Filomat 37:28 (2023), 9531–9542
https://doi.org/10.2298/FIL2328531A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

$(n_1, \cdots, n_d)$-quasi-(p, q)-isometric commuting tuple of operators

Naeem Ahmad[a], **E.S. Kamel**[a,b], **Sid Ahmed Ould Ahmed Mahmoud**[a,*]

[a]*Mathematics Department, College of Science, Jouf University, Sakaka P.O.Box 2014. Saudi Arabia*
[b]*Mathematics Department, College of Science, Fayoum University, P.O.Box:63514, Fayoum, Egypt*

Abstract. In this work we construct the concept based on the extension of n-quasi-p-isometric operators of a single operator studied in [11, 14] to the multi-dimentional operators. we are introducing some new interesting results of these family of tuples of operators that are expanding some results of recently published works based on a single operator.

1. Introduction

Let $\mathcal{X}$ (resp. $\mathcal{Y}$) be a complex Banach (resp. Hilbert) space and $\mathcal{B}[\mathcal{X}]$ (resp. $\mathcal{B}[\mathcal{Y}]$) be the algebra of bounded operators on $\mathcal{X}$ (resp.$\mathcal{Y}$) and say $I_{\mathcal{X}}$ (resp. $I_{\mathcal{Y}}$) be the identity of $\mathcal{B}[\mathcal{X}]$ (resp.$\mathcal{B}[\mathcal{Y}]$), then an operator $A \in \mathcal{B}[\mathcal{Y}]$, its range, kernel and adjoint of A represent by $\mathcal{R}(A)$, $\ker(A)$ and A^* respectively.

Any $A \in \mathcal{B}[\mathcal{Y}]$ is called p-isometry for $p > 0$ if

$$\sum_{l=0}^{p}(-1)^{p-l}\binom{p}{l}A^{*l}A^{l} = 0, \tag{1.1}$$

or equivalently,

$$\sum_{l=0}^{p}(-1)^{p-l}\binom{p}{l}\|A^{l}x\|^{2} = 0 \quad \forall \ x \in \mathcal{Y}, \tag{1.2}$$

where $\binom{p}{l} = \dfrac{p!}{l!(p-l)!}$ [1–3].

Such p-isometry was represent by Authors [1–3]. Several properties have been studied as nilpotent perturbations, products and tensor products of such operators (see [2, 3]. The extension of these family of operators to (p, q)-isometry on general Banach spaces has been studied by Sid Ahmed [13], Bayart [4] and

2020 *Mathematics Subject Classification*. Primary 47B15; Secondary 47B20, 47A15 optionally.
Keywords. p-isometries, n-quasi-p-isometries, p-isometric tuple of operators.
Received: 27 December 2022; Accepted: 02 June 2023
Communicated by Dragan S. Djordjević
Research supported by the Deanship of Scientific Research at Jouf University under grant No (DSR-2021-03-0335).
* Corresponding author: Sid Ahmed Ould Ahmed Mahmoud
Email addresses: naataullah@ju.edu.sa (Naeem Ahmad), essayed@ju.edu.sa (E.S. Kamel), sidahmed@ju.edu.sa (Sid Ahmed Ould Ahmed Mahmoud)

Hoffman et al. [9]. Given $p \in \mathbb{N}$ and $q \in (0,\infty)$, an operator $A \in \mathcal{B}[\mathcal{X}]$ is called an (p,q)-isometry if and only if

$$\sum_{l=0}^{p}(-1)^{p-l}\binom{p}{l}\|A^l x\|^q = 0 \ \ (\forall\ x \in \mathcal{X} \quad (see\ [4, 15]).$$

In [14], the authors has construct the concept of the family of n-quasi-p-isometry operators on Hilbert space which extended the family of p-isometry operators. An operator $A \in \mathcal{B}[\mathcal{Y}]$ is called n-quasi-p-isometry if it hold the following inequalities

$$A^{*n}\Big(\sum_{l=0}^{p}(-1)^{p-l}\binom{p}{l}A^{*l}A^l\Big)A^n = 0,$$

or if

$$\sum_{l=0}^{p}(-1)^{p-l}\binom{p}{l}\|A^{l+n}x\|^2 = 0, \ \ \forall\ x \in \mathcal{Y}.$$

We find in [11] that the products of some n-quasi-p-isometry are p-isometry, the powers of an n-quasi-p-isometry are n-quasi-p-isometry and the perturbation of n-quasi-p-isometry by nilpotent operators are n-quasi-p-isometries.

Recently, we have seen the concept of n-quasi-(p,q)-isometries on a Banach spaces has been introduced in [10] which gives and extension of these family of (p,q)-isometry on Banach spaces as follows: An $A \in \mathcal{B}[\mathcal{X}]$ is called an n-quasi-(p,q)-isometry if it is hold the relation

$$\sum_{l=0}^{p}(-1)^{p-l}\binom{p}{l}\|A^{l+n}x\|^q = 0, \ \ \forall\ x \in \mathcal{Y},$$

for some positive integers n, p and real $q \in (0,\infty)$.

For $d \in \mathbb{Z}_+$, let $\mathbf{A} = (A_1,\cdots,A_d) \in \mathcal{B}[\mathcal{X}]^d$ be c.t.o. (abbreviated c.t.o.) and $\gamma = (\gamma_1,\cdots,\gamma_d) \in \mathbb{Z}_+^d$ (multi-indices) and set $\gamma! := \gamma_1!\cdots\gamma_d!$. Further, we define $\mathbf{A}^\gamma := A_1^{\gamma_1}A_2^{\gamma_2}\cdots A_d^{\gamma_d}$ where $A_j^{\gamma_j}$ is represent the product of A_j times itself γ_j times.

