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Cesàro convergence of sequences of bi-complex numbers using BC-Orlicz function

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Abstract. In this article we have introduced the concept of Cesàro convergence, Cesàro null and Cesàro bounded sequences of bi-complex numbers defined by BC-Orlicz function having hyperbolic norm. we have investigated some of their algebraic and topological properties by defining a D-norm on these spaces. Also inclusion results involving these sequence spaces have been established.

1. Introduction

Bi-complex numbers are being studied for quite a long time now. Probably Italian school of Segre [12] introduced the bi-complex numbers. For more details on bi-complex numbers and bi-complex functional analysis see ([14], [16], [11]). The hyperbolic numbers studied by Cockle [2], Lie and Scheffers [7]. Hyperbolic number system has been studied for various reasons. Many research developed the hyperbolic numbers.

The sequence space has been investigated by different researchers from different aspects, such as Buck [1], Fast[5], Schoenberg [13], Fridy [6], Rath and Tripathy [10], Tripathy and Nath[15].
A real sequence $x = (x_k)$ is said to be Cesàro convergent to l if

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} x_k = l.$$

Definition 1.1. *An Orlicz function is a function $\mathcal{M} : [0,\infty) \to [0,\infty)$, which is continuous, non-decreasing and convex with $\mathcal{M}(0) = 0$, $\mathcal{M}(x) > 0$, for $x > 0$ and $\mathcal{M}(x) \to \infty$, as $x \to \infty$.*
Lindendstrauss and Tzafriri [8] used the idea of Orlicz function to construct the sequence space

$$\ell_M := \left\{ x \in \omega : \sum_{k=1}^{\infty} \mathcal{M}\left(\frac{|x_k|}{\rho}\right) < \infty, \textit{ for some } \rho > 0 \right\}.$$

The sequence space ℓ_M is Banach space with the norm

$$\|x\| := \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} \mathcal{M}\left(\frac{|x_k|}{\rho}\right) < 1 \right\}.$$

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The concept of Orlicz function has been applied for studying different classes of sequences by Datta and Tripathy[3], Nath and Tripathy[9] and many more. In this article we developed the Cesàro convergence using BC-Orlicz function. Throughout the article we denote C_0, C_1 and C_2 by set of real, complex and bi-complex numbers respectively also we denote by w^*, the sequences of all bi-complex numbers.

2. Definition and Preliminaries

2.1. Bi-complex Numbers

A bi-complex number ξ is of the form

$$\xi = z_1 + i_2 z_2 = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4,$$

where $z_1, z_2 \in C_1$ and $x_1, x_2, x_3, x_4 \in C_0$ and the independent units i_1, i_2 are such that $i_1^2 = i_2^2 = -1$ and $i_1 i_2 = i_2 i_1$, The set of bi-complex numbers C_2 is defined as:

$$C_2 = \{\xi : \xi = z_1 + i_2 z_2; z_1, z_2 \in C_1(i_1)\},$$

where $C_1(i_1) = \{x_1 + i_1 x_2 : x_1, x_2 \in C_0\}$. C_2 is a vector space over $C_1(i_1)$. Other than 0 and 1, there are two more idempotent elements in C_2 given by $e_1 = \frac{1+i_1 i_2}{2}$ and $e_2 = \frac{1-i_1 i_2}{2}$ such that $e_1 + e_2 = 1$ and $e_1 e_2 = 0$.
Every bi-complex number $\xi = z_1 + i_2 z_2$ can be uniquely expressed as the combination of e_1 and e_2, namely

$$\xi = z_1 + i_2 z_2 = (z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2 = \mu_1 e_1 + \mu_2 e_2,$$

where $\mu_1 = (z_1 - i_1 z_2)$ and $\mu_2 = (z_1 + i_1 z_2)$.
For $\xi = z_1 + i_2 z_2 \in C_2$, the norm is defined as

$$||\xi||_{C_2} = \sqrt{|z_1|^2 + |z_2|^2}.$$

The product of two bi-complex numbers is connected by the following inequality:

$$||\xi \cdot \eta||_{C_2} \leq \sqrt{2}||\xi||_{C_2}.||\eta||_{C_2}.$$

C_2 together with the norm defined above form a generalized algebra. Since $C_2 \simeq C_0^4$ and C_0^4 is complete with respect to usual metric, it follows that C_2 forms a generalized Banach algebra.
The bi-complex number $\xi = z_1 + i_2 z_2$ is called singular if $|z_1^2 + z_2^2| = 0$.
The set of all singular numbers is denoted by $\mathbb{O}_2$.

