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Hermite-Hadamard type inequalities for exponential type multiplicatively convex functions

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Abstract. In this paper, we defined and studied the concept of exponential type multiplicatively convex functions and some of their algebraic properties. We derived Hermite-Hadamard inequalities for this class of functions. We also established new Hermite-Hadamard type inequalities for the product and quotient of exponential type multiplicatively convex functions. In addition, we obtained new multiplicative integral-based inequalities for the quotient and product of exponential type multiplicatively convex functions and convex functions. The results in this study could potentially inspire further research in various scientific fields.

1. Introduction

Multiplicative calculus, also called non-Newtonian calculus, is a novel approach to differentiation and integration that employs multiplication and division instead of addition and subtraction. The concept was initially explored by Grossman and Katz [12] in the 1970s. This new calculus deviates from the classical calculus developed by Newton and Leibniz in the 17th century. Differentiation and integration are fundamental operations that have extensive applications in calculus and analysis, allowing for infinitesimal addition and subtraction. It is only applicable to positive functions, making its use more restrictive than the calculus of Newton and Leibnitz.

Therefore, it is not as well-known as the classical calculus. Despite its limited application area, multiplicative calculus has several intriguing applications in various fields. For example, in [4], Bashirov et al. established a fundamental theorem of multiplicative calculus. In [5], Bashirov and Riza introduced complex multiplicative calculus. In [9] and [18], properties of stochastic multiplicative calculus have been studied by Daletskii and Karandikar.For more information on applications and other aspects of multiplicative calculus, see [3, 6, 15–17, 26, 27, 29, 31], and the references cited therein.

Recall that the concept of multiplicative integral is denoted by $\int_a^b (f(x))^{dx}$ while the ordinary integral is denoted by $\int_a^b (f(x))\, dx$. This is because the sum of the terms of product is used in the definition of a classical Riemann integral of f on $[a, b]$, the product of terms raised to certain powers is used in the definition of multiplicative integral of f on $[a, b]$.

There is the following relation between Riemann integral and multiplicative integral [4].

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Proposition 1.1. *If f is Riemann integrable on $[a,b]$, then f is multiplicative integrable on $[a,b]$ and*

$$\int_a^b (f(x))^{dx} = e^{\int_a^b \ln(f(x))dx}.$$

In [4], Bashirov et al. show that multiplicative integral has the following results and notations:

Proposition 1.2. *If f is positive and Riemann integrable on $[a,b]$, then f is multiplicative integrable on $[a,b]$ and*

1. $\int_a^b \left((f(x))^p\right)^{dx} = \int_a^b \left((f(x))^{dx}\right)^p$,
2. $\int_a^b (f(x)g(x))^{dx} = \int_a^b (f(x))^{dx} \cdot \int_a^b (g(x))^{dx}$,
3. $\int_a^b \left(\frac{f(x)}{g(x)}\right)^{dx} = \frac{\int_a^b (f(x))^{dx}}{\int_a^b (g(x))^{dx}}$,
4. $\int_a^b (f(x))^{dx} = \int_a^\mu (f(x))^{dx} \cdot \int_\mu^b (f(x))^{dx}$, $a \le \mu \le b$.
5. $\int_a^a (f(x))^{dx} = 1$ and $\int_a^b (f(x))^{dx} = \left(\int_b^a (f(x))^{dx}\right)^{-1}$.

1.1. Preliminaries

A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex, if

$$f(a + t(b-a)) \le f(a) + t(f(b) - f(a)), \quad \forall t \in [0,1].$$

for all $a, b \in I$ and $t \in [0,1]$.

Convexity theory is a crucial aspect of various fields, such as mathematical finance, economics, engineering, management sciences, and optimization theory. The use of convexity has led to the discovery of numerous extensions and generalizations of integral inequalities, along with their useful applications (refer to [1, 7, 11, 21, 23]). It has been shown through several studies that many of the results obtained regarding these inequalities are direct consequences of the applications of convex functions. The Hermite-Hadamard inequality is one of the most famous inequalities related to the integral mean of a convex function. This double inequality is stated as follows (see, [10, 13, 30]):

Let $f : I \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}. \tag{1}$$

Hermite-Hadamard inequality can be considered as a refinement of the concept of convexity. Recently, several generalizations and extensions have been considered for classical convexity and also, the numerous studies have focused on to obtain new bounds for left and right hand sides of the inequality (1). For some examples, please refer the monographs [8, 19, 20, 22, 24, 25, 28, 32–34].

