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Essentially left and right generalized Drazin invertible operators and generalized Saphar decomposition

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Abstract. In this paper we define and study the classes of the essentially left and right generalized Drazin invertible operators and of the left and right Weyl-g-Drazin invertible operators by means of the analytical core and the quasinilpotent part of an operator. We show that the essentially left (right) generalized Drazin invertible operator can be represented as a sum of a left (right) Fredholm and a quasinilpotent operator. Analogously, the left (right) Weyl-g-Drazin invertible operator can be represented as a sum of a left (right) Weyl and a quasinilpotent operator. We also characterize these operators in terms of their generalized Saphar decompositions, accumulation and interior points of various spectra of operator pencils. Furthermore, we expand the results from [10], on the left and right generalized Drazin invertible operators. Special attention is devoted to the investigation of the corresponding spectra of operator pencils.

1. Introduction

Let $L(X)$ denote the Banach algebra of all bounded linear operators acting on an infinite-dimensional Banach space X. If $T \in L(X)$ and M and N are two closed T-invariant subspaces of X such that $X = M \oplus N$, we say that T is completely reduced by the pair (M, N) and it is denoted by $(M, N) \in Red(T)$. In this case we write $T = T_M \oplus T_N$ and say that T is the *direct sum* of T_M and T_N. A closed subspace M of X is said to be complemented if there is a closed subspace N of X such that $X = M \oplus N$.

The concept of Drazin inverse, first defined in 1958. for semigroups [9], has since developed considerably and gained a large the number of applications as well as generalizations. An operator $T \in L(X)$ is Drazin invertible if there exists $S \in L(X)$ that satisfies

$$ST = TS,\ STS = S \text{ and } T - TST \text{ is nilpotent.}$$

Koliha [15] generalized this concept by replacing the third condition of the previous definition with the condition that the operator $T - TST$ is quasinilpotent, thus defining a generalized Drazin inverse of T. For $T \in L(X)$, $H_0(T)$ is the quasinilpotent part of T and $K(T)$ is the analytical core of T [2]. The most important properties of generalized Drazin invertible operators are listed in the following theorem.

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Theorem 1.1. [8, 11, 15, 18] *For $T \in L(X)$ the following statements are equivalent:*

(i) *T is generalized Drazin invertible;*

(ii) $0 \notin \operatorname{acc}\sigma(T)$;

(iii) *$X = H_0(T) \oplus K(T)$ with at least one of the component spaces closed;*

(iv) *Both $H_0(T)$ and $K(T)$ are closed and $X = H_0(T) \oplus K(T)$, $T_{K(T)}$ is invertible, $T_{H_0(T)}$ is quasinilpotent;*

(v) *There exists $(M, N) \in Red(T)$ such that T_M is invertible, T_N is quasinilpotent;*

(vi) *There exists a projection $P \in L(X)$ such that $PT = TP$, $T + P$ is invertible, TP is quasinilpotent.*

One of the generalizations of this class of operators are the left (right) generalized Drazin invertible operators introduced by D. E. Ferreyra, F. E. Levis and N. Thome in [10]. An operator $T \in L(X)$ is called left generalized Drazin invertible if $H_0(T)$ is closed and there exists a closed subspace M of X such that $(M, H_0(T)) \in Red(T)$ and $T(M)$ is a complemented subspace of M. If $K(T)$ is closed and there exists a closed subspace N of X, $N \subset H_0(T)$, such that $(K(T), N) \in Red(T)$ and $K(T) \cap N(T)$ is complemented in $K(T)$, then T is called right generalized Drazin invertible. D. E. Ferreyra, F. E. Levis and N. Thome proved that every left (right) generalized Drazin invertible operator can be decomposed as a sum of a left (right) invertible operator and a quasinilpotent one. The essentially left (right) Drazin invertible operators are recently defined and characterized in [22], where it is shown that they can be represented as a sum of a left (right) Fredholm and a nilpotent operator.

In the third section we consider new classes of operators called the essentially left (right) generalized Drazin invertible operators. An operator $T \in L(X)$ is essentially left generalized Drazin invertible if there exists $(M, N) \in Red(T)$ such that $N \subset H_0(T)$, $N(T) \cap M$ is finite-dimensional and $T(M)$ is complemented in M, while T is essentially right generalized Drazin invertible if there exists $(M, N) \in Red(T)$ such that $N \subset H_0(T)$, $M \supset K(T)$, $R(T) \cap M$ is of finite codimension in M and $N(T) \cap M$ is complemented in M. We show that every essentially left (right) generalized Drazin invertible operator can be decomposed as a sum of a left (right) Fredholm and a quasinilpotent operator. In the same manner we generalize the class of the left (right) Weyl-Drazin invertible operators from [22] by defining the left (right) Weyl-g-Drazin invertible operators, and show that they can be represented as a sum of a left (right) Weyl and a quasinilpotent operator. For the newly defined classes of operators we proceed to investigate properties analogue to those in Theorem 1.1.

If for an operator $T \in L(X)$ there exists a pair $(M, N) \in Red(T)$ such that T_M is Saphar and T_N is quasinilpotent, we say that T admits a generalized Saphar decomposition. Using generalized Saphar decomposition we give some characterizations of essentially left (right) generalized Drazin invertible operators, as well as left (right) Weyl-g-Drazin invertible operators. We also observe the similar characteristics of the left (right) generalized Drazin invertible operators, thus extending the results of [10]. By comparing Theorems 3.2 and 3.19, reader should note how many "nice" properties of the left generalized Drazin invertible operators no longer hold for the essentially left generalized Drazin invertible operators. Moreover, we show that if $T \in L(X)$ admits a generalized Saphar decomposition, then its dual operator $T' \in L(X')$ also admits a generalized Saphar decomposition, which is the improvement of [1, Theorem 1.43]. We further apply this result to the observed operators. Theorem 3.23 at the end of the third section illustrates the importance of SVEP by showing how adding a request for a SVEP at a point erases the differences between some classes of operators. Throughout the paper we use various types of spectra of bounded linear operator pencils which have the form $T - \lambda S$, where $\lambda \in \mathbb{C}$, $T, S \in L(X)$.

By applying the results from the third section, in the forth section we establish relations between some known types of spectra of linear operator pencils and the newly defined ones, by observing their boundaries, convex hulls, accumulation points and isolated points. We devote special attention to the S-generalized Saphar spectrum $\sigma_{gS}(T, S)$ and its relation to the S-essential spectra, especially in the context of isolated points.

The paper is organized into four sections. Section 2 contains basic terminology and notations, including some important results that we often refer to in our later work. Our main results concerning operators, their definitions and characterizations, are gathered in Section 3, while in Section 4 we observe various types of spectra of operator pencils and how they relate to each other.

2. Basic notation

Throughout this paper we use $\mathbb{N}$ ($\mathbb{N}_0$) to denote the set of all positive (non-negative) integers and $\mathbb{C}$ to denote the set of all complex numbers. If $K \subset \mathbb{C}$, then ∂K is the boundary of K and $\operatorname{acc} K$, $\operatorname{int} K$ and $\operatorname{iso} K$ are the sets of accumulation points, interior points and isolated points of K, respectively. The *connected hull* of a compact subset K of the complex plane $\mathbb{C}$, denoted by ηK, is the complement of the unbounded component of $\mathbb{C} \setminus K$ [13, Definition 7.10.1]. A hole of K is a bounded component of $\mathbb{C} \setminus K$, and so a hole of K is a component of $\eta K \setminus K$. We racall that for compact subsets $H, K \subset \mathbb{C}$, the following implication holds ([13, Theorem 7.10.3]):

$$\partial H \subset K \subset H \Longrightarrow \partial H \subset \partial K \subset K \subset H \subset \eta K = \eta H\,. \tag{1}$$

For $T \in L(X)$ we use $N(T)$ and $R(T)$, respectively, to denote the null-space and the range of T. It is well-known that $T \in L(X)$ is left invertible if and only if T is injective and $R(T)$ is a complemented subspace of X. Meanwhile, $T \in L(X)$ is right invertible if and only if T is onto and $N(T)$ is a complemented subspace of X. We use $\mathcal{G}_l(X)$ and $\mathcal{G}_r(X)$, respectively, to denote the semigroups of left and right invertible operators on X.

If $S \in L(X)$ such that $S \neq 0$, then the S-spectrum of T, the S-left spectrum of T, the S-right spectrum of T, the S-point spectrum of T, the S-approximate point spectrum of T and the S-surjective spectrum of T, are defined respectively as

$$\begin{aligned}
\sigma(T,S) &= \{\lambda \in \mathbb{C} : T - \lambda S \text{ is not invertible}\},\\
\sigma_l(T,S) &= \{\lambda \in \mathbb{C} : T - \lambda S \text{ is not left invertible}\},\\
\sigma_r(T,S) &= \{\lambda \in \mathbb{C} : T - \lambda S \text{ is not right invertible}\},\\
\sigma_p(T,S) &= \{\lambda \in \mathbb{C} : T - \lambda S \text{ is not injective}\},\\
\sigma_{ap}(T,S) &= \{\lambda \in \mathbb{C} : T - \lambda S \text{ is not bounded below}\},\\
\sigma_{cp}(T,S) &= \{\lambda \in \mathbb{C} : T - \lambda S \text{ does not have dense range}\},\\
\sigma_{su}(T,S) &= \{\lambda \in \mathbb{C} : T - \lambda S \text{ is not surjective}\}.
\end{aligned}$$

Nullity of $T \in L(X)$ is defined by $\alpha(T) = \dim N(T)$ in case of a finite dimensional null-space and by $\alpha(T) = \infty$ when $N(T)$ is infinite dimensional. Similarly, defect of T is defined as $\beta(T) = \dim Y/R(T) = \operatorname{codim R(T)}$ if $Y/R(T)$ is finite dimensional, and $\beta(T) = \infty$ otherwise. An operator $T \in L(X)$ is called upper semi-Fredholm, or $T \in \Phi_+(X)$, if $\alpha(T) < \infty$ and $R(T)$ is closed, while $T \in L(X)$ is called lower semi-Fredholm, or $T \in \Phi_-(X)$, if $\beta(T) < \infty$. The set of semi-Fredholm operators is defined by $\Phi_\pm(X) = \Phi_+(X) \cup \Phi_-(X)$, while the set of Fredholm operators is defined by $\Phi(X) = \Phi_+(X) \cap \Phi_-(X)$.

If $T \in \Phi_\pm(X)$, the index of T is defined by $i(T) = \alpha(T) - \beta(T)$. The set of upper semi-Weyl operators, denoted by $\mathcal{W}_+(X)$, is the set of upper semi-Fredholm operators with non-positive index. The set of lower semi-Weyl operators, denoted by $\mathcal{W}_-(X)$, is the set of lower semi-Fredholm operators with non-negative index. The set of Weyl operators is defined by $\mathcal{W}(X) = \mathcal{W}_+(X) \cap \mathcal{W}_-(X) = \{T \in \Phi(X) : i(T) = 0\}$.

An operator $T \in L(X)$ is relatively regular (or g-invertible) if there exists $S \in L(X)$ such that $TST = T$. It is well-known that T is relatively regular if and only if $R(T)$ and $N(T)$ are complemented subspaces of X. An operator $T \in L(X)$ is called left Fredholm, or $T \in \Phi_l(X)$, if T is relatively regular upper semi-Fredholm. Also, $T \in L(X)$ is called right Fredholm, or $T \in \Phi_r(X)$, if T is relatively regular lower semi-Fredholm. If T is left or right Fredholm, it belongs to the set $\Phi_{l,r}(X) = \Phi_l(X) \cup \Phi_r(X)$. An operator $T \in L(X)$ is left (right) Weyl if T is left (right) Fredholm operator with non-positive (non-negative) index. We use $\mathcal{W}_l(X)$ ($\mathcal{W}_r(X)$) to denote the set of all left (right) Weyl operators. Evidently, T is left (right) Weyl if and only if T is upper (lower) semi-Weyl and relatively regular.

