem 2.15. *Let $\{\mathsf{Q}_s\}_{s\geq 0}$ be the Vietoris' generalized quaternionic sequence. The following order-2 relations hold:*

(i) $\mathsf{Q}_{2n+1}\mathsf{Q}_{2n+2} = \mathsf{Q}_{2n+1}^2\Phi_R(2n+2)$,

(ii) $\mathsf{Q}_{2n}\mathsf{Q}_{2n+1} = \mathsf{Q}_{2n}^2\Psi_R(2n)$,

(iii) $\mathsf{Q}_{2n}\mathsf{Q}_{2n+2} = \mathsf{Q}_{2n}^2\Theta_R(2n)$,

(iv) $\mathsf{Q}_{2n-1}\mathsf{Q}_{2n+1} = \mathsf{Q}_{2n-1}^2\Omega_R(2n)$,

(v) $\mathsf{Q}_{2n+2}\mathsf{Q}_{2n+1} = \Phi_L(2n+2)\mathsf{Q}_{2n+1}^2$,

(vi) $\mathsf{Q}_{2n+1}\mathsf{Q}_{2n} = \Psi_L(2n)\mathsf{Q}_{2n}^2$,

(vii) $\mathsf{Q}_{2n+2}\mathsf{Q}_{2n} = \Theta_L(2n)\mathsf{Q}_{2n}^2$,

(viii) $\mathsf{Q}_{2n+1}\mathsf{Q}_{2n-1} = \Omega_L(2n)\mathsf{Q}_{2n-1}^2$.

Proof. Theorems 2.9–2.12 allow us to prove these multiplicative relations easily. Applying the recurrence relations again and again, equivalent relations can also be calculated. For instance, for item (i) we have:

$$\mathsf{Q}_{2n+1}\mathsf{Q}_{2n+2} = \Psi_L(2n)\mathsf{Q}_{2n}\mathsf{Q}_{2n+2} = \Psi_L(2n)\mathsf{Q}_{2n}^2\Theta_R(2n) = \mathsf{Q}_{2n+1}\mathsf{Q}_{2n}\Theta_R(2n).$$

□

Example 2.16. *From Theorems 2.9–2.12, we get the following:*

- $\mathsf{Q}_{2n+1}\mathsf{Q}_{2m} - \mathsf{Q}_{2n}\mathsf{Q}_{2m+1} = \Psi_L(2n)\mathsf{Q}_{2n}\mathsf{Q}_{2m} - \mathsf{Q}_{2n}\mathsf{Q}_{2m}\Psi_R(2m)$,
- $\mathsf{Q}_{2n+1}\mathsf{Q}_{2m+1} - \mathsf{Q}_{2n+2}\mathsf{Q}_{2m} = \Psi_L(2n)\mathsf{Q}_{2n}\mathsf{Q}_{2m}\Psi_R(2m) - \Theta_L(2n)\mathsf{Q}_{2n}\mathsf{Q}_{2m}$.

Theorem 2.17. *The Catalan-like identity for the Vietoris' generalized quaternionic sequence $\{\mathsf{Q}_s\}_{s\geq 0}$ is:*

$$\mathsf{Q}_s^2 - \mathsf{Q}_{s-p}\mathsf{Q}_{s+p} = \mathsf{Q}_p^2\mathsf{T}(s,p), \quad s > p,$$

where $\mathsf{T}(s,p) = \Upsilon(s,p)\kappa(s,p)$ *having*

$$\Upsilon(s,p) = \begin{cases} \left(\frac{\overline{\mathsf{X}}(p)}{\mathsf{X}(p)\overline{\mathsf{X}}(p)} \prod_{l=1}^{\left\lfloor \frac{s+1-p}{2} \right\rfloor} d\left(2\left\lfloor \frac{s+1}{2} \right\rfloor - 2l\right)\right)^2, & p \text{ even} \\ \left(\frac{\overline{\mathsf{Y}}(p+1)}{\mathsf{Y}(p+1)\overline{\mathsf{Y}}(p+1)} \prod_{l=1}^{\left\lfloor \frac{s-1-p}{2} \right\rfloor} d\left(2\left\lfloor \frac{s}{2} \right\rfloor - 2l\right)\right)^2, & p \text{ odd} \end{cases}$$