In [7], the authors has construct the family of p-isometry c.t.o. on a Hilbert space. An $\mathbf{S} = (S_1,\cdots,S_d) \in \mathcal{B}[\mathcal{Y}]^d$ is called p-isometry c.t.o. if

$$\sum_{l=0}^{p}(-1)^{p-l}\binom{p}{l}\left(\sum_{|\gamma|=l}\frac{l!}{\gamma!}\mathbf{S}^{*\gamma}\mathbf{S}^\gamma\right) = 0 \tag{1.3}$$

or

$$\sum_{l=0}^{p}(-1)^{p-l}\binom{p}{l}\left(\sum_{|\gamma|=l}\frac{l!}{\gamma!}\|\mathbf{S}^\gamma x\|^2\right) = 0 \ \text{ for all } x \in \mathcal{X}. \tag{1.4}$$

An extension of p-isometry c.t.o. to $(n_1,\cdots,n_d)$-quasi-p-isometry c.t.o. was construct in [5] as follows. If $\mathbf{A} = \big(A_1,\cdots,A_d\big) \in \mathcal{B}[\mathcal{Y}]^d$ be an $(n_1,\cdots,n_d)$-quasi-p-isometry c.t.o. if

$$\prod_{j=1}^{d}A_j^{*n_j}\Big(\sum_{l=0}^{p}(-1)^l\binom{p}{l}\sum_{|\gamma|=l}\frac{l!}{\gamma!}\mathbf{A}^{*\gamma}\mathbf{A}^\gamma\Big)\prod_{j=1}^{d}A_j^{n_j} = 0,$$

or

$$\sum_{l=0}^{p}(-1)^l\binom{p}{l}\Big(\sum_{|\gamma|=l}\frac{l!}{\gamma!}\|\mathbf{A}^{\gamma+\widetilde{\mathbf{n}}}x\|^2\Big) = 0,$$

for all $x \in \mathcal{Y}$, where $\widetilde{\mathbf{n}} = (n_1, \cdots, n_d)$.

In this work, we see an extension of the n-quasi-(p,q)-isometry operators to the multi-dimensional operator theory on Banach spaces.

This work based on the family of multivariable operators on Banach space which shows a extended of n-quasi-(p,q)-isometry operators and the concept of $(n_1, \cdots, n_d)$-quasi-(p,q)-isometry c.t.o.

2. $(n_1, \cdots, n_d)$-quasi- (p,q)-isometric tuples

On Banach space, the idea of a $(n_1, \cdots, n_d)$-quasi-(p,q)-isometry-(c.t.o.) has been developed

Definition 2.1. *Let $\mathbf{A} = (A_1, \cdots, A_d) \in \mathcal{B}[\mathcal{X}]^d$, $\widetilde{\mathbf{n}} = (n_1, \cdots, n_d) \in \mathbb{Z}_+^d$ and p is a positive integer. $\mathbf{A}$ is called $(n_1, \cdots, n_d)$-quasi-(p,q)-isometry c.t.o. if*

$$\sum_{l=0}^{p}(-1)^{p-l}\binom{p}{l}\Big(\sum_{|\gamma|=l}\frac{l!}{\gamma!}\|\mathbf{A}^{\gamma+\widetilde{\mathbf{n}}}x\|^q\Big) = 0. \tag{2.1}$$

If $d = 1$, then (2.1) coincides with (1.3).

Remark 2.2. (1) *If $\mathcal{R}(A_l^{n_l}) \subset \ker(A_k^{n_k})$ for $n_l n_k \neq 0$, for some l and k, then $\mathbf{A}$ is a $(n_1, \cdots, n_d)$-quasi-(p,q)-isometry c.t.o.*

(2) *Every (p,q)-isometric c.t.o. is an $\widetilde{\mathbf{n}}$-quasi-(p,q)-isometry c.t.o. for $(n_1, \cdots, n_d) \in \mathbb{N}^d$.*

Remark 2.3. *Let $\mathbf{A} = (A_1, A_2) \in \mathcal{B}[\mathcal{X}]^2$ be a c.t.o. Then we observe that*

(i) $\mathbf{A}$ is $(1,1)$-quasi-(1,q)-isometry c.t.o. if

$$\|A_1A_2x\|^q - \|A_1^2A_2x\|^q - \|A_1A_2^2x\|^q = 0.$$

(ii) $\mathbf{A}$ is an $(1,1)$-quasi-$(2,q)$-isometry c.t.o. if

$$\|A_1A_2x\|^q - 2\Big(\|A_1^2x\|^q + \|A_2^2x\|^q - \|A_1^2A_2^2x\|^q\Big) + \|A_1^3x\|^q + \|A_2^3x\|^q = 0.$$

Remark 2.4. *Let $\mathbf{A} = (A_1, \cdots, A_d) \in \mathcal{B}[\mathcal{X}]^d$ be c.t.o. Then*

(i) $\mathbf{A}$ is an $(1, \cdots, 1)$-quasi-(1,q)-isometry c.t.o. if and only if

$$\|A_1 \cdots A_dx\|^q - \sum_{j=1}^{d}\|A_1 \cdots A_{j-1}A_{j+1} \cdots A_dA_j^2x\|^q = 0.$$

(ii) $\mathbf{A}$ is an $(1, \cdots, 1)$-quasi-$(2,q)$-isometry c.t.o.e if and only if

$$\begin{aligned}\|A_1 \cdots A_dx\|^q \quad - \quad & 2\sum_{j=1}^{d}\|A_1 \cdots A_{j-1}A_{j+1} \cdots A_dA_j^2x\|^q + \sum_{j=1}^{d}\|A_1 \cdots A_{j-1}A_{j+1} \cdots A_dA_j^3x\|^q \\ & + \sum_{1\le j\neq l\le d}\|A_1 \cdots A_{j-1}^2A_{j+1}^2 \cdots A_{l-1}^2A_{l+1}^2 \cdots A_dA_l^2A_j^2x\|^q = 0.\end{aligned}$$

(iii) $\mathbf{A}$ is an $\widetilde{n} = (n_1, \cdots, n_d)$-quasi-$(2, q)$-isometry c.t.o. if and only if

$$\|A^{\widetilde{n}}x\|^q - 2\sum_{j=1}^{d} \|A^{\widetilde{n}}A_jx\|^q + \sum_{j=1}^{d} \|A^{\widetilde{n}}A_j^2x\|^q + 2\sum_{1\leq j<l\leq d} \|A^{\widetilde{n}}A_lA_jx\|^q = 0. \tag{2.2}$$