For $S \in L(X)$ such that $S \neq 0$ and $H = \Phi_+, \Phi_-, \Phi_l, \Phi_r, \Phi_{l,r}, \Phi, \mathcal{W}_+, \mathcal{W}_-, \mathcal{W}_l, \mathcal{W}_r, \mathcal{W}$ the corresponding S-spectrum of $T \in L(X)$ is defined by

$$\sigma_H(T,S) = \{\lambda \in \mathbb{C} :\ T - \lambda S \notin H(X)\}.$$

For a bounded linear operator T and $n \in \mathbb{N}_0$ define T_n as the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$ (in particular, $T_0 = T$). If $T \in L(X)$ and if there exists an integer n for which the range space $R(T^n)$ is closed and T_n is Fredholm (resp. upper semi-Fredholm, lower semi-Fredholm, Weyl, upper semi-Weyl, lower semi-Weyl), then T is called a *B-Fredholm* (resp. *upper semi-B-Fredholm, lower semi-B-Fredholm, B-Weyl, upper semi-B-Weyl, lower semi-B-Weyl*) operator [4–6]. If $S \in L(X)$, $S \neq 0$, the S-B-Fredholm spectrum, the S-upper semi-B-Fredholm spectrum, the S-lower semi-B-Fredholm spectrum, the S-B-Weyl spectrum, the S-upper semi-B-Weyl spectrum, the S-lower semi-B-Weyl spectrum are denoted by $\sigma_{B\Phi}(T,S)$, $\sigma_{B\Phi_+}(T,S)$, $\sigma_{B\Phi_-}(T,S)$, $\sigma_{B\mathcal{W}}(T,S)$, $\sigma_{B\mathcal{W}_+}(T,S)$ and $\sigma_{B\mathcal{W}_-}(T,S)$, respectively.

We define the infimum of the empty set to be ∞. The ascent of an operator $T \in L(X)$ is defined by $a(T) = \inf\{n \in \mathbb{N}_0 : N(T^n) = N(T^{n+1})\}$, and the descent of T is defined by $d(T) = \inf\{n \in \mathbb{N}_0 : R(T^n) = R(T^{n+1})\}$.

For $T \in L(X)$ and $n \in \mathbb{N}_0$ we set

$$\alpha_n(T) = \dim N(T^{n+1})/N(T^n) \text{ and } \beta_n(T) = \dim R(T^n)/R(T^{n+1}).$$

From [14, Lemmas 3.1 and 3.2] it follows that $\alpha_n(T) = \dim(N(T) \cap R(T^n))$ and $\beta_n(T) = \operatorname{codim}(R(T) + N(T^n))$.

For each $n \in \mathbb{N}_0$, T induced a linear transformation from the vector space $R(T^n)/R(T^{n+1})$ to the space $R(T^{n+1})/R(T^{n+2})$ and $k_n(T)$ denotes the dimension of the null space of the induced map. We recall from [12] that

$$k_n(T) = \dim(R(T^n) \cap N(T))/(R(T^{n+1}) \cap N(T))$$

and

$$k_n(T) = \dim(R(T) + N(T^{n+1}))/(R(T) + N(T^n)).$$

This implies that $k_n(T) = \alpha_n(T) - \alpha_{n+1}(T)$ whenever $\alpha_{n+1}(T) < \infty$, and $k_n(T) = \beta_n(T) - \beta_{n+1}(T)$ whenever $\beta_{n+1}(T) < \infty$. If there is $d \in \mathbb{N}_0$ for which $k_n(T) = 0$ for $n \geq d$, then T is said to have *uniform descent* for $n \geq d$.

For $T \in L(X)$ and every $d \in \mathbb{N}_0$, the operator range topology on $R(T^d)$ is defined by the norm $\|\cdot\|_d$ such that for every $y \in R(T^d)$,

$$\|y\|_d = \inf\{\|x\| : x \in X,\ y = T^d x\}.$$

For $T \in L(X)$ if there is $d \in \mathbb{N}_0$ for which T has uniform descent for $n \geq d$ and if $R(T^n)$ is closed in the operator range topology of $R(T^d)$ for $n \geq d$, then we say that T has *eventual topological uniform descent* and, more precisely, that T has *topological uniform descent for* (TUD for brevity) $n \geq d$ [12].

For $T \in L(X)$ we say that it is *Kato* if $R(T)$ is closed and $N(T) \subset R(T^n)$ for every $n \in \mathbb{N}$. Every Kato operator has TUD for $n \geq 0$. An operator $T \in L(X)$ is said to be *Saphar* if it is a relatively regular Kato operator.

The *essential ascent* $a_e(T)$ and *essential descent* $d_e(T)$ of T are defined by $a_e(T) = \inf\{n \in \mathbb{N}_0 : \alpha_n(T) < \infty\}$ and $d_e(T) = \inf\{n \in \mathbb{N}_0 : \beta_n(T) < \infty\}$. We remark that $a_e(T) = 0$ if and only if $\alpha(T) < \infty$, and $d_e(T) = 0$ if and only if $\beta(T) < \infty$. So, $T \in L(X)$ is Fredholm if and only if $a_e(T) = d_e(T) = 0$.

If $T, S \in L(X)$ such that $S \neq 0$, the S-descent spectrum of T, the S-essential descent spectrum of T are defined, respectively, by:

$$\begin{aligned}\sigma_{dsc}(T,S) &= \{\lambda \in \mathbb{C} : d(T - \lambda S) = \infty\},\\ \sigma^e_{dsc}(T,S) &= \{\lambda \in \mathbb{C} : d_e(T - \lambda S) = \infty\}.\end{aligned}$$

It is well known that $T \in L(X)$ is Drazin invertible if and only if $a(T) < \infty$ and $d(T) < \infty$. An operator $T \in L(X)$ is called *upper Drazin invertible* operator if $a(T) < \infty$ and $R(T^{a(T)+1})$ is closed. If $d(T) < \infty$ and $R(T^{d(T)})$ is closed, then T is called *lower Drazin invertible*. An operator $T \in L(X)$ is an *essentially upper Drazin invertible* operator if $a_e(T) < \infty$ and $R(T^{a_e(T)+1})$ is closed. If $d_e(T) < \infty$ and $R(T^{d_e(T)})$ is closed, then T is called *essentially lower Drazin invertible*.

If $T, S \in L(X)$ such that $S \neq 0$, the S-upper Drazin spectrum of T, the S-lower Drazin spectrum of T, the S-Drazin spectrum of T, the S-essentially upper Drazin spectrum of T, the S-essentially lower Drazin spectrum of T are denoted as $\sigma_{D_+}(T,S)$, $\sigma_{D_-}(T,S)$, $\sigma_D(T,S)$, $\sigma^e_{D_+}(T,S)$, $\sigma^e_{D_-}(T,S)$, respectively.

The following two subspaces we use to define the new sets of operators. The *quasinilpotent part* of an operator $T \in L(X)$ is defined by

$$H_0(T) = \{x \in X : \lim_{n\to\infty} \|T^n x\|^{1/n} = 0\}.$$

Obviously, $N(T) \subset H_0(T)$ and it is well known that an operator $T \in L(X)$ is quasinilpotent if and only if $H_0(T) = X$. The *analytical core* of T, denoted by $K(T)$, is the set of all $x \in X$ for which there exist $\delta > 0$ and a sequence $(u_n)_n$ in X satisfying

$$Tu_1 = x, \;\; Tu_{n+1} = u_n \;\text{ for all }\; n \in \mathbb{N}, \;\; \|u_n\| \leq c^n\|x\| \;\text{ for all }\; n \in \mathbb{N}.$$

Clearly, $K(T)$ is a subset of $R(T)$. In general, the quasinilpotent part and the analytical core are not closed.

An operator $T \in L(X)$ has the single-valued extension property at $\lambda_0 \in \mathbb{C}$, SVEP at λ_0, if for every open disc D_{λ_0} centered at λ_0 the only analytic function $f : D_{\lambda_0} \to X$ which satisfies $(T-\lambda I)f(\lambda) = 0$ for all $\lambda \in D_{\lambda_0}$, is the function $f \equiv 0$.

If $\mathcal{K} \subset L(X)$ the commutant of $\mathcal{K}$ is defined by

$$\text{comm}(\mathcal{K}) = \{A \in L(X) : AB = BA \text{ for every } B \in \mathcal{K}\}.$$

The commutant of $T \in L(X)$ is $\text{comm}(T) = \text{comm}(\mathcal{K})$ with $\mathcal{K} = \{T\}$, and the double commutant is defined as $\text{comm}^2(T) = \text{comm}(\text{comm}(T))$.

The following lemmas are repeatedly used throughout the paper.

Lemma 2.1. [21, 22] *Let $T \in L(X)$ and let there exist a pair $(M, N) \in Red(T)$. Then the following statements hold:*
(i) *T is g-invertible if and only if T_M and T_N are g-invertible.*
(ii) *T is left (right) Fredholm if and only if T_M and T_N are left (right) Fredholm, and in that case $i(T) = i(T_M) + i(T_N)$.*
(iii) *If T_M and T_N are left (right) Weyl, then T is left (right) Weyl.*
(iv) *If T is left (right) Weyl and T_M is Weyl, then T_N is left (right) Weyl.*

Lemma 2.2. [22] *For $T \in L(X)$ let there exist a pair $(M, N) \in Red(T)$. Then T is Saphar if and only if T_M and T_N are Saphar.*

Lemma 2.3. *Let E and F be sets of the complex plane. Then:*
(i) *If $\partial F \subset E \subset F$, then* $\text{iso}\, F \subset \text{iso}\, E$.
(ii) *If $\partial F \subset E$ and F is closed, then $\partial F \cap \text{iso}\, E \subset \text{iso}\, F$.*

Proof. See [7, Lemma 2.2]. □

The dual space of X and the dual operator of $T \in L(X)$ are denoted respectively by X' and $T' \in L(X')$. If M is the subspace of X, the annihilator of M is the closed subspace of X', denoted by $M^\perp$ and defined by

$$M^\perp = \{f \in X' : f(x) = 0 \text{ for every } x \in M\}.$$

Lemma 2.4. *[22] Let $X = X_1 \oplus X_2 \oplus \cdots \oplus X_n$ where $X_1, X_2, \ldots, X_n$ are closed subspaces of X and let M_i be a subspace of X_i, $i = 1, \ldots, n$. Then the subspace $M_1 \oplus M_2 \oplus \cdots \oplus M_n$ is a complemented subspace of X if and only if M_i is a complemented subspace of X_i for each $i \in \{1, \ldots, n\}$.*

Lemma 2.5. *[22] Let M be complemented subspace of X and let M_1 be a closed subspace of X such that $M \subset M_1$. Then M is complemented in M_1.*

Lemma 2.6. *Let M be a complemented subspace of X. Then $M^\perp$ is a complemented subspace of X'.*

Proof. Let N be a closed subspace of X such that $X = M \oplus N$, and let $P \in \mathcal{B}(X)$ be the projection of X such that $R(P) = M$ and $N(P) = N$. Then $P' \in \mathcal{B}(X')$ is a projection, $N(P') = R(P)^\perp = M^\perp$ is closed, and since $R(P)$ is closed then $R(P') = N(P)^\perp = N^\perp$ is closed. Thus $X' = R(P') \oplus N(P') = N^\perp \oplus M^\perp$, and hence $M^\perp$ is complemented in X'. □

3. The essentially left and right generalized Drazin invertible operators

If for an operator $T \in L(X)$ there exists a pair $(M, N) \in Red(T)$ such that T_M is Kato and T_N is quasinilpotent, we say that T admits a *generalized Kato decomposition*, or shortly T admits a GKD(M, N). Furthermore, if T_M is Saphar we say that T admits a *generalized Saphar decomposition*, or T admits a GSD(M, N).

Definition 3.1. *An operator $T \in L(X)$ is essentially left generalized Drazin invertible if there exists $(M, N) \in Red(T)$ such that $N \subset H_0(T)$, $N(T) \cap M$ is finite-dimensional and $T(M)$ is complemented in M.*

If the operator $T \in L(X)$ is essentially left generalized Drazin invertible, we will write $T \in gD\Phi_l(X)$. This notation is justified by part (ii) of the following theorem.

Theorem 3.2. *Let $T, S \in L(X)$, and let S be invertible and $S \in \mathrm{comm}^2(T)$. The following statements are equivalent:*
(i) *T is essentially left generalized Drazin invertible;*
(ii) *There exists $(M, N) \in Red(T)$ such that T_M is a left Fredholm operator and T_N is quasinilpotent;*
(iii) *There exists a projection $P \in L(X)$ such that $TP = PT$, $T + P$ is left Fredholm and TP is quasinilpotent;*
(iv) *T admits a GSD and $0 \notin \mathrm{acc}\, \sigma_{\Phi_l}(T, S)$;*
(v) *T admits a GSD and $0 \notin \mathrm{int}\, \sigma_{\Phi_l}(T, S)$;*
(vi) *T admits a GSD and $0 \notin \mathrm{acc}\, \sigma_{\Phi_+}(T, S)$;*
(vii) *T admits a GSD and $0 \notin \mathrm{int}\, \sigma_{\Phi_+}(T, S)$;*
(viii) *T admits a GSD and $0 \notin \mathrm{acc}\, \sigma^e_{D_+}(T, S)$;*
(ix) *T admits a GSD and $0 \notin \mathrm{int}\, \sigma^e_{D_+}(T, S)$.*

Proof. (i)$\Longrightarrow$(ii) Let $N \subset H_0(T)$ and let M be a closed subspace of X such that $(M, N) \in Red(T)$, $N(T) \cap M$ is finite-dimensional and $T(M)$ is complemented in M. The operator T_N is quasinilpotent since $H_0(T_N) = H_0(T) \cap N = N$. For the operator T_M we have $\alpha(T_M) = \dim N(T_M) = \dim(N(T) \cap M) < \infty$ and $R(T_M) = T(M)$ is closed and complemented in M. Therefore, T_M is left Fredholm.