Definition 1.3. *[30] A function $f : I \to (0,\infty)$ is said to be logarithmically or multiplicatively convex on set F, if*

$$f(a + t(b-a)) \le (f(a))^{1-t} \cdot (f(b))^t, \quad \forall t \in [0,1].$$

In [2], Ali et al. established Hermite-Hadamard inequality for multiplicatively convex functions as follows:

Theorem 1.4. *Let f be a positive and multiplicatively convex function on interval $[a,b]$. Then*

$$f\left(\frac{a+b}{2}\right) \le \left(\int_a^b (f(x))^{dx}\right)^{\frac{1}{b-a}} \le G(f(a), f(b))$$

where $G(.,.)$ is a geometric mean.

In [14], Kadakal et al. introduced the concept of exponential type convexity and established Hermite-Hadamard inequality for this class of functions:

Definition 1.5. *A nonnegative function $f : I \to \mathbb{R}$ is said to be exponential type convex, if*

$$f(tb + (1-t)a) \leq \left(e^{1-t} - 1\right) f(a) + (e^t - 1) f(b)$$

for all $a, b \in I$ and $t \in [0, 1]$.

Theorem 1.6. *Let $f : [a, b] \to \mathbb{R}$ be an exponential type convex function. If $a < b$ and $f \in L[a, b]$, then*

$$\frac{1}{2\left(\sqrt{e}-1\right)} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq (e-2)\left[f(a)+f(b)\right].$$

2. Main Results

In this section we give a new definition, which is called exponential type multiplicatively convex function and study some of its basic algebraic properties.

Definition 2.1. *A positive function $f : I \to \mathbb{R}$ is called exponential type multiplicatively convex if,*

$$f(tb + (1-t)a) \leq (f(a))^{e^{1-t}-1} . (f(b))^{e^t-1}$$

holds for all $a, b \in I$ and $t \in [0, 1]$.

Remark 2.2. *The range of the exponential type multiplicatively convex functions is* $[1, \infty)$.

Proof. Using the definition of the exponential type multiplicatively convex function for $t = 1$ and $c \in I$, we have

$$f(c) \leq (f(c))^{e-1} \Longrightarrow (f(c))^{e-2} \geq 1 \Longrightarrow f(c) \geq 1.$$

□

Lemma 2.3. *For all $t \in [0, 1]$, the following inequalities hold:*

$$e^t - 1 \geq t \;\text{ and }\; e^{1-t} - 1 \geq 1 - t.$$

Proof. The proof is obvious. □

Proposition 2.4. *Every positive multiplicatively convex function is exponential type multiplicatively convex function.*

Proof. According to Lemma 2.3 and Definition 2.1, we have

$$f(tb + (1-t)a) \leq (f(a))^{1-t} . (f(b))^t \leq (f(a))^{e^{1-t}-1} . (f(b))^{e^t-1}.$$

□

Theorem 2.5. *Let $f, g : [a, b] \to \mathbb{R}$. If f and g are exponential type multiplicatively convex functions, then $f.g$ is exponential type multiplicatively convex function.*

Proof. Let f and g be exponential type multiplicatively convex funtions. Then

$$\begin{aligned}(f.g)(tb+(1-t)a) &= f(tb+(1-t)a).g(tb+(1-t)a)\\ &\leq (f(a))^{e^{1-t}-1}.(f(b))^{e^{t}-1}.(g(a))^{e^{1-t}-1}.(g(b))^{e^{t}-1}\\ &= (f(a).g(a))^{e^{1-t}-1}.(f(b).g(b))^{e^{t}-1}\\ &= ((f.g)(a))^{e^{1-t}-1}.((f.g)(b))^{e^{t}-1}.\end{aligned}$$

□

Theorem 2.6. *If $f: I \to \Im$ is convex and $g: \Im \to \mathbb{R}$ is an exponential type multiplicatively convex function and nondecreasing, then $g \circ f: I \to \mathbb{R}$ is an exponential type multiplicatively convex function.*

Proof. For $a, b \in I$ and $t \in [0,1]$, we get

$$\begin{aligned}(g\circ f)(tb+(1-t)a) &= g(f(tb+(1-t)a))\\ &\leq g(tf(b)+(1-t)f(a))\\ &\leq [g(f(a))]^{e^{1-t}-1}.[g(f(b))]^{e^{t}-1}\\ &= [(g\circ f)(a)]^{e^{1-t}-1}.[(g\circ f)(b)]^{e^{t}-1}.\end{aligned}$$

□

Theorem 2.7. *Let $f_i: [a,b] \to \mathbb{R}$ be an arbitrary family of exponential type multiplicatively convex functions and let $f(x) = \sup_i f_i(x)$. If $J = \{\theta \in [a,b] : f(\theta) < \infty\} \neq \emptyset$, then J is an interval and f is an exponential type multiplicatively convex function on J.*

