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Matrix-valued Gabor frames over LCA groups for operators

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Abstract. Găvruta studied atomic systems in terms of frames for range of operators (that is, for subspaces), namely Θ-frames, where the lower frame condition is controlled by the Hilbert-adjoint of a bounded linear operator Θ. For a locally compact abelian group G and a positive integer n, we study frames of matrix-valued Gabor systems in the matrix-valued Lebesgue space $L^2(G, \mathbb{C}^{n\times n})$, where a bounded linear operator Θ on $L^2(G, \mathbb{C}^{n\times n})$ controls not only lower but also the upper frame condition. We term such frames matrix-valued (Θ, Θ^*)-Gabor frames. Firstly, we discuss frame preserving mapping in terms of hyponormal operators. Secondly, we give necessary and sufficient conditions for the existence of matrix-valued (Θ, Θ^*)- Gabor frames in terms of hyponormal operators. It is shown that if Θ is adjointable hyponormal operator, then $L^2(G, \mathbb{C}^{n\times n})$ admits a λ-tight (Θ, Θ^*)-Gabor frame for every positive real number λ. A characterization of matrix-valued (Θ, Θ^*)-Gabor frames is given. Finally, we show that matrix-valued (Θ, Θ^*)-Gabor frames are stable under small perturbation of window functions. Several examples are given to support our study.

1. Introduction

In [10], Gabor introduced a fundamental approach to signal decomposition in terms of elementary signals. Duffin and Schaeffer [8] in 1952, while addressing some deep problems in non-harmonic Fourier series, abstracted Gabor's method to define frames for Hilbert spaces. To be exact, they introduced frames of exponentials for the space $L^2(-\delta, \delta)$ under the name Fourier frame. Let $\mathcal{H}$ be a complex separable Hilbert space with an inner product $\langle ., . \rangle$. A countable collection of vector $\Phi := \{\varphi_k\}_{k=1}^{\infty}$ in a separable Hilbert space $\mathcal{H}$ is called a *frame* (or *Hilbert frame*) for $\mathcal{H}$ if there exist finite positive scalars A_o B_o such that

$$A_o\|\varphi\|^2 \leq \sum_{k=1}^{\infty} |\langle \varphi, \varphi_k \rangle|^2 \leq B_o\|\varphi\|^2 \tag{1}$$

for all $\varphi \in \mathcal{H}$. The scalars A_o and B_o are called the *lower* and *upper frame bounds* of Φ, respectively. Ineq. (1) is called the *frame inequality* of Φ. The frame inequality guarantee invertibility of the frame

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operator $\mathcal{S}:\mathcal{H}\to\mathcal{H}$ given by $\mathcal{S}\varphi=\sum_{k=1}^{\infty}\langle\varphi,\varphi_k\rangle\varphi_k$. This gives the stable reconstruction of each φ in $\mathcal{H}$: $\varphi=\mathcal{S}\mathcal{S}^{-1}\varphi=\sum_{k=1}^{\infty}\langle\mathcal{S}^{-1}\varphi,\varphi_k\rangle\varphi_k=\sum_{k=1}^{\infty}\langle\varphi,\mathcal{S}^{-1}\varphi_k\rangle\varphi_k$. This decomposition is useful in signal processing [23, 27], in particular, in lost of coefficients, see [4, 14] for technical details. Nowadays, frames are used in sampling [3], iterated function system [26], distributed signal processing [7], operator theory [2, 5, 19, 21], application of wavelets [13], quantum physics [25]. We refer to texts [4, 12, 14, 29] for basic theory of frames.

Găvruta in [11] introduced the notion of Θ-frames, where Θ is a linear bounded operator acting on the underlying Hilbert space $\mathcal{H}$.

Definition 1.1. *Let $\Theta\in\mathcal{B}(\mathcal{H})$, the space of bounded linear operators on $\mathcal{H}$. A sequence $\Phi:=\{\varphi_k\}_{k\in\mathbb{I}}\subset\mathcal{H}$ is called a* Θ-frame *for $\mathcal{H}$ if there exist constants $0<a_o,\ b_o<\infty$ such that*

$$a_o\|\Theta^*\varphi\|^2\leq\sum_{k=1}^{\infty}|\langle\varphi_k,\varphi\rangle|^2\leq b_o\|\varphi\|^2 \text{ for all } \varphi\in\mathcal{H}. \tag{2}$$

[11, p. 142] The numbers a_o and b_o are collectively known as Θ*-frame bounds*. If $\Theta=I$, the identity operator on $\mathcal{H}$, then Θ-frames are the ordinary Hilbert frames. However, a Θ-frame need not be a frame when $\Theta\neq I$. To be exact, Θ-frames are generalization of frames, which allow the reconstruction of elements from the range Ran(Θ) of Θ. Note that a Θ-frame for $\mathcal{H}$ is a Bessel sequence, so its frame operator is well defined. But, in general, it is not invertible on $\mathcal{H}$. However, the frame operator of a Θ-frame is invertible on the subspace Ran(Θ) of $\mathcal{H}$, whenever the Ran(Θ) is closed. In [11], Găvruta characterized Θ-frames in separable Hilbert spaces by using bounded linear operators on the underlying space. Θ-frames are also related to atomic systems and Găvruta in [11] characterized atomic systems in terms of Θ-frames in separable Hilbert spaces. She also observed many differences between Θ-frames and ordinary frames in separable Hilbert spaces. More precisely, Θ-frames gives stable analysis and reconstruction of functions from a subspace, e.g., range of operators. Xiao, Zhu, and Găvruta [30] gave various methods to construct Θ-frames in separable Hilbert spaces. They also discussed stability of Θ-frames under small perturbation. Recently, Θ-frames in distributed signal processing are studied in [6, 18].

Frames in matrix-valued signal spaces have potential applications in signal processing as most of the application areas involve matrix-valued signals. Xia and Suter in [28] studied vector-valued wavelets which play important role in multivariate signals. It is worth observing that frame properties, in general, not carried from a signal space to its associated matrix-valued signal space. In this direction, the authors of [20] studied an interplay between frames and matrix-valued frames, where they considered the wave packet structure in the euclidean matrix-valued space $L^2(\mathbb{R}^d,\mathbb{C}^{s\times r})$. They also gave some classes of matrix-valued window functions which can generate frames. Frame properties of WH-packets which is generalized Aldroubi's model [1] for construction of new frames from a given frame studied in [16] and sufficient conditions for finite sums of matrix-valued wave packet frames can be found in [17]. Two authors in [18] introduced and studied matrix-valued frames for range of operators. Recently, matrix-valued Gabor frames over locally compact abelian groups studied by authors of [15]. Motivated by applications of matrix-valued frames and differences between ordinary frames and Θ-frames, we study matrix-valued Gabor frames over locally compact abelian (LCA) groups, where both the lower frame condition and upper frame condition are controlled by bounded linear operators, in particular hyponormal operators, on the matrix-valued signal space over LCA groups. Notable contribution in this work include frame preserving mapping in terms of hyponormal operators, existence of tight matrix-valued Gabor frames over LCA groups for hyponormal operators. A characterization of matrix-valued Gabor frames over LCA groups and new stability results for matrix-valued Gabor frames over LCA groups under small perturbation.

This paper is organized as follows. In Section 2, we set the basic notions and definitions on the matrix-valued signal space and matrix-valued Gabor frames over locally compact abelian (LCA) groups and frames for operators to the make the paper self-contained. We introduce matrix-valued (Θ,Θ^*)-Gabor frames in the matrix-valued signal space $L^2(G,\mathbb{C}^{n\times n})$ over LCA groups in Section 3, where G is a LCA group, n is a positive integer and Θ is a bounded linear operator acting on $L^2(G,\mathbb{C}^{n\times n})$. In (Θ,Θ^*)-Gabor frames both the lower frame condition and upper frame condition are controlled by Θ. Proposition 3.4 gives sufficient condition for a matrix-valued Gabor frame to be (Θ, Θ^*)-Gabor frame in terms of bounded belowness of Θ. Frame preserving maps in terms of hyponormal operators are given in Proposition 3.5 and Proposition 3.7. Theorem 3.8 provides existence of tight matrix-valued (Θ, Θ^*)-Gabor frames in $L^2(G, \mathbb{C}^{n\times n})$. Proposition 3.12 shows that (Θ, Θ^*)-Gabor frames are preserved under adjointable hyponormal operators. A characterization for the existence of (Θ, Θ^*)-Gabor frames in $L^2(G, \mathbb{C}^{n\times n})$ is given in Theorem 3.14. Two different perturbation results which gives stability of frame conditions in terms of window functions and operators are given in Theorem 4.1 and Theorem 4.3. Examples and counter-examples are given to illustrate our results.

2. Preliminaries

Throughout the paper, symbol $\mathbb{Z}$ and $\mathbb{C}$ denote the set of integers and complex numbers, respectively. $\mathbb{T}$ denote the unit circle group. Let G be a second countable locally compact abelian group equipped with the Hausdorff topology. We recall that a character on G is the map γ G into itself which satisfies $\gamma(x+y) = \gamma(x)\gamma(y)$ for all $x, y \in G$. The *dual group* of G, denoted by $\hat{G}$, is the collection of all continuous characters on G which forms a locally compact abelian group under the operation defined by $(\gamma+\gamma')(x) := \gamma(x)\gamma'(x)$, where $\gamma, \gamma' \in \hat{G}$ and $x \in G$ and an appropriate topology. It is well known that on a LCA group G there exists a Haar measure which is unique upto a positive scalar multiple, see [9] for details. The symbols μ_G and $\mu_{\hat{G}}$ denote the Haar measure on G and $\hat{G}$, respectively. A *lattice* of G is a discrete subgroup Λ of G for which G/Λ is compact. The annihilator of Λ, denoted by $\Lambda^\perp$, is defined by $\Lambda^\perp = \{\gamma \in \hat{G} \mid \gamma(x) = 1,\ x \in \Lambda\}$. Note that $\Lambda^\perp$ is a lattice in $\hat{G}$. The *fundamental domain* associated with the lattice $\Lambda^\perp$ of $\hat{G}$, denoted by V, is a Borel measurable relatively compact set in $\hat{G}$ such that $\hat{G} = \cup_{w\in\Lambda^\perp}(w + V)$, $(w + V) \cap (w' + V) = \emptyset$ for $w \neq w', w, w' \in \Lambda^\perp$. The collection of all continuous automorphisms on G is denoted by Aut$G$. As is standard $L^2(G)$ denote the space of measurable square integrable functions over G. The *Fourier transform* of a function f in $L^1 \bigcap L^2(G)$ is defined as

$$\widehat{f}(\gamma) = \int_G f(x)\overline{\gamma(x)}d\mu_G(x),\ \ \gamma \in \widehat{G}$$

Note that the Fourier transform can be extended isometrically to $L^2(G)$, see [9].