(ii)$\Longrightarrow$(i) Suppose that there exists a pair $(M, N) \in Red(T)$ such that T_M is a left Fredholm operator and T_N is quasinilpotent. Since T_N is quasinilpotent, we have that $N = H_0(T_N) \subset H_0(T)$ is closed and complemented subspace of X. Furthermore, if T_M is left Fredholm, we have that $\dim(N(T) \cap M) = \dim N(T_M) = \alpha(T_M) < \infty$ and $T(M) = R(T_M)$ is closed and complemented in M.

(ii)$\Longrightarrow$(iii) Suppose that there exists $(M, N) \in Red(T)$ such that T_M is left Fredholm and T_N is quasinilpotent. Let $P \in L(X)$ be the projection such that $N(P) = M$ and $R(P) = N$. Obviously, $TP = PT$ since M and N are T-invariant. Both TP and $T + P$ are reduced by the pair (M, N) and we get the following decompositions

$$TP = 0 \oplus T_N \;\text{ and }\; T + P = T_M \oplus (T_N + I_N). \tag{2}$$

Operator TP is quasinilpotent as a direct sum of quasinilpotent operators. This we can acquire by calculating its spectrum $\sigma(TP) = \sigma(0) \cup \sigma(T_N) = \{0\}$. Moreover, since T_N is quasinilpotent we know that $T_N + I_N$ is invertible. Hence, by Lemma 2.1(ii) we conclude that $T + P$ is left Fredholm.

(iii)$\Longrightarrow$(ii) Let $P \in L(X)$ be the projection such that $TP = PT$, TP is quasinilpotent and $T + P$ is a left Fredholm operator. If $M = N(P)$ and $N = R(P)$, then $(M, N) \in Red(T)$. From (2) we have that T_N is quasinilpotent on N since $\{0\} = \sigma(TP) = \sigma(0) \cup \sigma(T_N) = \{0\} \cup \sigma(T_N)$ and T_M is left Fredholm by Lemma 2.1(ii).

(ii)$\Longrightarrow$(iv) Suppose that there exists $(M, N) \in Red(T)$ such that T_M is left Fredholm and T_N is quasinilpotent. Lemma 2.1(i) and [17, Theorem 16.21] imply that there exists $(M_1, M_2) \in Red(T_M)$ such that $\dim M_2 < \infty$, T_{M_1} is Saphar and T_{M_2} is nilpotent. Then, $(M_1, M_2 \oplus N) \in Red(T)$, T_{M_1} is Saphar and $T_{M_2 \oplus N}$ is quasinilpotent. Hence, T admits a GSD.

Let $P \in L(X)$ be the projection such that $N(P) = M$ and $R(P) = N$. Then $TP = PT$, and hence $SP = PS$, which implies that $(M, N) \in Red(S)$. As S is invertible, it follows that S_M and S_N are invertible. Since $T_N S_N = S_N T_N$, from [17, Theorem 2.11] it follows that

$$\sigma(T_N - \lambda S_N) \subset \sigma(T_N) - \lambda \sigma(S_N) = -\lambda \sigma(S_N), \text{ for every } \lambda \in \mathbb{C}. \tag{3}$$

Since $0 \notin \sigma(S_N)$, from (3) it follows that $T_N - \lambda S_N$ is invertible for every $\lambda \in \mathbb{C}$, $\lambda \neq 0$. From the openness of the set $\Phi_l(M)$ follows the existence of $\epsilon > 0$ such that $T_M - \lambda S_M$ is left Fredholm for $|\lambda| < \epsilon$. Now, for $0 < |\lambda| < \epsilon$, from the decomposition

$$T - \lambda S = (T_M - \lambda S_M) \oplus (T_N - \lambda S_N), \tag{4}$$

and Lemma 2.1(ii), we get that $T - \lambda S$ is left Fredholm for $0 < |\lambda| < \epsilon$. Hence, $0 \notin \operatorname{acc} \sigma_{\Phi_l}(T, S)$.

Implications (iv)$\Longrightarrow$(vi)$\Longrightarrow$(viii)$\Longrightarrow$(ix) and (iv)$\Longrightarrow$(v)$\Longrightarrow$(vii)$\Longrightarrow$(ix) are clear.

(ix)$\Longrightarrow$(ii) Suppose that T admits a GSD and $0 \notin \operatorname{int} \sigma^e_{D_+}(T, S)$. Then there exists a decomposition $(M, N) \in Red(T)$ such that T_M is Saphar and T_N is quasinilpotent. Since T_M has TUD for $n \geq 0$, according to [12, Theorem 4.7] we conclude that there exists an $\epsilon > 0$ such that for every $\lambda \in \mathbb{C}$, the following implication holds:

$$0 < |\lambda| < \epsilon \Longrightarrow \alpha_n(T_M - \lambda S_M) = \alpha(T_M), \text{ for every } n \in \mathbb{N}_0. \tag{5}$$

Also from [12, Theorem 4.7] it follows that $\sigma^e_{D_+}(T, S)$ is closed. Since $0 \notin \operatorname{int} \sigma^e_{D_+}(T, S)$, we conclude that there exists $\mu \in \mathbb{C}$ such that $0 < |\mu| < \epsilon$ and $T - \mu S$ is essentially upper Drazin invertible. Hence there is $n \in \mathbb{N}_0$ such that $\alpha_n(T_M - \mu S_M) < \infty$. Now according to (5) we obtain that $\alpha(T_M) < \infty$. As T_M is Saphar we conclude that T_M is left Fredholm. □

Remark 3.3. Suppose that $T \in L(X)$ is essentially left generalized Drazin invertible, i.e. there exists $(M, N) \in Red(T)$ such that $N \subset H_0(T)$, $N(T) \cap M$ is finite-dimensional and $T(M)$ is complemented in M. Notice that if $N = H_0(T)$ then $N(T) \cap M \subset H_0(T) \cap M = \{0\}$ since $(N, M) \in Red(T)$. In this case, T is a left generalized Drazin invertible operator, defined in [10], decomposable to a sum of a left invertible and a quasinilpotent operator. •

Example 3.4. Observe a backward unilateral shift operator $V \in \ell^2(\mathbb{N})$ defined by

$$V(x_1, x_2, \dots) = (x_2, x_3, \dots).$$

Obviously, V is not injective, and yet from [23, Theorem 3.5] we see that $0 \notin \sigma_{\Phi_l}(V)$, so V is left Fredholm. Therefore, V is essentially left generalized Drazin invertible, but is not left generalized Drazin invertible.

Definition 3.5. *An operator $T \in L(X)$ is essentially right generalized Drazin invertible if there exists $(M, N) \in Red(T)$ such that $N \subset H_0(T)$, $M \supset K(T)$, $R(T) \cap M$ is of finite codimension in M and $N(T) \cap M$ is complemented in M.*

We denote by $gD\Phi_r(X)$ the set of essentially right generalized Drazin invertible operators acting on X.

Theorem 3.6. *Let $T, S \in L(X)$, and let S be invertible and $S \in \operatorname{comm}^2(T)$. The following statements are equivalent:*

(i) *T is essentially right generalized Drazin invertible;*

(ii) *There exists $(M, N) \in Red(T)$ such that T_M is a right Fredholm operator and T_N is quasinilpotent;*

(iii) *There exists a projection $P \in L(X)$ such that $TP = PT$, $T + P$ is right Fredholm and TP is quasinilpotent;*

(iv) *T admits a GSD and $0 \notin \operatorname{acc} \sigma_{\Phi_r}(T, S)$;*

(v) *T admits a GSD and $0 \notin \operatorname{int} \sigma_{\Phi_r}(T, S)$;*

(vi) *T admits a GSD and $0 \notin \operatorname{acc} \sigma_{\Phi_-}(T, S)$;*

(vii) *T admits a GSD and $0 \notin \operatorname{int} \sigma_{\Phi_-}(T, S)$;*

(viii) *T admits a GSD and $0 \notin \operatorname{acc} \sigma^e_{D_-}(T, S)$;*

(ix) *T admits a GSD and $0 \notin \operatorname{int} \sigma^e_{D_-}(T, S)$;*

(x) *T admits a GSD and $0 \notin \operatorname{acc} \sigma^e_{dsc}(T, S)$;*

(xi) *T admits a GSD and $0 \notin \operatorname{int} \sigma^e_{dsc}(T, S)$.*

Proof. (i)$\Longrightarrow$(ii) Suppose that there exist closed subspaces $N \subset H_0(T)$ and $M \supset K(T)$ such that $(M, N) \in Red(T)$, $R(T) \cap M$ is of finite codimension in M and $N(T) \cap M$ is complemented in M. Then $T = T_M \oplus T_N$ and T_N is quasinilpotent. For the operator T_M we have $\beta(T_M) = \operatorname{codim} R(T_M) = \dim M/(R(T) \cap M) < \infty$ and $N(T_M) = N(T) \cap M$ is complemented in M. Therefore, T_M is right Fredholm.

(ii)$\Longrightarrow$(i) Suppose that there exists a pair $(M, N) \in Red(T)$ such that T_M is a right Fredholm operator and T_N is quasinilpotent. Then $N \subset H_0(T)$ and $\operatorname{codim} R(T_M) = \beta(T_M) < \infty$, i.e. $R(T) \cap M$ is of finite codimension in M. Easily we see that $N(T) \cap M = N(T_M)$ is complemented in M. Since $(M, N) \in Red(T)$ and T_N is quasinilpotent from the proof of [1, Theorem 1.41 (i)] it follows that $K(T) = K(T_M) \subset M$.

Proofs of (ii)$\Longrightarrow$(iii), (iii)$\Longrightarrow$(ii) and (ii)$\Longrightarrow$(iv) can be derived analogously to the proof of Theorem 3.2.

Implications (iv)$\Longrightarrow$(vi)$\Longrightarrow$(viii)$\Longrightarrow$(x)$\Longrightarrow$(xi) and (iv)$\Longrightarrow$(v)$\Longrightarrow$(vii)$\Longrightarrow$(ix)$\Longrightarrow$(xi) are clear.

(xi)$\Longrightarrow$(ii) Suppose that T admits a GSD and $0 \notin \operatorname{int} \sigma^e_{dsc}(T, S)$. Then there exists a decomposition $(M, N) \in Red(T)$ such that T_M is Saphar and T_N is quasinilpotent. From [12, Theorem 4.7] it follows that $\sigma^e_{dsc}(T, S)$ is closed. Again according to [12, Theorem 4.7] we conclude that there exists $\epsilon > 0$ such that for every $\lambda \in \mathbb{C}$, the following implication holds:

$$0 < |\lambda| < \epsilon \Longrightarrow \beta_n(T_M - \lambda S_M) = \beta(T_M), \text{ for every } n \in \mathbb{N}_0. \tag{6}$$

Since $0 \notin \operatorname{int} \sigma^e_{dsc}(T, S)$, there exists $\mu \in \mathbb{C}$ such that $0 < |\mu| < \epsilon$ and $T - \mu S$ has finite essential descent. Hence there is $n \in \mathbb{N}_0$ such that $\beta_n(T_M - \mu S_M) < \infty$. Now according to (6) we obtain that $\beta(T_M) < \infty$. As T_M is Saphar we conclude that T_M is right Fredholm. □

Remark 3.7. Let $T \in L(X)$ be essentially right generalized Drazin invertible, i.e. there exists $(M, N) \in Red(T)$ such that $N \subset H_0(T)$, $M \supset K(T)$, $R(T) \cap M$ is of finite codimension in M and $N(T) \cap M$ is complemented in M. If $K(T) = M$, then T is a right generalized Drazin invertible operator, defined in [10], decomposed as a sum of a right invertible and a quasinilpotent operator. Indeed, $K(T) \cap N(T) = M \cap N(T)$ is complemented in $K(T)$ and hence T is right generalized Drazin invertible. •

Example 3.8. The forward unilateral shift $U \in \ell^2(\mathbb{N})$ defined by

$$U(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$$

is obviously not surjective. However, from [23, Theorem 3.4] we can see that U is right Fredholm. Therefore, U is essentially right generalized Drazin invertible, but is not right generalized Drazin invertible.