Proof. For all $a, b \in J$ and $t \in [0,1]$, we have

$$\begin{aligned}f(tb+(1-t)a) &= \sup_i f_i(tb+(1-t)a)\\ &\leq \sup_i\left[(f_i(a))^{e^{1-t}-1}.(f_i(b))^{e^{t}-1}\right]\\ &\leq \sup_i(f_i(a))^{e^{1-t}-1}.\sup_i(f_i(b))^{e^{t}-1}\\ &= (f(a))^{e^{1-t}-1}.(f(b))^{e^{t}-1}<\infty.\end{aligned}$$

□

Theorem 2.8. *If $f: [a,b] \to \mathbb{R}$ is exponential type multiplicatively convex function, then f is bounded on $[a,b]$.*

Proof. Let $x \in [a,b]$ and $L = \max\{f(a), f(b)\}$. Then, there exists $t \in [0,1]$ such that $x = tb + (1-t)a$. Thus, since $e^t \leq e$ and $e^{1-t} \leq e$, we have

$$\begin{aligned}f(x) &\leq f(tb+(1-t)a)\\ &\leq (f(a))^{e^{1-t}-1}.(f(b))^{e^{t}-1}\\ &\leq L^{e^t+e^{1-t}-2}\\ &\leq L^{2(e-1)}=K.\end{aligned}$$

□

Also, for all $x \in [a,b]$, there exists $\mu \in \left[0, \frac{b-a}{2}\right]$ such that $x = \frac{a+b}{2} + \mu$ or $x = \frac{a+b}{2} - \mu$. Without loss of generality, we suppose $x = \frac{a+b}{2} + \mu$. So, we have

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{1}{2}\left[\frac{a+b}{2} + \mu\right] + \frac{1}{2}\left[\frac{a+b}{2} - \mu\right]\right) \\ &\leq \left[f(x) . f\left(\frac{a+b}{2} - \mu\right)\right]^{\sqrt{e}-1}. \end{aligned}$$

Since K is the upper bound, we have

$$\begin{aligned} f(x) &\geq \frac{\left[f\left(\frac{a+b}{2}\right)\right]^{\frac{1}{\sqrt{e}-1}}}{f\left(\frac{a+b}{2} - \mu\right)} \\ &\geq \frac{\left[f\left(\frac{a+b}{2}\right)\right]^{\frac{1}{\sqrt{e}-1}}}{K} = k. \end{aligned}$$

3. Hermite-Hadamard inequalities for exponential type multiplicatively convex functions

In this section we derive integral inequalities of Hermite-Hadamard type for exponential type multiplicatively convex functions and convex functions in the framework of multiplicative calculus.

Theorem 3.1. *Let f be an exponential type multiplicatively convex function on $[a,b]$. Then*

$$f\left(\frac{a+b}{2}\right) \leq \left(\int_a^b f(x)^{dx}\right)^{\frac{2(\sqrt{e}-1)}{b-a}} \leq [f(a) f(b)]^{2(\sqrt{e}-1)(e-2)}.$$

The above inequality is called Hermite-Hadamard integral inequality for exponential type multiplicatively convex functions.

Proof. Note that

$$\begin{aligned} \ln f\left(\frac{a+b}{2}\right) &= \ln\left(f\left(\frac{(1-t)a + tb + ta + (1-t)b}{2}\right)\right) \\ &= \ln\left(f\left(\frac{(1-t)a + tb}{2} + \frac{ta + (1-t)b}{2}\right)\right) \\ &\leq \ln\left[\left(f\left(\frac{(1-t)a + tb}{2}\right)\right)^{\sqrt{e}-1} . \left(f\left(\frac{ta + (1-t)b}{2}\right)\right)^{\sqrt{e}-1}\right] \\ &= \left(\sqrt{e} - 1\right) \ln f\left(\frac{(1-t)a + tb}{2}\right) + \left(\sqrt{e} - 1\right) \ln f\left(\frac{ta + (1-t)b}{2}\right). \end{aligned}$$