2.1. *The Space* $L^2(G, \mathbb{C}^{n\times n})$

Throughout the paper, the matrix-valued functions are denoted by bold letters. Let n be a positive integer. The space of matrix-valued functions over G, denoted by $L^2(G, \mathbb{C}^{n\times n})$, is defined as

$$L^2(G, \mathbb{C}^{n\times n}) := \Big\{\mathbf{f} = \big[f_{ij}\big]_{1\leq i,j\leq n} : f_{ij} \in L^2(G)\ (1 \leq i,\ j \leq n)\Big\},$$

where $\big[f_{ij}\big]_{1\leq i,j\leq n}$ is matrix of order n with entries f_{ij}. The functions f_{ij} are called *components* or *atoms* of $\mathbf{f}$. The Frobenius norm on $L^2(G, \mathbb{C}^{n\times n})$ is given by

$$\|\mathbf{f}\| = \Big(\sum_{i,j=1}^{n} \int_G |f_{ij}|^2 d\mu_G\Big)^{\frac{1}{2}}. \tag{3}$$

It is easy to see that $L^2(G, \mathbb{C}^{n\times n})$ is a Banach space with respect to the Frobenius norm given in (3).

The integral of a function $\mathbf{f} = \big[f_{ij}\big]_{1\leq i,j\leq n} \in L^2(G, \mathbb{C}^{n\times n})$ is defined as

$$\int_G \mathbf{f} d\mu_G = \Big[\int_G f_{ij} d\mu_G\Big]_{1\leq i,j\leq n}$$

For $\mathbf{f}, \mathbf{g} \in L^2(G, \mathbb{C}^{n\times n})$, the matrix-valued inner product is defined as

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_G \mathbf{f}(x)\mathbf{g}^*(x) d\mu_G. \tag{4}$$

Here, $*$ denotes the transpose and the complex conjugate. One may observe that the matrix-valued inner product given in (4) is not an inner product in usual sense. Further, a bounded linear operator on $L^2(G, \mathbb{C}^{n\times n})$ may not be adjointable with respect to the matrix-valued product given in (4).

Let tr$A$ denotes trace of the matrix A. The space $L^2(G, \mathbb{C}^{n\times n})$ becomes a Hilbert space with respect to the inner product $\langle \cdot, \cdot \rangle_o$ defined by

$$\langle \mathbf{f}, \mathbf{g} \rangle_o = \mathrm{tr}\langle \mathbf{f}, \mathbf{g} \rangle, \ \ \mathbf{f}, \ \mathbf{g} \in L^2(G, \mathbb{C}^{n\times n}),$$

and $\langle \cdot, \cdot \rangle_o$ generates the Frobenius norm: $\|\mathbf{f}\|^2 = \langle \mathbf{f}, \mathbf{f} \rangle_o$, $\mathbf{f} \in L^2(G, \mathbb{C}^{n\times n})$.

Definition 2.1. *A bounded linear operator U on $L^2(G, \mathbb{C}^{n\times n})$ is said to be* hyponormal *if $tr\langle UU^*\mathbf{f}, \mathbf{f} \rangle \leq tr\langle U^*U\mathbf{f}, \mathbf{f} \rangle$, for all $\mathbf{f} \in L^2(G, \mathbb{C}^{n\times n})$. That is, $\|U^*\mathbf{f}\| \leq \|U\mathbf{f}\|$ for all $\mathbf{f} \in L^2(G, \mathbb{C}^{n\times n})$.*

For fundamental properties of hyponormal operators, we refer to [24].

2.2. Matrix-Valued Gabor Frames in $L^2(G, \mathbb{C}^{n\times n})$

Let Λ_0 be a finite subset of $\mathbb{N}$, $B \in \mathrm{Aut}G$, $C \in \mathrm{Aut}\widehat{G}$, Λ be a lattice in G and Λ' a lattice in $\widehat{G}$. Write

$$\Phi_{\Lambda_0} := \{\Phi_l\}_{l\in\Lambda_0} \subset L^2(G, \mathbb{C}^{n\times n}),$$
$$\mathsf{G}(C, B, \Phi_{\Lambda_0}) := \{E_{Cm}T_{Bk}\Phi_l\}_{l\in\Lambda_0, k\in\Lambda, m\in\Lambda'} \subset L^2(G, \mathbb{C}^{n\times n}).$$

For $a \in G$ and $\eta \in \hat{G}$, we consider following operators on $L^2(G, \mathbb{C}^{n\times n})$.

$$T_a\mathbf{f}(x) = \mathbf{f}(xa^{-1}) \quad \text{(Translation operator)},$$
$$E_\eta\mathbf{f}(x) = \eta(x)\mathbf{f}(x) \quad \text{(Modulation operator)}.$$

For $l \in \Lambda_0$, let $\Phi_l \in L^2(G, \mathbb{C}^{n\times n})$ be given by $\Phi_l(x) = \left[\phi_{ij}^{(l)}(x)\right]_{n\times n}$. Let $B \in \mathrm{Aut}G$ and $C \in \mathrm{Aut}\widehat{G}$. A collection of the form

$$\mathsf{G}(C, B, \Phi_{\Lambda_0}) := \{E_{Cm}T_{Bk}\Phi_l\}_{l\in\Lambda_0, k\in\Lambda, m\in\Lambda'}$$

is called the *matrix-valued Gabor system* in the space $L^2(G, \mathbb{C}^{n\times n})$ over LCA group G. The functions Φ_l are called the *matrix-valued Gabor window functions*.

Definition 2.2. *A frame of the form $\mathsf{G}(C, B, \Phi_{\Lambda_0})$ for $L^2(G, \mathbb{C}^{n\times n})$ is called a* matrix-valued Gabor frame. *That is, the inequality (*frame inequality*)*

$$a_o\|\mathbf{f}\|^2 \leq \sum_{l\in\Lambda_0} \sum_{k\in\Lambda, m\in\Lambda'} \left\| \left\langle E_{Cm}T_{Bk}\Phi_l, \mathbf{f} \right\rangle \right\|^2 \leq b_o\|\mathbf{f}\|^2, \ \mathbf{f} \in L^2(G, \mathbb{C}^{n\times n}),$$

holds for some positive scalars a_o and b_o. As in case of ordinary frames, a_o and b_o are called frame bounds.

Let $\mathcal{M}_n(\mathbb{C})$ be the complex vector space of all $n \times n$ complex matrices. The space

$$\ell^2(\Lambda_0 \times \Lambda \times \Lambda', \mathcal{M}_n(\mathbb{C})) := \left\{ \{M_{l,j,k}\}_{l\in\Lambda_0, j\in\Lambda, k\in\Lambda'} \subset \mathcal{M}_n(\mathbb{C}) : \sum_{l\in\Lambda_0} \sum_{j\in\Lambda, k\in\Lambda'} \|M_{l,j,k}\|^2 < \infty \right\}$$

is a Hilbert space and its related norm is given by

$$\|\{M_{l,j,k}\}_{l\in\Lambda_0,j\in\Lambda,k\in\Lambda'}\| = \Big(\sum_{l\in\Lambda_0}\sum_{j\in\Lambda,k\in\Lambda'}\|M_{l,j,k}\|^2\Big)^{\frac{1}{2}}.$$

If $\mathtt{G}(C,B,\Phi_{\Lambda_0})$ is a frame for $L^2(G,\mathbb{C}^{n\times n})$, then the map

$$V:\ell^2(\Lambda_0\times\Lambda\times\Lambda',\mathcal{M}_n(\mathbb{C}))\to L^2(G,\mathbb{C}^{n\times n}) \text{ defined by } V:\{M_{l,k,m}\}_{\substack{l\in\Lambda_0\\ k\in\Lambda,m\in\Lambda'}}\mapsto\sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'}M_{l,k,m}E_{Cm}T_{Bk}\Phi_l$$

is called the *synthesis operator* (or the *pre-frame operator*), associated with $\mathtt{G}(C,B,\Phi_{\Lambda_0})$. The *analysis operator* is the map

$$W:L^2(G,\mathbb{C}^{n\times n})\to\ell^2(\Lambda_0\times\Lambda\times\Lambda',\mathcal{M}_n(\mathbb{C})) \text{ given by } W:\mathbf{f}\mapsto\Big\{\langle\mathbf{f},E_{Cm}T_{Bk}\Phi_l\rangle\Big\}_{l\in\Lambda_0,k\in\Lambda,m\in\Lambda'}.$$

The frame operator of $\mathtt{G}(C,B,\Phi_{\Lambda_0})$ is the composition $S=VW$ on the space $L^2(G,\mathbb{C}^{n\times n})$ which is given by

$$S:\mathbf{f}\mapsto\sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'}\langle\mathbf{f},E_{Cm}T_{Bk}\Phi_l\rangle E_{Cm}T_{Bk}\Phi_l,$$

$\mathbf{f}\in L^2(G,\mathbb{C}^{n\times n})$. The frame operator is bounded, linear and invertible on $L^2(G,\mathbb{C}^{n\times n})$. We refer to [4, 12] for basic theory of Gabor frames.

The following example will be used in illustration of results.

Example 2.3. *[15, Example 3.1] Let G be the torus group. Its dual group is $\widehat{G}=\mathbb{Z}$. Fix a lattice $\Lambda=\left\{0,\frac{1}{8},\dots,\frac{7}{8}\right\}$. Then $\Lambda^\perp=8\mathbb{Z}$ with fundamental domain $V=\mathbb{Z}_8=\{0,1,\dots,7\}$. Let $\phi_1,\phi_2\in L^2(\mathbb{T})$ be such that*

$$\widehat{\phi_1}(\gamma)=\chi_{\mathbb{Z}_8}(\gamma)\quad\textit{and}\quad\widehat{\phi_2}(\gamma)=\frac{1}{2}\chi_{\mathbb{Z}_8}(\gamma)\textit{ in } L^2(\mathbb{Z})\textit{ for }\gamma\in\mathbb{Z}.$$

For $B\in AutG$ and $C\in Aut\widehat{G}$, consider the Gabor system $\{E_{Cm}T_{Bk}\phi_1\}_{\substack{k\in\Lambda\\ m\in\Lambda^\perp}}=\{E_{8m}T_k\phi_1\}_{k\in\Lambda,m\in\mathbb{Z}}$. Set $\phi_m^{(1)}(\xi)=E_{8m}\phi_1(\xi)$, $m\in\mathbb{Z}$, $\xi\in[0,1[$. Since $E_{8m}T_k\phi_1(\xi)=T_kE_{8m}\phi_1(\xi)$, thus by taking $\Lambda_m:=\Lambda$, one can write $\{T_k\phi_m^{(1)}\}_{k\in\Lambda,m\in\mathbb{Z}}=\{E_{8m}T_k\phi_1\}_{k\in\Lambda,m\in\mathbb{Z}}$.

Define

$$G_0(\gamma)=\sum_{m\in\mathbb{Z}}\mu_{\widehat{G}}(V)\Big|\widehat{\phi_m^{(1)}}(\gamma)\Big|^2,\ \gamma\in\mathbb{Z},$$

and

$$G_1(\gamma)=\sum_{m\in\mathbb{Z}}\mu_{\widehat{G}}(V)\sum_{w\in\Lambda^\perp\setminus\{0\}}|\widehat{\phi_m^{(1)}}(\gamma)\widehat{\phi_m^{(1)}}(\gamma+w)|,\ \gamma\in\mathbb{Z}.$$

Then, using $\widehat{\phi_m^{(1)}}(\gamma)=\widehat{E_{8m}\phi_1}(\gamma)=T_{8m}\widehat{\phi_1}(\gamma)$, $\gamma\in\mathbb{Z}$, we have

$$G_0(\gamma)=\sum_{m\in\mathbb{Z}}8|\chi_{\mathbb{Z}_8}(\gamma-8m)|^2=8\textit{ for }\gamma\in\mathbb{Z},$$

and

$$G_1(\gamma)=\sum_{m\in\mathbb{Z}}8\sum_{a\in\mathbb{Z}\setminus\{0\}}|\chi_{\mathbb{Z}_8}(\gamma-8m)\chi_{\mathbb{Z}_8}(\gamma+8a-8m)|^2=0\textit{ for }\gamma\in\mathbb{Z}.$$

Therefore, by [4, Theorem 21.6.1], the Gabor system $\{E_{8m}T_k\phi_1\}_{k\in\Lambda,m\in\mathbb{Z}}$ is a 8-tight frame for $L^2(G)$. Similarly, $\{E_{8m}T_k\phi_2\}_{k\in\Lambda,m\in\mathbb{Z}}$ is a 2-tight frame for $L^2(G)$.