(iv) $\mathbf{A}$ is an $\widetilde{n} = (n_1, \cdots, n_d)$-quasi-$(1, q)$-isometry c.t.o. if and only if

$$\|A^{\widetilde{n}}x\|^q - \sum_{j=1}^{d} \|A_jA^{\widetilde{n}}x\|^q = 0. \tag{2.3}$$

Example 2.5. *Let $\mathbf{A} = (A_1, A_2) \in \mathcal{B}[\mathbb{R}^3]^2$, where*

$$A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}.$$

by showing that $\|x\| = |x_1| + |x_2| + |x_3|$. Direct calculations show that

$$\|A_1x_0\| + \|A_2x_0\| - \|x_0\| \neq 0, \text{for some } x_0 \in \mathbb{R}^3.$$

Moreover

$$\|A_1A_2x\| - \|A_1^2A_2x\| - \|A_1A_2^2x\| = 0, \forall\ x \in \mathbb{R}^3.$$

Therefore $\mathbf{A} = (A_1, A_2)$ is a $(1, 1)$-quasi-$(1, 1)$-isometry, but $\mathbf{A}$ is not a $(1, 1)$-isometry.

Remark 2.6. *From the above example, we observe that the family of $\widetilde{n}$-quasi-(p, q)-isometry contain the family of (p, q)-isometry c.t.o. as a proper subfamily.*

Example 2.7. *Let $A \in \mathcal{B}[\mathcal{X}]$ be an n-quasi (p, q)-isometry and set $\mathbf{A} = \left(\frac{1}{\sqrt[q]{d}}A, \cdots \frac{1}{\sqrt[q]{d}}A\right)$, then $\mathbf{A}$ is an $\left(n, \cdots n\right)$-quasi-*

(p,q)-isometry.

$$\begin{aligned}
&\sum_{l=0}^{p}(-1)^l\binom{p}{l}\sum_{|\gamma|=l}\frac{l!}{\gamma!}\left\|\Big(\frac{1}{\sqrt[q]{d}}A\Big)^{\gamma_1+n},\cdots,\Big(\frac{1}{\sqrt[q]{d}}A\Big)^{\gamma_d+n}x\right\|^q\\
=&\sum_{l=0}^{p}(-1)^l\binom{p}{l}\sum_{|\gamma|=l}\frac{l!}{\gamma!}\left\|\Big(\frac{1}{\sqrt[q]{d}}A\Big)^{|\gamma|+nd}x\right\|^q\\
=&\sum_{l=0}^{p}(-1)^l\binom{p}{l}\sum_{|\gamma|=l}\frac{l!}{\gamma!}\left\|\Big(\frac{1}{\sqrt[q]{d}}A\Big)^{l}\Big(\frac{1}{\sqrt[q]{d}}A\Big)^{nd}x\right\|^q\\
=&\sum_{l=0}^{p}(-1)^l\binom{p}{l}d^l\left\|\Big(\frac{1}{\sqrt[q]{d}}A\Big)^{l}\Big(\frac{1}{\sqrt[q]{d}}A\Big)^{nd}x\right\|^q\\
=&\sum_{l=0}^{p}(-1)^l\binom{p}{l}\Big(\sqrt[q]{d}\Big)^{ld}\left\|\Big(\frac{1}{\sqrt[q]{d}}A\Big)^{l}\Big(\frac{1}{\sqrt[q]{d}}A\Big)^{nd}x\right\|^q\\
=&\sum_{l=0}^{p}(-1)^l\binom{p}{l}\left\|A^l\Big(\frac{1}{\sqrt[q]{d}}A\Big)^{nd}x\right\|^q\\
=&\sum_{l=0}^{p}(-1)^l\binom{p}{l}\left\|A^l\Big(\frac{1}{\sqrt[q]{d}}\Big)^{nd}A^{nd}x\right\|^q\\
=&\sum_{l=0}^{p}(-1)^l\binom{p}{l}\left\|A^{l+n}A^{(n)(d-1)}\Big(\frac{1}{\sqrt[q]{d}}\Big)^{nd}x\right\|^q\\
=&\sum_{l=0}^{p}(-1)^l\binom{p}{l}\left\|A^{l+n}y\right\|^q=0.
\end{aligned}$$

where $y=A^{(n)(d-1)}\Big(\frac{1}{\sqrt[q]{d}}\Big)^{nd}x$. Therefore $\mathbf{A}$ is an $\Big(n,\cdots n\Big)$-quasi-(p,q)-isometry tuple as required.

Proposition 2.8. *Let $\mathbf{A}=(A_1,\cdots,A_d)\in\mathcal{B}[\mathcal{X}]^d$, is an $(n_1,\cdots,n_d)$-quasi-(p,q)-isometry c.t.o., if and only if $\mathbf{A}$ is an (p,q)-isometry on $\overline{\mathcal{R}(\mathbf{A}^{\widetilde{\mathbf{n}}})}$.*

Proof. Let $\mathbf{A}=(A_1,\cdots,A_d)\in\mathcal{B}[\mathcal{X}]^d$, $\mathbf{A}$ is an $(n_1,\cdots,n_d)$-quasi-(p,q)-isometry c.t.o. if and only if, $\forall\ x\in\mathcal{X}$.

$$\begin{aligned}
0 &= \sum_{l=0}^{p}(-1)^l\binom{p}{l}\sum_{|\gamma|=l}\frac{l!}{\gamma!}\|\mathbf{A}^{\gamma+\widetilde{\mathbf{n}}}x\|^q\\
&= \sum_{l=0}^{p}(-1)^l\binom{p}{l}\sum_{|\gamma|=l}\frac{l!}{\gamma!}\|\mathbf{A}^{\gamma}(\mathbf{A}^{\widetilde{\mathbf{n}}}x)\|^q\\
&= \sum_{l=0}^{p}(-1)^l\binom{p}{l}\sum_{|\gamma|=l}\frac{l!}{\gamma!}\|\mathbf{A}^{\gamma}x\|^q,\ \forall x\in\overline{\mathcal{R}(\mathbf{A}^{\widetilde{\mathbf{n}}})}.
\end{aligned}$$

Equivalently, $\mathbf{A}$ is an (p,q)-isometry on $\overline{\mathcal{R}(\mathbf{A}^{\widetilde{\mathbf{n}}})}$. □

For $\widetilde{\mathbf{n}}=(n_1,\cdots,n_d)$, $\widetilde{\mathbf{n}}'=(\mathbf{n}_1',\cdots,\mathbf{n}_d')\in\mathbb{Z}_+^d$, we will say that $\widetilde{\mathbf{n}}\le\widetilde{\mathbf{n}}'$, if $\mathbf{n}_i\le\mathbf{n}_i'$, $i=1,\cdots,d$.