where $\mathsf{X}(p)\overline{\mathsf{X}}(p) \neq 0$ *and* $\mathsf{Y}(p+1)\overline{\mathsf{Y}}(p+1) \neq 0$*, and*

$$\kappa(s,p) = \begin{cases} \mathsf{X}^2(s) - \tau(s,p)\mathsf{X}(s-p)\mathsf{X}(s+p), & s = 2n, \quad p = 2k \\ \mathsf{X}^2(s) - \tau(s,p)d(s)\mathsf{Y}(s+1-p)\mathsf{Y}(s+1+p), & s = 2n, \quad p = 2k+1 \\ \mathsf{Y}^2(s+1) - \tau(s,p)\mathsf{Y}(s+1-p)\mathsf{Y}(s+1+p), & s = 2n+1, \quad p = 2k \\ d^2(s-1)\mathsf{Y}^2(s+1) - \tau(s,p)d(s-1)\mathsf{X}(s-p)\mathsf{X}(s+p), & s = 2n+1, \quad p = 2k+1 \end{cases}$$

with

$$\tau(s,p) = \begin{cases} \prod_{l=1}^{\left\lfloor \frac{p}{2} \right\rfloor} \frac{d(s+p-2l)}{d(s-p+2l-2)}, & s+p \text{ even} \\ \prod_{l=1}^{\left\lfloor \frac{p}{2} \right\rfloor} \frac{d(s+1+p-2l)}{d(s-1-p+2l)}, & s+p \text{ odd.} \end{cases}$$

Proof. The proof is divided into four parts according to the values s and p.

- Let $s = 2n$ and $p = 2k$. From equations (2) and (14), we have:

$$\mathsf{Q}_{2n+2k} = v_{2n+2k}\mathsf{X}(2n+2k) = \prod_{l=1}^{k} d(2n+2k-2l)v_{2n}\mathsf{X}(2n+2k), \quad n > k, \tag{24}$$

  and

$$\mathsf{Q}_{2n-2k} = v_{2n-2k}\mathsf{X}(2n-2k) = \prod_{l=1}^{k} \frac{1}{d(2n-2k+2l-2)} v_{2n}\mathsf{X}(2n-2k), \quad n > k. \tag{25}$$

  By utilizing equations (24) and (25), we get:

$$\begin{aligned} \mathsf{Q}_{2n}^2 - \mathsf{Q}_{2n-2k}\mathsf{Q}_{2n+2k} &= v_{2n}^2\mathsf{X}^2(2n) - v_{2n-2k}v_{2n+2k}\mathsf{X}(2n-2k)\mathsf{X}(2n+2k) \\ &= v_{2n}^2\left(\mathsf{X}^2(2n) - \prod_{l=1}^{k}\frac{d(2n+2k-2l)}{d(2n-2k+2l-2)}\mathsf{X}(2n-2k)\mathsf{X}(2n+2k)\right) \\ &= v_{2k}^2 \prod_{l=1}^{n-k} d^2(2n-2l)\left(\mathsf{X}^2(2n) - \tau(2n,2k)\mathsf{X}(2n-2k)\mathsf{X}(2n+2k)\right) \\ &= v_{2k}^2\mathsf{X}^2(2k)\Upsilon(2n,2k)\kappa(2n,2k) \\ &= \mathsf{Q}_{2k}^2\mathsf{T}(2n,2k). \end{aligned}$$

- Let $s = 2n$ and $p = 2k+1$. From equations (2) and (16), we have:

$$\mathsf{Q}_{2n+2k+1} = v_{2n+2k+2}\mathsf{Y}(2n+2k+2) = \prod_{l=1}^{k} d(2n+2k+2-2l)v_{2n+2}\mathsf{Y}(2n+2k+2), \quad n > k, \tag{26}$$

  and

$$\mathsf{Q}_{2n-2k-1} = v_{2n-2k}\mathsf{Y}(2n-2k) = \prod_{l=1}^{k} \frac{1}{d(2n-2k+2l-2)} v_{2n}\mathsf{Y}(2n-2k), \quad n > k. \tag{27}$$