3. Matrix-Valued (Θ, Θ^*)-Gabor Frames

We begin this section with the definition of a matrix-valued (Θ, Θ^*)-Gabor frame in the matrix-valued function space $L^2(G, \mathbb{C}^{n\times n})$.

Definition 3.1. *Let Θ be a bounded linear operator acting on $L^2(G, \mathbb{C}^{n\times n})$. A countable family of vectors $\mathcal{G}(C, B, \Phi_{\Lambda_0}) := \{E_{Cm}T_{Bk}\Phi_l\}_{l\in\Lambda_0, k\in\Lambda, m\in\Lambda'}$ in $L^2(G, \mathbb{C}^{n\times n})$ is called a* matrix-valued (Θ, Θ^*)-Gabor frame *for $L^2(G, \mathbb{C}^{n\times n})$ if for all $\mathbf{f} \in L^2(G, \mathbb{C}^{n\times n})$,*

$$\alpha_o\|\Theta^*\mathbf{f}\|^2 \leq \sum_{l\in\Lambda_0}\sum_{k\in\Lambda, m\in\Lambda'} \left\|\left\langle E_{Cm}T_{Bk}\Phi_l, \mathbf{f}\right\rangle\right\|^2 \leq \beta_o\|\Theta\mathbf{f}\|^2 \tag{5}$$

holds for some positive scalars α_o and β_o.

The positive scalars α_o and β_o are called *lower* and *upper frame bounds* of the (Θ, Θ^*)-Gabor frame $\mathcal{G}(C, B, \Phi_{\Lambda_0})$. If $\alpha_o = \beta_o$, then we say that $\mathcal{G}(C, B, \Phi_{\Lambda_0})$ is a α_o-(Θ, Θ^*)-tight matrix-valued Gabor frame for $L^2(G, \mathbb{C}^{n\times n})$.

Remark 3.2. *If Θ is the identity operator on $L^2(G, \mathbb{C}^{n\times n})$, then a matrix-valued (Θ, Θ^*)-Gabor frame for $L^2(G, \mathbb{C}^{n\times n})$ is the standard matrix-valued Gabor frame for $L^2(G, \mathbb{C}^{n\times n})$. However, if Θ is a non-identity operator on $L^2(G, \mathbb{C}^{n\times n})$, then a matrix-valued (Θ, Θ^*)-Gabor frame for $L^2(G, \mathbb{C}^{n\times n})$ need not be the standard matrix-valued Gabor frame for $L^2(G, \mathbb{C}^{n\times n})$. For example, consider the tight Gabor frames $\{E_{8m}T_k\phi_l\}_{k\in\Lambda, m\in\mathbb{Z}}, (l = 1, 2)$ for $L^2(G)$ given in Example 2.3. Let $\Phi_1 = \begin{bmatrix} 0 & \phi_1 \\ 0 & \phi_1 \end{bmatrix}, \Phi_2 = \begin{bmatrix} 0 & \phi_2 \\ 0 & \phi_2 \end{bmatrix}$. Then, $\Phi_1, \Phi_2 \in L^2(G, \mathbb{C}^{2\times 2})$. For any $\mathbf{f} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$ in $L^2(G, \mathbb{C}^{2\times 2})$, we have*

$$\begin{aligned}
&\sum_{l\in\{1,2\}}\sum_{k\in\Lambda, m\in\mathbb{Z}} \left\|\left\langle E_{8m}T_k\Phi_l, \mathbf{f}\right\rangle\right\|^2 \\
&= \sum_{l\in\{1,2\}}\sum_{k\in\Lambda, m\in\mathbb{Z}} 2\Big(\Big|\int_G E_{8m}T_k\phi_l\overline{f_{12}}d\mu_G\Big|^2 + \Big|\int_G E_{8m}T_k\phi_l\overline{f_{22}}d\mu_G\Big|^2\Big) \\
&= 20\Big(\|f_{12}\|^2 + \|f_{22}\|^2\Big).
\end{aligned}$$

Therefore, for $\mathbf{f}_o = \begin{bmatrix} f & 0 \\ f & 0 \end{bmatrix}$, where $0 \neq f \in L^2(G)$, we have

$$\sum_{l\in\{1,2\}}\sum_{k\in\Lambda, m\in\mathbb{Z}} \left\|\left\langle E_{8m}T_k\Phi_l, \mathbf{f}_o\right\rangle\right\|^2 = 0.$$

Thus, $\{E_{8m}T_k\Phi_l\}_{l\in\{1,2\}, k\in\Lambda, m\in\mathbb{Z}}$ is not a matrix-valued Gabor frame for $L^2(G, \mathbb{C}^{2\times 2})$. But the family $\{E_{8m}T_k\Phi_l\}_{l\in\{1,2\}, k\in\Lambda, m\in\mathbb{Z}}$ is a (Θ_o, Θ_o^*)-Gabor frame for $L^2(G, \mathbb{C}^{2\times 2})$, where Θ_o is a bounded linear operator on $L^2(G, \mathbb{C}^{2\times 2})$ given by

$$\Theta_o : \mathbf{f} \mapsto \begin{bmatrix} 0 & f_{12} \\ 0 & f_{22} \end{bmatrix}, \ \mathbf{f} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \in L^2(G, \mathbb{C}^{2\times 2}).$$

It is easy to see that $\Theta_o^* = \Theta_o$. Therefore, for any $\mathbf{f} \in L^2(G, \mathbb{C}^{2\times 2})$, we have

$$20\|\Theta_o^*\mathbf{f}\|^2 = \sum_{l\in\{1,2\}}\sum_{k\in\Lambda, m\in\mathbb{Z}} \left\|\left\langle E_{8m}T_k\Phi_l, \mathbf{f}\right\rangle\right\|^2 = 20\|\Theta_o\mathbf{f}\|^2.$$

Hence, $\{E_{8m}T_k\Phi_l\}_{l\in\{1,2\}, k\in\Lambda, m\in\mathbb{Z}}$ is a matrix-valued (Θ_o, Θ_o^*)-Gabor frame for $L^2(G, \mathbb{C}^{2\times 2})$.

Remark 3.3. *It is mentioned in [15] that a matrix-valued Gabor frame for $L^2(G, \mathbb{C}^{n\times n})$ is always a Θ-Gabor frame for $L^2(G, \mathbb{C}^{n\times n})$ where Θ is a bounded linear operator on $L^2(G, \mathbb{C}^{n\times n})$. However, this is not true in the case of (Θ, Θ^*)-matrix-valued Gabor frame. Precisely, a matrix-valued Gabor frame for $L^2(G, \mathbb{C}^{n\times n})$ need not be a (Θ, Θ^*)-Gabor* *frame for $L^2(G, \mathbb{C}^{n\times n})$. For example, let G be the torus group and $\{E_{8m}T_k\phi_l\}_{k\in\Lambda,m\in\mathbb{Z}}$ $(l = 1, 2)$ be the tight Gabor frames for $L^2(G)$ given in Example 2.3. Let Φ_1, $\Phi_2 \in L^2(G, \mathbb{C}^{2\times 2})$ be given by*

$$\Phi_1 = \begin{bmatrix} 0 & \phi_1 \\ \phi_2 & 0 \end{bmatrix} \quad and \quad \Phi_2 = \begin{bmatrix} 0 & \phi_2 \\ \phi_1 & 0 \end{bmatrix}.$$

Then, $\{E_{8m}T_k\Phi_l\}_{l\in\{1,2\},k\in\Lambda,m\in\mathbb{Z}}$ is a 10-tight matrix-valued Gabor frame for $L^2(G, \mathbb{C}^{2\times 2})$. Define $\Theta : L^2(G, \mathbb{C}^{2\times 2}) \to L^2(G, \mathbb{C}^{2\times 2})$ by

$$\Theta\colon \mathbf{f} \mapsto \begin{bmatrix} f_{11} & 0 \\ 0 & 0 \end{bmatrix}, \ \mathbf{f} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \in L^2(G, \mathbb{C}^{2\times 2}).$$

Then, Θ is a bounded linear operator. If possible, let $\{E_{8m}T_k\Phi_l\}_{l\in\{1,2\},k\in\Lambda,m\in\mathbb{Z}}$ be a (Θ, Θ^*)- Gabor frame for $L^2(G, \mathbb{C}^{2\times 2})$ with bounds a, b. Then, for $\mathbf{f}_o = \begin{bmatrix} 0 & f \\ f & f \end{bmatrix}$, where $0 \neq f \in L^2(G)$, we have

$$\sum_{l\in\{1,2\}} \sum_{k\in\Lambda,m\in\mathbb{Z}} \left\| \left\langle E_{8m}T_k\Phi_l, \mathbf{f}_o \right\rangle \right\|^2 = 10\|\mathbf{f}_o\|^2 = 30\|f\|^2 > 0 = b\|\Theta\mathbf{f}_o\|^2,$$

which is a contradiction. Hence, $\{E_{8m}T_k\Phi_l\}_{l\in\{1,2\},k\in\Lambda,m\in\mathbb{Z}}$ is not a (Θ, Θ^*)- Gabor frame for $L^2(G, \mathbb{C}^{2\times 2})$.

Now, we show that a matrix-valued Gabor frame for $L^2(G, \mathbb{C}^{n\times n})$ becomes a (Θ, Θ^*)-Gabor frame for $L^2(G, \mathbb{C}^{n\times n})$ provided Θ is bounded below.