Theorem 3.9. *Let $T \in L(X)$. If T admits a GSD(M, N), then T' admits a GSD$(N^\perp, M^\perp)$.*

Proof. There exists a pair $(M, N) \in Red(T)$ such that T_M is Saphar and T_N is quasinilpotent. Let P_M be the projection of X onto M along N. Then $TP_M = P_M T$, and hence $T'P'_M = P'_M T'$. As $R(P'_M) = N^\perp$ and $N(P'_M) = M^\perp$, we obtain that $(N^\perp, M^\perp) \in Red(T')$. From the proof of [1, Theorem 1.43] it follows that $T'_{N^\perp}$ is Kato. Moreover, we have that

$$\begin{aligned} R(T'_{N^\perp}) &= R(T') \cap N^\perp = N(T)^\perp \cap N^\perp = (N(T) + N)^\perp \\ &= (N(T_M) \oplus N)^\perp \end{aligned} \tag{7}$$

and

$$\begin{aligned} N(T'_{N^\perp}) &= N(T') \cap N^\perp = R(T)^\perp \cap N^\perp = (R(T) + N)^\perp \\ &= (R(T_M) \oplus N)^\perp \end{aligned} \tag{8}$$

Since T_M is Saphar, it follows that $N(T_M)$ and $R(T_M)$ are complemented in M. According to Lemma 2.4 we conclude that $N(T_M) \oplus N$ and $R(T_M) \oplus N$ are complemented in X. Lemma 2.6 ensures that $(N(T_M) \oplus N)^\perp$ and $(R(T_M) \oplus N)^\perp$ are complemented in X'. As $N^\perp$ is a closed subspace of X' which contains $(N(T_M) \oplus N)^\perp$ and $(R(T_M) \oplus N)^\perp$, applying Lemma 2.5 we conclude that $(N(T_M) \oplus N)^\perp$ and $(R(T_M) \oplus N)^\perp$ are complemented in $N^\perp$. Now according to (7) and (8) we have that $R(T'_{N^\perp})$ and $N(T'_{N^\perp})$ are complemented in $N^\perp$, and hence $T'_{N^\perp}$ is Saphar.

If $P_N = I - P_M$, then $(M,N) \in Red(TP_N)$, $TP_N = P_N T$, $TP_N = 0_M \oplus T_N$, and so TP_N is quasinilpotent. Consequently, $T'P'_N = P'_N T'$ is quasinilpotent and $(N^\perp, M^\perp) \in Red(T'P'_N)$. As $R(P'_N) = N(P_N)^\perp = M^\perp$ and $N(P'_N) = R(P_N)^\perp = N^\perp$, we conclude that $T'P'_N = (T'P'_N)_{N^\perp} \oplus (T'P'_N)_{M^\perp} = 0_{N^\perp} \oplus T'_{M^\perp}$. Hence $T'_{M^\perp}$ is quasinilpotent. Consequently, T' admits a $GSD(N^\perp, M^\perp)$. □

Proposition 3.10. *Let $T \in L(X)$. If T is essentially left generalized Drazin invertible then T' is essentially right generalized Drazin invertible.*

Proof. If T is essentially left generalized Drazin invertible, by (i)⟺(iv) in Theorem 3.2 it admits a GSD(M,N) for some closed T-invariant subspaces M and N and $0 \notin \operatorname{acc}\sigma_{\Phi_l}(T,S)$. From Theorem 3.9 it follows that T' admits a GSD$(N^\perp, M^\perp)$.

If $0 \notin \operatorname{acc}\sigma_{\Phi_l}(T,S)$ then there exists $\epsilon > 0$ such that for every $0 < |\lambda| < \epsilon$ the operator $T - \lambda S$ is left Fredholm. Hence, $T - \lambda S$ is upper semi-Fredholm and relatively regular. From [19, Lemma 2.8] it follows that $T' - \lambda S'$ is lower semi-Fredholm. It is a known fact that if $T - \lambda S$ is relatively regular then $T' - \lambda S'$ is also relatively regular. Therefore, $T' - \lambda S'$ is right Fredholm for every $0 < |\lambda| < \epsilon$ and we conclude that $0 \notin \operatorname{acc}\sigma_{\Phi_r}(T', S')$.

From (i)⟺(iv) in Theorem 3.6 it follows that T' is essentially right generalized Drazin invertible. □

For $T \in L(X)$ we say that T is *Fredholm-g-Drazin invertible*, and write $T \in gD\Phi(X)$, if there exists a pair $(M,N) \in Red(T)$ such that T_M is Fredholm and T_N is quasinilpotent.

Proposition 3.11. *Let $T \in L(X)$. Then $T \in L(X)$ is essentially left and right generalized Drazin invertible if and only if T is a Fredholm-g-Drazin invertible.*

Proof. Suppose that T is essentially left and right generalized Drazin invertible. From the equivalences (i)⟺(ii) in Theorems 3.2 and 3.6 it follows that there exists $(M_1, N_1) \in Red(T)$ such that T_{M_1} is left Fredhom and T_{N_1} is quasinilpotent, T_{M_2} is right Fredholm and T_{N_2} is quasinilpotent. From [3, Proposition 2.5] (i) it follows that T_{M_1} and T_{M_2} are Fredholm, and so $T \in gD\Phi(X)$.

The converse follows again from the equivalences (i)⟺(ii) in Theorems 3.2 and 3.6. □

Definition 3.12. *Operator $T \in L(X)$ is left Weyl-g-Drazin invertible if there exists $(M,N) \in Red(T)$ such that $N \subset H_0(T)$, $T(M)$ is complemented in M and $N(T) \cap M$ is of finite dimension no greater than the dimension of $M/T(M)$.*

The set of left Weyl-g-Drazin invertible operators on X will be denoted by $gD\mathcal{W}_l(X)$.

Theorem 3.13. *Let $T, S \in L(X)$, and let S be invertible and $S \in \operatorname{comm}^2(T)$. The following statements are equivalent:*

(i) *T is left Weyl-g-Drazin invertible;*

(ii) *There exists $(M,N) \in Red(T)$ such that T_M is a left Weyl operator and T_N is quasinilpotent;*

(iii) *There exists a projection $P \in L(X)$ such that $TP = PT$, $T + P$ is left Weyl and TP is quasinilpotent;*

(iv) *T admits a GSD and $0 \notin \operatorname{acc}\sigma_{\mathcal{W}_l}(T,S)$;*

(v) *T admits a GSD and $0 \notin \operatorname{int}\sigma_{\mathcal{W}_l}(T,S)$;*

(vi) *T admits a GSD and $0 \notin \operatorname{acc}\sigma_{\mathcal{W}_+}(T,S)$;*

(vii) *T admits a GSD and $0 \notin \operatorname{int}\sigma_{\mathcal{W}_+}(T,S)$;*

(viii) *T admits a GSD and $0 \notin \operatorname{acc}\sigma_{B\mathcal{W}_+}(T,S)$;*

(ix) *T admits a GSD and $0 \notin \operatorname{int}\sigma_{B\mathcal{W}_+}(T,S)$.*

Proof. (i)⟹(ii) Let $N \subset H_0(T)$ and let M be a closed subspace of X such that $(M,N) \in Red(T)$, $T(M)$ is complemented in M and $N(T) \cap M$ is finite-dimensional subspace of M, for which $\dim(N(T) \cap M) \leq \dim M/T(M)$. Then the operator T_N is quasinilpotent and from Theorem 3.2 T_M is left Fredholm. We also have

$$i(T_M) = \alpha(T_M) - \beta(T_M) = \dim(N(T) \cap M) - \dim M/T(M) \leq 0.$$

Therefore, T_M is left Weyl.

(ii)$\Longrightarrow$(i) Suppose that there exists a pair $(M, N) \in Red(T)$ such that T_M is a left Weyl operator and T_N is quasinilpotent. Then from $i(T_M) \le 0$ we get that

$$\dim(N(T) \cap M) = \dim N(T_M) = \alpha(T_M) \le \beta(T_M) = \dim M/T(M).$$

The rest of the proof is the same as in Theorem 3.2.

(ii)$\Longrightarrow$(iii) Suppose there exists $(M, N) \in Red(T)$ such that T_M is left Weyl and T_N is quasinilpotent. Let $P \in L(X)$ be the projection such that $N(P) = M$ and $R(P) = N$. Decompositions (2) hold, TP is quasinilpotent and from Lemma 2.1(iii) it follows that $T + P$ is left Weyl.

(iii)$\Longrightarrow$(ii) Let $P \in L(X)$ be the projection such that $TP = PT$, TP is quasinilpotent and $T + P$ is a left Weyl operator. If $M = N(P)$ and $N = R(P)$, then from (2) and Lemma 2.1(iv) we get that T_N is quasinilpotent and T_M is left Weyl.

(ii)$\Longrightarrow$(iv) Follows from the openness of the set of left Weyl operators and Lemma 2.1(iii), analogously to the proof of Theorem 3.2.

Implications (iv)$\Longrightarrow$(vi)$\Longrightarrow$(viii)$\Longrightarrow$(ix) and (iv)$\Longrightarrow$(v)$\Longrightarrow$(vii)$\Longrightarrow$(ix) are clear.

(ix)$\Longrightarrow$(ii) Suppose that T admits a GSD and $0 \notin \operatorname{int} \sigma_{B\mathcal{W}_+}(T, S)$. Then there exists $(M, N) \in Red(T)$ such that T_M is Saphar and T_N is quasinilpotent. Operator T_M has a TUD for $n \ge 0$, so according to [12, Theorem 4.7] there exists an $\epsilon > 0$ such that for every $\lambda \in \mathbb{C}$, the following implication holds:

$$0 < |\lambda| < \epsilon \Longrightarrow \alpha_n(T_M - \lambda S_M) = \alpha(T_M) \tag{9}$$

$$\beta_n(T_M - \lambda S_M) = \beta(T_M), \text{ for every } n \in \mathbb{N}_0. \tag{10}$$

From [12, Theorem 4.7] it follows that $\sigma_{B\mathcal{W}_+}(T, S)$ is closed. Hence the assumption $0 \notin \operatorname{int} \sigma_{B\mathcal{W}_+}(T, S)$ implies the existence of $\mu \in \mathbb{C}$, $0 < |\mu| < \epsilon$ such that $T - \mu S \in B\mathcal{W}_+(X)$. Therefore, there exists $m \in \mathbb{N}_0$ such that $R((T - \mu S)^m)$ is closed and the operator $(T - \mu S)_m : R((T - \mu S)^m) \to R((T - \mu S)^m)$ is upper semi-Weyl.

Since $T_N - \mu S_N$ is invertible, $(T_N - \mu S_N)^n$ is also invertible for each $n \in \mathbb{N}$ and we have the equality

$$\alpha_n(T - \mu S) = \alpha_n(T_M - \mu S_M) + \alpha_n(T_N - \mu S_N) = \alpha_n(T_M - \mu S_M),$$
$$\beta_n(T - \mu S) = \beta_n(T_M - \mu S_M) + \beta_n(T_N - \mu S_N) = \beta_n(T_M - \mu S_M).$$

Now we get

$$\alpha((T - \mu S)_m) = \dim\Big(N(T - \mu S) \cap R((T - \mu S)^m\Big) = \alpha_m(T - \mu S) = \alpha_m(T_M - \mu S_M) \tag{11}$$

and

$$\beta((T - \mu S)_m) = \dim\Big(R((T_M - \mu S_M)^m)/R((T_M - \mu S_M)^{m+1})\Big) = \beta_m(T - \mu S) = \beta_m(T_M - \mu S_M). \tag{12}$$

Using (9), (11), (12) and the fact that $(T - \mu S)_m$ is upper semi-Weyl we get

$$\alpha(T_M) = \alpha_m(T_M - \mu S_M) = \alpha((T - \mu S)_m) < \infty,$$
$$\beta(T_M) = \beta_m(T_M - \mu S_M) = \beta((T - \mu S)_m)$$
$$i(T_M) = \alpha(T_M) - \beta(T_M) = i((T - \mu S)_m) \le 0.$$

Since T_M is Saphar, we have proved that T_M is left Weyl. □

Definition 3.14. *Operator $T \in L(X)$ is right Weyl-g-Drazin invertible if there exist closed subspaces $N \subset H_0(T)$ and $M \supset K(T)$ such that $(M, N) \in Red(T)$, $N(T) \cap M$ is complemented in M and $R(T) \cap M$ is of finite codimension in M, no greater then the dimension of $N(T) \cap M$.*

By $gD\mathcal{W}_r(X)$ we denote the set of right Weyl-g-Drazin invertible operators on X.