2.2. Hyperbolic Numbers

The hyperbolic number is of the form

$$\alpha = x_1 + i_1 i_2 x_2; x_1, x_2 \in C_0.$$

The idempotent representation of any hyperbolic number $\alpha = x_1 + i_1 i_2 x_2$ is

$$\alpha = v_1 e_1 + v_2 e_2,$$

where $v_1 = x_1 + x_2, v_2 = x_2 - x_1$.
The set of hyperbolic numbers is given by

$$D = \{v_1 e_1 + v_2 e_2 : v_1, v_2 \in C_0\}.$$

The set of positive hyperbolic numbers is given by

$$D_+ = \{v_1 e_1 + v_2 e_2 : v_1, v_2 \geq 0\}.$$

Let $\xi \in C_2$, then hyperbolic norm(D- valued) norm on C_2 is given by

$$|\xi|_D = |\mu_1|e_1 + |\mu_2|e_2 \in D_+.$$

If $\xi, \eta \in C_2$, then

$$|\xi + \eta|_D \leq' |\xi_D + |\eta|_D \text{ and } |\xi\eta|_D = |\xi|_D|\eta|_D.$$

Let S be a subset of D. Consider the two sets $D_1 = \{v_1 : v_1e_1 + v_2e_2 \in S\}$ and $D_2 = \{v_2 : v_1e_1 + v_2e_2 \in S\}$. Then supremum of the set S is given by

$$\sup_D S = e_1 \sup D_1 + e_2 \sup D_2.$$

Similarly, infimum of the set S is given by

$$\inf_D S = e_1 \inf D_1 + e_2 \inf D_2.$$

The partial order relation on D is given by

$$\alpha \leq' \beta \text{ if and only if } \beta - \alpha \in D_+ \forall \alpha, \beta \in D.$$

Remark 2.1. *Denote D^*_+, by the the non negative extended hyperbolic numbers*

$$D^*_+ = \{\mu_1e_1 + \mu_2e_2, \mu_1, \mu_2 > 0\} \cup \{\infty\} \cup \{-\infty\} \cup \{\infty e_1 + \mu_2e_2\} \cup \{\mu_1e_1 - \infty e_2\}$$

Throughout the article we denote

$$0_D = 0 + 0i_1i_2.$$

Definition 2.2. *A function $\Upsilon_D : D \to D^*_+$ is called D-valued convex function if for every $\xi, \eta \in D$ with $0 \leq' \alpha \leq' 1$ such that*

$$\Upsilon_D(\alpha\xi + (1 - \alpha)\eta) \leq' \alpha\Upsilon_D(\xi) + (1 - \alpha)\Upsilon_D(\eta).$$

Definition 2.3. *[4] A convex function $\Upsilon_D : D_+ \to D^*_+$ is said to be BC-Orlicz function if it satisfies the following conditions*
(i) $\Upsilon_D(0_D) = 0_D$;
(ii) $\lim_{\xi\to\infty} \Upsilon_D(\xi) = \infty^*$, where $\infty^* = \mu_1e_1 + \infty e_2 = \infty e_1 + \mu_2e_2 = \infty e_1 + \infty e_2$ and $\lim_{\xi\to\infty} \Upsilon_D(\xi)$ must exist along any line in the hyperbolic plane and must be equal.
We denote the BC-Orlicz function by $\mathcal{M}_D$.

Definition 2.4. *An BC-Orlicz function $\mathcal{M}_D$ is said to satisfy the Δ^2_D-condition denoted by $\mathcal{M}_D \in \Delta^2_D$ if there exist some hyperbolic constants $K \geq' 0$ and ξ_0(depending upon K) such that*

$$\mathcal{M}_D((2e_1 + 2e_2)\xi) \leq' K\mathcal{M}_D(\xi), \forall\ 0 \leq' \xi \leq' \xi_0.$$

Definition 2.5. *A function $g : C_2 \to D^*_+$ is called D-norm if the following conditions are satisfied;*
$p_1 : g(\xi) \geq' 0_D$, for all $\xi \in C_2$;
$p_2 : g(-\xi) = g(\xi)$, for all $\xi \in C_2$;
$p_3 : g(\xi + \eta) \leq' g(\xi) + g(\eta)$, for all $\xi, \eta \in C_2$;
$p_4 : \alpha_k \to \alpha, |x_k - x|_D \to 0_D$, then $|\alpha_k\xi_k - \alpha\xi|_D \to 0_D$.