Proposition 3.4. *Let $\{E_{Cm}T_{Bk}\Phi_l\}_{l\in\Lambda_0,k\in\Lambda,m\in\Lambda'}$ be a matrix-valued Gabor frame for $L^2(G, \mathbb{C}^{n\times n})$. Let Θ be a bounded linear operator acting on the space $L^2(G, \mathbb{C}^{n\times n})$ which is bounded below. Then, the collection $\{E_{Cm}T_{Bk}\Phi_l\}_{l\in\Lambda_0,k\in\Lambda,m\in\Lambda'}$ is a matrix-valued (Θ, Θ^*)-Gabor frame for $L^2(G, \mathbb{C}^{n\times n})$.*

Proof. Let γ and δ be frame bounds of $\{E_{Cm}T_{Bk}\Phi_l\}_{l\in\Lambda_0,k\in\Lambda,m\in\Lambda'}$. Let Θ be bounded below by a constant α, that is, $\|\Theta\mathbf{f}\| \geq \alpha\|\mathbf{f}\|$ for all $\mathbf{f}$ in $L^2(G, \mathbb{C}^{n\times n})$. Then, for any $\mathbf{f} \in L^2(G, \mathbb{C}^{n\times n})$, we have

$$\frac{\gamma}{\|\Theta^*\|^2}\|\Theta^*\mathbf{f}\|^2 \leq \gamma\|\mathbf{f}\|^2 \leq \sum_{l\in\Lambda_0} \sum_{k\in\Lambda,m\in\Lambda'} \|\langle E_{Cm}T_{Bk}\Phi_l, \mathbf{f}\rangle\|^2,$$

and

$$\sum_{l\in\Lambda_0} \sum_{k\in\Lambda,m\in\Lambda'} \|\langle E_{Cm}T_{Bk}\Phi_l, \mathbf{f}\rangle\|^2 \leq \delta\|\mathbf{f}\|^2 \leq \frac{\delta}{\alpha^2}\|\Theta\mathbf{f}\|^2.$$

Thus, $\{E_{Cm}T_{Bk}\Phi_l\}_{l\in\Lambda_0,k\in\Lambda,m\in\Lambda'}$ is a matrix-valued (Θ, Θ^*)-Gabor frame for the space $L^2(G, \mathbb{C}^{n\times n})$ with frame bounds $\frac{\gamma}{\|\Theta^*\|^2}$ and $\frac{\delta}{\alpha^2}$. □

Now, we discuss relations between hyponormal operators on $L^2(G, \mathbb{C}^{n\times n})$ and matrix-valued λ_o-(Θ, Θ^*)-tight frames for $L^2(G, \mathbb{C}^{n\times n})$. By Definition 3.1, one may observe that a bounded linear operator Θ on $L^2(G, \mathbb{C}^{n\times n})$ is hyponormal if there exists a matrix-valued λ_o-(Θ, Θ^*)-tight frame for the space $L^2(G, \mathbb{C}^{n\times n})$. Indeed, if $\{\mathbf{f}_k\}_{k\in I}$ is a matrix-valued λ_o-(Θ, Θ^*)-tight frame for $L^2(G, \mathbb{C}^{n\times n})$, then by Definition 3.1, we have $\|\Theta^*\mathbf{f}\| \leq \|\Theta\mathbf{f}\|$, for all $\mathbf{f} \in L^2(G, \mathbb{C}^{n\times n})$. Hence, Θ is a hyponormal operator on $L^2(G, \mathbb{C}^{n\times n})$.

In order to see the other way round relationship, we first discuss some frame preserving properties of (Θ, Θ^*)-frames in $L^2(G, \mathbb{C}^{n\times n})$. The following result says that the image of a frame in $L^2(G)$ under a hyponormal operator Θ is a (Θ, Θ^*)-frame for $L^2(G)$.

Proposition 3.5. *Let $\{E_{Cm}T_{Bk}\phi_l\}_{l\in\Lambda_0,k\in\Lambda,m\in\Lambda'}$ be a Gabor frame for $L^2(G)$ and let Θ be a hyponormal operator on $L^2(G)$. Then, $\{\Theta E_{Cm}T_{Bk}\phi_l\}_{\substack{l\in\Lambda_0,k\in\Lambda \\ m\in\Lambda'}}$ is a (Θ, Θ^*)-frame for $L^2(G)$.*

Proof. Let λ and μ be lower and upper frame bounds of $\{E_{Cm}T_{Bk}\phi_l\}_{\substack{l\in\Lambda_0,k\in\Lambda\\ m\in\Lambda'}}$. Then, using the hyponormality of Θ, for any $f \in L^2(G)$, we have

$$\begin{aligned}\sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'}|\langle\Theta E_{Cm}T_{Bk}\phi_l, f\rangle|^2 &= \sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'}|\langle E_{Cm}T_{Bk}\phi_l, \Theta^* f\rangle|^2\\ &\leq \mu\|\Theta^* f\|^2\\ &\leq \mu\|\Theta f\|^2.\end{aligned}$$

Also

$$\begin{aligned}\lambda\|\Theta^* f\|^2 &\leq \sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'}|\langle E_{Cm}T_{Bk}\phi_l, \Theta^* f\rangle|^2\\ &= \sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'}|\langle\Theta E_{Cm}T_{Bk}\phi_l, f\rangle|^2\end{aligned}$$

for all $f \in L^2(G)$. Hence, $\{\Theta E_{Cm}T_{Bk}\phi_l\}_{l\in\Lambda_0,k\in\Lambda,m\in\Lambda'}$ is a (Θ,Θ^*)-frame for $L^2(G)$ with frame bounds λ and μ. □

Remark 3.6. *Proposition 3.5 is not true for matrix-valued frames in matrix-valued signal spaces $L^2(G,\mathbb{C}^{n\times n})$. This problem is related to adjointable operators on matrix-valued signal spaces with respect to matrix-valued inner product on the underlying space. For example, consider the tight Gabor frames $\{E_{8m}T_k\phi_l\}_{\substack{k\in\Lambda\\ m\in\mathbb{Z}}}$ $(l = 1,2)$ for $L^2(G)$ given in Example 2.3. Let $\Phi_1, \Phi_2 \in L^2(G,\mathbb{C}^{2\times 2})$ be given by*

$$\Phi_1 = \begin{bmatrix} 0 & \phi_1\\ \phi_2 & 0\end{bmatrix} \quad \text{and} \quad \Phi_2 = \begin{bmatrix} 0 & \phi_2\\ \phi_1 & 0\end{bmatrix}.$$

Then, $\{E_{8m}T_k\Phi_l\}_{l\in\{1,2\},k\in\Lambda,m\in\mathbb{Z}}$ is a 10-tight matrix-valued Gabor frame for $L^2(G,\mathbb{C}^{2\times 2})$.

Define $\Theta : L^2(G,\mathbb{C}^{2\times 2}) \to L^2(G,\mathbb{C}^{2\times 2})$ by

$$\Theta\colon \mathbf{f} \mapsto \begin{bmatrix} f_{11} & 0\\ 0 & 0\end{bmatrix}, \ \mathbf{f} = \begin{bmatrix} f_{11} & f_{12}\\ f_{21} & f_{22}\end{bmatrix} \in L^2(G,\mathbb{C}^{2\times 2}).$$

Then, Θ is a bounded linear operator with adjoint $\Theta^* = \Theta$ and hence a hyponormal operator. But Θ is not adjointable with respect to matrix-valued inner product on $L^2(G,\mathbb{C}^{2\times 2})$. That is, $\langle\Theta\mathbf{f},\mathbf{g}\rangle \neq \langle\mathbf{f},\Theta^*\mathbf{g}\rangle$ for all $\mathbf{f},\mathbf{g}$ in $L^2(G,\mathbb{C}^{2\times 2})$. Furthermore, $\Theta E_{8m}T_k\Phi_l = \mathbf{O}$ for $l \in \{1,2\}, k \in \Lambda, m \in \mathbb{Z}$. Hence, $\{\Theta E_{8m}T_k\Phi_l\}_{l\in\{1,2\},k\in\Lambda,m\in\mathbb{Z}}$ is not a (Θ,Θ^*)-frame for $L^2(G,\mathbb{C}^{2\times 2})$.

The following result gives sufficient conditions on matrix-valued Θ-frame preserving transformations acting on matrix-valued signal spaces in terms of adjointability of Θ.

Proposition 3.7. *Let $\{E_{Cm}T_{Bk}\Phi_l\}_{l\in\Lambda_0,k\in\Lambda,m\in\Lambda'}$ be a matrix-valued frame for the space $L^2(G,\mathbb{C}^{n\times n})$ with frame bounds γ and δ. Let Θ be a hyponormal operator acting on $L^2(G,\mathbb{C}^{n\times n})$ which is adjointable with respect to the matrix-valued inner product. Then, $\{\Theta E_{Cm}T_{Bk}\Phi_l\}_{l\in\Lambda_0,k\in\Lambda,m\in\Lambda'}$ is a matrix-valued (Θ,Θ^*)-frame for $L^2(G,\mathbb{C}^{n\times n})$ with frame bounds γ and δ.*

Proof. For any $\mathbf{f} \in L^2(G,\mathbb{C}^{n\times n})$, we have

$$\begin{aligned}\gamma\|\Theta^*\mathbf{f}\|^2 &\leq \sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'}\|\langle E_{Cm}T_{Bk}\Phi_l, \Theta^*\mathbf{f}\rangle\|^2\\ &= \sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'}\|\langle\Theta E_{Cm}T_{Bk}\Phi_l, \mathbf{f}\rangle\|^2\\ &= \sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'}\|\langle E_{Cm}T_{Bk}\Phi_l, \Theta^*\mathbf{f}\rangle\|^2\\ &\leq \delta\|\Theta^*\mathbf{f}\|^2\\ &\leq \delta\|\Theta\mathbf{f}\|^2.\end{aligned}$$

Thus, $\{\Theta E_{Cm}T_{Bk}\Phi_l\}_{l\in\Lambda_0,k\in\Lambda,m\in\Lambda'}$ is a matrix-valued (Θ,Θ^*)-frame for the space $L^2(G,\mathbb{C}^{n\times n})$ with the desired frame bounds. □

Now, we have enough knowledge to discuss the conditions on an operator Θ acting on $L^2(G,\mathbb{C}^{n\times n})$ such that the existence of λ_o-(Θ,Θ^*)-tight frames for $L^2(G,\mathbb{C}^{n\times n})$ is guaranteed. We give the following result regarding this.

Theorem 3.8. *Let Θ be a hyponormal operator on $L^2(G,\mathbb{C}^{n\times n})$. If Θ is adjointable with respect to the matrix-valued inner product, then there exists a matrix-valued λ_o-(Θ,Θ^*)-tight frame for $L^2(G,\mathbb{C}^{n\times n})$ for every positive real number λ_o.*

Proof. Let $\{E_{Cm}T_{Bk}\phi_l\}_{l\in\Lambda_0,k\in\Lambda,m\in\Lambda'}$ be a Parseval frame for $L^2(G)$. For each $l\in\Lambda_0$, define the matrix-valued function $\Phi_l\in L^2(G,\mathbb{C}^{n\times n})$ as

$$\Phi_l=\begin{bmatrix}\sqrt{\lambda_o}\,\phi_l & 0 & \cdots & 0\\ 0 & \sqrt{\lambda_o}\,\phi_l & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \sqrt{\lambda_o}\,\phi_l\end{bmatrix}.$$

Then

$$E_{Cm}T_{Bk}\Phi_l=\begin{bmatrix}E_{Cm}T_{Bk}(\sqrt{\lambda_o}\,\phi_l) & 0 & \cdots & 0\\ 0 & E_{Cm}T_{Bk}(\sqrt{\lambda_o}\,\phi_l) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & E_{Cm}T_{Bk}(\sqrt{\lambda_o}\,\phi_l)\end{bmatrix}.$$