Proposition 2.9. *Let* $\mathbf{A} = (A_1, \cdots, A_d) \in \mathcal{B}[\mathcal{X}]^d$ *be an* $\widetilde{\mathbf{n}}$*-quasi-*(p,q)*-isometry c.t.o., then* $\mathbf{A}$ *is an* $\widetilde{\mathbf{n}}'$*-quasi-*(p,q)*-isometry c.t.o.,* $\forall\ \widetilde{\mathbf{n}}' \geq \widetilde{\mathbf{n}}$.

Proof. Consider $\mathbf{A}$ is an $\widetilde{\mathbf{n}}$-quasi-(p,q)-isometry c.t.o. By Proposition 2.8, we shows that $\mathbf{A}$ is an (p,q)-isometry c.t.o. on $\overline{\mathcal{R}(\mathbf{A}^{\widetilde{\mathbf{n}}})}$. It is obvious that

$$\overline{\mathcal{R}(\mathbf{A}^{\widetilde{\mathbf{n}}})} \supset \overline{\mathcal{R}(\mathbf{A}^{\widetilde{\mathbf{n}}'})} \quad \forall \quad \widetilde{\mathbf{n}}' \geq \widetilde{\mathbf{n}}.$$

This shows that $\mathbf{A}$ is an (p,q)-isometry c.t.o. on $\overline{\mathcal{R}(\mathbf{A}^{\widetilde{\mathbf{n}}'})}$. Hence, $\mathbf{A}$ is an $\widetilde{\mathbf{n}}'$-quasi-(p,q)-isometry c.t.o. $\forall\, \widetilde{\mathbf{n}}' \geq \widetilde{\mathbf{n}}$. □

The following theorem extended [10, Theorem 2.2].

Theorem 2.10. *Let* $\mathbf{A} = (A_1, \cdots, A_d) \in \mathcal{B}[\mathcal{X}]^d$ *be an* $\widetilde{\mathbf{n}}$*-quasi-*(p,q)*-isometry c.t.o., then* $\mathbf{A}$ *is an* $\widetilde{\mathbf{n}}$*-quasi-*$(p+l,q)$*-isometry c.t.o.,* $\forall\ l$.

Proof. we need to show that that $\mathbf{A}$ is an $\widetilde{\mathbf{n}}$-quasi-$(p+1,q)$-isometry c.t.o. .

Indeed, we have

$$\begin{aligned}
&\sum_{l=0}^{p+1}(-1)^l\binom{p+1}{l}\sum_{|\gamma|=k}\frac{l!}{\gamma!}\|\mathbf{A}^{\gamma}\mathbf{A}^{\widetilde{\mathbf{n}}}x\|^q \\
= &\ \|\mathbf{A}^{\widetilde{\mathbf{n}}}x\|^q + \sum_{l=0}^{p}(-1)^l\Big[\binom{p}{l}+\binom{p}{l-1}\Big]\sum_{|\gamma|=l}\frac{l!}{\gamma!}\|\mathbf{A}^{\gamma}\mathbf{A}^{\widetilde{\mathbf{n}}}x\|^q - (-1)^p\sum_{|\gamma|=p+1}\frac{(p+1)!}{\gamma!}\|\mathbf{A}^{\gamma}\mathbf{A}^{\widetilde{\mathbf{n}}}x\|^q \\
= &\ \sum_{l=0}^{p}(-1)^l\binom{p}{l}\sum_{|\gamma|-l}\frac{l!}{\gamma!}\|\mathbf{A}^{\gamma}\mathbf{A}^{\widetilde{\mathbf{n}}}x\|^q - \sum_{l=0}^{p-1}(-1)^l\binom{p}{l}\sum_{|\gamma|=l+1}\frac{(l+1)!}{\gamma!}\|\mathbf{A}^{\gamma}\mathbf{A}^{\widetilde{\mathbf{n}}}x\|^q \\
&-(-1)^p\sum_{|\gamma|=p+1}\frac{(p+1)!}{\gamma!}\|\mathbf{A}^{\gamma}\mathbf{A}^{\widetilde{\mathbf{n}}}x\|^q \\
= &\ -\sum_{l=0}^{p-1}(-1)^l\binom{p}{l}\sum_{|\gamma|=l+1}\frac{l!(\gamma_1+\cdots+\gamma_d)}{\gamma_1!.\gamma_2!\cdots\gamma_d!}\|\mathbf{A}^{\gamma}\mathbf{A}^{\widetilde{\mathbf{n}}}x\|^q \\
&-(-1)^p\sum_{|\gamma|=p+1}\frac{p!(\gamma_1+\cdots+\gamma_d)}{\gamma_1!.\gamma_2!\cdots\gamma_d!}\|\mathbf{A}^{\gamma}\mathbf{A}^{\widetilde{\mathbf{n}}}x\|^q \\
= &\ -\sum_{j=1}^{d}\sum_{l=0}^{p-1}(-1)^l\binom{p}{l}\sum_{|\gamma|=l+1}(-1)^l\binom{p}{l}\frac{l!\gamma_j}{\gamma_1!.\gamma_2!\cdots\gamma_d!}\|A^{\gamma_1}\cdots A_j^{\gamma_j-1}A_{j+1}^{\gamma_{j+1}}\cdots A_d^{\gamma_d}A_j\mathbf{A}^{\widetilde{\mathbf{n}}}x\|^q \\
&-(-1)^p\sum_{j=1}^{d}\sum_{|\gamma|=p+1}\frac{p!\gamma_j}{\gamma_1!.\gamma_2!\cdots\gamma_d!}\|A^{\gamma_1}\cdots A_j^{\gamma_j-1}A_{j+1}^{\gamma_{j+1}}\cdots A_d^{\gamma_d}A_j\mathbf{A}^{\widetilde{\mathbf{n}}}x\|^q \\
= &\ -\sum_{j=1}^{d}\sum_{l=0}^{p-1}(-1)^l\binom{p}{l}\sum_{|\delta|=l}(-1)^l\binom{p}{l}\frac{l!}{\delta!}\|\mathbf{A}^{\delta}A_j\mathbf{A}^{\widetilde{\mathbf{n}}}x\|^q \\
&-(-1)^p\sum_{j=1}^{d}\sum_{|\gamma|=p}\frac{p!}{\delta!}\|\mathbf{A}^{\delta}A_j\mathbf{A}^{\widetilde{\mathbf{n}}}x\|^q \\
= &\ -\sum_{j=1}^{d}\sum_{l=0}^{p}(-1)^l\binom{p}{l}\sum_{|\delta|=l}\frac{l1}{\delta!}\|\mathbf{A}^{\delta}A_j\mathbf{A}^{\widetilde{\mathbf{n}}}x\|^q \\
= &\ 0.
\end{aligned}$$