By substituting equations (26) and (27), we get:

$$\begin{aligned}
\mathsf{Q}_{2n}^2-\mathsf{Q}_{2n-2k-1}\mathsf{Q}_{2n+2k+1} &= v_{2n}^2\mathsf{X}^2(2n)-v_{2n-2k}v_{2n+2k+2}\mathsf{Y}(2n-2k)\mathsf{Y}(2n+2k+2)\\
&= v_{2n}^2\mathsf{X}^2(2n)-v_{2n+2}^2\prod_{l=1}^{k}\tfrac{d(2n+2k+2-2l)}{d(2n-2k+2l-2)d(2n)}\mathsf{Y}(2n-2k)\mathsf{Y}(2n+2k+2)\\
&= v_{2k+2}^2\tfrac{1}{d^2(2n)}\prod_{l=1}^{n-k}d^2(2n+2-2l)\\
&\quad\left(\mathsf{X}^2(2n)-\tau(2n,2k+1)d(2n)\mathsf{Y}(2n-2k)\mathsf{Y}(2n+2k+2)\right)\\
&= v_{2k+2}^2\mathsf{Y}^2(2k+2)\Upsilon(2n,2k+1)\kappa(2n,2k+1)\\
&= \mathsf{Q}_{2k+1}^2\mathsf{T}(2n,2k+1).
\end{aligned}$$

- Let $s=2n+1$ and $p=2k$. From equations (2) and (16), we have:

$$\mathsf{Q}_{2n-2k+1}= v_{2n-2k+2}\mathsf{Y}(2n-2k+2)=\prod_{l=1}^{k}\tfrac{1}{d(2n-2k+2l)}v_{2n+2}\mathsf{Y}(2n-2k+2),\quad n>k. \tag{28}$$

Considering equations (26) and (28), we obtain:

$$\begin{aligned}
\mathsf{Q}_{2n+1}^2-\mathsf{Q}_{2n-2k+1}\mathsf{Q}_{2n+2k+1} &= v_{2n+2}^2\mathsf{Y}^2(2n+2)-v_{2n-2k+2}v_{2n+2k+2}\mathsf{Y}(2n-2k+2)\mathsf{Y}(2n+2k+2)\\
&= v_{2n+2}^2\mathsf{Y}^2(2n+2)-v_{2n+2}^2\prod_{l=1}^{k}\tfrac{d(2n+2k+2-2l)}{d(2n-2k+2l)}\mathsf{Y}(2n-2k+2)\mathsf{Y}(2n+2k+2)\\
&= v_{2k}^2\prod_{l=1}^{n-k}d^2(2n-2l)d^2(2n)\\
&\quad\left(\mathsf{Y}^2(2n+2)-\tau(2n+1,2k)\mathsf{Y}(2n-2k+2)\mathsf{Y}(2n+2k+2)\right)\\
&= v_{2k}^2\mathsf{X}^2(2k)\Upsilon(2n+1,2k)\kappa(2n+1,2k)\\
&= \mathsf{Q}_{2k}^2\mathsf{T}(2n+1,2k).
\end{aligned}$$

- Let $s=2n+1$ and $p=2k+1$. From equations (2) and (14), we have:

$$\mathsf{Q}_{2n+2k+2}= v_{2n+2k+2}\mathsf{X}(2n+2k+2)=\prod_{l=1}^{k}d(2n+2k+2-2l)v_{2n+2}\mathsf{X}(2n-2k+2),\quad n>k. \tag{29}$$

By using equations (25) and (29), we obtain:

$$\begin{aligned}
\mathsf{Q}_{2n+1}^2-\mathsf{Q}_{2n-2k}\mathsf{Q}_{2n+2k+2} &= v_{2n+2}^2\mathsf{Y}^2(2n+2)-v_{2n-2k}v_{2n+2k+2}\mathsf{X}(2n-2k)\mathsf{X}(2n+2k+2)\\
&= v_{2n+2}^2\left(\mathsf{Y}^2(2n+2)-\prod_{l=1}^{k}\tfrac{d(2n+2k+2-2l)}{d(2n-2k+2l-2)d(2n)}\mathsf{X}(2n-2k)\mathsf{X}(2n+2k+2)\right)\\
&= v_{2k+2}^2\prod_{l=1}^{n-k}d^2(2n+2-2l)\\
&\quad\left(\mathsf{Y}^2(2n+2)-\tfrac{\tau(2n+1,2k+1)}{d(2n)}\mathsf{X}(2n-2k)\mathsf{Y}(2n+2k+2)\right)\\
&= v_{2k+2}^2\mathsf{Y}^2(2k+2)\Upsilon(2n+1,2k+1)\kappa(2n+1,2k+1)\\
&= \mathsf{Q}_{2k+1}^2\mathsf{T}(2n+1,2k+1),
\end{aligned}$$