Therefore, for any $\mathbf{f}=\begin{bmatrix}f_{11} & f_{12} & \cdots & f_{1n}\\ f_{21} & f_{22} & \cdots & f_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ f_{n1} & f_{n2} & \cdots & f_{nn}\end{bmatrix}\in L^2(G,\mathbb{C}^{n\times n})$, we have

$$\begin{aligned}&\sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'}\|\langle\mathbf{f},E_{Cm}T_{Bk}\Phi_l\rangle\|^2\\ &=\sum_{l\in\Lambda_0}\sum_{\substack{k\in\Lambda\\ m\in\Lambda'}}\left\|\begin{bmatrix}\langle f_{11},E_{Cm}T_{Bk}(\sqrt{\lambda_o}\,\phi_l)\rangle & \cdots & \langle f_{1n},E_{Cm}T_{Bk}(\sqrt{\lambda_o}\,\phi_l)\rangle\\ \langle f_{21},E_{Cm}T_{Bk}(\sqrt{\lambda_o}\,\phi_l)\rangle & \cdots & \langle f_{2n},E_{Cm}T_{Bk}(\sqrt{\lambda_o}\,\phi_l)\rangle\\ \vdots & \ddots & \vdots\\ \langle f_{n1},E_{Cm}T_{Bk}(\sqrt{\lambda_o}\,\phi_l)\rangle & \cdots & \langle f_{nn},E_{Cm}T_{Bk}(\sqrt{\lambda_o}\,\phi_l)\rangle\end{bmatrix}\right\|^2\\ &=\lambda_o\sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'}\sum_{1\le i,j\le n}|\langle f_{ij},E_{Cm}T_{Bk}\,\phi_l\rangle|^2\\ &=\lambda_o\sum_{1\le i,j\le n}\|f_{ij}\|^2\\ &=\lambda_o\|\mathbf{f}\|^2.\end{aligned}$$

Hence, $\{E_{Cm}T_{Bk}\Phi_l\}_{l\in\Lambda_0,k\in\Lambda,m\in\Lambda'}$ is a matrix-valued λ_o-tight Gabor frame for $L^2(G,\mathbb{C}^{n\times n})$. Further, by Proposition 3.7, $\{\Theta E_{Cm}T_{Bk}\Phi_l\}_{l\in\Lambda_0,k\in\Lambda,m\in\Lambda'}$ is a (Θ,Θ^*)-frame for $L^2(G,\mathbb{C}^{n\times n})$ with λ_o as lower and upper frame bounds. Hence, the existence of a matrix-valued λ_o-(Θ,Θ^*)-tight frame for the space $L^2(G,\mathbb{C}^{n\times n})$ is proved. □

We illustrate Theorem 3.8 by giving the following example regarding the existence of λ_o-(Θ,Θ^*)-tight frames for $L^2(\mathbb{R},\mathbb{C}^{3\times 3})$.

Example 3.9. *Let $G = \mathbb{R}$ be the additive group of real numbers. The characters on $\mathbb{R}$ are the functions $\eta_y : \mathbb{R} \to \mathbb{C}$ defined by*

$$\eta_y(x) = e^{2\pi iyx},\ x \in \mathbb{R}$$

for fixed $y \in \mathbb{R}$. That is, the dual group $\widehat{G}$ can be identified with $\mathbb{R}$, see [9] for technical details. Consider the lattice $\Lambda = \mathbb{Z}$ and $\Lambda' = \mathbb{Z}$. Then, for $\phi = \chi_{[0,1]}$, the Gabor system $\{E_m T_k \phi\}_{m,k\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$, see [4, p. 96] for details.

Define a matrix-valued function $\Phi \in L^2(\mathbb{R}, \mathbb{C}^{3\times 3})$ as

$$\Phi = \begin{bmatrix} \sqrt{3}\,\phi & 0 & 0 \\ 0 & \sqrt{3}\,\phi & 0 \\ 0 & 0 & \sqrt{3}\,\phi \end{bmatrix}.$$

Then, for any $\mathbf{f} = \left[f_{i,j}\right]_{1\le i,j\le n} \in L^2(\mathbb{R}, \mathbb{C}^{3\times 3})$, we have

$$\sum_{m,k\in\mathbb{Z}} \|\langle \mathbf{f}, E_m T_k \Phi\rangle\|^2 = 3\|\mathbf{f}\|^2,$$

which implies that $\{E_m T_k \Phi\}_{m,k\in\mathbb{Z}}$ is a matrix-valued 3-tight Gabor frame for $L^2(\mathbb{R}, \mathbb{C}^{3\times 3})$. Define $\Theta : L^2(\mathbb{R}, \mathbb{C}^{3\times 3}) \to L^2(\mathbb{R}, \mathbb{C}^{3\times 3})$ by

$$\Theta \colon \mathbf{f} \mapsto \begin{bmatrix} f_{11} & 0 & f_{13} \\ f_{21} & 0 & f_{23} \\ f_{31} & 0 & f_{33} \end{bmatrix},\ \mathbf{f} = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \in L^2(\mathbb{R}, \mathbb{C}^{3\times 3}).$$

Then, Θ is a bounded linear operator with adjoint $\Theta^* = \Theta$. Also, Θ is adjointable with respect to matrix-valued inner product on $L^2(\mathbb{R}, \mathbb{C}^{3\times 3})$. That is, $\langle \Theta\mathbf{f}, \mathbf{g}\rangle = \langle \mathbf{f}, \Theta^*\mathbf{g}\rangle$, $\mathbf{f}, \mathbf{g} \in L^2(\mathbb{R}, \mathbb{C}^{3\times 3})$. Then, by Proposition 3.7, the matrix-valued system $\{\Theta E_m T_k \Phi\}_{m,k\in\mathbb{Z}}$ is a matrix-valued 3-(Θ, Θ^*)-tight frame for $L^2(\mathbb{R}, \mathbb{C}^{3\times 3})$.

Remark 3.10. *In Theorem 3.8, the condition of adjointability of Θ with respect to matrix-valued inner product is not a necessary condition.*

Next, we discuss frame properties of the image of a (Θ, Θ^*)-Gabor frame in $L^2(G, \mathbb{C}^{n\times n})$ under a bounded linear operator Ξ. It is proved in [15, Proposition 4.2] that the image of a Θ-Gabor frame for $L^2(G, \mathbb{C}^{n\times n})$ under an operator $\Xi \in \mathcal{B}(L^2(G, \mathbb{C}^{n\times n}))$ becomes a $\Xi\Theta$-frame for $L^2(G, \mathbb{C}^{n\times n})$ provided Ξ is adjointable with respect to matrix-valued inner product. But, this is not true for the case of (Θ, Θ^*)-Gabor frames in $L^2(G, \mathbb{C}^{n\times n})$. That is, if $\mathcal{G}(C, B, \Phi_{\Lambda_0})$ is a (Θ, Θ^*)-Gabor frame for $L^2(G, \mathbb{C}^{n\times n})$ and $\Xi \in \mathcal{B}(L^2(G, \mathbb{C}^{n\times n}))$ is adjointable with respect to matrix-valued inner product, then $\Xi(\mathcal{G}(C, B, \Phi_{\Lambda_0}))$ may not be a $(\Xi\Theta, (\Xi\Theta)^*)$-frame for $L^2(G, \mathbb{C}^{n\times n})$. This is justified in the following example.

Example 3.11. *Consider tight Gabor frames $\{E_{8m} T_k \phi_1\}_{\substack{k\in\Lambda \\ m\in\mathbb{Z}}}$ and $\{E_{8m} T_k \phi_2\}_{\substack{k\in\Lambda \\ m\in\mathbb{Z}}}$ for $L^2(G)$ given in Example 2.3. Define $\Theta : L^2(G, \mathbb{C}^{2\times 2}) \to L^2(G, \mathbb{C}^{2\times 2})$ by*

$$\Theta \colon \mathbf{f} \mapsto \begin{bmatrix} f_{22} & f_{21} \\ f_{12} & f_{11} \end{bmatrix},\ \mathbf{f} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \in L^2(G, \mathbb{C}^{2\times 2}).$$

Then, $\{E_{8m} T_k \Phi_l\}_{l\in\{1,2\},k\in\Lambda,m\in\mathbb{Z}}$ is a matrix-valued 10-tight (Θ, Θ^*)-frame for $L^2(G, \mathbb{C}^{2\times 2})$. In fact, for any $\mathbf{f} \in L^2(G, \mathbb{C}^{2\times 2})$, we have

$$10\|\Theta^*\mathbf{f}\|^2 = 10\|\mathbf{f}\|^2 \le \sum_{l\in\{1,2\}} \sum_{k\in\Lambda, m\in\mathbb{Z}} \|\langle \mathbf{f}, E_{8m} T_k \Phi_l\rangle\|^2 \le 10\|\mathbf{f}\|^2 = 10\|\Theta\mathbf{f}\|^2.$$

Define $\Xi : L^2(G, \mathbb{C}^{2\times 2}) \to L^2(G, \mathbb{C}^{2\times 2})$ *by*

$$\Xi\colon \mathbf{f} \mapsto \begin{bmatrix} 0 & f_{12} \\ 0 & f_{22} \end{bmatrix}, \ \mathbf{f} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \in L^2(G, \mathbb{C}^{2\times 2}).$$

Then, Ξ is a bounded linear operator with adjoint $\Xi^* = \Xi$. Also, Ξ is adjointable with respect to matrix-valued inner product on $L^2(G, \mathbb{C}^{2\times 2})$. That is, $\langle \Xi\mathbf{f}, \mathbf{g}\rangle = \langle \mathbf{f}, \Xi^*\mathbf{g}\rangle$, $\mathbf{f}, \mathbf{g} \in L^2(G, \mathbb{C}^{2\times 2})$. However, $\{\Xi E_{8m} T_k \Phi_l\}_{l\in\{1,2\},k\in\Lambda, m\in\mathbb{Z}}$ is not a $(\Xi\Theta, (\Xi\Theta)^*)$-frame. If possible, let $\{\Xi E_{8m} T_k \Phi_l\}_{l\in\{1,2\},k\in\Lambda,m\in\mathbb{Z}}$ be a $(\Xi\Theta, (\Xi\Theta)^*)$-frame with bounds γ, δ. Then, for $\mathbf{f}_o = \begin{bmatrix} 0 & f \\ 0 & f \end{bmatrix}$, where f is a non-zero function in $L^2(G)$, we have

$$\sum_{l\in\{1,2\}} \sum_{k\in\Lambda, m\in\mathbb{Z}} \left\| \left\langle \Xi E_{8m} T_k \Phi_l, \mathbf{f}_o \right\rangle \right\|^2 = 20\|f\|^2 > 0 = \delta \|\Xi\Theta\mathbf{f}_o\|^2,$$

which is a contradiction.

In the following result, we give some additional conditions on Ξ so that $\Xi(\mathcal{G}(C, B, \Phi_{\Lambda_0}))$ becomes a $(\Xi\Theta, (\Xi\Theta)^*)$-frame. This result can be seen as a generalization of Proposition 3.7.

Proposition 3.12. *Let $\{E_{Cm} T_{Bk} \Phi_l\}_{l\in\Lambda_0, k\in\Lambda, m\in\Lambda'}$ be a matrix-valued (Θ, Θ^*)-Gabor frame for $L^2(G, \mathbb{C}^{n\times n})$ with frame bounds γ and δ. Suppose*

(i) $\Xi \in \mathcal{B}(L^2(G, \mathbb{C}^{n\times n}))$ is adjointable with respect to matrix-valued inner product.
(ii) Ξ is hyponormal on $Ran(\Theta)$ such that $\Theta\Xi^* = \Xi^*\Theta$.

Then, $\{\Xi E_{Cm} T_{Bk} \Phi_l\}_{l\in\Lambda_0,k\in\Lambda,m\in\Lambda'}$ is a $(\Xi\Theta, (\Xi\Theta)^*)$-frame for $L^2(G, \mathbb{C}^{n\times n})$ with the same frame bounds.