□

Proposition 2.11. *Let $\mathbb{S}_d$ be a symmetry group on d symbols $\{1,2,...,d\}$ and let $\mathbf{A} = (A_1, \cdots, A_d) \in \mathcal{B}[\mathcal{X}]^d$. If $\mathbf{A}$ is an $\widetilde{\mathbf{n}} = (n_1, \cdots, n_d)$-quasi-$(p,q)$-isometry c.t.o., then for every $\sigma \in \mathbb{S}_d$, $\mathbf{A}_\sigma := (A_{\sigma(1)}, \cdots, A_{\sigma(d)})$ is an $\widetilde{\mathbf{n}}_\sigma = (n_{\sigma(1)}, \cdots, n_{\sigma(d)})$-quasi-$(p,q)$-isometry c.t.o.*

Proof. It follows from the condition that $\prod_{j=1}^{d} A_j = \prod_{j=1}^{d} A_{\sigma(j)}$ and the identity

$$\begin{aligned}&\sum_{l=0}^{p}(-1)^l\binom{p}{l}\sum_{|\alpha|=l}\frac{l!}{\gamma!}\|\mathbf{A}_\sigma{}^{\gamma+\widetilde{\mathbf{n}}_\sigma}x\|^q\\ &\sum_{l=0}^{p}(-1)^l\binom{p}{l}\sum_{|\alpha|=l}\frac{l!}{\gamma!}\|\mathbf{A}^{\gamma+\widetilde{\mathbf{n}}}x\|^q\\ =\ & 0.\end{aligned}$$

□

The following theorem describe the condition under which an $(n_1, \cdots, n_d)$-quasi-(p,q)-isometry c.t.o. to be any $(1, \cdots, 1)$-quasi-(p,q)-isometry c.t.o.

Theorem 2.12. *If $\mathbf{A} = (A_1, \cdots, A_d) \in \mathcal{B}[\mathcal{X}]^d$ is an $\widetilde{\mathbf{n}}$-quasi-(p,q)-isometric c.t.o. such that $\mathcal{R}(A_j) = \mathcal{R}(A_j^{n_j})$ for each j, $1 \le j \le d$, then $\mathbf{A}$ is an quasi-(p,q)-isometric c.t.o.*

Proof. Since $\mathbf{A}$ is an $\widetilde{\mathbf{n}}$-quasi-(p,q)-isometric c.t.o. for $j = 1, \cdots, d$, then we have

$$\begin{aligned}&\sum_{l=0}^{p}(-1)^l\binom{p}{l}\sum_{|\gamma|=l}\frac{l!}{\gamma!}\|\mathbf{A}^{\gamma+\widetilde{\mathbf{n}}}x\|^q = 0\\ \Rightarrow\ &\sum_{l=0}^{p}(-1)^l\binom{p}{l}\sum_{|\gamma|=l}\frac{l!}{\gamma!}\|\mathbf{A}^{\gamma}A_1{}^{n_1}\cdots A_d{}^{n_d}x\|^q = 0\\ \Rightarrow\ &\sum_{l=0}^{p}(-1)^l\binom{p}{l}\sum_{|\gamma|=l}\frac{l!}{\gamma!}\|\mathbf{A}^{\gamma}A_1{}^{n_1}\cdots A_{d-1}{}^{n_{d-1}}A_d y\|^q = 0\\ \Rightarrow\ &\sum_{l=0}^{p}(-1)^l\binom{p}{l}\sum_{|\gamma|=l}\frac{l!}{\gamma!}\|\mathbf{A}^{\gamma}A_d A_1{}^{n_1}\cdots A_{d-1}{}^{n_{d-1}}y\|^q = 0\\ \Rightarrow\ &\vdots\\ \Rightarrow\ &\sum_{l=0}^{p}(-1)^l\binom{p}{l}\sum_{|\gamma|=l}\frac{l!}{\gamma!}\|\mathbf{A}^{\gamma}A_1\cdots A_d x\|^q = 0.\end{aligned}$$

Hence, $\mathbf{A}$ is an quasi-(p,q)-isometric c.t.o. □

Corollary 2.13. *If $\mathbf{A} = (A_1, \cdots, A_d) \in \mathcal{B}[\mathcal{X}]^d$ is an $\widetilde{\mathbf{n}}$-quasi-(p,q)-isometric c.t.o. such that each $A_j^2 = A_j$ for $j = 1, \cdots, d$, then $\mathbf{A}$ is a quasi-(p,q)-isometric c.t.o.*

Proof. Each $A_j^2 = A_j$, then $\mathcal{R}(A_j) = \mathcal{R}(A_j^{n_j})$ $\forall j = 1, ..., d$. Hence the prove is consequence Theorem 2.12. □