Integrating the above inequality with respect to t on $[0,1]$, we have

$$\begin{aligned} &\ln f\left(\frac{a+b}{2}\right) \\ &\leq \left(\sqrt{e} - 1\right) \int_0^1 \ln f\left(\frac{(1-t)a + tb}{2}\right) + \left(\sqrt{e} - 1\right) \int_0^1 \ln f\left(\frac{ta + (1-t)b}{2}\right) \\ &= \left(\sqrt{e} - 1\right) \left[\frac{1}{b-a} \int_a^b \ln(f(x))\, dx + \frac{1}{a-b} \int_b^a \ln(f(x))\, dx\right] \end{aligned}$$

$$
\begin{aligned}
&= \left(\sqrt{e}-1\right)\left[\frac{1}{b-a}\int_a^b \ln\left(f(x)\right)dx + \frac{1}{b-a}\int_a^b \ln\left(f(x)\right)dx\right] \\
&= \frac{2\left(\sqrt{e}-1\right)}{b-a}\int_a^b \ln\left(f(x)\right)dx.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right) &\leq e^{\left(\frac{2(\sqrt{e}-1)}{b-a}\int_a^b \ln\left(f(x)\right)dx\right)} \\
&= \left(\int_a^b \left(f(x)\right)^{dx}\right)^{\frac{2(\sqrt{e}-1)}{b-a}}.
\end{aligned}
$$

Hence,

$$
f\left(\frac{a+b}{2}\right) \leq \left(\int_a^b \left(f(x)\right)^{dx}\right)^{\frac{2(\sqrt{e}-1)}{b-a}}. \tag{2}
$$

Consider the second inequality:

$$
\begin{aligned}
\left(\int_a^b \left(f(x)\right)^{dx}\right)^{\frac{1}{b-a}} &= \left(e^{\left(\int_a^b \ln\left(f(x)\right)dx\right)}\right)^{\frac{1}{b-a}} \\
&= e^{\frac{1}{b-a}\left(\int_a^b \ln\left(f(x)\right)dx\right)} \\
&= e^{\left(\int_0^1 \ln\left(f(a+t(b-a))\right)dt\right)} \\
&\leq e^{\int_0^1 \ln\left[\left(f(b)\right)^{e^t-1}.\left(f(a)\right)^{e^{1-t}-1}\right]} \\
&= e^{\int_0^1 \left[\left(e^t-1\right)\ln f(b)+\left(e^{1-t}-1\right)\ln\left(f(a)\right)\right]dt} \\
&= e^{\ln\left[f(a)f(b)\right]^{e-2}} \\
&= \left[f(a)f(b)\right]^{e-2}.
\end{aligned}
$$

Hence,

$$
\left(\int_a^b \left(f(x)\right)^{dx}\right)^{\frac{1}{b-a}} \leq \left[f(a)f(b)\right]^{e-2}. \tag{3}
$$

Combining the inequalities (2) and (3), we have

$$
f\left(\frac{a+b}{2}\right) \leq \left(\int_a^b \left(f(x)\right)^{dx}\right)^{\frac{2(\sqrt{e}-1)}{b-a}} \leq \left[f(a)f(b)\right]^{2\left(\sqrt{e}-1\right)(e-2)}.
$$

□

Theorem 3.2. *Let f and g be two exponential type multiplicatively convex functions on $[a,b]$. Then*

$$
f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leq \left(\int_a^b \left(f(x)\right)^{dx}\int_a^b \left(g(x)\right)^{dx}\right)^{\frac{2(\sqrt{e}-1)}{b-a}} \leq \left[f(a)f(b)g(a)g(b)\right]^{2\left(\sqrt{e}-1\right)(e-2)}.
$$

Proof. Note that,

$$
\begin{aligned}
& \ln\left(f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\right) \\
= & \ln\left(f\left(\frac{a+b}{2}\right)\right)+\ln\left(g\left(\frac{a+b}{2}\right)\right) \\
= & \ln\left(f\left(\frac{(1-t)a+tb+ta+(1-t)b}{2}\right)\right) \\
& +\ln\left(g\left(\frac{(1-t)a+tb+ta+(1-t)b}{2}\right)\right) \\
= & \ln\left(f\left(\frac{(1-t)a+tb}{2}+\frac{ta+(1-t)b}{2}\right)\right) \\
& +\ln\left(g\left(\frac{(1-t)a+tb}{2}+\frac{ta+(1-t)b}{2}\right)\right) \\
\leq & \ln\left[\left(f\left(\frac{(1-t)a+tb}{2}\right)\right)^{\sqrt{e}-1}\cdot\left(f\left(\frac{ta+(1-t)b}{2}\right)\right)^{\sqrt{e}-1}\right] \\
& +\ln\left[\left(g\left(\frac{(1-t)a+tb}{2}\right)\right)^{\sqrt{e}-1}\cdot\left(g\left(\frac{ta+(1-t)b}{2}\right)\right)^{\sqrt{e}-1}\right] \\
= & \left(\sqrt{e}-1\right)\ln f\left(\frac{(1-t)a+tb}{2}\right)+\left(\sqrt{e}-1\right)\ln f\left(\frac{ta+(1-t)b}{2}\right) \\
& +\left(\sqrt{e}-1\right)\ln g\left(\frac{(1-t)a+tb}{2}\right)+\left(\sqrt{e}-1\right)\ln g\left(\frac{ta+(1-t)b}{2}\right).
\end{aligned}
$$