Proof. For any $\mathbf{f} \in L^2(G, \mathbb{C}^{n\times n})$, we have

$$\begin{aligned} \sum_{l\in\Lambda_0} \sum_{k\in\Lambda, m\in\Lambda'} \left\| \left\langle \Xi E_{Cm} T_{Bk} \Phi_l, \mathbf{f} \right\rangle \right\|^2 &= \sum_{l\in\Lambda_0} \sum_{k\in\Lambda, m\in\Lambda'} \left\| \left\langle E_{Cm} T_{Bk} \Phi_l, \Xi^*\mathbf{f} \right\rangle \right\|^2 \\ &\leq \delta \|\Theta\Xi^*\mathbf{f}\|^2 \\ &= \delta \|\Xi^*\Theta\mathbf{f}\|^2 \\ &\leq \delta \|\Xi\Theta\mathbf{f}\|^2. \end{aligned} \tag{6}$$

Similarly

$$\begin{aligned} \sum_{l\in\Lambda_0} \sum_{k\in\Lambda, m\in\Lambda'} \left\| \left\langle \Xi E_{Cm} T_{Bk} \Phi_l, \mathbf{f} \right\rangle \right\|^2 &= \sum_{l\in\Lambda_0} \sum_{k\in\Lambda, m\in\Lambda'} \left\| \left\langle E_{Cm} T_{Bk} \Phi_l, \Xi^*\mathbf{f} \right\rangle \right\|^2 \\ &\geq \gamma \|\Theta^*\Xi^*\mathbf{f}\|^2 \\ &= \gamma \|(\Xi\Theta)^*\mathbf{f}\|^2 \ \mathbf{f} \in L^2(G, \mathbb{C}^{n\times n}). \end{aligned} \tag{7}$$

By (6) and (7), we conclude that $\{\Xi E_{Cm} T_{Bk} \Phi_l\}_{l\in\Lambda_0,k\in\Lambda,m\in\Lambda'}$ is a matrix-valued $(\Xi\Theta, (\Xi\Theta)^*)$-frame for $L^2(G, \mathbb{C}^{n\times n})$ with frame bounds γ and δ. This completes the proof. □

Remark 3.13. *The condition that the operator Θ commutes with Ξ^* in Theorem 3.12 cannot be relaxed. Consider the operators Θ, Ξ defined on $L^2(G, \mathbb{C}^{2\times 2})$ and the system $\{E_{8m} T_k \Phi_l\}_{l\in\{1,2\},k\in\Lambda,m\in\mathbb{Z}}$ which is a (Θ, Θ^*)-Gabor frame for $L^2(G, \mathbb{C}^{2\times 2})$ given in Example 3.11. As mentioned in Example 3.11, the operator Ξ is adjointable with respect to matrix-valued inner product, and Ξ is hyponormal on $Ran(\Theta)$ since $\Xi^* = \Xi$. But, $\Theta\Xi^* \neq \Xi^*\Theta$. In fact, for any $\mathbf{f} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \in L^2(G, \mathbb{C}^{2\times 2})$, we have*

$$\Theta\Xi^*\mathbf{f} = \Theta\Xi\mathbf{f} = \begin{bmatrix} f_{22} & 0 \\ f_{12} & 0 \end{bmatrix} \quad \text{and} \quad \Xi^*\Theta\mathbf{f} = \Xi\Theta\mathbf{f} = \begin{bmatrix} 0 & f_{21} \\ 0 & f_{11} \end{bmatrix}.$$

Therefore, the system $\{\Xi E_{8m} T_k \Phi_l\}_{l\in\{1,2\},k\in\Lambda,m\in\mathbb{Z}}$ not being a $(\Xi\Theta, (\Xi\Theta)^*)$-frame, details in Example 3.11, supports our argument.

Next, we give a characterization for matrix-valued (Θ, Θ^*)-Gabor frames in $L^2(G, \mathbb{C}^{n\times n})$. This is inspired by a fundamental result due to Găvruta in [11, Theorem 4] for ordinary K-frames in separable Hilbert spaces. This is also related with the concept of atomic systems in Hilbert spaces. The matrix-valued atomic system in matrix-valued function spaces can be studied in terms of (Θ, Θ^*)-Gabor frames.

Theorem 3.14. *Let Θ be a bounded linear operator acting on $L^2(G, \mathbb{C}^{n\times n})$. A matrix-valued Gabor system $\mathcal{G}(C, B, \Phi_{\Lambda_0})$ is a (Θ, Θ^*)-Gabor frame for the space $L^2(G, \mathbb{C}^{n\times n})$ if and only if there exists a bounded linear operator Ω from $\ell^2(\Lambda_0 \times \Lambda \times \Lambda', \mathcal{M}_n(\mathbb{C}))$ into $L^2(G, \mathbb{C}^{n\times n})$ such that*

(*i*) *$E_{Cm}T_{Bk}\Phi_l = \Omega\chi_{l,k,m}$, $l \in \Lambda_0, k \in \Lambda, m \in \Lambda'$, where $\{\chi_{l,k,m}\}_{l\in\Lambda_0, k\in\Lambda, m\in\Lambda'}$ is an orthonormal basis of $\ell^2(\Lambda_0 \times \Lambda \times \Lambda', \mathcal{M}_n(\mathbb{C}))$,*

(*ii*) *there exist finite positive numbers α and β satisfying*

$$\alpha\, tr\langle \Theta\Theta^* \mathbf{f}, \mathbf{f}\rangle \leq tr\langle \Omega\Omega^* \mathbf{f}, \mathbf{f}\rangle \leq \beta\, tr\langle \Theta^*\Theta \mathbf{f}, \mathbf{f}\rangle,\ \mathbf{f} \in L^2(G, \mathbb{C}^{n\times n}).$$

Proof. Suppose first that $\mathcal{G}(C, B, \Phi_{\Lambda_0})$ is a matrix-valued (Θ, Θ^*)-Gabor frame for the space $L^2(G, \mathbb{C}^{n\times n})$ with frame bounds a_o, b_o.

Define $\Xi : L^2(G, \mathbb{C}^{n\times n}) \to \ell^2(\Lambda_0 \times \Lambda \times \Lambda', \mathcal{M}_n(\mathbb{C}))$ by

$$\Xi(\mathbf{f}) = \sum_{l\in\Lambda_0} \sum_{k\in\Lambda, m\in\Lambda'} \langle \mathbf{f}, E_{Cm}T_{Bk}\Phi_l\rangle \chi_{l,k,m},\ \mathbf{f} \in L^2(G, \mathbb{C}^{n\times n}).$$

Then, Ξ is a bounded linear operator and $\|\Xi\| \leq \sqrt{b_o}\|\Theta\|$.

Now, for any $l \in \Lambda_0, k \in \Lambda, m \in \Lambda'$, we have

$$\begin{aligned}
\mathrm{tr}\langle \chi_{l,k,m}, \Xi\mathbf{f}\rangle &= \mathrm{tr}\Big\langle \chi_{l,k,m}, \sum_{l'\in\Lambda_0} \sum_{k'\in\Lambda, m'\in\Lambda'} \langle \mathbf{f}, E_{Cm'}T_{Bk'}\Phi_{l'}\rangle \chi_{l',k',m'}\Big\rangle \\
&= \mathrm{tr}\langle \mathbf{f}, E_{Cm}T_{Bk}\Phi_l\rangle^* \\
&= \mathrm{tr}\langle E_{Cm}T_{Bk}\Phi_l, \mathbf{f}\rangle \text{ for all } \mathbf{f} \in L^2(G, \mathbb{C}^{n\times n}).
\end{aligned}$$

Thus, $\Xi^*\chi_{l,k,m} = E_{Cm}T_{Bk}\Phi_l$, for all $l \in \Lambda_0$, $k \in \Lambda$ and $m \in \Lambda'$. If we take $\Omega = \Xi^*$, then we obtain (i). To prove (ii), let $\mathbf{f} \in L^2(G, \mathbb{C}^{n\times n})$ be arbitrary. Then,

$$a_o\|\Theta^*\mathbf{f}\|^2 \leq \sum_{l\in\Lambda_0} \sum_{k\in\Lambda, m\in\Lambda'} \left\|\langle \mathbf{f}, E_{Cm}T_{Bk}\Phi_l\rangle\right\|^2 = \|\Xi\mathbf{f}\|^2,$$

and

$$\|\Xi\mathbf{f}\|^2 = \sum_{l\in\Lambda_0} \sum_{k\in\Lambda, m\in\Lambda'} \left\|\langle \mathbf{f}, E_{Cm}T_{Bk}\Phi_l\rangle\right\|^2 \leq b_o\|\Theta\mathbf{f}\|^2.$$

This imply that

$$a_o\mathrm{tr}\langle \Theta\Theta^*\mathbf{f}, \mathbf{f}\rangle \leq \mathrm{tr}\langle \Xi^*\Xi\mathbf{f}, \mathbf{f}\rangle = \mathrm{tr}\langle \Omega\Omega^*\mathbf{f}, \mathbf{f}\rangle \leq b_o\mathrm{tr}\langle \Theta^*\Theta\mathbf{f}, \mathbf{f}\rangle.$$

This gives (ii), where $\alpha = a_o$ and $\beta = b_o$.

To prove the converse, assume that conditions (i) and (ii) hold. Then, using condition (i), for any $l \in \Lambda_0, k \in \Lambda, m \in \Lambda'$ and any $\mathbf{f} \in L^2(G, \mathbb{C}^{n\times n})$, we have

$$\begin{aligned}
\mathrm{tr}\langle \chi_{l,k,m}, \Omega^*\mathbf{f}\rangle &= \mathrm{tr}\langle \Omega\chi_{l,k,m}, \mathbf{f}\rangle \\
&= \mathrm{tr}\langle E_{C_m}T_{Bk}\Phi_l, \mathbf{f}\rangle \\
&= \mathrm{tr}\langle \mathbf{f}, E_{Cm}T_{Bk}\Phi_l\rangle^* \\
&= \mathrm{tr}\Big\langle \chi_{l,k,m}, \sum_{l'\in\Lambda_0} \sum_{k'\in\Lambda, m'\in\Lambda'} \langle \mathbf{f}, E_{Cm'}T_{Bk'}\Phi_{l'}\rangle \chi_{l',k',m'}\Big\rangle,
\end{aligned}$$

which entails

$$\Omega^*\mathbf{f} = \sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'} \langle \mathbf{f}, E_{Cm}T_{Bk}\Phi_l\rangle \chi_{l,k,m}, \ \mathbf{f} \in L^2(G, \mathbb{C}^{n\times n}).$$

Therefore

$$\|\Omega^*\mathbf{f}\|^2 = \sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'} \left\|\langle \mathbf{f}, E_{Cm}T_{Bk}\Phi_l\rangle\right\|^2, \ \mathbf{f} \in L^2(G, \mathbb{C}^{n\times n}).$$

Using condition (ii), we have $\alpha\|\Theta^*\mathbf{f}\|^2 \leq \|\Omega^*\mathbf{f}\|^2 = \sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'} \left\|\langle \mathbf{f}, E_{Cm}T_{Bk}\Phi_l\rangle\right\|^2 \leq \beta\|\Theta\mathbf{f}\|^2$ for all $\mathbf{f} \in L^2(G, \mathbb{C}^{n\times n})$. Hence, $\mathcal{G}(C, B, \Phi_{\Lambda_0})$ is a matrix-valued (Θ, Θ^*)-Gabor frame for $L^2(G, \mathbb{C}^{n\times n})$. This completes the proof. □

4. Perturbation of (Θ, Θ^*)-Gabor Frames

In this section, we show that matrix-valued (Θ, Θ^*)-Gabor frames are stable under small perturbation. Perturbation theory plays a significant role in both pure mathematics and applied science, see e.g. [22]. For applications of perturbation theory for frames in various directions, we refer to [14]. The following result shows that multivariate (Θ, Θ^*)-Gabor frames in matrix-valued signal spaces are stable under small perturbations.