Proposition 2.14. *Let* $\mathbf{A} = (A_1, \cdots, A_d) \in \mathcal{B}[\mathcal{X}]^d$ *be an* $\widetilde{\mathbf{n}}$*-quasi-*$(2, q)$*-isometry c.t.o., then the following axioms are true.*

$$\sum_{|\gamma|=l} \frac{l!}{\gamma!} \|\mathbf{A}^{\gamma+\widetilde{\mathbf{n}}} x\|^q = (1-l)\|\mathbf{A}^{\widetilde{\mathbf{n}}} x\|^q + l\Big(\sum_{j=1}^{d} \|\mathbf{A}^{\widetilde{\mathbf{n}}} A_j x\|^q\Big), \ \ \forall \ \ x \in \mathcal{X}, \ \forall \ l \in \mathbb{N}. \tag{2.4}$$

$$\sum_{j=1}^{d} \|\mathbf{A}^{\widetilde{\mathbf{n}}} A_j x\|^q \geq \frac{l}{l-1} \|\mathbf{A}^{\widetilde{\mathbf{n}}} x\|^q \quad \forall \ x \in \mathcal{X}, \ l \in \mathbb{N}, l \neq 1. \tag{2.5}$$

$$\sum_{j=1}^{d} \|\mathbf{A}^{\widetilde{\mathbf{n}}} A^{n_j} x\|^q \geq \|\mathbf{A}^{\widetilde{\mathbf{n}}} x\|^q \quad \forall \ x \in \mathcal{X}. \tag{2.6}$$

$$\lim_{l \to \infty} \Big(\sum_{|\gamma|=l} \frac{l!}{\gamma!} \|\mathbf{A}^{\gamma+\widetilde{\mathbf{n}}} x\|^q\Big)^{\frac{1}{l}} = 1. \tag{2.7}$$

Proof. We us induction on l to prove inequality (2.4). For $l \in \{0, 1\}$ it is obvious. presume that (2.4) hold l and prove it for $l+1$. In view of [9, Lemma 2.1], it follows that

Therefore by the induction hypothesis, we get

$$\begin{aligned}
&\sum_{|\gamma|=l+1} \frac{(l+1)!}{\gamma!} \|\mathbf{A}^{\alpha+\widetilde{\mathbf{n}}} x\|^q \\
= \ & (1-l) \sum_{i=1}^{d} \|A_i \mathbf{A}^{\widetilde{\mathbf{n}}} x\|^q + l \sum_{i=1}^{d} \Big(\sum_{j=1}^{d} \|A_j A_i \mathbf{A}^{\widetilde{\mathbf{n}}} x\|^q\Big) \\
= \ & (1-l) \sum_{i=1}^{d} \|A_i \mathbf{A}^{\widetilde{\mathbf{n}}} x\|^q + k \sum_{j=1}^{d} \|A_j^2 \mathbf{A}^{\widetilde{\mathbf{n}}} x\|^q + 2l\Big(\sum_{1 \leq j \neq i \leq d} \|A_j A_i \mathbf{A}^{\widetilde{\mathbf{n}}} x\|^q\Big).
\end{aligned}$$

Since $\mathbf{A}$ is an $\widetilde{\mathbf{n}}$-quasi-$(2, q)$-isometry c.t.o., it follows from (2.2)

$$\begin{aligned}
&\sum_{|\gamma|=l+1} \frac{(l+1)!}{\gamma!} \|\mathbf{A}^{\alpha+\widetilde{\mathbf{n}}} x\|^q \\
= \ & (1-l) \sum_{i=1}^{d} \|A_i \mathbf{A}^{\widetilde{\mathbf{n}}} x\|^q + l\Big(-\|\mathbf{A}^{\widetilde{\mathbf{n}}} x\|^q + 2 \sum_{i=1}^{d} \|\mathbf{A}^{\widetilde{\mathbf{n}}} A_i x\|^q\Big) \\
= \ & -l\|\mathbf{A}^{\widetilde{\mathbf{n}}} x\|^q + (l+1)\Big(\sum_{1 \leq i \leq d} \|A_i \mathbf{A}^{\widetilde{\mathbf{n}}} x\|^q\Big),
\end{aligned}$$

so that (2.4) holds for $l+1$.

The inequality (2.5) follows from (2.4) and the inequality (2.6) follows from (2.5) by taking $l \longrightarrow \infty$. To proof (2.7), it follows from (2.4)

$$\lim_{l \to \infty} \sup \Big(\sum_{|\gamma|=l} \frac{l!}{\gamma!} \|\mathbf{A}^{\gamma+\widetilde{\mathbf{n}}} x\|^q\Big)^{\frac{1}{l}} \leq 1.$$

However according to (2.6), $\Big(\sum_{|\gamma|=l}\frac{l!}{\gamma!}\|\mathbf{A}^{\gamma+\widetilde{\mathbf{n}}}x\|^q\Big)^{\frac{1}{l}}$ is monotonically increasing, so that

$$\liminf_{l\to\infty}\Big(\sum_{|\gamma|=l}\frac{l!}{\gamma!}\|\mathbf{A}^{\gamma+\widetilde{\mathbf{n}}}x\|^q\Big)^{\frac{1}{l}} \geq \lim_{l\to\infty}\Big(\|\mathbf{A}^{\gamma+\widetilde{\mathbf{n}}}x\|^q\Big) = 1.$$

□

Definition 2.15. *Let* $\mathbf{A} = (A_1, \cdots, A_d) \in \mathcal{B}[\mathcal{X}]^d$ *be a c.t.o. Then* $\mathbf{A}$ *is called bounded if*

$$\sup\Big\{\sum_{|\gamma|=l}\frac{l!}{\gamma!}\big\|A^\gamma x\|^q \ , \ \forall\ l \in \mathbb{N}\Big\} < \infty$$

for all $x \in \mathcal{X}$.