Integrating the above inequality with respect to t on $[0,1]$, we have

$$
\begin{aligned}
& \ln\left(f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\right) \\
\leq & \int_0^1\left[\left(\sqrt{e}-1\right)\ln f\left(\frac{(1-t)a+tb}{2}\right)+\left(\sqrt{e}-1\right)\ln f\left(\frac{ta+(1-t)b}{2}\right)\right] \\
& +\int_0^1\left[\left(\sqrt{e}-1\right)\ln g\left(\frac{(1-t)a+tb}{2}\right)+\left(\sqrt{e}-1\right)\ln g\left(\frac{ta+(1-t)b}{2}\right)\right] \\
= & \frac{\sqrt{e}-1}{b-a}\int_a^b \ln(f(x))\,dx+\frac{\sqrt{e}-1}{a-b}\int_b^a \ln(f(x))\,dx \\
& +\frac{\sqrt{e}-1}{b-a}\int_a^b \ln(g(x))\,dx+\frac{\sqrt{e}-1}{a-b}\int_b^a \ln(g(x))\,dx \\
= & \frac{2\left(\sqrt{e}-1\right)}{b-a}\int_a^b \ln(f(x))\,dx+\frac{2\left(\sqrt{e}-1\right)}{b-a}\int_a^b \ln(g(x))\,dx,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) & \leq e^{\left(\frac{2\left(\sqrt{e}-1\right)}{b-a}\int_a^b \ln\left(f(x)\right)dx+\frac{2\left(\sqrt{e}-1\right)}{b-a}\int_a^b \ln\left(g(x)\right)dx\right)} \\
& = \left(e^{\int_a^b \ln\left(f(x)\right)dx+\int_a^b \ln\left(g(x)\right)dx}\right)^{\frac{2\left(\sqrt{e}-1\right)}{b-a}}
\end{aligned}
$$

$$= \left(e^{\int_a^b \ln(f(x))dx} . e^{\int_a^b \ln(g(x))dx}\right)^{\frac{2(\sqrt{e}-1)}{b-a}}$$
$$= \left(\int_a^b (f(x))^{dx} \int_a^b (g(x))^{dx}\right)^{\frac{2(\sqrt{e}-1)}{b-a}}.$$

Hence,

$$f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \leq \left(\int_a^b (f(x))^{dx} \int_a^b (g(x))^{dx}\right)^{\frac{2(\sqrt{e}-1)}{b-a}}. \tag{4}$$

Consider the second inequality:

$$\begin{aligned}
& \left(\int_a^b (f(x))^{dx} \int_a^b (g(x))^{dx}\right)^{\frac{1}{b-a}} \\
= \; & \left(e^{\int_a^b \ln(f(x))dx + \int_a^b \ln(g(x))dx}\right)^{\frac{1}{b-a}} \\
= \; & e^{\frac{1}{b-a}\left(\int_a^b \ln(f(x))dx + \int_a^b \ln(g(x))dx\right)} \\
= \; & e^{\left(\int_0^1 \ln(f(a+t(b-a)))dt + \int_0^1 \ln(g(a+t(b-a)))dt\right)} \\
\leq \; & e^{\int_0^1 \ln\left[(f(b))^{e^t-1} . (f(a))^{e^{1-t}-1}\right] + \int_0^1 \ln\left[(g(b))^{e^t-1} . (g(a))^{e^{1-t}-1}\right]} \\
= \; & e^{\int_0^1 \left[(e^t-1)\ln f(b) + (e^{1-t}-1)\ln(f(a))\right]dt + \int_0^1 \left[(e^t-1)\ln g(b) + (e^{1-t}-1)\ln(g(a))\right]dt} \\
= \; & e^{\ln[f(a)f(b)]^{e-2} + \ln[g(a)g(b)]^{e-2}} \\
= \; & [f(a)f(b)g(a)g(b)]^{e-2}.
\end{aligned}$$

Hence,

$$\left(\int_a^b (f(x))^{dx} \int_a^b (g(x))^{dx}\right)^{\frac{1}{b-a}} \leq [f(a)f(b)g(a)g(b)]^{e-2}. \tag{5}$$

Combining (4) and (5), we have

$$f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \leq \left(\int_a^b (f(x))^{dx} \int_a^b (g(x))^{dx}\right)^{\frac{2(\sqrt{e}-1)}{b-a}} \leq [f(a)f(b)g(a)g(b)]^{2(\sqrt{e}-1)(e-2)}.$$