which completes the proof. □

Theorem 2.18. *The generating function for the Vietoris' generalized quaternionic sequence* $\{\mathsf{Q}_s\}_{s\geq 0}$ *is:*

$$\mathsf{G}(x)=\frac{1}{x^3}\left(g(x)(x^3+x^2\mathsf{i}+x\mathsf{j}+\mathsf{k})-\mathsf{H}(x)\right),\quad 0<|x|<1$$

where $\mathsf{H}(x)=\frac{1}{2}\left(2x^2\mathsf{i}+(2x+x^2)\mathsf{j}+(2+x+x^2)\mathsf{k}\right)$.

Proof. Let us recall the generating function in equation (6) for the Vietoris' sequence $\{v_n\}_{n\geq 0}$. Then, suppose that $\mathsf{G}(x) = \sum_{s=0}^{\infty} \mathsf{Q}_s\, x^s$ be the generating function of generalized quaternionic sequence $\{\mathsf{Q}_s\}_{s\geq 0}$. Then, by multiplying it with x^3 gives:

$$
\begin{aligned}
x^3\mathsf{G}(x) = & \sum_{s=0}^{\infty} \mathsf{Q}_s\, x^{s+3} \\
= & \sum_{s=0}^{\infty} (v_s + v_{s+1}\mathsf{i} + v_{s+2}\mathsf{j} + v_{s+3}\mathsf{k})\; x^{s+3} \\
= & x^3 \sum_{s=0}^{\infty} v_s x^s + x^2 \sum_{s=0}^{\infty} v_{s+1}x^{s+1}\mathsf{i} + x \sum_{s=0}^{\infty} v_{s+2}x^{s+2}\mathsf{j} + \sum_{s=0}^{\infty} v_{s+3}x^{s+3}\mathsf{k} \\
= & x^3 g(x) + x^2(-v_0 + v_0 + \sum_{s=0}^{\infty} v_{s+1}x^{s+1})\mathsf{i} + x(-v_0 - v_1 x + v_0 + v_1 x + \sum_{s=0}^{\infty} v_{s+2}x^{s+2})\mathsf{j} \\
& +(-v_0 - v_1 x - v_2 x^2 + v_0 + v_1 x + v_2 x^2 + \sum_{s=0}^{\infty} v_{s+3}x^{s+3})\mathsf{k} \\
= & x^3 g(x) + x^2(-v_0 + g(x))\mathsf{i} + x(-v_0 - v_1 x + g(x))\mathsf{j} + (-v_0 - v_1 x - v_2 x^2 + g(x))\mathsf{k} \\
= & g(x)\left(x^3 + x^2\mathsf{i} + x\mathsf{j} + \mathsf{k}\right) - \tfrac{1}{2}\left(2x^2\mathsf{i} + (2x + x^2)\mathsf{j} + (2 + x + x^2)\mathsf{k}\right) \\
= & g(x)\left(x^3 + x^2\mathsf{i} + x\mathsf{j} + \mathsf{k}\right) - \mathsf{H}(x).
\end{aligned}
$$

□

Theorem 2.19. *The Binet-like formula for the Vietoris' generalized quaternionic sequence $\{\mathsf{Q}_s\}_{s\geq 0}$ is:*

$$
\mathsf{Q}_s = \rho_1(s) r_1^{2\lfloor \frac{s+1}{2} \rfloor}\left(2\left\lfloor \frac{s+1}{2}\right\rfloor\right) + \rho_2(s) r_2^{2\lfloor \frac{s+1}{2} \rfloor}\left(2\left\lfloor \frac{s+1}{2}\right\rfloor\right),
$$

where

$$
\rho_i(s) = \begin{cases} c_i(s)\mathsf{X}(s), & s = 2n \\ c_i(s+1)\mathsf{Y}(s+1), & s = 2n+1 \end{cases}
$$

with $r_i(s), c_i(s)$ are defined in the Binet-like formula for the Vietoris' sequence $\{v_n\}_{n\geq 0}$ (see in equations (8) (9)*) for $i = 1, 2$.*