Theorem 3.15. *Let $T, S \in L(X)$, and let S be invertible and $S \in \mathrm{comm}^2(T)$. The following statements are equivalent:*
(i) *T is right Weyl-g-Drazin invertible;*
(ii) *There exist $(M, N) \in Red(T)$ such that T_M is a right Weyl operator and T_N is quasinilpotent;*
(iii) *There exists a projection $P \in L(X)$ such that $TP = PT$, $T + P$ is right Weyl and TP is quasinilpotent;*
(iv) *T admits a GSD and $0 \notin \mathrm{acc}\, \sigma_{\mathcal{W}_r}(T, S)$;*
(v) *T admits a GSD and $0 \notin \mathrm{int}\, \sigma_{\mathcal{W}_r}(T, S)$;*
(vi) *T admits a GSD and $0 \notin \mathrm{acc}\, \sigma_{\mathcal{W}_-}(T, S)$;*
(vii) *T admits a GSD and $0 \notin \mathrm{int}\, \sigma_{\mathcal{W}_-}(T, S)$;*
(viii) *T admits a GSD and $0 \notin \mathrm{acc}\, \sigma_{B\mathcal{W}_-}(T, S)$;*
(ix) *T admits a GSD and $0 \notin \mathrm{int}\, \sigma_{B\mathcal{W}_-}(T, S)$.*

Proof. Analogously to Theorem 3.13. □

Proposition 3.16. *Let $T \in L(X)$. If T is left Weyl-g-Drazin invertible, then T' is right Weyl-g-Drazin invertible.*

Proof. Suppose that T is left Weyl-g-Drazin invertible. From (i)$\Longleftrightarrow$(iv) in Theorem 3.13 it follows that T admits a GSD(M, N) and $0 \notin \mathrm{acc}\, \sigma_{\mathcal{W}_l}(T, S)$. From Theorem 3.9 we get that T' admits a GSD$(N^\perp, M^\perp)$. If $0 \notin \mathrm{acc}\, \sigma_{\mathcal{W}_l}(T, S)$ then there exists $\epsilon > 0$ such that $T - \lambda S$ is left Weyl for every $0 < |\lambda| < \epsilon$. Hence, $T - \lambda S$ is left Fredholm with nonpositive index. From the proof of Proposition 3.10 we know that $T' - \lambda S'$ is a right Fredholm operator. By applying [19, Lemma 2.8] we get $i(T' - \lambda S') = -i(T - \lambda S) \geq 0$. Therefore, $T' - \lambda S'$ is a right Weyl operator for every $0 < |\lambda| < \epsilon$ and we have proved that $0 \notin \mathrm{acc}\, \sigma_{\mathcal{W}_r}(T', S')$. From (i)$\Longleftrightarrow$(iv) in Theorem 3.15 T' is a right Weyl-g-Drazin invertible operator. □

For $T \in L(X)$ we say that T is *Weyl-g-Drazin invertible*, and write $T \in gD\mathcal{W}(X)$, if there exists a pair $(M, N) \in Red(T)$ such that T_M is Weyl and T_N is quasinilpotent.

Proposition 3.17. *Let $T \in L(X)$. Then $T \in L(X)$ is left and right Weyl-g-Drazin invertible if and only if T is a Weyl-g-Drazin invertible operator.*

Proof. Follows from [3, Proposition 2.5] (ii) and the equivalence (i)$\Longleftrightarrow$(ii) in Theorems 3.13 and 3.15, analogously to the proof of Proposition 3.11. □

We say that $T \in gD\Phi_{l,r}(X)$ if there exists a pair $(M, N) \in Red(T)$ such that $T_M \in \Phi_{l,r}(X)$ and T_N is quasinilpotent.

The following theorem can be proved analogously to Theorems 3.2 and 3.13.

Theorem 3.18. *Let $H \in \{\Phi, \mathcal{W}, \Phi_{l,r}\}$, $T, S \in L(X)$ and let S be invertible and $S \in \mathrm{comm}^2(T)$. The following statements are equivalent:*
(i) *$T \in gDH(X)$;*
(ii) *There exists a projection $P \in L(X)$ such that $TP = PT$, $T + P \in H(X)$ and TP is quasinilpotent;*
(iii) *T admits a GSD and $0 \notin \mathrm{acc}\, \sigma_H(T, S)$;*
(iv) *T admits a GSD and $0 \notin \mathrm{int}\, \sigma_H(T, S)$.*

The following two theorems provide some characterizations of left and right generalized Drazin invertible operators introduced in [10].

Theorem 3.19. *Let $T, S \in L(X)$, and let S be invertible and $S \in \text{comm}^2(T)$. The following statements are equivalent:*
(i) *T is left generalized Drazin invertible;*
(ii) *T admits a GSD and T has SVEP at 0;*
(iii) *T admits a GSD(M, N) and there exists $p \in \mathbb{N}$ such that $H_0(T) = N(T^p)$;*
(iv) *T admits a GSD and $H_0(T)$ is closed;*
(v) *T admits a GSD and $H_0(T) \cap K(T) = \{0\}$;*
(vi) *T admits a GSD and $H_0(T) \cap K(T)$ is closed;*
(vii) *T admits a GSD and $0 \notin \text{acc}\, \sigma_l(T, S)$;*
(viii) *T admits a GSD and $0 \notin \text{int}\, \sigma_l(T, S)$;*
(ix) *T admits a GSD and $0 \notin \text{acc}\, \sigma_{ap}(T, S)$;*
(x) *T admits a GSD and $0 \notin \text{int}\, \sigma_{ap}(T, S)$;*
(xi) *T admits a GSD and $0 \notin \text{acc}\, \sigma_p(T, S)$;*
(xii) *T admits a GSD and $0 \notin \text{int}\, \sigma_p(T, S)$;*
(xiii) *T admits a GSD and $0 \notin \text{acc}\, \sigma_{D_+}(T, S)$;*
(xiv) *T admits a GSD and $0 \notin \text{int}\, \sigma_{D_+}(T, S)$.*

Proof. (i)$\Longrightarrow$(ii) Suppose that T is left generalized Drazin invertible. According to [10, Theorem 3.3] there exist a pair $(M, N) \in Red(T)$ such that T_M is left invertible and T_N is quasinilpotent. Then T_M is Saphar, and hence T admits a GSD. From [1, Theorem 3.14] it follows that T has SVEP at 0.

(ii)$\Longrightarrow$(i) Suppose that T admits a GSD(M, N) and T has SVEP at 0. From [1, Theorem 2.49] it follows that T_M is injective, a since T_M is Saphar, we obtain that T_M is left invertible. From [10, Theorem 3.3] it follows that T is left generalized Drazin invertible.

The equivalences (ii)$\Longleftrightarrow$(iii)$\Longleftrightarrow$(iv)$\Longleftrightarrow$(v)$\Longleftrightarrow$(vi) follow from [1, Theorem 3.14].

(i)$\Longrightarrow$(vii) Let T be left generalized Drazin invertible. Then there exist a pair $(M, N) \in Red(T)$ such that T_M is left invertible and T_N is quasinilpotent, and so T admits a GSD. Let $P \in L(X)$ be the projection such that $N(P) = M$ and $R(P) = N$. As in the proof of Theorem 3.2, we draw the conclusion from the openness of the set of left invertible operators and the equality (4), bearing in mind that the sum of an invertible and a left invertible operator is left invertible.

The implications (vii)$\Longrightarrow$(viii)$\Longrightarrow$(x)$\Longrightarrow$(xii), (vii)$\Longrightarrow$(viii)$\Longrightarrow$(x)$\Longrightarrow$(xiv), (vii)$\Longrightarrow$(ix)$\Longrightarrow$(xi)$\Longrightarrow$(xii), (vii)$\Longrightarrow$(ix) $\Longrightarrow$(xiii)$\Longrightarrow$(xiv) are clear.

(xii)$\Longrightarrow$(i): Suppose that T admits a GSD and $0 \notin \text{int}\, \sigma_p(T, S)$. Then there exists a decomposition $(M, N) \in Red(T)$ such that T_M is Saphar and T_N is quasinilpotent. As before, $(M, N) \in Red(S)$, T_M has TUD for $n \geq 0$, and so by [12, Theorem 4.7] we obtain that there exists an $\epsilon > 0$ such that for every $\lambda \in \mathbb{C}$ it holds:

$$0 < |\lambda| < \epsilon \Longrightarrow \alpha(T_M - \lambda S_M) = \alpha(T_M). \tag{13}$$

From $0 \notin \text{int}\, \sigma_p(T, S)$ it follows that there exists $\mu \in \mathbb{C}$ such that $|\mu| < \epsilon$ and $T - \mu S$ is injective, and hence $T_M - \mu S_M$ is injective. If $\mu = 0$ we have that T_M is injective. If $\mu \neq 0$, from (13) it follows that $\alpha(T_M) = 0$, i.e. T_M is injective. Consequently, T_M is left invertible, and according to [10, Theorem 3.3] it follows that T is left generalized Drazin invertible.

(xiv)$\Longrightarrow$(i): Suppose that T admits a GSD and $0 \notin \text{int}\, \sigma_{D_+}(T, S)$. Then there exists a decomposition $(M, N) \in Red(T)$ such that T_M is Saphar and T_N is quasinilpotent. According to [12, Theorem 4.7] we conclude that there exists an $\epsilon > 0$ such that for every $\lambda \in \mathbb{C}$, the following implication holds:

$$0 < |\lambda| < \epsilon \Longrightarrow \alpha_n(T_M - \lambda S_M) = \alpha(T_M), \text{ for every } n \in \mathbb{N}_0. \tag{14}$$

Also from [12, Theorem 4.7] it follows that $\sigma_{D_+}(T, S)$ is closed, and since $0 \notin \text{int}\, \sigma_{D_+}(T, S)$, there exists a $\mu \in \mathbb{C}$ such that $0 < |\mu| < \epsilon$ and $T - \mu S$ is upper Drazin invertible. Hence there is $n \in \mathbb{N}_0$ such that $\alpha_n(T_M - \mu S_M) = 0$. Now according to (14) we obtain that $\alpha(T_M) = 0$. As T_M is Saphar we conclude that T_M is left invertible. Consequently, T is left generalized Drazin invertible. $\square$

Theorem 3.20. *Let $T, S \in L(X)$, and let S be invertible and $S \in \operatorname{comm}^2(T)$. The following statements are equivalent:*
(i) *T is right generalized Drazin invertible;*
(ii) *T admits a GSD and T' has SVEP at 0;*
(iii) *T admits a GSD(M, N) and there exists $q \in \mathbb{N}$ such that $K(T) = R(T^q)$;*
(iv) *T admits a GSD and $H_0(T) + K(T) = X$;*
(v) *T admits a GSD and $H_0(T) + K(T)$ is norm dense in X;*
(vi) *T admits a GSD and $0 \notin \operatorname{acc} \sigma_r(T, S)$;*
(vii) *T admits a GSD and $0 \notin \operatorname{int} \sigma_r(T, S)$;*
(viii) *T admits a GSD and $0 \notin \operatorname{acc} \sigma_{su}(T, S)$;*
(ix) *T admits a GSD and $0 \notin \operatorname{int} \sigma_{su}(T, S)$;*
(x) *T admits a GSD and $0 \notin \operatorname{acc} \sigma_{cp}(T, S)$;*
(xi) *T admits a GSD and $0 \notin \operatorname{int} \sigma_{cp}(T, S)$;*
(xii) *T admits a GSD and $0 \notin \operatorname{acc} \sigma_{dsc}(T, S)$;*
(xiii) *T admits a GSD and $0 \notin \operatorname{int} \sigma_{dsc}(T, S)$.*

Proof. (i)$\Longrightarrow$(ii) Suppose that T is right generalized Drazin invertible. According to [10, Theorem 3.4] there exist a pair $(M, N) \in Red(T)$ such that T_M is right invertible and T_N is quasinilpotent. Then T_M is Saphar, and hence T admits a GSD. From [1, Theorem 3.15] it follows that T' has SVEP at 0.

(ii)$\Longrightarrow$(i) Suppose that T admits a GSD(M, N) and T' has SVEP at 0. From [1, Theorem 3.15] it follows that T_M is surjective, a since T_M is Saphar, we obtain that T_M is right invertible. From [10, Theorem 3.4] it follows that T is right generalized Drazin invertible.

The equivalences (ii)$\Longleftrightarrow$(iii)$\Longleftrightarrow$(iv)$\Longleftrightarrow$(v) follow from [1, Theorem 3.14].

The proof of the implication (i)$\Longrightarrow$(vi) is similar to the proof of the implication (i)$\Longrightarrow$(vii) in Theorem 3.19.

The implications (vi)$\Longrightarrow$(vii)$\Longrightarrow$(ix)$\Longrightarrow$(xi), (vi)$\Longrightarrow$(vii)$\Longrightarrow$(ix)$\Longrightarrow$(xiii), (vi)$\Longrightarrow$(viii)$\Longrightarrow$(x)$\Longrightarrow$(xi), (vi)$\Longrightarrow$(viii) $\Longrightarrow$(xii)$\Longrightarrow$(xiii) are clear.