□

Theorem 3.3. *Let f and g be an exponential type multiplicatively convex functions on $[a, b]$. Then*

$$\frac{f\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)} \leq \left(\frac{\int_a^b (f(x))^{dx}}{\int_a^b (g(x))^{dx}}\right)^{\frac{2(\sqrt{e}-1)}{b-a}} \leq \left(\frac{f(a)f(b)}{g(a)g(b)}\right)^{2(\sqrt{e}-1)(e-2)}.$$

Proof. Note that

$$\begin{aligned}
& \ln\left(\frac{f\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)}\right) \\
= \; & \ln\left(f\left(\frac{a+b}{2}\right)\right) - \ln\left(g\left(\frac{a+b}{2}\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \ln\left(f\left(\frac{(1-t)a+tb+ta+(1-t)b}{2}\right)\right) \\
&\quad -\ln\left(g\left(\frac{(1-t)a+tb+ta+(1-t)b}{2}\right)\right) \\
&= \ln\left(f\left(\frac{(1-t)a+tb}{2}+\frac{ta+(1-t)b}{2}\right)\right) \\
&\quad -\ln\left(g\left(\frac{(1-t)a+tb}{2}+\frac{ta+(1-t)b}{2}\right)\right) \\
&\leq \ln\left[\left(f\left(\frac{(1-t)a+tb}{2}\right)\right)^{\sqrt{e}-1}\cdot\left(f\left(\frac{ta+(1-t)b}{2}\right)\right)^{\sqrt{e}-1}\right] \\
&\quad -\ln\left[\left(g\left(\frac{(1-t)a+tb}{2}\right)\right)^{\sqrt{e}-1}\cdot\left(g\left(\frac{ta+(1-t)b}{2}\right)\right)^{\sqrt{e}-1}\right] \\
&= \left(\sqrt{e}-1\right)\ln f\left(\frac{(1-t)a+tb}{2}\right)+\left(\sqrt{e}-1\right)\ln f\left(\frac{ta+(1-t)b}{2}\right) \\
&\quad -\left(\sqrt{e}-1\right)\ln g\left(\frac{(1-t)a+tb}{2}\right)-\left(\sqrt{e}-1\right)\ln g\left(\frac{ta+(1-t)b}{2}\right).
\end{aligned}
$$

Integrating the above inequality with respect to t on $[0,1]$, we have

$$
\begin{aligned}
&\ln\left(\frac{f\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)}\right) \\
&\leq \int_0^1\left[\left(\sqrt{e}-1\right)\ln f\left(\frac{(1-t)a+tb}{2}\right)+\left(\sqrt{e}-1\right)\ln f\left(\frac{ta+(1-t)b}{2}\right)\right] \\
&\quad -\int_0^1\left[\left(\sqrt{e}-1\right)\ln g\left(\frac{(1-t)a+tb}{2}\right)+\left(\sqrt{e}-1\right)\ln g\left(\frac{ta+(1-t)b}{2}\right)\right] \\
&= \frac{\sqrt{e}-1}{b-a}\int_a^b \ln\left(f(x)\right)dx+\frac{\sqrt{e}-1}{a-b}\int_b^a \ln\left(f(x)\right)dx \\
&\quad -\frac{\sqrt{e}-1}{b-a}\int_a^b \ln\left(g(x)\right)dx-\frac{\sqrt{e}-1}{a-b}\int_b^a \ln\left(f(x)\right)dx \\
&= \frac{2\left(\sqrt{e}-1\right)}{b-a}\int_a^b \ln\left(f(x)\right)dx-\frac{2\left(\sqrt{e}-1\right)}{b-a}\int_a^b \ln\left(g(x)\right)dx,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\frac{f\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)} &\leq e^{\left(\frac{2\left(\sqrt{e}-1\right)}{b-a}\int_a^b \ln\left(f(x)\right)dx-\frac{2\left(\sqrt{e}-1\right)}{b-a}\int_a^b \ln\left(g(x)\right)dx\right)} \\
&= \left(e^{\int_a^b \ln\left(f(x)\right)dx-\int_a^b \ln\left(g(x)\right)dx}\right)^{\frac{2\left(\sqrt{e}-1\right)}{b-a}} \\
&= \left(\frac{e^{\int_a^b \ln\left(f(x)\right)dx}}{e^{\int_a^b \ln\left(g(x)\right)dx}}\right)^{\frac{2\left(\sqrt{e}-1\right)}{b-a}}
\end{aligned}
$$

$$= \left(\frac{\int_a^b (f(x))^{dx}}{\int_a^b (g(x))^{dx}} \right)^{\frac{2(\sqrt{e}-1)}{b-a}}.$$