Proof. Let us recall the Binet-like formula in equation (7) for the Vietoris' sequence $\{v_n\}_{n\geq 0}$. For $s = 2n$, using equation (14), we have:

$$
\begin{aligned}
\mathsf{Q}_{2n} = & \; v_{2n}\mathsf{X}(2n) \\
= & \left(c_1(2n)\, r_1^{2n}(2n) + c_2(2n)\, r_2^{2n}(2n)\right)\mathsf{X}(2n) \\
= & \; c_1(2n)\,\mathsf{X}(2n) r_1^{2n}(2n) + c_2(2n)\,\mathsf{X}(2n) r_2^{2n}(2n) \\
= & \; \rho_1(2n) r_1^{2n}(2n) + \rho_2(2n) r_2^{2n}(2n).
\end{aligned}
$$

The case $s = 2n + 1$ can also be proved in similar manner. □

3. A Determinantal Approach to $\{\mathsf{Q}_s\}_{s\geq 0}$

The studies [6, 7, 21] motivate the rest of the paper. We investigate the determinant[3)] of some special tridiagonal matrices that generate $\{\mathsf{Q}_s\}_{s\geq 0}$. In the sequel, we always assume Λ and Γ are functions in Theorems 2.13 and 2.14.

[3)] The determinant of the matrix with quaternion entries can be calculated by using the Laplace expansion starting always with all entries of the last column. For any $M = \left[m_{ij}\right]_{n\times n}$, $\det M = \sum_{i=1}^{n} c_{in} m_{in}$, with $c_{in} = (-1)^{i+n} \det Y_{in}$ where $\det Y_{in}$ is the i, n minor of M, [7].

Theorem 3.1. *Let* M_{n+1} *be a tridiagonal matrix of order* $n+1$ *with the generalized quaternionic sequence with Vietoris' number entries and given by:*

$$\mathsf{M}_{n+1}=\begin{bmatrix} \mathsf{Q}_1 & -\mathsf{Q}_0 & & & & & \\ \Lambda_0(0) & \Lambda_1(1) & -1 & & & & \\ & \Lambda_0(1) & \Lambda_1(2) & -1 & & & \\ & & \Lambda_0(2) & \Lambda_1(3) & -1 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & \Lambda_0(n-2) & \Lambda_1(n-1) & -1 \\ & & & & & \Lambda_0(n-1) & \Lambda_1(n) \end{bmatrix}.$$

Then, we have $\det(\mathsf{M}_{n+1})=\mathsf{Q}_{n+1}$.

Proof. The proof depends on an induction.

- Let $n=0$. It is clear that $\det(\mathsf{M}_1)=\mathsf{Q}_1$.
- Let $n=1$. We get $\mathsf{M}_2=\begin{bmatrix} \mathsf{Q}_1 & -\mathsf{Q}_0 \\ \Lambda_0(0) & \Lambda_1(1) \end{bmatrix}$. By using Theorem 2.13, we have:

$$\det(\mathsf{M}_2)=\mathsf{Q}_1\Lambda_1(1)+\mathsf{Q}_0\Lambda_0(0)=\mathsf{Q}_2.$$

- Let $n=2$. We obtain $\mathsf{M}_3=\begin{bmatrix} \mathsf{Q}_1 & -\mathsf{Q}_0 & 0 \\ \Lambda_0(0) & \Lambda_1(1) & -1 \\ 0 & \Lambda_0(1) & \Lambda_1(2) \end{bmatrix}$ and so

$$\det(\mathsf{M}_3)=\ (-1)^{3+3}\det(\mathsf{M}_2)\Lambda_1(2)-(-1)^{3+2}\det(\mathsf{M}_1)\Lambda_0(1)=\mathsf{Q}_2\Lambda_1(2)+\mathsf{Q}_1\Lambda_0(1)=\mathsf{Q}_3.$$