3. Main result

In this section we introduce the notion of different types of Cesàro convergence sequences of bi-complex numbers defined by BC-Orlicz function. We investigate their different properties and we define the following sets

$$[b_1^*, \mathcal{M}_D] := \left\{\xi \in \omega^* : \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathcal{M}_D\left(\frac{|\xi_k - \xi^*|_D}{\alpha}\right) = 0_D, \text{ for some hyperbolic number } \alpha >' 0\right\}$$

$$[b_0^*, \mathcal{M}_D] := \left\{\xi \in \omega^* : \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathcal{M}_D\left(\frac{|\xi_k|_D}{\alpha}\right) = 0_D, \text{ for some hyperbolic number } \alpha >' 0\right\}$$

$$[b_\infty^*, \mathcal{M}_D] := \left\{\xi \in \omega^* : \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathcal{M}_D\left(\frac{|\xi_k|_D}{\alpha}\right) <' \infty, \text{ for some hyperbolic number } \alpha >' 0\right\}.$$

Theorem 3.1. *The sets $[b_1^*, \mathcal{M}_D], [b_0^*, \mathcal{M}_D]$ and $[b_\infty^*, \mathcal{M}_D]$ are linear space over $C_2 \setminus \mathbb{O}_2$.*

Proof. Let $\xi, \eta \in [b_\infty^*, \mathcal{M}_D]$, then for some small hyperbolic numbers $\alpha_1, \alpha_2 >' 0$ such that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathcal{M}_D\left(\frac{|\xi_k|_D}{\alpha_1}\right) <' \infty$$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathcal{M}_D\left(\frac{|\eta_k|_D}{\alpha_2}\right) <' \infty.$$

Let $k_1, k_2 \in C_2 \setminus \mathbb{O}_2$. and $\alpha = \max\{|k_1|_D\alpha_1, |k_2|_D\alpha_2\}$.
Now

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathcal{M}_D\left(\frac{|k_1\xi_k + k_2\eta_k|_D}{\alpha}\right)\\
&\leq' \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathcal{M}_D\left(\frac{|k_1\xi_k|_D}{\alpha}\right) + \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathcal{M}_D\left(\frac{|k_2\eta_k|_D}{\alpha}\right)\\
&= \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathcal{M}_D\left(\frac{|k_1|_D|\xi_k|_D}{\alpha}\right) + \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathcal{M}_D\left(\frac{|k_2|_D|\eta_k|_D}{\alpha}\right)\\
&= \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathcal{M}_D\left(\frac{|\xi_k|_D}{\alpha_1}\right) + \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathcal{M}_D\left(\frac{|\eta_k|_D}{\alpha_2}\right) <' \infty.
\end{aligned}$$

Therefore, $[b_\infty^*, \mathcal{M}_D]$ is linear space over $C_2 \setminus \mathbb{O}_2$.

Result 3.2. *Let $\mathcal{M}_D$ be BC-Orlicz function then*

$$[b_0^*, \mathcal{M}_D] \subset [b_1^*, \mathcal{M}_D] \subset [b_\infty^*, \mathcal{M}_D].$$

Theorem 3.3. *The spaces $[b_0^*, \mathcal{M}_D]$ and $[b_\infty^*, \mathcal{M}_D]$ are solid.*

Proof. Let $\xi = (\xi_k) \in [b_\infty^*, \mathcal{M}_D]$, then

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathcal{M}_D\left(\frac{|\xi_k|_D}{\alpha}\right) <' \infty.$$

Let us consider a sequence of bi-complex scalars (ζ_k) with $|\zeta_k|_D \leq' 1$.
Now

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathcal{M}_D\left(\frac{|\zeta_k \xi_k|_D}{\alpha}\right)$$
$$= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathcal{M}_D\left(\frac{|\zeta_k|_D |\xi_k|_D}{\alpha}\right)$$
$$<' \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathcal{M}_D\left(\frac{|\xi_k|_D}{\alpha}\right) <' \infty.$$

Hence, $[b_\infty^*, \mathcal{M}_D]$ is solid.
Similarly other cases can be proved.

Result 3.4. *The spaces $[b_1^*, \mathcal{M}_D]$, $[b_0^*, \mathcal{M}_D]$ and $[b_\infty^*, \mathcal{M}_D]$ are not convergence free.*

Theorem 3.5. *Let $\mathcal{M}_D^1$ and $\mathcal{M}_D^2$ be two BC-Orlicz functions with Δ_D^2-condition, then*

$$[b_p^*, \mathcal{M}_D^1] \cup [b_p^*, \mathcal{M}_D^2] \subseteq [b_p^*, \mathcal{M}_D^1 + \mathcal{M}_D^2],$$

where $p = o, 1, \infty$.

Theorem 3.6. *Let $\mathcal{M}_D^1$ and $\mathcal{M}_D^2$-be two BC-Orlicz functions with Δ_D^2-condition, then*

$$[b_\infty^*, \mathcal{M}_D^2] \subset [b_\infty^*, \mathcal{M}_D^1 * \mathcal{M}_D^2].$$

Proof. Let $\xi \in [b_\infty^*, \mathcal{M}_D^2]$, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathcal{M}_D^2\left(\frac{|\xi_k|_D}{\alpha}\right) <' \infty.$$