Hence,

$$\frac{f\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)} \le \left(\frac{\int_a^b (f(x))^{dx}}{\int_a^b (g(x))^{dx}} \right)^{\frac{2(\sqrt{e}-1)}{b-a}}. \tag{6}$$

Now, consider the second inequality:

$$\begin{aligned}
& \left(\frac{\int_a^b (f(x))^{dx}}{\int_a^b (g(x))^{dx}} \right)^{\frac{1}{b-a}} \\
= & \left(\frac{e^{\int_a^b \ln(f(x))dx}}{e^{\int_a^b \ln(g(x))dx}} \right)^{\frac{1}{b-a}} \\
= & \left(e^{\int_a^b \ln(f(x))dx - \int_a^b \ln(g(x))dx} \right)^{\frac{1}{b-a}} \\
= & e^{\left(\int_0^1 \ln(f(a+t(b-a)))dt - \int_0^1 \ln(g(a+t(b-a)))dt \right)} \\
\le & e^{\int_0^1 \ln\left[(f(b))^{e^t-1} . (f(a))^{e^{1-t}-1} \right] - \int_0^1 \ln\left[(g(b))^{e^t-1} . (g(a))^{e^{1-t}-1} \right]} \\
= & e^{\int_0^1 \left[(e^t-1)\ln f(b) + (e^{1-t}-1)\ln(f(a)) \right] dt - \int_0^1 \left[(e^t-1)\ln g(b) + (e^{1-t}-1)\ln(g(a)) \right] dt} \\
= & e^{\ln[f(a)f(b)]^{e-2} - \ln[g(a)g(b)]^{e-2}} \\
= & \left(\frac{f(a) f(b)}{g(a) g(b)} \right)^{e-2}
\end{aligned}$$

Hence,

$$\left(\frac{\int_a^b (f(x))^{dx}}{\int_a^b (g(x))^{dx}} \right)^{\frac{1}{b-a}} \le \left(\frac{f(a) f(b)}{g(a) g(b)} \right)^{e-2}. \tag{7}$$

Combining the inequalities (6) and (7), we have

$$\frac{f\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)} \le \left(\frac{\int_a^b (f(x))^{dx}}{\int_a^b (g(x))^{dx}} \right)^{\frac{2(\sqrt{e}-1)}{b-a}} \le \left(\frac{f(a) f(b)}{g(a) g(b)} \right)^{2(\sqrt{e}-1)(e-2)}.$$

□

Theorem 3.4. *Let f and g be convex and exponential type multiplicatively convex functions, respectively. Then*

$$\left(\frac{\int_a^b (f(x))^{dx}}{\int_a^b (g(x))^{dx}} \right)^{\frac{1}{b-a}} \le \frac{\left(\frac{(f(b))^{f(b)}}{(f(a))^{f(a)}} \right)^{\frac{1}{f(b)-f(a)}}}{e.\,(g(a)\,g(b))^{e-2}}.$$

Proof. Note that

$$\begin{aligned}
\left(\frac{\int_a^b (f(x))^{dx}}{\int_a^b (g(x))^{dx}}\right)^{\frac{1}{b-a}} &= \left(\frac{e^{\int_a^b \ln(f(x))dx}}{e^{\int_a^b \ln(g(x))dx}}\right)^{\frac{1}{b-a}} \\
&= \left(e^{\int_a^b \ln(f(x))dx-\int_a^b \ln(g(x))dx}\right)^{\frac{1}{b-a}} \\
&= e^{\left(\int_0^1 \ln(f(a+t(b-a)))dt-\int_0^1 \ln(g(a+t(b-a)))dt\right)} \\
&\leq e^{\int_0^1 \ln(f(a)+t(f(b)-f(a)))dt-\int_0^1 \ln\left((g(b))^{e^t-1}.(g(a))^{e^{1-t}-1}\right)dt} \\
&= e^{\ln\left(\frac{(f(b))^{f(b)}}{(f(a))^{f(a)}}\right)^{\frac{1}{f(b)-f(a)}}-1-\ln(g(a)g(b))^{\int_0^1 (e^t-1)dt}} \\
&= \frac{\left(\frac{(f(b))^{f(b)}}{(f(a))^{f(a)}}\right)^{\frac{1}{f(b)-f(a)}}}{e.(g(a)g(b))^{e-2}}.
\end{aligned}$$