- For $n-1$, assume that $\det(\mathsf{M}_n)=\mathsf{Q}_n$.
- By applying Laplace expansion and Theorem 2.13 for n, we get:

$$\begin{aligned}\det(\mathsf{M}_{n+1}) &= (-1)^{2n+2}\det(\mathsf{M}_n)\Lambda_1(n)-(-1)^{2n+1}\det(\mathsf{M}_{n-1})\Lambda_0(n-1)\\ &= \mathsf{Q}_n\Lambda_1(n)+\mathsf{Q}_{n-1}\Lambda_0(n-1)=\mathsf{Q}_{n+1}.\end{aligned}$$

□

Theorem 3.2. *Let* $\mathcal{M}_{n+1}$ *be a tridiagonal matrix of order* $n+1$ *with the generalized quaternionic sequence with Vietoris' number entries and given by:*

$$\mathcal{M}_{n+1}=\begin{bmatrix} \mathsf{Q}_0 & & & & & & \\ -1 & 2\Lambda_1(0) & \Lambda_0(0) & & & & \\ & -1 & \Lambda_1(1) & \Lambda_0(1) & & & \\ & & -1 & \Lambda_1(2) & \Lambda_0(2) & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & -1 & \Lambda_1(n-2) & \Lambda_0(n-2) \\ & & & & & -1 & \Lambda_1(n-1) \end{bmatrix}.$$

Then, we have $\det(\mathcal{M}_{n+1})=\mathsf{Q}_n$.

Proof. The proof depends on an induction.

- Let $n=0$. It is clear that $\det(\mathcal{M}_1)=\mathsf{Q}_0$.

- Let $n = 1$. We get $\mathcal{M}_2 = \begin{bmatrix} \mathsf{Q}_0 & 0 \\ -1 & 2\Lambda_1(0) \end{bmatrix}$. By using the proof of Theorem 2.13, we find:

$$\det(\mathcal{M}_2) = 2\mathsf{Q}_0\Lambda_1(0) = \mathsf{Q}_1.$$

- Let $n = 2$. We have $\mathcal{M}_3 = \begin{bmatrix} \mathsf{Q}_0 & 0 & 0 \\ -1 & 2\Lambda_1(0) & \Lambda_0(0) \\ 0 & -1 & \Lambda_1(1) \end{bmatrix}$. Then,

$$\det(\mathcal{M}_3) = (-1)^{3+3}\det(\mathcal{M}_2)\,\Lambda_1(1) - (-1)^{3+2}\det(\mathcal{M}_1)\,\Lambda_0(0) = \mathsf{Q}_1\Lambda_1(1) + \mathsf{Q}_0\Lambda_0(0) = \mathsf{Q}_2.$$

- For $n-1$, assume that $\det(\mathcal{M}_n) = \mathsf{Q}_{n-1}$.

- From Laplace expansion and Theorem 2.13 for n, we obtain:

$$\begin{aligned} \det(\mathcal{M}_{n+1}) &= (-1)^{2n+2}\det(\mathcal{M}_n)\,\Lambda_1(n-1) - (-1)^{2n+1}\det(\mathcal{M}_{n-1})\,\Lambda_0(n-2) \\ &= \mathsf{Q}_{n-1}\Lambda_1(n-1) + \mathsf{Q}_{n-2}\Lambda_0(n-2) \\ &= \mathsf{Q}_n. \end{aligned}$$

This completes the proof. □

The following theorems can be proved in a similar manner by induction.

Theorem 3.3. *Let $\mathtt{M}_{n+1}$ be a tridiagonal matrix of order $n+1$ with the generalized quaternionic sequence with Vietoris' number entries and given by:*

$$\mathtt{M}_{n+1} = \begin{bmatrix} \mathsf{Q}_0 & & & & & & \\ -1 & 2\Gamma_1(0) & \Gamma_0(0) & & & & \\ & -1 & \Gamma_1(2) & \Gamma_0(2) & & & \\ & & -1 & \Gamma_1(4) & \Gamma_0(4) & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & -1 & \Gamma_1(2n-4) & \Gamma_0(2n-4) \\ & & & & & -1 & \Gamma_1(2n-2) \end{bmatrix}.$$

Then, we have $\det(\mathtt{M}_{n+1}) = \mathsf{Q}_{2n}$.