Theorem 2.16. *Let* $\mathbf{A} = (A_1, \cdots, A_d)$ *be an* $\widetilde{\mathbf{n}}$*-quasi-*(p,q)*-isometry c.t.o. and bounded. Then*

$$\sum_{i=1}^{d}\|A_i\mathbf{A}^{\widetilde{\mathbf{n}}}x\|^q = \|\mathbf{A}^{\widetilde{\mathbf{n}}}x\|^q,$$

i.e., $\mathbf{A}$ *is an* $\widetilde{\mathbf{n}}$*-quasi-*$(1,q)$*-isometry c.t.o.*

Proof. Since $\mathbf{A}$ is an $\widetilde{\mathbf{n}}$-quasi-(p,q)-isometry c.t.o., then $\mathbf{A}$ is an (p,q)-isometric on $\overline{\mathcal{R}(\mathbf{A}^{\widetilde{\mathbf{n}}})}$ (Proposition 2.8). Therefore

$$\sum_{|\gamma|=l}\frac{l!}{\gamma!}\|A^\gamma\mathbf{A}^{\widetilde{\mathbf{n}}}x\|^q = \sum_{j=1}^{p}\binom{l}{j}\Psi_j(A,\mathbf{A}^{\widetilde{\mathbf{n}}}x)$$

Hence there exist real numbers $\delta_0(x), \delta_1(x), ..., \delta_{p-1}(x)$ such that

$$\sum_{i=1}^{d}\|A_i^l\mathbf{A}^{\widetilde{\mathbf{n}}}x\|^q = \sum_{j=0}^{p-1}\delta_j(x)\, l^j. \tag{2.8}$$

Since $\mathbf{A}$ is power bounded, for $x \in \mathcal{X}$,

$$M = \sup\Big\{\sum_{|\gamma|=l}\frac{l!}{\alpha!}\|A^\gamma x\|^l,\ \ l = 0,1,2,...\Big\} < \infty.$$

Then we have

$$0 \leq \sup\Big\{\sum_{j=0}^{p-1}\delta_j(x)l^j \ :\ l = 0,1,2,...\Big\} \leq M^q.$$

From the fact that l is arbitrary, we obtain $\delta_1(x) = \delta_2(x) = \cdots = \delta_{p-1}(x) = 0$. Hence

$$\sum_{i=0}^{d}\|A_i^l\mathbf{A}^{\widetilde{\mathbf{n}}}x\|^q = \|\mathbf{A}^{\widetilde{\mathbf{n}}}x\|^q.$$

Since l is arbitrary, by showing $l = 1$ we get equality. Hence $\mathbf{A}$ is an $\widetilde{\mathbf{n}}$-quasi-$(1,q)$-isometry c.t.o. by (2.4). □

3. Spectral properties of $\widetilde{\mathbf{n}}$-quasi-(p, q)-isometry

In this section we study some spetral properties of c.t.o. Our inspiration can be from referred to [6–8, 12, 15].

Definition 3.1. *([6]). Let $\mathbf{A} = (A_1, \cdots, A_d) \in \mathcal{B}[\mathcal{X}]^d$ be any c.t.o. on complex Banach space $\mathcal{X}$.*
Let $\mu = (\mu_1, \cdots, \mu_d) \in \mathbb{C}^d$, then μ is said to be point eigenvalue of $\mathbf{A}$ if

$$\bigcap_{j=1}^{d} \ker(A_j - \mu_j) \neq 0,$$

the set of all joint spectrum of $\mathbf{A}$ by $\sigma_p(\mathbf{A})$.

Definition 3.2. *([6]).Let $\mathbf{A} = (A_1 \cdots, A_d) \in \mathcal{B}[\mathcal{X}]^d$ be c.t.o, A number $\mu = (\mu_1, \cdots, \mu_d) \in \mathbb{C}^d$ is in the joint approximate point spectrum $\sigma_{ap}(\mathbf{A})$ if*

$$\exists \ (x_n)_n \in \mathcal{X}, \ \ \|x_n\| = 1$$

such that

$$(A_j - \mu_j)x_n \longrightarrow 0 \ \text{ as } \ n \longrightarrow \infty \ \text{ for every } \ \ j = 1, \cdots, d.$$

We denote by

$$\mathbb{B}_q(\mathbb{C}^d) := \{\mu = (\mu_1, \cdots, \mu_d) \in \mathbb{C}^d \ / \ \|\mu\|_q = \Big(\sum_{j=1}^{d} |\mu_j|^q\Big)^{\frac{1}{q}} < 1 \ \}$$

and

$$\partial\mathbb{B}_q(\mathbb{C}^d) := \{\mu = (\mu_1, ..., \mu_d) \in \mathbb{C}^d \ / \ \|\mu\|_q = \Big(\sum_{j=1}^{d} |\mu_j|^q\Big)^{\frac{1}{q}} = 1 \ \}$$

Theorem 3.3. *Let $\mathbf{A} = (A_1 \cdots, A_d) \in \mathcal{B}[\mathcal{X}]^d$ be an $\widetilde{\mathbf{n}}$-quasi-(p, q)-isometry c.t.o. Then $\sigma_{ap}(\mathbf{A}) \subset \Big[0\Big] \cup \Big\{(\mu_1, \cdots, \mu_d) \in \mathbb{C}^d / \Big(\sum_{l=1}^{d} |\mu_l|^q\Big)^{\frac{1}{q}} = 1\Big\}$. where*

$$\Big[0\Big] := \{(\mu_1, \cdots, \mu_d) \in \mathbb{C}^d : \ \prod_{l=1}^{d} \mu_l = 0\}.$$

Proof. Let $\mu = (\mu_1, \cdots, \mu_d) \in \sigma_{ap}(\mathbf{A})$, then $\exists \ \{x_j\}_{j\geq 1} \subset \mathcal{X}$, with $\|x_j\| = 1$ such that $(A_j - \mu_j I)x_j \longrightarrow 0$ for all $j = 1, 2, ..., d$. Since for $\gamma_j > 1$,

$$A_j^{\gamma_j} - \mu_j^{\gamma_j} = (A_j - \mu_j) \sum_{l=1}^{\gamma_j} \mu_j^{l-1} A_j^{\gamma_j - l}$$