Theorem 4.1. *Let $\mathcal{G}(C, B, \Phi_{\Lambda_0})$ be a matrix-valued (Θ, Θ^*)-Gabor frame for $L^2(G, \mathbb{C}^{n\times n})$ with frame bounds γ_o, δ_o, and let $\{\widetilde{\Phi}_l\}_{l\in\Lambda_0} \subset L^2(G, \mathbb{C}^{n\times n})$. Assume that*

- (i) *Θ^* be bounded below by m_o.*
- (ii) *λ, μ, $\eta \geq 0$ be such that $\frac{(1-2\lambda)\gamma_o - 2\mu}{2\eta} > \frac{\|\Theta\|^2}{m_o^2}$.*
- (iii) *For all $\mathbf{f} \in L^2(G, \mathbb{C}^{n\times n})$,*

$$\begin{aligned}\sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'} \|\langle \mathbf{f}, E_{Cm}T_{Bk}(\Phi_l - \widetilde{\Phi}_l)\rangle\|^2 &\leq \lambda \sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'} \|\langle \mathbf{f}, E_{Cm}T_{Bk}\Phi_l\rangle\|^2 \\ &\quad + \mu\|\Theta^*\mathbf{f}\|^2 + \eta\|\Theta\mathbf{f}\|^2. \end{aligned} \tag{8}$$

Then, $\mathcal{G}(C, B, \widetilde{\Phi}_{\Lambda_0})$ is a matrix-valued (Θ, Θ^*)-Gabor frame for $L^2(G, \mathbb{C}^{n\times n})$ with frame bounds

$$\left(\left(\frac{1}{2} - \lambda\right)\gamma_o - \mu - \frac{\eta\|\Theta\|^2}{m_o^2}\right) \text{ and } 2\left(\left(1 + \lambda + \frac{\mu}{\gamma_o}\right)\delta_o + \eta\right).$$

Proof. By hypothesis (8), for any $\mathbf{f} \in L^2(G, \mathbb{C}^{n\times n})$, we have

$$\begin{aligned}
&\sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'} \|\langle \mathbf{f}, E_{Cm}T_{Bk}\widetilde{\Phi}_l\rangle\|^2 \\
&\leq 2\sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'} \|\langle \mathbf{f}, E_{Cm}T_{Bk}\Phi_l - E_{Cm}T_{Bk}\widetilde{\Phi}_l\rangle\|^2 + 2\sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'} \|\langle \mathbf{f}, E_{Cm}T_{Bk}\Phi_l\rangle\|^2 \\
&\leq (2\lambda + 2)\sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'} \|\langle \mathbf{f}, E_{Cm}T_{Bk}\Phi_l\rangle\|^2 + 2\mu\|\Theta^*\mathbf{f}\|^2 + 2\eta\|\Theta\mathbf{f}\|^2 \\
&\leq (2\lambda + 2)\sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'} \|\langle \mathbf{f}, E_{Cm}T_{Bk}\Phi_l\rangle\|^2 + \frac{2\mu}{\gamma_o}\sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'} \|\langle \mathbf{f}, E_{Cm}T_{Bk}\Phi_l\rangle\|^2 + 2\eta\|\Theta\mathbf{f}\|^2.
\end{aligned}$$

Therefore

$$\sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'}\|\langle \mathbf{f}, E_{Cm}T_{Bk}\widetilde{\Phi}_l\rangle\|^2 \leq 2\left(\left(1+\lambda+\frac{\mu}{\gamma_o}\right)\delta_o+\eta\right)\|\Theta\mathbf{f}\|^2,\ \mathbf{f}\in L^2(G,\mathbb{C}^{n\times n}). \tag{9}$$

Similarly,

$$\begin{aligned}
&\sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'}\|\langle \mathbf{f}, E_{Cm}T_{Bk}\Phi_l\rangle\|^2\\
&\leq 2\sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'}\|\langle \mathbf{f}, E_{Cm}T_{Bk}\Phi_l - E_{Cm}T_{Bk}\widetilde{\Phi}_l\rangle\|^2 + 2\sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'}\|\langle \mathbf{f}, E_{Cm}T_{Bk}\widetilde{\Phi}_l\rangle\|^2\\
&\leq 2\lambda\sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'}\|\langle \mathbf{f}, E_{Cm}T_{Bk}\Phi_l\rangle\|^2 + +2\mu\|\Theta^*\mathbf{f}\|^2 + 2\eta\|\Theta\mathbf{f}\|^2 + 2\sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'}\|\langle \mathbf{f}, E_{Cm}T_{Bk}\widetilde{\Phi}_l\rangle\|^2,
\end{aligned}$$

which entails

$$\begin{aligned}
&2\sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'}\|\langle \mathbf{f}, E_{Cm}T_{Bk}\widetilde{\Phi}_l\rangle\|^2\\
&\geq (1-2\lambda)\sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'}\|\langle \mathbf{f}, E_{Cm}T_{Bk}\Phi_l\rangle\|^2 - 2\mu\|\Theta^*\mathbf{f}\|^2 - 2\eta\|\Theta\mathbf{f}\|^2\\
&\geq (1-2\lambda)\gamma_o\|\Theta^*\mathbf{f}\|^2 - 2\mu\|\Theta^*\mathbf{f}\|^2 - 2\eta\|\Theta\|^2\|\mathbf{f}\|^2\\
&\geq (1-2\lambda)\gamma_o\|\Theta^*\mathbf{f}\|^2 - 2\mu\|\Theta^*\mathbf{f}\|^2 - \frac{2\eta\|\Theta\|^2}{m_o^2}\|\Theta^*\mathbf{f}\|^2 \quad \big(\text{using hypothesis (i)}\big)
\end{aligned}$$

That is

$$\sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'}\|\langle \mathbf{f}, E_{Cm}T_{Bk}\widetilde{\Phi}_l\rangle\|^2 \geq \left(\left(\frac{1}{2}-\lambda\right)\gamma_o - \mu - \frac{\eta\|\Theta\|^2}{m_o^2}\right)\|\Theta^*\mathbf{f}\|^2 \tag{10}$$

for all $\mathbf{f}\in L^2(G,\mathbb{C}^{n\times n})$. From (9) and (10), we conclude that $\mathcal{G}(C,B,\widetilde{\Phi}_{\Lambda_0})$ is a frame for $L^2(G,\mathbb{C}^{n\times n})$ with the desired frame bounds. □

Next is an applicative example of Theorem 4.1.

Example 4.2. *Let $\{E_{8m}T_k\Phi_l\}_{l\in\{1,2\},k\in\Lambda,m\in\mathbb{Z}}$ be the 10-tight matrix-valued Gabor frame for $L^2(G,\mathbb{C}^{2\times 2})$ given in Remark 3.3. Define Θ on $L^2(G,\mathbb{C}^{2\times 2})$ by*

$$\Theta:\mathbf{f}\mapsto\begin{bmatrix}2f_{22} & f_{21}\\ f_{12} & f_{11}\end{bmatrix},\ \mathbf{f}=\begin{bmatrix}f_{11} & f_{12}\\ f_{21} & f_{22}\end{bmatrix}\in L^2(G,\mathbb{C}^{2\times 2}).$$

Then, Θ is a bounded linear operator satisfying $\|\Theta\mathbf{f}\|\leq 2\|\mathbf{f}\|$, for all $\mathbf{f}\in L^2(G,\mathbb{C}^{2\times 2})$. In fact, we have $\|\Theta\|=2$. For any $\mathbf{f}$, $\mathbf{g}\in L^2(G,\mathbb{C}^{2\times 2})$, we have

$$\begin{aligned}
tr\langle\Theta\mathbf{f},\mathbf{g}\rangle &= tr\int_G\begin{bmatrix}2f_{22} & f_{21}\\ f_{12} & f_{11}\end{bmatrix}\begin{bmatrix}\overline{g_{11}} & \overline{g_{21}}\\ \overline{g_{12}} & \overline{g_{22}}\end{bmatrix}d\mu_G\\
&= tr\int_G\begin{bmatrix}f_{11} & f_{12}\\ f_{21} & f_{22}\end{bmatrix}\begin{bmatrix}\overline{g_{22}} & \overline{g_{12}}\\ \overline{g_{21}} & 2\,\overline{g_{11}}\end{bmatrix}d\mu_G,
\end{aligned}$$

which implies that Θ^* is given by

$$\Theta^*:\mathbf{g}\mapsto\begin{bmatrix}g_{22} & g_{21}\\ g_{12} & 2g_{11}\end{bmatrix},\ \mathbf{g}=\begin{bmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{bmatrix}\in L^2(G,\mathbb{C}^{2\times 2}).$$

It can be easily seen that Θ^* satisfies $\|\mathbf{f}\| \leq \|\Theta^*\mathbf{f}\| \leq 2\|\mathbf{f}\|$, for all $\mathbf{f} \in L^2(G, \mathbb{C}^{2\times 2})$. That is, Θ^* is bounded below by $m_o = 1$.

Now, for any $\mathbf{f} \in L^2(G, \mathbb{C}^{2\times 2})$, we have $\frac{5}{2}\|\Theta^*\mathbf{f}\|^2 \leq \sum_{l\in\{1,2\}}\sum_{k\in\Lambda,m\in\mathbb{Z}}\left\|\left\langle E_{8m}T_k\Phi_l, \mathbf{f}\right\rangle\right\|^2 = 10\|\mathbf{f}\|^2 \leq 10\|\Theta\mathbf{f}\|^2$. Therefore, $\{E_{8m}T_k\Phi_l\}_{l\in\{1,2\},k\in\Lambda,m\in\mathbb{Z}}$ is a matrix-valued (Θ,Θ^*)-Gabor frame for $L^2(G,\mathbb{C}^{2\times 2})$ with frame bounds $\gamma_1 = \frac{5}{2}$ and $\delta_1 = 10$.