□

Theorem 3.5. *Let f and g be exponential type multiplicatively convex and convex functions, respectively. Then*

$$\left(\frac{\int_a^b (f(x))^{dx}}{\int_a^b (g(x))^{dx}}\right)^{\frac{1}{b-a}} \leq \frac{e.(f(a)f(b))^{e-2}}{\left(\frac{(g(b))^{g(b)}}{(g(a))^{g(a)}}\right)^{\frac{1}{g(b)-g(a)}}}.$$

Proof. Note that

$$\begin{aligned}
\left(\frac{\int_a^b (f(x))^{dx}}{\int_a^b (g(x))^{dx}}\right)^{\frac{1}{b-a}} &= \left(\frac{e^{\int_a^b \ln(f(x))dx}}{e^{\int_a^b \ln(g(x))dx}}\right)^{\frac{1}{b-a}} \\
&= \left(e^{\int_a^b \ln(f(x))dx-\int_a^b \ln(g(x))dx}\right)^{\frac{1}{b-a}} \\
&= e^{\left(\int_0^1 \ln(f(a+t(b-a)))dt-\int_0^1 \ln(g(a+t(b-a)))dt\right)} \\
&\leq e^{\int_0^1 \ln\left((f(b))^{e^t-1}.(f(a))^{e^{1-t}-1}\right)dt-\int_0^1 \ln(g(a)+t(g(b)-g(a)))dt} \\
&= e^{\ln(f(a)f(b))^{\int_0^1 (e^t-1)dt}-\ln\left(\frac{(g(b))^{g(b)}}{(g(a))^{g(a)}}\right)^{\frac{1}{g(b)-g(a)}}+1} \\
&= \frac{e.(f(a)f(b))^{e-2}}{\left(\frac{(g(b))^{g(b)}}{(g(a))^{g(a)}}\right)^{\frac{1}{g(b)-g(a)}}}.
\end{aligned}$$

□

Theorem 3.6. *Let f and g be convex and exponential type multiplicatively convex functions, respectively. Then*

$$\left(\int_a^b (f(x))^{dx}\int_a^b (g(x))^{dx}\right)^{\frac{1}{b-a}} \leq \frac{\left(\frac{(f(b))^{f(b)}}{(f(a))^{f(a)}}\right)^{\frac{1}{f(b)-f(a)}}(g(a)g(b))^{e-2}}{e}.$$

Proof. Note that

$$
\begin{aligned}
\left(\int_a^b (f(x))^{dx} \int_a^b (g(x))^{dx}\right)^{\frac{1}{b-a}} &= \left(e^{\int_a^b \ln\left(f(x)\right)dx} . e^{\int_a^b \ln\left(g(x)\right)dx}\right)^{\frac{1}{b-a}} \\
&= \left(e^{\int_a^b \ln\left(f(x)\right)dx + \int_a^b \ln\left(g(x)\right)dx}\right)^{\frac{1}{b-a}} \\
&= e^{\left(\int_0^1 \ln\left(f(a+t(b-a))\right)dt + \int_0^1 \ln\left(g(a+t(b-a))\right)dt\right)} \\
&\leq e^{\int_0^1 \ln\left(f(a)+t\left(f(b)-f(a)\right)\right)dt + \int_0^1 \ln\left(\left(g(b)\right)^{e^t-1} . \left(g(a)\right)^{e^{1-t}-1}\right)dt} \\
&= e^{\ln\left(\frac{(f(b))^{f(b)}}{(f(a))^{f(a)}}\right)^{\frac{1}{f(b)-f(a)}} - 1 + \ln\left(g(a)g(b)\right)^{\int_0^1 \left(e^t-1\right)dt}} \\
&= \frac{\left(\frac{\left(f(b)\right)^{f(b)}}{\left(f(a)\right)^{f(a)}}\right)^{\frac{1}{f(b)-f(a)}} (g(a)\, g(b))^{e-2}}{e}.
\end{aligned}
$$

□

4. Conclusion

In this paper, we defined and investigated the class of exponential type multiplicatively convex functions. We established a new version of Hermite-Hadamard integral inequality for these functions in the context of multiplicative calculus, as well as several other integral inequalities for product and quotient of these functions. We also provided upper bounds for products and quotient of two exponential type multiplicatively convex functions. As a result, we established several new integral inequalities of Hermite-Hadamard type. In recent years, Hermite-Hadamard inequalities have received renewed attention and have become a significant tool for mathematical analysis, probability theory, optimization and other fields of mathematics. So, numerous researchers have focused on to bring new dimensions to the theory of inequalities. We believe that our new class of functions will provide an attractive and absorbing field for further research in pure and applied sciences. We also believe that our techniques and ideas will inspire future researchers in this field.

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