By induction, for $\gamma \in \mathbb{Z}_+^d$, we have

$$(A^\gamma - \mu^\gamma I) = \sum_{l=0}^{d} \Big(\prod_{i\leq l} \mu_i^{\gamma_i}\Big)\Big(A_j^{\gamma_j} - \mu_j^{\gamma_i}\Big) \prod_{i>l} A_i^{\gamma_i}.$$

$$
\begin{aligned}
0 &= \sum_{l=0}^{p}(-1)^l\binom{p}{l}\sum_{|\alpha|=l}\frac{l!}{\alpha!}\|\mathbf{A}^{\gamma+\widetilde{\mathbf{n}}}x_j\|^q \\
&= \sum_{l=0}^{p}(-1)^l\binom{p}{l}\sum_{|\alpha|=l}\frac{l!}{\alpha!}\|(\mathbf{A}^{\gamma+\widetilde{\mathbf{n}}}-\mu^{\gamma+\widetilde{\mathbf{n}}})x_j+\mu^{\gamma+\widetilde{\mathbf{n}}}x_j\|^q \quad \forall j \\
&= \sum_{l=0}^{p}(-1)^l\binom{p}{l}\sum_{|\alpha|=l}\frac{l!}{\alpha!}\|\mu^{\gamma+\widetilde{\mathbf{n}}}\|^q \\
&= |\mu|^{q\widetilde{\mathbf{n}}}\sum_{l=0}^{p}(-1)^l\binom{p}{l}\sum_{|\alpha|=l}\frac{l!}{\alpha!}|\mu|^{q\gamma} \\
0 &= |\mu|^{q\widetilde{\mathbf{n}}}\Big(1-\Big(\sum_{l=1}^{d}|\mu|^q\Big)^{\frac{1}{q}}\Big)^{\gamma}.
\end{aligned}
$$

Therefore $\mu=(\mu_1,\cdots,\mu_d)\in[0]$ or $(\mu_1,\cdots,\mu_d)\in\mathbb{C}^d/\Big(\sum_{l=1}^{d}|\mu_l|^q\Big)^{\frac{1}{q}}=1\Big\}$. □

Theorem 3.4. *If* $\mathbf{A}=(A_1\cdots,A_d)\in\mathcal{B}[\mathcal{X}]^d$ *is an* $\widetilde{\mathbf{n}}$*-quasi-*(p,q)*-isometry c.t.o. Then* $r(\mathbf{A})=1$*. In particular* $\sigma(\mathbf{A})\subset\Big[0\Big]$ *or* $\sigma(\mathbf{A})\subset\Big[0\Big]\cup\Big\{(\mu_1,\cdots,\mu_d)\in\mathbb{C}^d/\Big(\sum_{l=1}^{d}|\mu_l|^q\Big)^{\frac{1}{q}}=1\Big\}$.

Proof. From Theorem 3.3, we obtain $\sigma_a(\mathbf{A})\subset\partial\mathbb{B}_q(\mathbb{C}^d)$ .From which it follows that $r(\mathbf{A})=1$. On the other hand, $\rho(\mathbf{A})\cap\mathbb{B}_q(\mathbb{C}^d)$ is both open and closed subset of the domain $\mathbb{B}_q(\mathbb{C}^d)$. Consequently we find $\sigma(\mathbf{A})\subset\partial\mathbb{B}_q(\mathbb{C}^d)$ or $\sigma(\mathbf{A})=\overline{\mathbb{B}}_q(\mathbb{C}^d)$. □

Acknowledgment. This work was funded by the Deanship of Scientific Research at Jouf University under grant No (DSR-2021-03-0335).

The authors would like to extend their thanks to the Editor and the Reviewer for the devoted time and efforts on our manuscript.

References

[1] J. Agler and M. Stankus, *m-Isometric transformations of Hilbert space* I, Integral Equ. Oper. Theory, **21**(1995), 383-429.
[2] J. Agler M. Stankus, m-Isometric transformations of Hilbert space. *II*. Integral Equ. Oper. Theory, **23**(1)(1995), 1-48.
[3] J. Agler M. Stankus, m-Isometric transformations of Hilbert space. *III*. Integral Equ. Oper. Theory, **24**(4) (1996), 379-421.
[4] F. Bayart, *m-isometries on Banach spaces*, Math. Nachr. 284 (2011), 2141–2147.
[5] M.Chō O. B. El Moctar, O. A. Mahmoud Sid Ahmed, $(n_1,\cdots,n_p)$*-quasi-m-isometric commuting tuple of operators on a Hilbert space*, Annals of Functional Analysis (2021) 12:4 https://doi.org/10.1007/s43034-020-00093-7
[6] M. Chō, *Joint spetra of commuting normal operators on banach spaces*, Glasgow Math. J. 30 (1988) 339-345.
[7] J. Gleason and S. Richter, m-Isometric Commuting Tuples of Operators on a Hilbert Space, Integral Equ. Oper. Theory, **56**(2) (2006), 181-196.
[8] C. Gu, *Examples of m-isometric tuples of operators on a Hilbert spaces*, J. Korean Math. Soc. 0 (0), No. 0, pp. 1-0. https://doi.org/10.4134/JKMS.j170183.
[9] P. H. W. Hoffmann and M. Mackey, (m,p) and (m,∞)-isometric operator tuples on normed spaces, Asian-Eur. J. Math., Vol. 8, No. 2 (2015).
[10] G. Khadija, *On n-quasi-*(m,q)*-isometric operators on a Banach space*, DOI: https://doi.org/10.21203/rs.3.rs-1855042/v2.
[11] O. A. Mahmoud Sid Ahmed, A.Saddi and K. Gherairi, *Some results on higher orders quasi-isometries*. Hacet. J. Math. Stat.(2019). https://doi.org/10.15672/hujms.532964
[12] O. A. Mahmoud Sid Ahmed, M. Chō and Ji Eun Lee, *On* (m,C)*-Isometric Commuting Tuples of Operators on a Hilbert Space*, Results Math., 73:51(2018), 1-31.

[13] O. A. Mahmoud Sid Ahmed, *m-Isometric Operators on Banach Spaces*, Asian-European Journal of Mathematics Vol. 3, No. 1 (2010) 1-19.
[14] S. Mecheri and T. Prasad, *On n-quasi-m-isometric operators*, Asian-European Journal of Mathematics **9** (2016), (8 pages).
[15] Z. Slodkowski and W. Želazko, On joint spectra of commuting families of operators, Studia Math. 50 (1974), 127-148.