Consider $\widetilde{\Phi_1} = \begin{bmatrix} \frac{1}{5}\phi_1 & \phi_1 \\ \phi_2 & \frac{1}{5}\phi_2 \end{bmatrix}$, $\widetilde{\Phi_2} = \begin{bmatrix} \frac{1}{5}\phi_2 & \phi_2 \\ \phi_1 & \frac{1}{5}\phi_1 \end{bmatrix}$ in $L^2(G,\mathbb{C}^{2\times 2})$. Then, for any $\mathbf{f} = \left[f_{ij}\right]_{1\leq i,j\leq 2} \in L^2(G,\mathbb{C}^{2\times 2})$, we have

$$\begin{aligned}
&\sum_{l\in\{1,2\}}\sum_{k\in\Lambda,m\in\mathbb{Z}}\|\langle \mathbf{f}, E_{8m}T_k\Phi_l - E_{8m}T_k\widetilde{\Phi_l}\rangle\|^2 \\
&= \frac{1}{25}\sum_{l\in\{1,2\}}\sum_{k\in\Lambda,m\in\mathbb{Z}}\Big(\Big|\int_G E_{8m}T_k\phi_l\overline{f_{11}}\Big|^2 + \Big|\int_G E_{8m}T_k\phi_l\overline{f_{21}}\Big|^2 + \Big|\int_G E_{8m}T_k\phi_l\overline{f_{12}}\Big|^2 + \Big|\int_G E_{8m}T_k\phi_l\overline{f_{22}}\Big|^2\Big) \\
&= \frac{10}{25}\|f\|^2 \leq \frac{1}{5}\|\Theta^*\mathbf{f}\|^2 + \frac{1}{5}\|\Theta\mathbf{f}\|^2.
\end{aligned}$$

Thus, all the conditions in Theorem 4.1 are satisfied with $\lambda = 0, \mu = \frac{1}{5}, \eta = \frac{1}{5}$. Hence, the collection $\{E_{8m}T_k\widetilde{\Phi_l}\}_{l\in\{1,2\},k\in\Lambda,m\in\mathbb{Z}}$ is a matrix-valued (Θ,Θ^*)-Gabor frame for $L^2(G,\mathbb{C}^{2\times 2})$.

Theorem 4.1 shows that a matrix-valued Gabor system $\mathcal{G}(C,B,\widetilde{\Phi}_{\Lambda_0})$ becomes a (Θ,Θ^*)-Gabor frame for $L^2(G,\mathbb{C}^{n\times n})$ if its window functions $\widetilde{\Phi}_l, l \in \Lambda_0$ are sufficiently close to the window functions $\Phi_l, l \in \Lambda_0$ of a matrix-valued (Θ,Θ^*)-Gabor frame $\mathcal{G}(C,B,\Phi_{\Lambda_0})$. This can also be seen as a way of constructing new matrix-valued (Θ,Θ^*)-Gabor frames by altering the window functions of a known matrix-valued (Θ,Θ^*)-Gabor frame appropriately. In the direction of obtaining new matrix-valued (Θ,Θ^*)-Gabor frames from known matrix-valued (Θ,Θ^*)-Gabor frames, we give the following result which states that the perturbed matrix-valued Gabor systems (Θ,Θ^*)-Gabor frames, under suitable conditions, becomes a matrix-valued (Θ,Θ^*)-Gabor frame for $L^2(G,\mathbb{C}^{n\times n})$.

Theorem 4.3. *Let $\mathcal{G}(C,B,\Phi_{\Lambda_0})$ and $\mathcal{G}(C,B,\Psi_{\Lambda_0})$ be matrix-valued (Θ,Θ^*)-Gabor frames for $L^2(G,\mathbb{C}^{n\times n})$ with frame bounds γ_1, δ_1 and γ_2, δ_2, respectively. Suppose Θ^* is bounded below with constant m_o such that $\sqrt{\frac{\gamma_1}{\delta_2}} > \frac{\|\Theta\|}{m_o}$. Then, the perturbed matrix-valued Gabor system $\mathcal{G}\Big(C,B,(\Phi_{\Lambda_0}+\Psi_{\Lambda_0})\Big)$ is a (Θ,Θ^*)-Gabor frame for $L^2(G,\mathbb{C}^{n\times n})$ with frame bounds $\left(\sqrt{\gamma_1} - \frac{\sqrt{\delta_2}\|\Theta\|}{m_o}\right)^2$ and $2(\delta_1+\delta_2)$.*

Proof. For any $\mathbf{f} \in L^2(G,\mathbb{C}^{n\times n})$, we compute

$$\begin{aligned}
&\Big(\sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'}\|\langle \mathbf{f}, E_{Cm}T_{Bk}(\Phi_l+\Psi_l)\rangle\|^2\Big)^{\frac{1}{2}} \\
&= \Big(\sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'}\|\langle \mathbf{f}, E_{Cm}T_{Bk}\Phi_l + E_{Cm}T_{Bk}\Psi_l\rangle\|^2\Big)^{\frac{1}{2}} \\
&\geq \Big(\sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'}\|\langle \mathbf{f}, E_{Cm}T_{Bk}\Phi_l\rangle\|^2\Big)^{\frac{1}{2}} - \Big(\sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'}\|\langle \mathbf{f}, E_{Cm}T_{Bk}\Psi_l\rangle\|^2\Big)^{\frac{1}{2}} \\
&\geq \sqrt{\gamma_1}\|\Theta^*\mathbf{f}\| - \sqrt{\delta_2}\|\Theta\mathbf{f}\| \\
&\geq \sqrt{\gamma_1}\|\Theta^*\mathbf{f}\| - \sqrt{\delta_2}\|\Theta\|\|\mathbf{f}\| \\
&\geq \sqrt{\gamma_1}\|\Theta^*\mathbf{f}\| - \frac{\sqrt{\delta_2}\|\Theta\|}{m_o}\|\Theta^*\mathbf{f}\|.
\end{aligned}$$

This gives

$$\sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'}\|\langle\mathbf{f}\,,E_{Cm}T_{Bk}\Phi_l+E_{Cm}T_{Bk}\Psi_l\rangle\|^2\geq\Big(\sqrt{\gamma_1}-\frac{\sqrt{\delta_2}\|\Theta\|}{m_o}\Big)^2\|\Theta^*\mathbf{f}\|^2 \text{ for all } \mathbf{f}\in L^2(G,\mathbb{C}^{n\times n}). \tag{11}$$

Similarly,

$$\begin{aligned}&\sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'}\|\langle\mathbf{f}\,,E_{Cm}T_{Bk}(\Phi_l+\Psi_l)\rangle\|^2\\&\leq 2\Big(\sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'}\|\langle\mathbf{f}\,,E_{Cm}T_{Bk}\Phi_l\rangle\|^2+\sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'}\|\langle\mathbf{f}\,,E_{Cm}T_{Bk}\Psi_l\rangle\|^2\Big)\\&\leq 2(\delta_1+\delta_2)\|\Theta\mathbf{f}\|^2 \text{ for all } \mathbf{f}\in L^2(G,\mathbb{C}^{n\times n}).\end{aligned}\tag{12}$$

From (11) and (12), we conclude that $\mathcal{G}\Big(C,B,(\Phi_{\Lambda_0}+\Psi_{\Lambda_0})\Big)$ is a (Θ,Θ^*)-Gabor frame for $L^2(G,\mathbb{C}^{n\times n})$ with the desired frame bounds. This completes the proof. □

We end this paper by providing an application of Theorem 4.3.

Example 4.4. *Consider the (Θ,Θ^*)-Gabor frame $\{E_{8m}T_k\Phi_l\}_{l\in\{1,2\},k\in\Lambda,m\in\mathbb{Z}}$ for $L^2(G,\mathbb{C}^{2\times 2})$ with frame bounds $\gamma_1=\frac{5}{2}$ and $\delta_1=10$ given in Example 4.2.*

Let $\Psi_1=\begin{bmatrix}\frac{1}{5}\phi_1 & 0\\ 0 & \frac{1}{5}\phi_2\end{bmatrix}$, $\Psi_2=\begin{bmatrix}\frac{1}{5}\phi_2 & 0\\ 0 & \frac{1}{5}\phi_1\end{bmatrix}$. Then, $\Psi_1,\Psi_2\in L^2(G,\mathbb{C}^{2\times 2})$, and for any $\mathbf{f}\in L^2(G,\mathbb{C}^{2\times 2})$, we have

$$\begin{aligned}&\sum_{l\in\{1,2\}}\sum_{k\in\Lambda,m\in\mathbb{Z}}\|\langle\mathbf{f},E_{8m}T_k\Psi_l\rangle\|^2\\&=\frac{1}{25}\sum_{l\in\{1,2\}}\sum_{k\in\Lambda,m\in\mathbb{Z}}\Big(\Big|\int_G E_{8m}T_k\phi_l\overline{f_{11}}\Big|^2+\Big|\int_G E_{8m}T_k\phi_l\overline{f_{21}}\Big|^2+\Big|\int_G E_{8m}T_k\phi_l\overline{f_{12}}\Big|^2+\Big|\int_G E_{8m}T_k\phi_l\overline{f_{22}}\Big|^2\Big)\\&=\frac{10}{25}\|f\|^2\\&\leq\frac{2}{5}\|\Theta f\|^2.\end{aligned}$$

Also

$$\sum_{l\in\{1,2\}}\sum_{k\in\Lambda,m\in\mathbb{Z}}\|\langle\mathbf{f},E_{8m}T_k\Psi_l\rangle\|^2=\frac{2}{5}\|f\|^2\geq\frac{1}{10}\|\Theta^*f\|^2,\quad \mathbf{f}\in L^2(G,\mathbb{C}^{2\times 2}).$$

Thus, $\{E_{8m}T_k\Psi_l\}_{l\in\{1,2\},k\in\Lambda,m\in\mathbb{Z}}$ is a (Θ,Θ^*)-Gabor frame for $L^2(G,\mathbb{C}^{2\times 2})$ with frame bounds $\gamma_2=\frac{1}{10}$ and $\delta_2=\frac{2}{5}$. Further, Θ^* is bounded below by $m_o=1$ and $\frac{5}{2}=\sqrt{\frac{\gamma_1}{\delta_2}}>\frac{\|\Theta\|}{m_o}=2$. Hence, by Theorem 4.3, the perturbed matrix-valued Gabor system $\{E_{8m}T_k(\Phi_l+\Psi_l)\}_{l\in\{1,2\},k\in\Lambda,m\in\mathbb{Z}}$ is a (Θ,Θ^*)-Gabor frame for $L^2(G,\mathbb{C}^{2\times 2})$ with frame bounds

$$\left(\sqrt{\gamma_1}-\frac{\sqrt{\delta_2}\|\Theta\|}{m_o}\right)^2\quad \textit{and}\quad 2(\delta_1+\delta_2).$$

Remark 4.5. *Theorem 4.1 and Theorem 4.3 are not only ways of constructing new frames but also can be used to check if a matrix-valued Gabor system $\mathcal{G}(C,B,\widetilde{\Phi}_{\Lambda_0})$ is a (Θ,Θ^*)-Gabor frame for $L^2(G,\mathbb{C}^{n\times n})$, where the window functions $\widetilde{\Phi}_l, l\in\Lambda_0$ have complex structure leading to complicated calculations. In order to understand this better, we compare Example 4.2 and Example 4.4. In Example 4.2, to prove matrix-valued (Θ,Θ^*)-Gabor frame conditions of $\{E_{8m}T_k\widetilde{\Phi}_l\}_{l\in\{1,2\},k\in\Lambda,m\in\mathbb{Z}}$, a (Θ,Θ^*)-Gabor frame $\{E_{8m}T_k\Phi_l\}_{l\in\{1,2\},k\in\Lambda,m\in\mathbb{Z}}$ having simpler window functions is considered. However, in Example 4.4, $\Phi_l+\Psi_l=\widetilde{\Phi}_l$, $l\in\Lambda_0$. Hence, Example 4.4 can be seen as a method by which the collection $\{E_{8m}T_k\widetilde{\Phi}_l\}_{l\in\{1,2\},k\in\Lambda,m\in\mathbb{Z}}$ is proved to be a (Θ,Θ^*)-Gabor frame by splitting its window functions as a sum of the window functions (perturbed window functions) of two (Θ,Θ^*)-Gabor frames.*

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