(xi)$\Longrightarrow$(i): Suppose that T admits a GSD and $0 \notin \operatorname{int} \sigma_{cp}(T, S)$. Then there exists a decomposition $(M, N) \in Red(T)$ such that T_M is Saphar and T_N is quasinilpotent. Then $(M, N) \in Red(S)$. Using [12, Theorem 4.7] we obtain that there exists an $\epsilon > 0$ such that for every $\lambda \in \mathbb{C}$, it holds:

$$0 < |\lambda| < \epsilon \Longrightarrow R(T_M - \lambda S_M) \text{ is closed and } \beta(T_M - \lambda S_M) = \beta(T_M). \tag{15}$$

From $0 \notin \operatorname{int} \sigma_{cp}(T, S)$ it follows that there exists $\mu \in \mathbb{C}$ such that $|\mu| < \epsilon$ and $\overline{R(T - \mu S)} = X$. Then $\overline{R(T_M - \mu S_M)} = M$, and since $R(T_M - \mu S_M)$ is closed, we obtain that $R(T_M - \mu S_M) = M$. Now from $\beta(T_M - \mu S_M) = 0$ and (13) it follows that $\beta(T_M) = 0$, i.e. T_M is surjective. Hence T_M is right invertible, and according to [10, Theorem 3.4] we obtain that T is right generalized Drazin invertible.

(xiii)$\Longrightarrow$(i): Suppose that T admits a GSD and $0 \notin \operatorname{int} \sigma_{dsc}(T, S)$. Then there exists a decomposition $(M, N) \in Red(T)$ such that T_M is Saphar and T_N is quasinilpotent. According to [12, Theorem 4.7] we conclude that there exists an $\epsilon > 0$ such that for every $\lambda \in \mathbb{C}$, the following implication holds:

$$0 < |\lambda| < \epsilon \Longrightarrow \beta_n(T_M - \lambda S_M) = \beta(T_M), \text{ for every } n \in \mathbb{N}_0. \tag{16}$$

From [12, Theorem 4.7] it follows that $\sigma_{dsc}(T, S)$ is closed and since $0 \notin \operatorname{int} \sigma_{dsc}(T, S)$, there exists a $\mu \in \mathbb{C}$ such that $0 < |\mu| < \epsilon$ and $d(T - \mu S) < \infty$. Hence $d(T_M - \mu S_M) < \infty$ and there is $n \in \mathbb{N}_0$ such that $\beta_n(T_M - \mu S_M) = 0$. From (16) it follows that $\beta(T_M) = 0$, and so T_M is surjective. As T_M is Saphar, we conclude that T_M is right invertible, and hence T is right generalized Drazin invertible. □

By $gD\mathcal{G}_l(X)$ ($gD\mathcal{G}_r(X)$) we denote the set of left (right) generalized Drazin invertible operators on X.

Proposition 3.21. *Let $T \in L(X)$. If T is left generalized Drazin invertible, then T' is right generalized Drazin invertible.*

Proof. Suppose T is left generalized Drazin invertible. From (i)$\Longleftrightarrow$(vii) in Theorem 3.19 it follows that T admits a GSD(M, N) and $0 \notin \operatorname{acc}\sigma_l(T, S)$. Theorem 3.9 implies that T' admits a GSD$(N^{\perp}, M^{\perp})$. If $0 \notin \operatorname{acc}\sigma_l(T, S)$, there exists $\epsilon > 0$ such that $T - \lambda S$ is left invertible for every $0 < |\lambda| < \epsilon$. Then $T' - \lambda S'$ is right invertible for every $0 < |\lambda| < \epsilon$, implying that $0 \notin \operatorname{acc}\sigma_r(T', S')$. From (i)$\Longleftrightarrow$(vi) in Theorem 3.20 we have that T' is right generalized Drazin invertible. □

For $T, S \in L(X)$, $S \neq 0$, we define the S-generalized Drazin spectrum by

$$\sigma_{gD}(T, S) = \{\lambda \in \mathbb{C} : T - \lambda S \text{ is not generalized Drazin invertible}\}.$$

Theorem 3.22. *Let $T, S \in L(X)$ and let S be invertible and $S \in \operatorname{comm}^2(T)$. The following statements are equivalent:*

(i) *T is generalized Drazin invertible;*

(ii) *T admits a GSD and $0 \notin \operatorname{int}\sigma(T, S)$;*

(iii) $0 \notin \operatorname{acc}\sigma(T, S)$.

Proof. (i)$\Longleftrightarrow$(ii): It follows from Theorem 1.1 analogously to the proof of Theorem 3.19.

(i)$\Longleftrightarrow$(iii): Since S is invertible and $S \in \operatorname{comm}^2(T)$ we have that S^{-1} commutes with $T - \lambda S$ for every $\lambda \in \mathbb{C}$. As generalized Drazin invertible operators acting on X form a regularity [16, Theorem 1.2], from [17, Proposition 6.2(iii)] we conclude that

$$\begin{aligned} \lambda \notin \sigma_{gD}(T, S) &\Longleftrightarrow T - \lambda S \text{ is generalized Drazin invertible} \\ &\Longleftrightarrow TS^{-1} - \lambda \text{ is generalized Drazin invertible} \\ &\Longleftrightarrow \lambda \notin \sigma_{gD}(TS^{-1}). \end{aligned}$$

Consequently, by using the equivalence (i)$\Longleftrightarrow$(ii) in Theorem 1.1 we obtain that $\sigma_{gD}(T, S) = \sigma_{gD}(TS^{-1}) = \operatorname{acc}\sigma(TS^{-1}) = \operatorname{acc}\sigma(T, S)$. Therefore, T is generalized Drazin invertible if and only if $0 \notin \sigma_{gD}(T, S) = \operatorname{acc}\sigma(T, S)$. □

Theorem 3.23. *Let $T \in L(X)$.*

(i) *If T has the SVEP at 0 then $T \in gD\Phi_l(X) \Leftrightarrow T \in gD\mathcal{W}_l(X) \Leftrightarrow T \in gD\mathcal{G}_l(X)$.*

(ii) *If T' has the SVEP at 0 then $T \in gD\Phi_r(X) \Leftrightarrow T \in gD\mathcal{W}_r(X) \Leftrightarrow T \in gD\mathcal{G}_r(X)$.*

(iii) *If both T and T' have the SVEP at 0 then T is generalized Drazin invertible if and only if $T \in gD\Phi(X)$ if and only if $T \in gD\mathcal{W}(X)$.*

Proof. (i): The implications $T \in gD\mathcal{G}_l(X) \Longrightarrow T \in gD\mathcal{W}_l(X) \Longrightarrow T \in gD\Phi_l(X)$ follow from the equivalence (a)$\Longleftrightarrow$(b) in [10, Theorem 3.3], the equivalence (i)$\Longleftrightarrow$(ii) in Theorem 3.13 and the equivalence (i)$\Longleftrightarrow$(ii) in Theorem 3.2.

Suppose that T has the SVEP at 0 and that $T \in gD\Phi_l(X)$. From Theorem 3.2 it follows that T admits a GSD. Now from the equivalence (i)$\Longleftrightarrow$(ii) in Theorem 3.19 we conclude that $T \in gD\mathcal{G}_l(X)$.

(ii): It follows from [10, Theorem 3.4], Theorem 3.6 and Theorem 3.20, analogously to the proof of (i).

(iii): It follows from [10, Corollary 3.5], (i), (ii), Proposition 3.11 and Proposition 3.17. □

4. Spectra

If $T, S \in L(X)$ such that $S \neq 0$, the S-Saphar spectrum and the S-generalized Saphar spectrum are denoted respectively by $\sigma_S(T, S)$ and $\sigma_{gS}(T, S)$, and defined by

$$\begin{aligned} \sigma_S(T, S) &= \{\lambda \in \mathbb{C} : T - \lambda S \text{ is not Saphar}\}, \\ \sigma_{gS}(T, S) &= \{\lambda \in \mathbb{C} : T - \lambda S \text{ does not admit generalized Saphar decomposition}\}. \end{aligned}$$

For $T, S \in L(X)$, $S \neq 0$, and $H \in \{\mathcal{G}_l, \mathcal{G}_r, \Phi_l, \Phi_r, \Phi_{l,r}, \Phi, \mathcal{W}_l, \mathcal{W}_r, \mathcal{W}\}$, we define for each H the appropriate spectrum of operator pencil

$$\sigma_{gDH}(T, S) = \{\lambda \in \mathbb{C} : \ T - \lambda S \notin gDH(X)\}.$$

Theorem 4.1. *Let $T, S \in L(X)$, and let S be invertible and $S \in \mathrm{comm}^2(T)$. If T admits a GSD(M, N), then there exists $\epsilon > 0$ such that $T - \lambda S$ is Saphar for each λ such that $0 < |\lambda| < \epsilon$.*

Proof. Suppose that T admits a GSD(M, N). Then $T = T_M \oplus T_N$, T_M is Saphar and T_N is quasinilpotent. If $M = \{0\}$, then T is quasinilpotent. Since $TS = ST$, from [17, Theorem 2.11] it follows that

$$\sigma(T - \lambda S) \subset \sigma(T) - \lambda\sigma(S) = -\lambda\sigma(S), \text{ for every } \lambda \in \mathbb{C}. \tag{17}$$

As $0 \notin \sigma(S)$, from (17) it follows that $T - \lambda S$ is invertible for every $\lambda \in \mathbb{C}$, $\lambda \neq 0$. Therefore, $T - \lambda S$ is Saphar for all $\lambda \neq 0$.

Suppose that $M \neq \{0\}$. Let $P \in L(X)$ be the projection such that $N(P) = M$ and $R(P) = N$. Then $TP = PT$, and hence $SP = PS$, which implies that $(M, N) \in Red(S)$.

From [17, Corollary 12.4 and Lemma 13.6] it follows that there exists an $\epsilon > 0$ such that for $|\lambda| < \epsilon$, $T_M - \lambda S_M$ is Saphar. Since T_N is quasinilpotent and S_N is invertible and commutes with T_N, as in the previous part of the proof we can conclude that $T_N - \lambda S_N$ is invertible for all $\lambda \neq 0$. Thus $T_N - \lambda S_N$ is Saphar for all $\lambda \neq 0$. Lemma 2.2 provides that $T - \lambda S$ is Saphar for each λ such that $0 < |\lambda| < \epsilon$. $\square$

Corollary 4.2. *Let $T, S \in L(X)$, and let S be invertible and $S \in \mathrm{comm}^2(T)$. Then*

(i) *$\sigma_{gS}(T, S)$ is closed;*

(ii) *The set $\sigma_S(T, S) \setminus \sigma_{gS}(T, S)$ consists of at most countably many points.*

Proof. (i) It follows from Theorem 4.1.

(ii): Suppose that $\lambda_0 \in \sigma_S(T, S) \setminus \sigma_{gS}(T, S)$. Then $T - \lambda_0 S$ admits a GSD and according to Theorem 4.1 there exists $\epsilon > 0$ such that $T - \lambda S$ is Saphar for each $\lambda \in \mathbb{C}$ such that $0 < |\lambda - \lambda_0| < \epsilon$. This implies that $\lambda_0 \in \mathrm{iso}\,\sigma_S(T, S)$. Therefore, $\sigma_S(T, S) \setminus \sigma_{gS}(T, S) \subset \mathrm{iso}\,\sigma_S(T, S)$, which implies that $\sigma_S(T, S) \setminus \sigma_{gS}(T, S)$ is at most countable. $\square$

The following corollary is an improvement of [22, Corollary 5.6].

Corollary 4.3. *Let $T \in L(X)$.*

(i) *If T has the SVEP, then all accumulation points of $\sigma_l(T)$ belong to $\sigma_{gS}(T)$.*

(ii) *If T' has the SVEP, then all accumulation points of $\sigma_r(T)$ belong to $\sigma_{gS}(T)$.*

Proof. (i): It follows from the equivalence (ii)$\Longleftrightarrow$(vii) in Theorem 3.19.