Let

$$p = \mathcal{M}_D^2\left(\frac{|\xi_k|_D}{\alpha}\right).$$

Since $\mathcal{M}_D^1$ satisfies Δ_D^2-condition, so there exist $K \geq' 0$ and ξ_0(depending upon K) such that

$$\mathcal{M}_D^1(p) \leq' Kp\mathcal{M}_D^1(2e_1 + 2e_2), \forall\, 0 \leq' \xi \leq' \xi_0.$$

Now,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (\mathcal{M}_D^1 * \mathcal{M}_D^2)\left(\frac{|\xi_k|_D}{\alpha}\right)$$
$$= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathcal{M}_D^1\left(\mathcal{M}_D^2\left(\frac{|\xi_k|_D}{\alpha}\right)\right)$$
$$= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathcal{M}_D^1\,(p)$$
$$\leq' \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} KpM_D^1(2e_1 + 2e_2)$$
$$\leq' \infty.$$

Thus, $\xi \in [b_\infty^*, \mathcal{M}_D^1 * \mathcal{M}_D^2]$.
Hence, the theorem.

Theorem 3.7. *Let $\mathcal{M}_D$ be any BC-Orlicz function, the space $[b^*_\infty, \mathcal{M}^2_D]$ is a D-norm space with*

$$g(\xi) = \inf\left\{\alpha : \sum_{k=1}^{n}\left[\mathcal{M}_D\left(\frac{|\xi_k|_D}{\alpha}\right)\right] \leq' 1, \text{ for some hyperbolic number } \alpha >' 0\right\}.$$

Proof. Since $\alpha >' 0$, so $g(\xi) >' 0$ and $g(-\xi) = g(\xi), \forall \xi \in [b^*_\infty, \mathcal{M}^2_D]$.
Let $\xi, \eta \in [b^*_\infty, \mathcal{M}^2_D]$, then for some hyperbolic numbers $\alpha_1, \alpha_2 >' 0$ such that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathcal{M}_D\left(\frac{|\xi_k|_D}{\alpha_1}\right) <' \infty$$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathcal{M}_D\left(\frac{|\eta_k|_D}{\alpha_2}\right) <' \infty.$$

Let

$$S = \left\{\alpha : \sum_{k=1}^{n}\left[\mathcal{M}_D\left(\frac{|\xi_k+\eta_k|_D}{\alpha}\right)\right] \leq' 1\right\},$$
$$S_1 = \left\{\alpha_1 : \sum_{k=1}^{n}\left[\mathcal{M}_D\left(\frac{|\xi_k+\eta_k|_D}{\alpha_1}\right)\right] \leq' 1\right\},$$
$$S_2 = \left\{\alpha_2 : \sum_{k=1}^{n}\left[\mathcal{M}_D\left(\frac{|\xi_k+\eta_k|_D}{\alpha_2}\right)\right] \leq' 1\right\}.$$

Let $\alpha = (\alpha_1 + \alpha_2) \in S, \alpha_1 = v'_1 e_1 + v'_2 e_2 \in S_1, \alpha_2 = v''_1 e_1 + v''_2 e_2 \in S_2$ and $\alpha = v_1 e_1 + v_2 e_2$.
Now,

$$\begin{aligned} g(\xi+\eta) &= \inf\left\{\alpha : \sum_{k=1}^{n}\left[\mathcal{M}_D\left(\frac{|\xi_k+\eta_k|_D}{\alpha}\right)\right] \leq' 1\right\} \\ &= \inf\{v_1 : \alpha \in S\}e_1 + \inf\{v_2 : \alpha \in S\}e_2 \\ &= \inf\{v'_1 : \alpha_1 \in S_1\}e_1 + \inf\{v''_1 : \alpha_2 \in S_2\}e_1 + \inf\{v'_2 : \alpha_1 \in S_1\}e_2 + \inf\{v''_2 : \alpha_2 \in S_2\}e_2 \\ &= \inf\{v'_1 : \alpha_1 \in S_1\}e_1 + \inf\{v'_2 : \alpha_1 \in S_1\}e_2 + \inf\{v''_1 : \alpha_2 \in S_2\}e_1 + \inf\{v''_2 : \alpha_2 \in S_2\}e_2 \\ &= \inf\left\{\alpha_1 : \sum_{k=1}^{n}\left[\mathcal{M}_D\left(\frac{|\xi_k+\eta_k|_D}{\alpha_1}\right)\right] \leq' 1\right\} + \inf\left\{\alpha_2 : \sum_{k=1}^{n}\left[\mathcal{M}_D\left(\frac{|\xi_k+\eta_k|_D}{\alpha_2}\right)\right] \leq' 1\right\} \\ &= g(\xi) + g(\eta). \end{aligned}$$

Hence, the theorem.

Conclusion. In this article, we have introduced the notion of Cesàro convergence of sequences of bi-complex numbers defined by BC-Orlicz function. We have investigated its different algebraic and topological properties. There are very few articles on sequences of bi-complex numbers.

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