Theorem 3.4. *Let $\mathbf{M}_{n+1}$ be a tridiagonal matrix of order $n+1$ with the generalized quaternionic sequence with Vietoris' number entries and given by:*

$$\mathbf{M}_{n+1} = \begin{bmatrix} \mathsf{Q}_1 & & & & & & \\ -1 & 2\Gamma_1(1) & \Gamma_0(1) & & & & \\ & -1 & \Gamma_1(3) & \Gamma_0(3) & & & \\ & & -1 & \Gamma_1(5) & \Gamma_0(5) & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & -1 & \Gamma_1(2n-3) & \Gamma_0(2n-3) \\ & & & & & -1 & \Gamma_1(2n-1) \end{bmatrix}.$$

Then, we have $\det(\mathbf{M}_{n+1}) = \mathsf{Q}_{2n+1}$.

Theorem 3.5. *Let M_{n+1} be a tridiagonal matrix of order $n+1$ with the generalized quaternionic sequence with Vietoris' number entries and given by:*

$$M_{n+1} = \begin{bmatrix} \mathsf{Q}_2 & -\mathsf{Q}_0 & & & & & \\ \Gamma_0(0) & \Gamma_1(2) & -1 & & & & \\ & \Gamma_0(2) & \Gamma_1(4) & -1 & & & \\ & & \Gamma_0(4) & \Gamma_1(6) & -1 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & \Gamma_0(2n-4) & \Gamma_1(2n-2) & -1 \\ & & & & & \Gamma_0(2n-2) & \Gamma_1(2n) \end{bmatrix}.$$

Then, we have $\det(M_{n+1}) = \mathsf{Q}_{2n+2}$.

Theorem 3.6. *Let M_{n+1} be a tridiagonal matrix of order $n+1$ with the generalized quaternionic sequence with Vietoris' number entries and given by:*

$$M_{n+1} = \begin{bmatrix} \mathsf{Q}_3 & -\mathsf{Q}_1 & & & & & \\ \Gamma_0(1) & \Gamma_1(3) & -1 & & & & \\ & \Gamma_0(3) & \Gamma_1(5) & -1 & & & \\ & & \Gamma_0(5) & \Gamma_1(7) & -1 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & \Gamma_0(2n-3) & \Gamma_1(2n-1) & -1 \\ & & & & & \Gamma_0(2n-1) & \Gamma_1(2n+1) \end{bmatrix}.$$

Then, we have $\det(M_{n+1}) = \mathsf{Q}_{2n+3}$.

4. Conclusion and Vision of the Future Work

Based on the ideas given by Catarino and Almeida [7], and Pottman and Wallner [23], we investigate and discuss in detail the generalized quaternionic sequence with Vietoris' number components. In the framework of generalized quaternion structures, we have

- the real quaternionic sequence with Vietoris' for $\alpha = \beta = 1$, (see [7]),
- the split quaternionic sequence with Vietoris' for $\alpha = 1, \beta = -1$,
- the semi quaternionic sequence with Vietoris' for $\alpha = 1, \beta = 0$,
- the split-semi quaternionic sequence with Vietoris' for $\alpha = -1, \beta = 0$,
- the quasi quaternionic sequence with Vietoris' for $\alpha = \beta = 0$.

The Vietoris' number sequence has some relations with the Catalan number sequence, (see details in [8]). The Catalan number sequence is a very popular integer sequence and arising in many combinatorial problems closely related to different scientific areas and has many applications ranging from computer science to computational biology and mathematical physics. Now, the relation between Vietoris' generalized quaternionic sequence and the Catalan generalized quaternionic sequence is now an open problem for researchers.

In concluding the paper, we also want to draw the reader's attention toward the quaternion-valued functions. They have applications in many areas and have been gaining more attentions recently. One can examine whether short time SAFT as a part of quaternion-valued signals applies to Vietoris' generalized quaternionic sequence. Finding methods if any exists to define SAFT for Vietoris' generalized quaternionic sequence is the main question to answer that reqires a close attention (see [19, 31] for SAFT).

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