(ii): It follows from the equivalence (ii)$\Longleftrightarrow$(vi) in Theorem 3.20. $\square$

Theorem 4.4. *Let $T, S \in L(X)$, and let S be invertible and $S \in \mathrm{comm}^2(T)$. Then*

(i)

$$\begin{array}{ccccccccc}
 & & \partial\sigma_{gD\mathcal{G}_l}(T,S) & \subset & \partial\sigma_{gD\mathcal{W}_l}(T,S) & \subset & \partial\sigma_{gD\Phi_l}(T,S) & & \\
 & \rotatebox{45}{$\subset$} & & \rotatebox{45}{$\subset$} & & \rotatebox{45}{$\subset$} & & \rotatebox{-45}{$\subset$} & \\
\partial\sigma_{gD}(T,S) & \subset & \partial\sigma_{gD\mathcal{W}}(T,S) & \subset & \partial\sigma_{gD\Phi}(T,S) & \subset & & & \partial\sigma_{gD\Phi_{l,r}}(T,S) \subset \partial\sigma_{gS}(T,S), \\
 & \rotatebox{-45}{$\subset$} & & \rotatebox{-45}{$\subset$} & & \rotatebox{-45}{$\subset$} & & \rotatebox{45}{$\subset$} & \\
 & & \partial\sigma_{gD\mathcal{G}_r}(T,S) & \subset & \partial\sigma_{gD\mathcal{W}_r}(T,S) & \subset & \partial\sigma_{gD\Phi_r}(T,S) & &
\end{array}$$

(ii) $\eta\sigma_{gS}(T, S) = \eta\sigma_*(T, S) = \eta\sigma_{gD}(T, S)$ *where* $\sigma_* \in \{\sigma_{gD\Phi_l}, \sigma_{gD\Phi_r}, \sigma_{gD\mathcal{W}_l}, \sigma_{gD\mathcal{W}_r}, \sigma_{gD\Phi}, \sigma_{gD\mathcal{W}}, \sigma_{gD\Phi_{l,r}}, \sigma_{gD\mathcal{G}_l}, \sigma_{gD\mathcal{G}_r}\}$.

(iii) *The set $\sigma_*(T, S)$ consists of $\sigma_{gS}(T, S)$ and possibly some holes in $\sigma_{gS}(T, S)$ where $\sigma_* \in \{\sigma_{gD\Phi_l}, \sigma_{gD\Phi_r}, \sigma_{gD\mathcal{W}_l}, \sigma_{gD\mathcal{W}_r}, \sigma_{gD\Phi}, \sigma_{gD\mathcal{W}}, \sigma_{gD\Phi_{l,r}}, \sigma_{gD\mathcal{G}_l}, \sigma_{gD\mathcal{G}_r}, \sigma_{gD}\}$.*

The set $\sigma_{gD}(T, S)$ consists of $\sigma_*(T, S)$ and possibly some holes in $\sigma_*(T, S)$ where $\sigma_* \in \{\sigma_{gD\Phi_l}, \sigma_{gD\Phi_r}, \sigma_{gD\mathcal{W}_l}, \sigma_{gD\mathcal{W}_r}, \sigma_{gD\Phi}, \sigma_{gD\mathcal{W}}, \sigma_{gD\Phi_{l,r}}, \sigma_{gD\mathcal{G}_l}, \sigma_{gD\mathcal{G}_r}\}$.

Proof. From the equivalence (i)$\Longleftrightarrow$(iv) in Theorem 3.2 we have

$$\begin{aligned}\lambda \notin \sigma_{gD\Phi_l}(T,S) &\iff T-\lambda S \text{ admits a GSD } \wedge\ 0 \notin \operatorname{acc}\sigma_{\Phi_l}(T-\lambda S,S)\\ &\iff \lambda \notin \sigma_{gS}(T,S)\ \wedge\ \lambda \notin \operatorname{acc}\sigma_{\Phi_l}(T,S),\end{aligned}$$

which proves the equality

$$\sigma_{gD\Phi_l}(T,S) = \sigma_{gS}(T,S) \cup \operatorname{acc}\sigma_{\Phi_l}(T,S). \tag{18}$$

Similarly, from the equivalence (i)$\Longleftrightarrow$(v) in Theorem 3.2 we have

$$\sigma_{gD\Phi_l}(T,S) = \sigma_{gS}(T,S) \cup \operatorname{int}\sigma_{\Phi_l}(T,S). \tag{19}$$

From Corollary 4.2 (i) and (18) we conclude that $\sigma_{gD\Phi_l}(T,S)$ is closed. As $\sigma_{gD\Phi_l}(T,S) \subset \sigma(T,S) = \sigma(TS^{-1})$ we conclude that $\sigma_{gD\Phi_l}(T,S)$ is bounded, and hence $\sigma_{gD\Phi_l}(T,S)$ is compact.

We prove that

$$\operatorname{int}\sigma_{gD\Phi_l}(T,S) = \operatorname{int}\sigma_{\Phi_l}(T,S). \tag{20}$$

The equality (19) provides the inclusion $\operatorname{int}\sigma_{\Phi_l}(T,S) \subset \sigma_{gD\Phi_l}(T,S)$ and so $\operatorname{int}\sigma_{\Phi_l}(T,S) \subset \operatorname{int}\sigma_{gD\Phi_l}(T,S)$. It is obvious that $\sigma_{gD\Phi_l}(T,S) \subset \sigma_{\Phi_l}(T,S)$, from which follows that $\operatorname{int}\sigma_{gD\Phi_l}(T,S) \subset \operatorname{int}\sigma_{\Phi_l}(T,S)$.

Since $\sigma_{gD\Phi_l}(T,S)$ is closed we have that $\partial\sigma_{gD\Phi_l}(T,S) \subset \sigma_{gD\Phi_l}(T,S)$ and from the equalities (19) and (20) it follows that

$$\partial\sigma_{gD\Phi_l}(T,S) \subset \sigma_{gS}(T,S). \tag{21}$$

Analogously, for $H \in \{\Phi_r, \mathcal{W}_l, \mathcal{W}_r, \Phi, \mathcal{W}, \Phi_{l,r}, \mathcal{G}_l, \mathcal{G}_r\}$ from Theorems 3.6, 3.13, 3.15, 3.18, 3.19, 3.20 we have that

$$\begin{aligned}\sigma_{gDH}(T,S) &= \sigma_{gS}(T,S) \cup \operatorname{acc}\sigma_H(T,S) \qquad (22)\\ &= \sigma_{gS}(T,S) \cup \operatorname{int}\sigma_H(T,S). \qquad (23)\end{aligned}$$

From (22) we get that $\sigma_{gDH}(T,S)$ is closed, while from (23) it follows that $\operatorname{int}\sigma_{gDH}(T,S) = \operatorname{int}\sigma_H(T,S)$, and hence

$$\partial\sigma_{gDH}(T,S) \subset \sigma_{gS}(T,S). \tag{24}$$

Since S is invertible and $S \in \operatorname{comm}^2(T)$, according to the proof of Theorem 3.22 we have that $\sigma_{gD}(T,S) = \operatorname{acc}\sigma(T,S)$, and hence $\sigma_{gD}(T,S)$ is closed. From the equivalence (i)$\Longleftrightarrow$(ii) in Theorem 3.22 it follows that $\sigma_{gD}(T,S) = \sigma_{gS}(T,S) \cup \operatorname{int}\sigma(T,S)$ and $\operatorname{int}\sigma_{gD}(T,S) = \operatorname{int}\sigma(T,S)$, which implies that

$$\partial\sigma_{gD}(T,S) \subset \sigma_{gS}(T,S). \tag{25}$$

Since the following inclusions hold

$$\begin{array}{ccccccccccc}
 & & & & \sigma_{gD\Phi_l}(T,S) & \subset & \sigma_{gD\mathcal{W}_l}(T,S) & \subset & \sigma_{gD\mathcal{G}_l}(T,S) & & \\
 & & & \rotatebox{45}{$\subset$} & & \rotatebox{-90}{$\subset$} & & \rotatebox{-90}{$\subset$} & & \rotatebox{-45}{$\subset$} & \\
\sigma_{gS}(T,S) & \subset & \sigma_{gD\Phi_{l,r}}(T,S) & \subset & & \subset & \sigma_{gD\Phi}(T,S) & \subset & \sigma_{gD\mathcal{W}}(T,S) & \subset & \sigma_{gD}(T,S), \\
 & & & \rotatebox{-45}{$\subset$} & & \rotatebox{90}{$\subset$} & & \rotatebox{90}{$\subset$} & & \rotatebox{45}{$\subset$} & \\
 & & & & \sigma_{gD\Phi_r}(T,S) & \subset & \sigma_{gD\mathcal{W}_r}(T,S) & \subset & \sigma_{gD\mathcal{G}_r}(T,S) & &
\end{array}$$

and since all aforementioned sets are compact, according to (1) and by using (21), (24) and (25) we get the desired result. $\square$

Corollary 4.5. *Let $T, S \in L(X)$, and let S be invertible and $S \in \operatorname{comm}^2(T)$. If one of $\sigma_{gS}(T,S)$, $\sigma_{gD\Phi_{l,r}}(T,S)$, $\sigma_{gD\Phi}(T,S)$, $\sigma_{gD\mathcal{W}}(T,S)$, $\sigma_{gD}(T,S)$, $\sigma_{gD\Phi_*}(T,S)$, $\sigma_{gD\mathcal{W}_*}(T,S)$, $\sigma_{gD\mathcal{G}_*}(T,S)$, where $* \in \{l,r\}$, is at most countable, then all of them are equal.*

Proof. It follows from Theorem 4.4 (ii). □

Corollary 4.6. *Let $T, S \in L(X)$, and let S be invertible and $S \in \operatorname{comm}^2(T)$. Then there are inclusions:*

$$\begin{array}{ccccccccc}
 & & \operatorname{iso}\sigma_{gD\mathcal{G}_l}(T,S) & \subset & \operatorname{iso}\sigma_{gD\mathcal{W}_l}(T,S) & \subset & \operatorname{iso}\sigma_{gD\Phi_l}(T,S) & & \\
 & \rotatebox{45}{$\subset$} & & & \cup & & \cup & \rotatebox{-45}{$\subset$} & \\
\operatorname{iso}\sigma_{gD}(T,S) & \subset & \operatorname{iso}\sigma_{gD\mathcal{W}}(T,S) & \subset & \operatorname{iso}\sigma_{gD\Phi}(T,S) & \subset & & & \operatorname{iso}\sigma_{gD\Phi_{l,r}}(T,S) \subset \operatorname{iso}\sigma_{gS}(T,S). \\
 & \rotatebox{-45}{$\subset$} & & & \cap & & \cap & \rotatebox{45}{$\subset$} & \\
 & & \operatorname{iso}\sigma_{gD\mathcal{G}_r}(T,S) & \subset & \operatorname{iso}\sigma_{gD\mathcal{W}_r}(T,S) & \subset & \operatorname{iso}\sigma_{gD\Phi_r}(T,S) & &
\end{array}$$

Proof. It follows from Theorem 4.4 and Lemma 2.3 (i). □

Theorem 4.7. *Let $T, S \in L(X)$, and let S be invertible and $S \in \operatorname{comm}^2(T)$. Then*
(i) $\operatorname{iso}\sigma_{gD\Phi_l}(T,S) \subset \operatorname{iso}\sigma_{gD\Phi}(T,S) \cup \operatorname{int}\sigma^e_{dsc}(T,S)$;
(ii) $\operatorname{iso}\sigma_{gD\Phi_r}(T,S) \subset \operatorname{iso}\sigma_{gD\Phi}(T,S) \cup \operatorname{int}\sigma^e_{D_+}(T,S)$;
(iii) $\operatorname{iso}\sigma_{gD\mathcal{W}_l}(T,S) \subset \operatorname{iso}\sigma_{gD\mathcal{W}}(T,S) \cup \operatorname{int}\sigma_{B\mathcal{W}_-}(T,S)$;
(iv) $\operatorname{iso}\sigma_{gD\mathcal{W}_r}(T,S) \subset \operatorname{iso}\sigma_{gD\mathcal{W}}(T,S) \cup \operatorname{int}\sigma_{B\mathcal{W}_+}(T,S)$;
(v) $\operatorname{iso}\sigma_{gD\mathcal{G}_l}(T,S) \subset \operatorname{iso}\sigma(T,S) \cup \operatorname{int}\sigma_{dsc}(T,S)$;
(vi) $\operatorname{iso}\sigma_{gD\mathcal{G}_r}(T,S) \subset \operatorname{iso}\sigma(T,S) \cup \operatorname{int}\sigma_{D_+}(T,S)$.

Proof. (i) Let $\lambda_0 \in \operatorname{iso}\sigma_{gD\Phi_l}(T,S)\setminus\operatorname{int}\sigma^e_{dsc}(T,S)$. There exists a sequence (λ_n) converging to λ_0 such that $d_e(T-\lambda_n S) < \infty$ and $T-\lambda_n S$ is essentially left generalized Drazin invertible for every $n \in \mathbb{N}$. Fix an arbitrary $n \in \mathbb{N}$. By Theorem 3.2, there exists $(M_n, N_n) \in Red(T-\lambda_n S)$ such that $T-\lambda_n S = ((T-\lambda_n S)_{M_n}) \oplus ((T-\lambda_n S)_{N_n})$, where $(T-\lambda_n S)_{M_n}$ is left Fredholm and $(T-\lambda_n S)_{N_n}$ is quasinilpotent. From the equality

$$\beta_m(T-\lambda_n S) = \beta_m((T-\lambda_n S)_{M_n}) + \beta_m((T-\lambda_n S)_{N_n}) \tag{26}$$

for an arbitrary $m \in \mathbb{N}$, since $d_e(T-\lambda_n S) < \infty$, we know that $d_e((T-\lambda_n S)_{M_n}) < \infty$. As $(T-\lambda_n S)_{M_n}$ is left Fredholm then $\alpha((T-\lambda_n S)_{M_n}) < \infty$, which implies that $a_e((T-\lambda_n S)_{M_n}) = 0$. According to [17, Lemma 22.11], $d_e((T-\lambda_n S)_{M_n}) = a_e((T-\lambda_n S)_{M_n}) = 0$, i.e. $\beta((T-\lambda_n S)_{M_n}) < \infty$ and so $(T-\lambda_n S)_{M_n}$ is a Fredholm operator. Therefore, $T-\lambda_n S$ is Fredholm-g-Drazin invertible for every $n \in \mathbb{N}$ and hence $\lambda_0 \in \partial\sigma_{gD\Phi}(T,S)$. From Theorem 4.4 (i) we have that $\partial\sigma_{gD\Phi}(T,S) \subset \sigma_{gD\Phi_l}(T,S)$, which together with $\lambda_0 \in \operatorname{iso}\sigma_{gD\Phi_l}(T,S) \cap \partial\sigma_{gD\Phi}(T,S)$ implies that $\lambda_0 \in \operatorname{iso}\sigma_{gD\Phi}(T,S)$, by Lemma 2.3 (ii).

(ii) Follows similarly to the proof of (i), since the equality

$$\alpha_m(T-\lambda_n S) = \alpha_m((T-\lambda_n S)_{M_n}) + \alpha_m((T-\lambda_n S)_{N_n}) \tag{27}$$

holds for every $m \in \mathbb{N}$ and $\partial\sigma_{gD\Phi}(T,S) \subset \sigma_{gD\Phi_r}(T,S)$.

(iii) Let $\lambda_0 \in \operatorname{iso}\sigma_{gD\mathcal{W}_l}(T,S)\setminus\operatorname{int}\sigma_{B\mathcal{W}_-}(T,S)$. There exists a sequence (λ_n) converging to λ_0 such that $T-\lambda_n S$ is lower semi B-Weyl and left Weyl-g-Drazin invertible for every $n \in \mathbb{N}$. Take an arbitrary $n \in \mathbb{N}$. We can find $m_n \in \mathbb{N}$ such that $R((T-\lambda_n S)^{m_n})$ is closed and $(T-\lambda_n S)_{m_n} : R((T-\lambda_n S)^{m_n}) \to R((T-\lambda_n S)^{m_n})$ is a lower semi-Fredholm operator with nonnegative index. Also, we can find a pair of subspaces $(M_n, N_n) \in Red(T-\lambda_n S)$, such that the operator $(T-\lambda_n S)_{M_n}$ is left Weyl and $(T-\lambda_n S)_{N_n}$ is quasinilpotent.

As in Theorem 3.13, we have

$$\alpha_{m_n}(T-\lambda_n S) = \alpha((T-\lambda_n S)_{m_n}), \tag{28}$$

$$\beta_{m_n}(T-\lambda_n S) = \beta((T-\lambda_n S)_{m_n}), \tag{29}$$

and from (27) and (26) we have the inequalities

$$\alpha_{m_n}(T-\lambda_n S) \geq \alpha_{m_n}((T-\lambda_n S)_{M_n}), \tag{30}$$

$$\beta_{m_n}(T-\lambda_n S) \geq \beta_{m_n}((T-\lambda_n S)_{M_n}). \tag{31}$$

Since $\beta((T-\lambda_n S)_{m_n}) < \infty$, from (29) and (31) we conclude that $d_e((T-\lambda_n S)_{M_n}) \leq m_n < \infty$. We also have $\alpha((T-\lambda_n S)_{M_n}) < \infty$, as the operator is left Weyl. Therefore, $a_e((T-\lambda_n S)_{M_n}) = 0$ and so $d_e((T-\lambda_n S)_{M_n}) = a_e((T-\lambda_n S)_{M_n}) = 0$. Hence, $(T-\lambda_n S)_{M_n}$ is a Fredholm operator.

From [3, Proposition 2.22], because $T - \lambda_n S$ is lower semi B-Weyl, and a direct sum of a semi-Fredholm operator and a quasinilpotent one, we have

$$i(T - \lambda_n S) = i((T - \lambda_n S)_{M_n}) \leq 0. \tag{32}$$

Also, since $T - \lambda_n S$ is lower semi B-Weyl we have (see [3, Proposition 2.12, Definition 2.13])

$$i(T - \lambda_n S) = i((T - \lambda_n S)_{m_n}) \geq 0. \tag{33}$$

Equalities (32) and (33) imply that $T - \lambda_n S$ is Weyl-g-Drazin for every $n \in \mathbb{N}$ and hence $\lambda_0 \in \partial\sigma_{gD\mathcal{W}}(T, S)$. From Theorem 4.4 (i) we have that $\partial\sigma_{gD\mathcal{W}}(T, S) \subset \sigma_{gD\mathcal{W}_l}(T, S)$, which together with $\lambda_0 \in \operatorname{iso}\sigma_{gD\mathcal{W}_l}(T, S) \cap \partial\sigma_{gD\mathcal{W}}(T, S)$ implies that $\lambda_0 \in \operatorname{iso}\sigma_{gD\mathcal{W}}(T, S)$, by Lemma 2.3 (ii).

(iv) Similarly to the proof of (iii).

(v) Let $\lambda_0 \in \operatorname{iso}\sigma_{gD\mathcal{G}_l}(T, S) \setminus \operatorname{int}\sigma_{dsc}(T, S)$. There exists a sequence (λ_n) converging to λ_0 such that $T - \lambda_n S$ is left generalized Drazin invertible and $d(T - \lambda_n S) < \infty$. For an arbitrary fixed $n \in \mathbb{N}$, there exists a pair $(M_n, N_n) \in Red(T - \lambda_n S)$ such that $T - \lambda_n S = (T - \lambda_n S)_{M_n} \oplus (T - \lambda_n S)_{N_n}$, where $(T - \lambda_n S)_{M_n}$ is left invertible and $(T - \lambda_n S)_{N_n}$ is quasinilpotent. The ascent of the operator $(T - \lambda_n S)_{M_n}$ is zero since it is injective, and the descent is finite since $d((T - \lambda_n S)_{M_n}) \leq d(T - \lambda_n S) < \infty$. From [2, Theorem 1.20], $a((T - \lambda_n S)_{M_n}) = d((T - \lambda_n S)_{M_n}) = 0$, so the operator $(T - \lambda_n S)_{M_n}$ is invertible. Hence, $\lambda_0 \in \partial\sigma(T, S)$. Obviously, $\partial\sigma(T, S) \subset \sigma_{gD\mathcal{G}_l}(T, S)$ and $\lambda_0 \in \operatorname{iso}\sigma_{gD\mathcal{G}_l}(T, S) \cap \partial\sigma(T, S)$ implies that $\lambda_0 \in \operatorname{iso}\sigma(T, S)$, by Lemma 2.3 (ii).

(vi) Similarly to the proof of (v). □

Corollary 4.8. *Let $T, S \in L(X)$, and let S be invertible and $S \in \operatorname{comm}^2(T)$. Then*
(i) $\sigma_{gD\Phi}(T, S) = \sigma_{gD\Phi_l}(T, S) \cup \operatorname{int}\sigma^e_{dsc}(T, S)$;
(ii) $\sigma_{gD\Phi}(T, S) = \sigma_{gD\Phi_r}(T, S) \cup \operatorname{int}\sigma^e_{D_+}(T, S)$;
(iii) $\sigma_{gD\mathcal{W}}(T, S) = \sigma_{gD\mathcal{W}_l}(T, S) \cup \operatorname{int}\sigma_{B\mathcal{W}_-}(T, S)$;
(iv) $\sigma_{gD\mathcal{W}}(T, S) = \sigma_{gD\mathcal{W}_r}(T, S) \cup \operatorname{int}\sigma_{B\mathcal{W}_+}(T, S)$.

Proof. (i) From the equivalence (i)$\Longleftrightarrow$(ix) in Theorem 3.6 we have that $\operatorname{int}\sigma^e_{dsc}(T, S) \subset \sigma_{gD\Phi_r}(T, S) \subset \sigma_{gD\Phi}(T, S)$ and since $\sigma_{gD\Phi_l}(T, S) \subset \sigma_{gD\Phi}(T, S)$, it follows that $\sigma_{gD\Phi_l}(T, S) \cup \operatorname{int}\sigma^e_{dsc}(T, S) \subset \sigma_{gD\Phi}(T, S)$.

In order to prove the converse inclusion suppose that there exists some $\lambda_0 \in \sigma_{gD\Phi}(T, S)$ that does not belong to the set $\sigma_{gD\Phi_l}(T, S) \cup \operatorname{int}\sigma^e_{dsc}(T, S)$. Let (λ_n) be the sequence converging to λ_0 such that $T - \lambda_n S$ is essentially left generalized Drazin invertible and $d_e(T - \lambda_n) < \infty$. From the proof of Theorem 4.7(i) we can see that $\lambda_0 \in \partial\sigma_{gD\Phi}(T, S) \subset \sigma_{gD\Phi_l}(T, S)$ which contradicts the assumption that λ_0 does not belong to $\sigma_{gD\Phi_l}(T, S)$.

Remaining inclusions can be proved analogously. □

Corollary 4.9. *Let $T, S \in L(X)$, and let S be invertible and $S \in \operatorname{comm}^2(T)$. Then*
(i) $\operatorname{iso}\sigma_{gS}(T, S) \subset \operatorname{iso}\sigma_{gD\Phi_l}(T, S) \cup \operatorname{int}\sigma^e_{D_+}(T, S)$;
(ii) $\operatorname{iso}\sigma_{gS}(T, S) \subset \operatorname{iso}\sigma_{gD\Phi_r}(T, S) \cup \operatorname{int}\sigma^e_{dsc}(T, S)$;
(iii) $\operatorname{iso}\sigma_{gS}(T, S) \subset \operatorname{iso}\sigma_{gD\mathcal{W}_l}(T, S) \cup \operatorname{int}\sigma_{B\mathcal{W}_+}(T, S)$;
(iv) $\operatorname{iso}\sigma_{gS}(T, S) \subset \operatorname{iso}\sigma_{gD\mathcal{W}_r}(T, S) \cup \operatorname{int}\sigma_{B\mathcal{W}_-}(T, S)$;
(v) $\operatorname{iso}\sigma_{gS}(T, S) \subset \operatorname{iso}\sigma_{gD\mathcal{G}_l}(T, S) \cup \operatorname{int}\sigma_{D_+}(T, S)$;
(vi) $\operatorname{iso}\sigma_{gS}(T, S) \subset \operatorname{iso}\sigma_{gD\mathcal{G}_l}(T, S) \cup \operatorname{int}\sigma_p(T, S)$;
(vii) $\operatorname{iso}\sigma_{gS}(T, S) \subset \operatorname{iso}\sigma_{gD\mathcal{G}_r}(T, S) \cup \operatorname{int}\sigma_{dsc}(T, S)$;
(vii) $\operatorname{iso}\sigma_{gS}(T, S) \subset \operatorname{iso}\sigma_{gD\mathcal{G}_r}(T, S) \cup \operatorname{int}\sigma_{cp}(T, S)$.

Proof. (i) Let $\lambda_0 \in \operatorname{iso}\sigma_{gS}(T, S) \setminus \operatorname{int}\sigma^e_{D_+}(T, S)$. There exists a sequence (λ_n) that converges to λ_0 and for which $T - \lambda_n S$ admits a GSD, while $\lambda_n \notin \sigma^e_{D_+}(T, S)$. Then $0 \notin \operatorname{int}\sigma^e_{D_+}(T - \lambda_n S, S)$ for each $n \in \mathbb{N}$, hence according to Theorem 3.2, $T - \lambda_n S$ is essentially left generalized Drazin invertible. Therefore, we have that $\lambda_0 \in \partial\sigma_{gD\Phi_l}(T, S) \cap \operatorname{iso}\sigma_{gS}(T, S)$ which together with $\partial\sigma_{gD\Phi_l}(T, S) \subset \sigma_{gS}(T, S)$ from Theorem 4.4 (i), by Lemma 2.3 (ii) implies that $\lambda_0 \in \operatorname{iso}\sigma_{gD\Phi_l}(T, S)$.

All the remaining inclusions are proved similarly, by using Theorems 3.6, 3.13, 3.15, 3.19 and 3.20. □

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